LIVING to 100

SOCIETY OF ACTUARIES INTERNATIONAL SYMPOSIUM

2020 Symposium
Jan. 13–15
Lake Buena Vista, FL

3A – Aging Measurement and Mortality Modeling 1

New Developments in Mortality Modeling - Discussion

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3A: Aging Measurement and Mortality Modeling 1

January 13, 2020

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Two Excellent Papers

- Congratulations to authors
  - Mathematically challenging
- Will discuss separately
- Longevity Risk
  - Aggregate vs individual

Huang – Affine Mortality Models

- Continuous annuity: $$\bar{a}_x = \int_0^{\omega} e^{-\mu_{x+t}-\delta t}\, dt$$

- Both μ and δ are complex stochastic processes
- Need to model both to capture risk

Arbitrage-Free Nelson-Siegel model

- Continuous time force of mortality model
- Closed form
  - Components for Level, Slope, and Curvature
  - But otherwise not very intuitive
- Consistent modeling of mortality & interest
  - Appropriate for aggregate longevity risk
- “Winner” in out of sample forecasting (SMAPE)

Cohort Mortality Model

- Advantages
  - Mortality patterns likely to be more related
  - More relevant for long term obligations
    - Annuities, pensions
- Disadvantages:
  - Complete cohorts were born a long time ago
    - Out of sample data was 1916 birth cohort
  - How relevant to current pensioner mortality?

Huynh – Multi-population Model

- Many insurance applications!
  - Male versus female mortality
  - Preferred versus standard mortality
    - And multi-level preferred
  - Company mortality versus industry mortality
  - Insured mortality versus population mortality
- Lengthscale parameter θ determines closeness

Coherent mortality improvement models

- See heat maps on slide 15
  - Color scales are different
- Would have been nice for 2015 VBT development

But Wait!

- Computationally expensive
  - Only used ages 70 – 84 to fit models
  - Suggested limit of 5 populations
- Covariance stationarity constrains shape
  - Suggested modeling age groups separately
  - Implicit Gompertz-Makeham assumption?

Congratulations Again to Authors!

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The Application of Affine Processes in Cohort Mortality Risk Models

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Hilton Orlando Lake Buena Vista
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Introduction

- Mortality models have attracted research attention over recent years, particularly discrete time mortality models (Lee and Carter (1992), Cairns et al. (2006b), Cairns et al. (2009), Renshaw and Haberman (2006)).
    - focus on improvement trends,
    - impact of uncertainty or volatility of mortality, and
    - cohort effects.
- Continuous time affine cohort mortality models have attracted more recent research
    - single cohort models (Milevsky and Promislow (2001), Dahl and Møller (2006), Biffis (2005), Luciano et al. (2008), Schrager (2006), Cairns et al. (2006a), Blackburn and Sherris (2013))
    - multi-cohort models (Jevtic et al. (2013), Xu et al. (2019a), Chang and Sherris (2018), Huang et al. (2019)).

Why Continuous Time Affine Models?

- Analytical tractability - closed form survival curves for affine class,
- Consistency between mortality dynamics and functional form of the survival curve,
- Stability of parameter estimates,
- Use of mathematical finance methods for term structure and credit risk models familiar to financial market participants,
- Natural extensions to multi-factor models, capturing differing trends, volatility and correlations by age,
- Arbitrage-free formulation along with real world dynamics to allow calibration of prices of risk.

Coverage of this Presentation

- Introduce continuous-time affine cohort mortality models - closed-form expressions for survival curves, dynamics of mortality rates, AFNS mortality models with factors of level, slope and curvature for the mortality curve,
- Fitting with age-cohort data,
- Kalman filter and estimation of the models, highlighting how Poisson variation can be incorporated into the model estimation,
- Comparison of fits and cohort survival curve prediction using historical US mortality data.

Survival Curve - Continuous Time Affine Mortality Model

- Drawing on term structure of interest rate models - equivalence of average force of mortality rate to yield to maturity for zero coupon bond. Use of similar notation as in yield curve modelling.
- Survival probability $S(x, t, T)$ for single cohort aged x at time t for survival for a duration $(T - t)$ to age $x + (T - t)$, as an affine function of (latent) factors (3 factor case)

$$
\begin{aligned}
S(x, t, T) &= E[e^{-\int_t^T \mu^i(x,s)ds} | \mathcal{F}_t] \\
&= e^{-\bar{\mu}(t,T)(T-t)} \\
&= e^{B_1(t,T)X_1(t)+B_2(t,T)X_2(t)+B_3(t,T)X_3(t)+A(t,T)}, \qquad (1)
\end{aligned}
$$

- $B_j(t, T)$ are factor loadings (functional form derived from mortality dynamics for the latent factors, exponential terms) and $X_j(t)$ are the latent factors (stochastic parameters).

Mortality Rate - Continuous Time Affine Mortality Model

- Average mortality rate - age-period data or age-cohort data

$$\bar{\mu}(t, T) = -\frac{1}{T-t}\log\left[S(t, T)\right] = -\frac{B(t, T)'}{T-t}X_t - \frac{A(t, T)}{T-t}. \quad (2)$$

  where vector $B(t, T)$, the factor loadings, and $A(t, T)$ have explicit expressions (derivations similar to term structure models).

- Canonical form for these (Blackburn and Sherris, 2013), where δ_{jj} and σ_{jj} are parameters in the latent factor dynamics (estimated from historical data)

$$B_j(t, T) = -\frac{1 - e^{-\delta_{jj}(T-t)}}{\delta_{jj}}, \quad j = 1, 2, 3, \quad (3)$$

$$A(t, T) = \frac{1}{2}\sum_{j=1}^{3}\frac{\sigma_{jj}^2}{\delta_{jj}^3}\left[\frac{1}{2}\left(1 - e^{-2\delta_{jj}(T-t)}\right) - 2\left(1 - e^{-\delta_{jj}(T-t)}\right) + \delta_{jj}(T-t)\right]. \quad (4)$$

Mortality Rate - AFNS Mortality model

- Mortality model equivalent of the Nelson-Seigel term structure model in arbitrage-free dynamic implementation (Christensen et al., 2011)
- Independent AFNS mortality model - mortality rate curve has level, slope and curvature factors.

$$\begin{aligned} B^1(t,T) &= -(T-t), \quad B^2(t,T) = -\frac{1-e^{-\delta(T-t)}}{\delta}, \\ B^3(t,T) &= (T-t)e^{-\delta(T-t)} - \frac{1-e^{-\delta(T-t)}}{\delta}, \end{aligned} \tag{5}$$

$$\begin{aligned} \frac{A(t,T)}{T-t} &= \sigma_{11}^2 \frac{(T-t)}{6} + \sigma_{22}^2 \left[\frac{1}{2\delta^2} - \frac{1}{\delta^3} \frac{1-e^{-\delta(T-t)}}{T-t} + \frac{1}{4\delta^3} \frac{1-e^{-2\delta(T-t)}}{T-t} \right] + \\ & \sigma_{33}^2 \left[\frac{1}{2\delta^2} + \frac{1}{\delta^2} e^{-\delta(T-t)} - \frac{1}{4\delta}(T-t)e^{-2\delta(T-t)} - \frac{3}{4\delta^2} e^{-2\delta(T-t)} \right. \\ & \left. - \frac{2}{\delta^3} \frac{1-e^{-\delta(T-t)}}{T-t} + \frac{5}{8\delta^3} \frac{1-e^{-2\delta(T-t)}}{T-t} \right]. \end{aligned} \tag{6}$$

Affine Mortality Models - Q Pricing measure

- The price of a longevity zero-coupon bond on a specific cohort currently aged x is

$$\begin{aligned}\bar{P}_x(t,T) &= E^Q\left[e^{-\int_t^T (r(s)+\mu(x,s))ds}|\mathcal{F}(t)\right] \\ &= E^Q\left[e^{-\int_t^T r(s)ds}|\mathcal{G}(t)\right] E^Q\left[e^{-\int_t^T \mu(x,s)ds}|\mathcal{H}(t)\right] \\ &= P(t,T)S^Q(x,t,T), \end{aligned} \tag{7}$$

  where the dynamics of the mortality rates and the dynamics of the interest rates are independent.

- Dynamics of the mortality rate used to derive the (risk-neutral) survival probability $S^Q(x,t,T)$.
- Use the same methodology of term structure modelling applied to (pricing) survival probability.

Affine Mortality Models - Dynamics of Mortality Rates

- The affine dynamics of the latent factors X_t follow a system of stochastic differential equations (SDEs) under the risk-neutral measure Q (Duffie and Kan, 1996; Christensen et al., 2011):

$$dX_t = K^Q \left[\theta^Q - X_t\right] dt + \Sigma D\left(X_t, t\right) dW_t^Q, \tag{8}$$

  - where $K^Q \in \mathbb{R}^{n \times n}$ is the mean reversion matrix,
  - $\theta^Q \in \mathbb{R}^n$ is the long-term mean (usually zero in mortality models),
  - $\Sigma \in \mathbb{R}^{n \times n}$ is the volatility matrix,
  - $W_t^Q \in \mathbb{R}^n$ is a standard Brownian motion, and
  - $D\left(X_t, t\right)$ is a diagonal matrix with the ith diagonal entry as $\sqrt{\alpha^i\left(t\right) + \beta_1^i\left(t\right) x_t^1 + \ldots + \beta_n^i\left(t\right) x_t^n}$. α and β are bounded continuous functions.
- Continuous time equivalent of auto-regressive processes for factors.

Affine Mortality Models - ODEs for Factor Loadings

- Under these dynamics the (risk-neutral) survival probabilities for age x for survival from time t to time T are (see details in Blackburn and Sherris, 2013):

$$S(t, T) = \exp\left(B(t, T)' X_t + A(t, T)\right), \tag{9}$$

- Where $B(t, T)$ and $A(t, T)$ are the solutions to the following system of ordinary differential equations (ODEs):

$$\frac{dB(t, T)}{dt} = \rho_1 + \left(K^Q\right)' B(t, T), \tag{10}$$

$$\frac{dA(t, T)}{dt} = -B(t, T)' K^Q \theta^Q - \frac{1}{2} \sum_{j=1}^{3} \left(\Sigma' B(t, T) B(t, T)' \Sigma\right)_{j,j}, \tag{11}$$

with boundary conditions $B(T, T) = A(T, T) = 0$.

Affine Mortality Models - Gaussian and CIR Models

- Dynamics include Gaussian models (where there is a probability of negative mortality rates).
- These models
  - are readily estimated with (Gaussian) Kalman filter
  - are easily simulated
  - in practice, have very low probabilities of negative mortality rates.
- Dynamics also include square root process dynamics with potential to capture mortality heterogeneity (Cox-Ingersoll-Ross or CIR)
- These models
  - can capture mortality heterogeneity (gamma distributed mortality rates)
  - avoid probabilities of negative mortality rates
  - are more difficult to estimate with the Kalman filter (we use maximum quasi-likelihood).

Affine Mortality Models - Historical Mortality Rates

- To calibrate models to historical mortality rates (P measure) we need a link between the risk neutral dynamics and the historical dynamics - assumption for the price of risk.

- Assuming an essentially affine form for the risk premium (Duffee, 2002):

$$\Lambda_t = \begin{cases} \lambda^0 + \lambda^1 X_t, & \text{Gaussian processes;} \\ D\left(X_t, t\right)\lambda^0, & \text{the CIR model,} \end{cases} \tag{12}$$

  where $\Lambda_t \in \mathbb{R}^{n\times 1}$, $\lambda^0 \in \mathbb{R}^{n\times 1}$ and $\lambda^1 \in \mathbb{R}^{n\times n}$.

- The SDEs for factors under the measure P have the same (auto-regressive) form:

$$dX_t = \begin{cases} K^P\left[\theta^P - X_t\right]dt + \Sigma dW_t^P, & \text{Gaussian processes;} \\ K^P\left[\theta^P - X_t\right]dt + \Sigma D\left(X_t, t\right)dW_t^P, & \text{the CIR model.} \end{cases} \tag{13}$$

Affine Cohort Mortality Models - 3-factor Dynamics

The dynamics of the factors (Canonical and AFNS models) are (we estimate both P and Q measure dynamics):

- The independent Blackburn-Sherris model (Blackburn and Sherris, 2013)

$$\begin{pmatrix} dX_t^1 \\ dX_t^2 \\ dX_t^3 \end{pmatrix} = - \begin{pmatrix} \delta_{11} & 0 & 0 \\ 0 & \delta_{22} & 0 \\ 0 & 0 & \delta_{33} \end{pmatrix} \begin{pmatrix} X_t^1 \\ X_t^2 \\ X_t^3 \end{pmatrix} dt + \begin{pmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{pmatrix} \begin{pmatrix} dW_t^{1,Q} \\ dW_t^{2,Q} \\ dW_t^{3,Q} \end{pmatrix}. \tag{14}$$

- The independent AFNS model (Christensen et al., 2011). The dynamics of the factors under the Q-measure are given by:

$$\begin{pmatrix} dL_t \\ dS_t \\ dC_t \end{pmatrix} = - \begin{pmatrix} 0 & 0 & 0 \\ 0 & \delta & -\delta \\ 0 & 0 & \delta \end{pmatrix} \begin{pmatrix} L_t \\ S_t \\ C_t \end{pmatrix} dt + \begin{pmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{pmatrix} \begin{pmatrix} dW_t^{1,Q} \\ dW_t^{2,Q} \\ dW_t^{3,Q} \end{pmatrix}. \tag{15}$$

- The dependent Blackburn-Sherris model

$$\begin{pmatrix} dX_t^1 \\ dX_t^2 \\ dX_t^3 \end{pmatrix} = - \begin{pmatrix} \delta_{11} & 0 & 0 \\ \delta_{21} & \delta_{22} & 0 \\ \delta_{31} & \delta_{32} & \delta_{33} \end{pmatrix} \begin{pmatrix} X_t^1 \\ X_t^2 \\ X_t^3 \end{pmatrix} dt + \begin{pmatrix} \sigma_{11} & 0 & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \begin{pmatrix} dW_t^{1,Q} \\ dW_t^{2,Q} \\ dW_t^{3,Q} \end{pmatrix}. \tag{16}$$

Affine Cohort Mortality Models - 3-factor Dynamics

- The dependent AFNS model

$$\begin{pmatrix} dX_t^1 \\ dX_t^2 \\ dX_t^3 \end{pmatrix} = -\begin{pmatrix} \delta_{11} & 0 & 0 \\ \delta_{21} & \delta_{22} & 0 \\ \delta_{31} & \delta_{32} & \delta_{33} \end{pmatrix} \begin{pmatrix} X_t^1 \\ X_t^2 \\ X_t^3 \end{pmatrix} dt + \begin{pmatrix} \sigma_{11} & 0 & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \begin{pmatrix} dW_t^{1,Q} \\ dW_t^{2,Q} \\ dW_t^{3,Q} \end{pmatrix}. \tag{17}$$

- The CIR model

$$\begin{aligned} \begin{pmatrix} dX_t^1 \\ dX_t^2 \\ dX_t^3 \end{pmatrix} &= -\begin{pmatrix} \delta_{11} & 0 & 0 \\ 0 & \delta_{22} & 0 \\ 0 & 0 & \delta_{33} \end{pmatrix} \left[\begin{pmatrix} \theta_1^Q \\ \theta_2^Q \\ \theta_3^Q \end{pmatrix} - \begin{pmatrix} X_t^1 \\ X_t^2 \\ X_t^3 \end{pmatrix} \right] dt \\ &+ \begin{pmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{pmatrix} \begin{pmatrix} \sqrt{X_t^1} & 0 & 0 \\ 0 & \sqrt{X_t^2} & 0 \\ 0 & 0 & \sqrt{X_t^3} \end{pmatrix} \begin{pmatrix} dW_t^{1,Q} \\ dW_t^{2,Q} \\ dW_t^{3,Q} \end{pmatrix}. \end{aligned} \tag{18}$$

Model Estimation - Kalman Filter

- We model mortality rates but observe deaths - we need a measurement equation capturing the effects of Poisson variation and heterogeneity.
- Mortality rate curve changes stochastically through time, driven by latent factors with trend and uncertainty - we need a state transition equation for the dynamics.
- We then filter the values of latent factors from historical data - deriving means and covariances which are functions of the parameters in the dynamics.
- We can then construct the likelihood (Gaussian) in terms of means and covariance (a function of parameters to be estimated).
- Then numerically select the parameter set that maximises the likelihood using an iterative process.

Mortality Data - Estimating Mortality Models

- Mortality models are usually estimated with age-period historical data (life tables) - US data from 1933 to 2015 at ages from 50 to 100 is shown below.
- Cohort mortality rates are required in practice. Age-period models require forecasting of age-period curves and derivation of cohort mortality rates from the diagonal as the cohort ages.

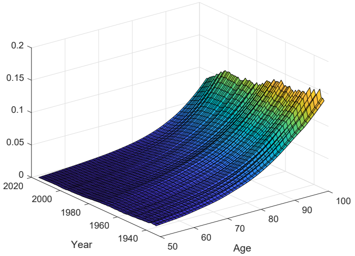


Figure 1: Average force of mortality of US Males Using Age-Period data, from 1933 to 2015

Mortality Data - Estimating Mortality Models

- Age-cohort data allows fitting of age-cohort curves directly but incomplete data for more recent cohorts - see US cohort data below.

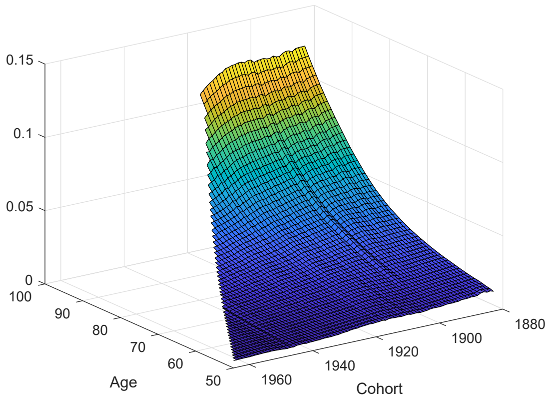


Figure 2: Average Force of Mortality for Males Born from 1883 to 1965

Mortality Data - Estimating Mortality Models

- Complete age-cohort data for cohorts born in earlier years. US complete cohort data below.

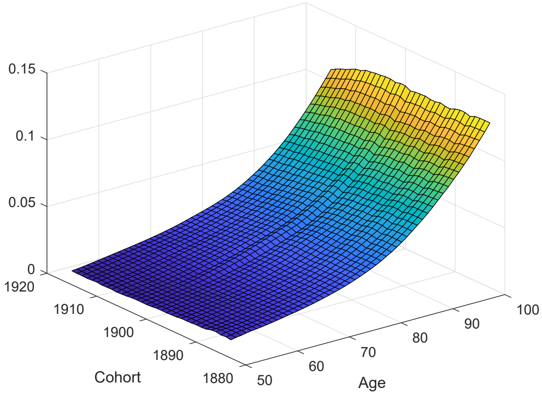


Figure 3: Average Force of Mortality for Males Born from 1883 to 1915

Mortality Data - Calibrating Affine Mortality Models

- US mortality age-cohort data from the Human Mortality Database (2017) (HMD) to calibrate and compare the mortality models.
- Mortality data of males from ages 50 to 100 for the cohorts born from 1883 to 1915.
- Historical survival probability, $S^i(x; t, T)$, and the historical average forces of mortality $\bar{\mu}^i(x; t, T)$ over the period $\tau = T - t$ for each cohort i aged x at time t from the data, using:

$$S^i(x; t, T) = \prod_{s=1}^{T-t} \left[1 - q^i(x + s - 1, t + s - 1)\right], \tag{19}$$

$$\bar{\mu}^i(x; t, T) = -\frac{1}{T - t} \log \left[S^i(x; t, T)\right], \tag{20}$$

where $q^i(x, t)$ is the one year death probability for an individual aged x at time t in cohort i.

Affine Cohort Mortality Models - Goodness of Fit

Table 1: Comparison of Affine Mortality Models

	The Blackburn-Sherris Model		The AFNS Model		The CIR Model
	Independent-Factor	Dependent-Factor	Independent-Factor	Dependent-Factor	
Log Likelihood	9896.419	9938.696	9665.801	9887.878	10045.70
RMSE	0.00250	7.601e-04	6.856e-04	9.160e-04	5.227e-04
No. of Parameters	12	18	10	13	18
AIC	-19570.837	-19643.392	-19113.602	-19551.757	-19857.40
BIC	-18968.292	-19008.277	-18521.914	-18943.783	-19222.29
Probability of Negative Mortality	0.02700	1.011e-32	1.722e-31	4.34e-14	-

- AFNS model fits historical age-cohort data well. Low negative mortality probabilities. CIR the best fit.

Canonical Age-Period Mortality Curve Factors

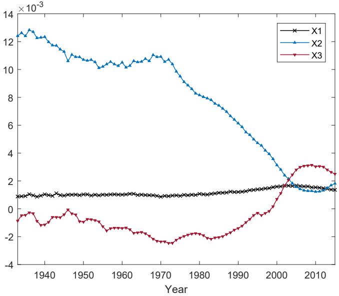
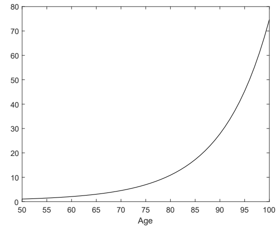


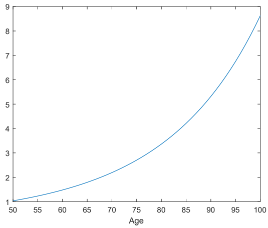
Figure 4: Factors in the Blackburn-Sherris Model with Age-Period Data

- Factor X_2 captures trend change around 1970's.

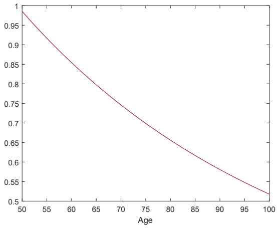
Canonical Age-Period Mortality Curve Factor Loadings



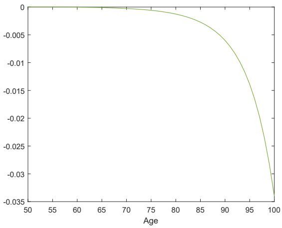
(a) $B_1 = -\frac{B^1(t,T)}{T-t}$



(b) $B_2 = -\frac{B^2(t,T)}{T-t}$



(c) $B_3 = -\frac{B^3(t,T)}{T-t}$



(d) $A = -\frac{A(t,T)}{T-t}$

AFNS Age-Cohort Mortality Curve Factors

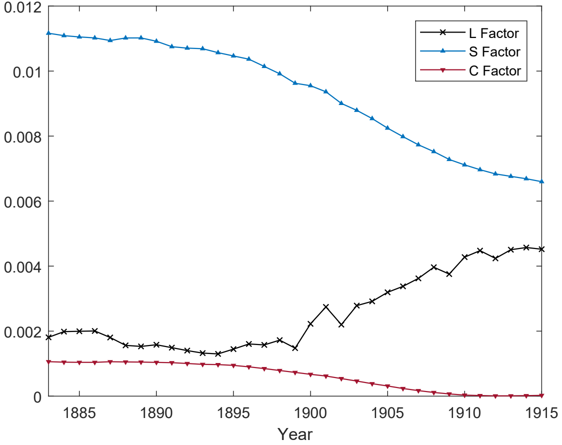
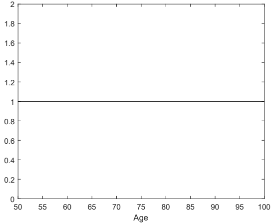
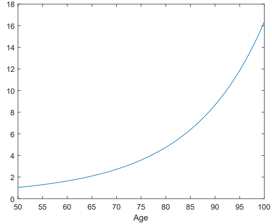


Figure 6: Factors in the Independent AFNS Model

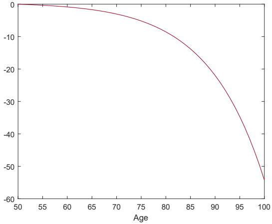
AFNS Age-Cohort Mortality Curve Factor Loadings



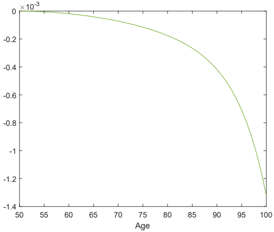
(a) $B_1 = -\frac{B^1(t,T)}{T-t}$



(b) $B_2 = -\frac{B^2(t,T)}{T-t}$

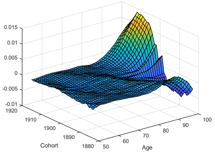


(c) $B_3 = -\frac{B^3(t,T)}{T-t}$

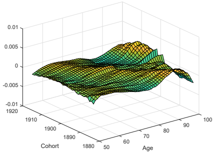


(d) $A = -\frac{A(t,T)}{T-t}$

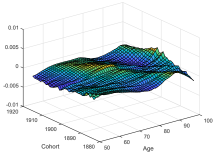
Affine Cohort Mortality Models - Residual Analysis



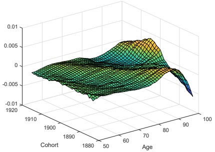
(a) The Independent Blackburn-Sherris Model



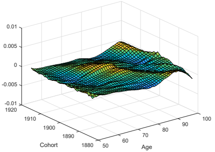
(b) The Dependent Blackburn-Sherris Model



(c) The Independent AFNS Model



(d) The Dependent AFNS Model



(e) The CIR Model

Figure 8: Residuals of Affine Mortality Models

Affine Cohort Mortality Models - MAPE

Mean Absolute Percentage Error (MAPE) for each age, across all cohorts

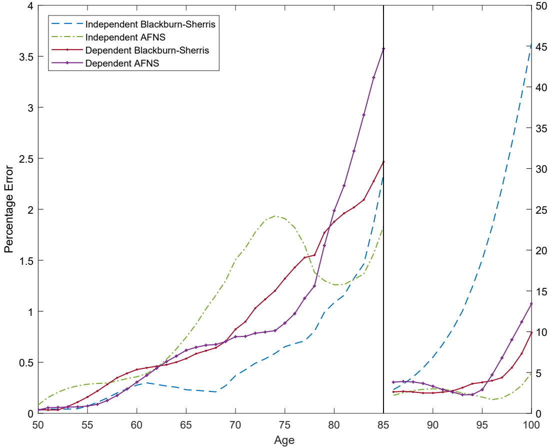


Figure 9: The Models with Gaussian Processes

Affine Cohort Mortality Models - MAPE

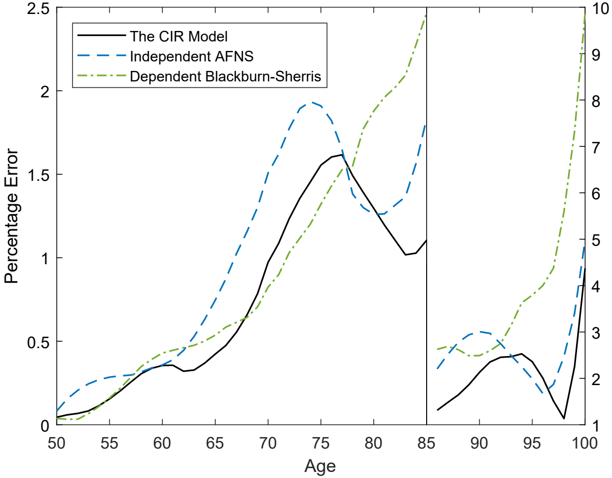


Figure 10: The CIR Model, the Dependent Blackburn-Sherris Model and the Independent AFNS Model

Forecasting Survival Curves

- Optimal forecasts, or best-estimate forecasts, are used to project average forces of mortality and survival probabilities for future cohorts.
- At time t, the one step ahead forecast of the average force of mortality is

$$\bar{\mu}(t+1, T+1) = -\frac{B(t,T)'}{T-t} E[X_{t+1}|X_t] - \frac{A(t,T)}{T-t}, \tag{21}$$

  where $B(t,T)$ and $A(t,T)$ depend on the model.
- The forcasts of survival probabilities are then:

$$S(t+1, T+1) = \exp\Big(B(t,T)' E[X_{t+1}|X_t] + A(t,T)\Big). \tag{22}$$

Forecasting Survival Curves

- The factor dynamics under measure P in the independent Blackburn-Sherris model and the 3-factor independent AFNS model are the same. The conditional expectation of state variables for these two models are as follows:

$$E\left[X_{t+1}^1 | X_t^1\right] = e^{-k_{11}^P} X_t^1, \quad E\left[X_{t+1}^2 | X_t^2\right] = e^{-k_{22}^P} X_t^2,$$
$$E\left[X_{t+1}^3 | X_t^3\right] = e^{-k_{33}^P} X_t^3. \tag{23}$$

- For the independent AFNS model, the conditional mean has the same structure but with $X_t = (L_t, S_t, C_t)$.

- The conditional mean of the CIR mortality model is:

$$E\left[X_{t+1}^1 | X_t^1\right] = e^{-k_{11}^P} X_t^1 + \theta_1^P \left(1 - e^{-k_{11}^P}\right),$$
$$E\left[X_{t+1}^2 | X_t^2\right] = e^{-k_{22}^P} X_t^2 + \theta_2^P \left(1 - e^{-k_{22}^P}\right), \tag{24}$$
$$E\left[X_{t+1}^3 | X_t^3\right] = e^{-k_{33}^P} X_t^3 + \theta_3^P \left(1 - e^{-k_{33}^P}\right).$$

Affine Cohort Mortality Models - Forecast RMSE

Table 2: RMSE by Comparing the Actual and Best-Estimate Survival Probabilities of the 1916 Cohort

	The Blackburn-Sherris Model		The AFNS Model		The CIR Model
	Independent	Dependent	Independent	Dependent	
RMSE	0.03197	0.00726	0.00668	0.00754	0.01835

- AFNS model performs well. CIR model has poorer forecasting performance. Forecast for a single cohort.

Affine Cohort Mortality Models - Forecast RMSE

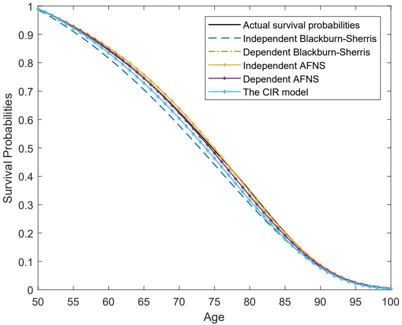


Figure 11: Actual and Best-Estimate Survival Probabilities of the 1916 Cohort

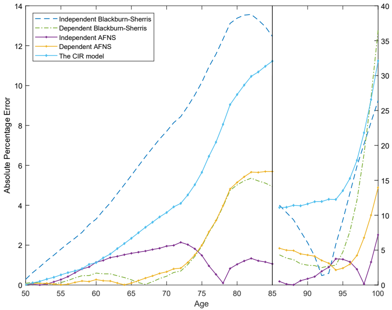


Figure 12: Absolute Percentage Errors between Actual and Best-Estimate Survival Probabilities

Wrap Up

- Introduced continuous-time mortality models including an AFNS cohort mortality model with interpretable latent stochastic factors for level, slope and curvature of the survival curve.
  - The model is based on factor loadings multiplied by (latent) factors, where the factors are equivalent to stochastic parameters and the factor loadings determine how the factors impact different ages.
- Outlined the dynamics of the mortality rates and the affine survival curves.
- Outlined the estimation of the models using the Gaussian Kalman filter.
- Outlined how the models can capture Poisson variation in the estimation. .

Some comments on the Models

- Empirical results show that the independent-factor AFNS cohort mortality model:
  - Is parsimonious, captures the variation in cohort mortality rates in US data, producing a better fit at older ages than the independent-factor Blackburn-Sherris model, and has good predictive performance.
  - Is easy to implement with closed-form expressions for survival probabilities, and as a Gaussian model is easy to estimate using the Kalman filter. Negative mortality rates have very low probability.
  - Has factors that fit historical data dynamics and have intuitive factor interpretation (Level, Slope, Curvature).
  - Multi-factor age-cohort models, and particularly the AFNS model, is well suited for financial and insurance applications - see for example Xu et al. (2019b).
- Work to be done: incorporating imcomplete cohorts inrto estimation, better capturing Poisson variation, age-dependence in trend and covariance, CIR model estimation and forecasting.

Acknowledgements

- Financial support from:
  - Society of Actuaries Center of Actuarial Excellence Research Grant 2017–2020: Longevity Risk: Actuarial and Predictive Models, Retirement Product Innovation, and Risk Management Strategies.
  - CEPAR Australian Research Council Centre of Excellence in Population Ageing Research project number CE170100005 for 2018–2024.

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Appendix

- Discretization of Continuous Model Dynamics
- Kalman Filter estimation

Model estimation - Measurement equation

- The measurement equation, based on the average force of mortality, for a given current age to different future survival ages is:

$$\bar{\mu}(t, T) = -\frac{B(t, T)^{'}}{T - t} X_t - \frac{A(t, T)}{T - t} + \varepsilon_t, \tag{25}$$

  where the measurement error ε_t is independently and identically distributed noise and X_t are the latent factors.

- We can write this as

$$y_t = BX_t + A + \varepsilon_t. \tag{26}$$

Model estimation - Measurement equation

- For a 3-factor affine mortality model, the measurement equation with N observed average forces of mortality for ages $x+1$ to $x+N$ is:

$$\begin{pmatrix} \bar{\mu}(t,t+1) \\ \bar{\mu}(t,t+2) \\ \vdots \\ \bar{\mu}(t,t+N) \end{pmatrix} = \begin{pmatrix} -B^1(t,t+1) & -B^2(t,t+1) & -B^3(t,t+1) \\ -\frac{B^1(t,t+2)}{2} & -\frac{B^2(t,t+2)}{2} & -\frac{B^3(t,t+2)}{2} \\ \vdots & \vdots & \vdots \\ -\frac{B^1(t,t+N)}{N} & -\frac{B^2(t,t+N)}{N} & -\frac{B^3(t,t+N)}{N} \end{pmatrix} \begin{pmatrix} X_t^1 \\ X_t^2 \\ X_t^3 \end{pmatrix} \tag{27}$$

$$+ \begin{pmatrix} -A(t,t+1) \\ -\frac{A(t,t+2)}{2} \\ \vdots \\ -\frac{A(t,t+N)}{N} \end{pmatrix} + \begin{pmatrix} \varepsilon_t(1) \\ \varepsilon_t(2) \\ \vdots \\ \varepsilon_t(N) \end{pmatrix}, \tag{28}$$

Kalman Filter - State Transition Equation

- The state transition equation is a discretized version of the SDE dynamics and is given by:

$$X_t = \exp\left(-K^P\right) X_{t-1} + \eta_t, \tag{29}$$

  where η_t is the transition error vector.

- The structure of stochastic error terms is assumed to be:

$$\begin{pmatrix} \eta_t \\ \varepsilon_t \end{pmatrix} \sim N\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} R & 0 \\ 0 & H \end{pmatrix} \right], \tag{30}$$

  where both the matrix H and matrix R are diagonal, with R being the covariance matrix of the measurement error and H being the covariance matrix of the transition error.

Kalman Filter - Error Assumptions

- The error matrix R for the state transitions, derived from the dynamics of the latent factors, in discrete time form is

$$R = \int_{t-1}^{t} e^{-K^P(t-s)} \Sigma\Sigma' e^{-\left(K^P\right)'(t-s)} ds. \tag{31}$$

- Poisson variation is captured in the diagonal of the covariance matrix H, assumed to have exponential form (reflecting exponential increase in mortality rate) given by

$$H(t, T) = \frac{1}{T-t} \sum_{i=1}^{T-t} \left[r_c + r_1 e^{r_2 i} \right], \tag{32}$$

where the values of r_c, r_1 and r_2 are estimated as part of the optimal parameter set.

Kalman Filter - Forecasting step

- Denote the (average) mortality rates at time t by $Y_t = (y_1, \ldots, y_t)$ and the parameters by ψ.
- In the forecasting step we first update the state, X_{t-1}, and its mean square error, Σ_{t-1},

$$X_{t|t-1} = E\left[X_t | Y_{t-1}\right] = \Phi\left(\psi\right) X_{t-1}, \tag{33}$$

$$\Sigma_{t|t-1} = \Phi\left(\psi\right) \Sigma_{t-1} \Phi\left(\psi\right)^{'} + R\left(\psi\right), \tag{34}$$

where $\Phi = \exp\left(-K^P\right)$ and $R = \int_{t-1}^{t} e^{-K^P(t-s)} \Sigma\Sigma' e^{-\left(K^P\right)^{'}(t-s)} ds$.

Kalman Filter - Forecasting step

- We then use the historical mortality rate information at time t to update the forecasts to obtain:

$$X_t = E\left[X_t | Y_t\right] = X_{t|t-1} + \Sigma_{t|t-1} B\left(\psi\right)' F_t^{-1} \nu_t, \tag{35}$$

$$\Sigma_t = \Sigma_{t|t-1} - \Sigma_{t|t-1} B\left(\psi\right)' F_t^{-1} B\left(\psi\right) \Sigma_{t|t-1}, \tag{36}$$

where

$$\nu_t = y_t - E\left[y_t | Y_{t-1}\right] = y_t - A\left(\psi\right) - B\left(\psi\right) X_{t|t-1}, \tag{37}$$

$$F_t = cov\left(\nu_t\right) = B\left(\psi\right) \Sigma_{t|t-1} B\left(\psi\right)' + H\left(\psi\right). \tag{38}$$

Kalman Filter - Log-likelihood

- The log-likelihood function is then computed as:

$$\log L\left(y_1, \ldots, y_t; \psi\right) = \sum_{t=1}^{T}\left(-\frac{N}{2}\log\left(2\pi\right) - \frac{1}{2}\log\left|F_t\right| - \frac{1}{2}\nu_t^{'}F_t\nu_t\right), \tag{39}$$

  where N is the number of ages with observed average forces of mortality.

- The log-likelihood function is maximized with respect to ψ to obtain the optimal parameter set using an iterative process
- Start with initial values, use Kalman filter to determine likelihood of data, update parameter values and iterate until maximum of likelihood is derived.

Multi-Population Longevity Models: A Spatial Random Field Approach

Nhan Huynh

Department of Statistics & Applied Probability, UC Santa Barbara

Joint with Prof. Michael Ludkovski

2020 Living to 100 Symposium

January 13, 2020

Outline

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Multi-population Mortality Modeling

- Mortality changes over time for populations with similar geographic characteristics are correlated.
  - Countries in a region, states in a country, males/females in a country.

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Challenges:

- Model complexity and computational challenges.
- Data availability.
  - Used aggregated mortality datasets from *Human Mortality Database*.
  - Ages 50–84 and Years 1990–2016 from 10 European countries.

Lee & Li Method for Two-population Model

- Treats Age and Year as factors.
- Aggregated data $\rightarrow$ estimate global age and year factors $+$ country-specific age and year trend.
- Employs time-series method (e.g.: AR(1)) for mortality projection.

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- Aggregated data $\rightarrow$ estimate global age and year factors $+$ country-specific age and year trend.
- Employs time-series method (e.g.: AR(1)) for mortality projection.
  - (+) Simple and the parameters are easily interpretable.
  - (+) Stochastic forecasts with probabilistic prediction intervals.
  - (−) Smoothing via point estimators, no credible bands.
  - (−) Number of populations is limited to 2.
  - (−) Hundreds of parameters needed to be estimated.

Main Contributions

- Employ spatial statistical framework of **Gaussian Process** (GP) regression as a machine learning method for multi-population modeling.

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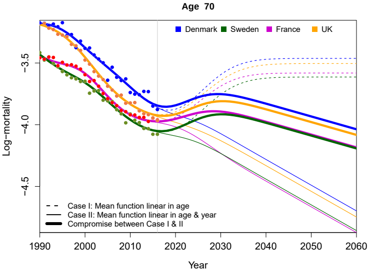

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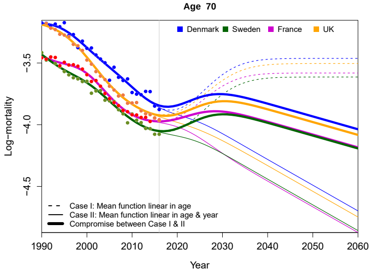

- Non-parametric → smoothed mortality surfaces over Age & Year dimensions.
- Bayesian approach → quantify predictive uncertainty and generate stochastic trajectories for predictions.

- Captures the cross-population dependence for mortality from multiple populations.
- Number of parameters estimated is substantially smaller.
- Can handle more than 2 populations.

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Multi-population GP Regression

- Assume the output-input relationship:

$$y^n = f(x^n) + \epsilon^n$$

  - Input $= x^n =$ (Age, Year, Indicators for population) & Output $= y^n =$ log-mortality.
  - $f(.)$: true log-mortality surface as a random variable.
  - ϵ^n: errors from i.i.d. Gaussian with zero mean and constant variance.

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- Specify prior distribution, then compute the posterior distribution after collecting data: $p(f|\mathcal{D}) \propto \; p(f).p(\mathbf{y}|\mathbf{x}, f) = \{prior\}.\{likelihood\}$.
  - Gaussian prior + Guassian likelihood → Gaussian posterior.

- A Gaussian process defines a prior over functions: any finite sample is a realization of a **Multivariate normal distribution**.
  - All properties specified via the mean and covariance function.

Covariance Kernel

- Covariance function describes the influences between data points.

  "Everything is related to everything else, but near things are more related than distant things" – Tobler's First Law of Geography.

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- Consider a squared-exponential kernel:

$$C(x^i, x^j) = \eta^2 \exp\left[-\frac{(x^i_{ag} - x^j_{ag})^2}{2\theta^2_{ag}} - \frac{(x^i_{yr} - x^j_{yr})^2}{2\theta^2_{yr}} \right] \prod_{\{l_1, l_2\}} \exp\left[-\theta_{l_1,l_2} \delta^{ij}_{l_1,l_2} \right]$$

where $\delta^{ij}_{l_1,l_2}$: an indicator whether ith & jth obs. from populations l_1 and l_2.

- Cross-population correlation is an exponential function of θ_{l_1,l_2}. Large value of $\theta_{l_1,l_2} \rightarrow$ low correlation between two populations.
- η^2: process variance, controls amplitude of f;
- θ_{ag} & θ_{yr}: characteristic lengthscales, determine the spatial smoothness in Age and Year dimensions.
- Kernel hyperparameters Θ are $\eta, \theta_{ag}, \theta_{yr}, \theta_{l_1,l_2}$, etc.

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- **MLE**: optimization of the marginal likelihood function.
  - Used packages `DiceKriging` and `kergp` in R.

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- Fitting $\equiv$ learning the hyperparameters.
- **MLE**: optimization of the marginal likelihood function.
  - Used packages `DiceKriging` and `kergp` in R.
- **Hierarchical approach**: specification of the priors for the hyperparameters (a fully Bayesian approach).
  - Quantify model risk - range of GP models consistent with the data via Bayesian framework.
  - Computationally more intensive but quantifies how well the spatial structure is being learned.
  - `Stan`: free, open-source software for Bayesian statistical inferences.

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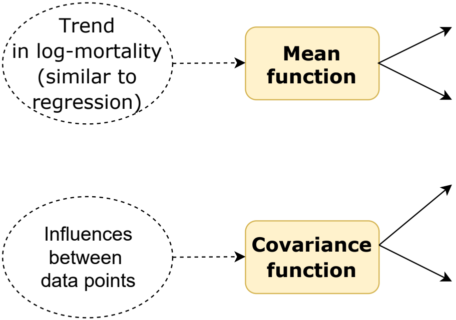
Joint model for European Nations

- Incorporate more than 2 populations in the model.
  - ✎ Joint model for Male mortality from 4 European countries: {Denmark, France, Sweden, UK} ≡ {DNK, FRA, SWE, UK}.

    Fitted on Ages 70–84 and calendar Years 1990–2012:



10% increases in log-mortality for each additional unit in Age.

Sweden has lowest trend while Denmark has highest trend. Mean difference is 3%.

Common hyper-parameters that determine spatial smoothness in Age and Year

Cross-population correlation

	DNK	FRA	SWE
FRA	0.28		
SWE	0.23	0.60	
UK	0.91	0.61	0.30

Multi-population Models via Bayesian GP

- Utilize aggregated data from multiple populations $\rightarrow$ provides tighter hyper-parameter posteriors (reduce the model risk).

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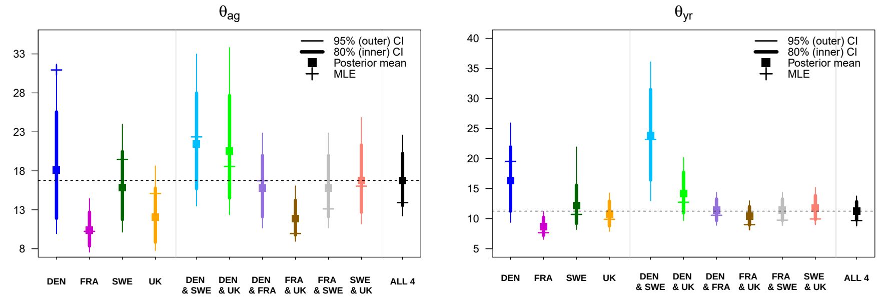

Length-scale in Age dimension Length-scale in Year dimension

Prediction Performance in Multi-population Models

- Out-of-sample predictions via multi-pop. model are more accurate.
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Multi-population model for two Nordic countries: Denmark & Sweden (Male populations, Ages 70–84 and Years 1990-2012).

Prediction accuracy for Ages 70–84

SMAPE	2013 (1-yr out)		2016 (4-yr out)	
	Single-pop	Multi-pop	Single-pop	Multi-pop
Denmark	1.58	**1.52**	1.26	**1.22**
Sweden	1.05	**0.82**	2.53	**0.83**

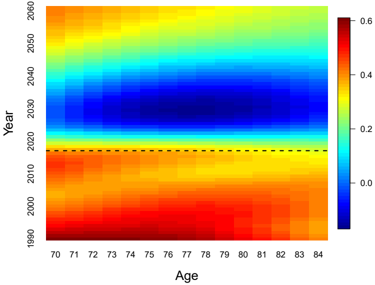
Posterior S.D. for Sweden

$s_{SWE}(.)$	2013 (1-yr out)		2016 (4-yr out)	
	Single-pop	Multi-pop	Single-pop	Multi-pop
Age 75	0.0300	**0.0292**	0.0399	**0.0317**
Age 85	0.0330	**0.0311**	0.0461	**0.0351**

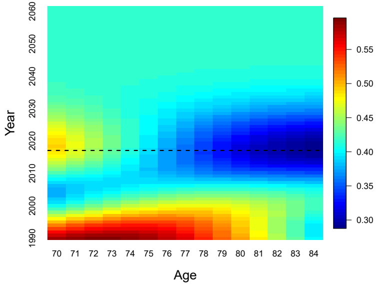
Limitation of Single-population Models

- Often generate divergent long-term forecasts that are inconsistent with historical patterns.
  - Implausible difference in mean forecasts, or excessively fast changes in relative mortality.

Log-mortality difference in mean forecast between Danish Males and Females



(a) Single-population models

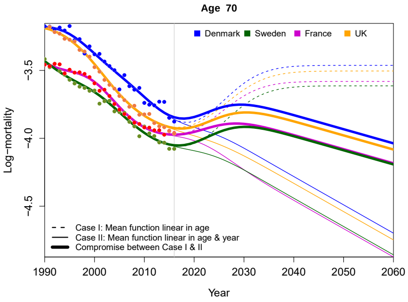


(b) Two-population model

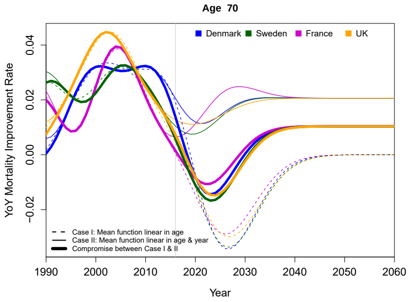
Coherence Forecasts

- Forecasts via multi-pop. models maintain historical characteristics observed in the data into the future.
- Use the mean function to generate long-term forecasts and enforce desired coherence.
  - Mortality across populations move in unison → strong coherence.

Long-term mortality forecasting in 1990–2060



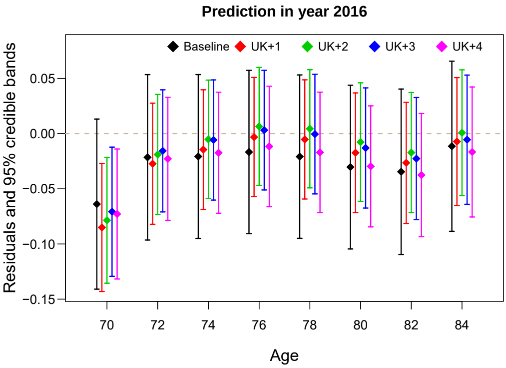
(a) Log-mortality



(b) Improvement factors

Borrowing latest data

- Opportunities to borrow latest information from other populations to improve prediction about the recent domestic mortality.
  - In HMD, data from different countries arrives non-synchronously.
  - 2016 (one-year out) forecast in UK:
    Baseline ≡ single model for UK up to 2015.
    Multi-pop'n models with UK up to 2015 & other populations up to 2016.



Borrowing latest data (Cont.)

- Improvement in recent mortality prediction via multi-population models depend on the cross-population correlation.
  - SMAPE: difference between forecast and observed log-mortality.
  - Continuous Ranked Probability Score (CRPS): difference between forecast and empirical cumulative distribution function of the observation.

Correlation vs. prediction performance in one-year out cross validation

2-pop'n model with UK	Correlation	Improvement in SMAPE	Improvement in CRPS
Switzerland	0.89	−22.0%	−1.7%
Czech	0.84	−12.1%	−3.2%
Sweden	0.54	−8.0%	−1.6%
Hungary	0.34	−1.7%	1.6%
Latvia	0.03	44.4%	6.9%

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Next steps

- Cluster methods to identify groups of populations with similar characteristics in multi-population models.
- Implementation on 50-state US Mortality Database.
- Modeling cause-of-death mortality.
- Investigate other kernel families to explore spatial covariance structure.
- Computational speed-up to handle larger data sets.


THANK
YOU

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