QFI QF Model Solutions
Fall 2023

1.

Learning Objectives:

1. The candidate will understand the foundations of quantitative finance.

3. The candidate will understand:
   - How to apply the standard models for pricing financial derivatives.
   - The implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory.
   - How to evaluate risk exposures and the issues in hedging them.

Learning Outcomes:

(1a) Understand and apply concepts of probability and statistics important in mathematical finance.

(1b) Understand the importance of the no-arbitrage condition in asset pricing.

(1c) Understand Ito integral and stochastic differential equations.

(1e) Understand the Black Scholes Merton PDE (partial differential equation).

(3a) Demonstrate an understanding of option pricing techniques and theory for equity derivatives.

Sources:

An Introduction to the Mathematics of Financial Derivatives, Hirsa, Ali and Neftci, Salih N., 3rd Edition 2nd Printing, 2014 (Ch. 3, 10, 12, 13)

Commentary on Question:

Commentary listed underneath question component.

Solution:

(a) Derive $\frac{\partial V}{\partial t}$ and $\frac{\partial V}{\partial S}$ in terms of the partial derivatives of the vanilla call.

Commentary on Question:

Most candidates were successful in this part. It is a straightforward application of the derivative operator. The most common error was not invoking the chain rule properly for $\frac{\partial V}{\partial S}$.

1. Continued

For simplicity, write $C\left(\frac{B^2}{S},K,t\right) = C\left(\frac{B^2}{S}\right)$.

$$\frac{\partial V}{\partial t} = S^\alpha \frac{\partial}{\partial t}\left[C\left(\frac{B^2}{S}\right)\right] = S^\alpha \frac{\partial C}{\partial t}$$

$$\frac{\partial V}{\partial S} = \frac{\partial}{\partial S}\left[S^\alpha C\left(\frac{B^2}{S}\right)\right]$$
$$= \frac{\partial}{\partial S}[S^\alpha]C\left(\frac{B^2}{S}\right) + S^\alpha \frac{\partial}{\partial S}\left[C\left(\frac{B^2}{S}\right)\right]$$
$$= \alpha S^{\alpha-1}C\left(\frac{B^2}{S}\right) + S^\alpha\left(\frac{-B^2}{S^2}\right)\frac{\partial C}{\partial S}$$
$$= \alpha S^{\alpha-1}C - B^2 S^{\alpha-2}\frac{\partial C}{\partial S}$$

(b) Show that $\frac{\partial^2 V}{\partial S^2} = \alpha(\alpha-1)S^{\alpha-2}C - B^2(2\alpha-2)S^{\alpha-3}\frac{\partial C}{\partial S} + B^4 S^{\alpha-4}\frac{\partial^2 C}{\partial S^2}$

Commentary on Question:
Candidates who were successful on part (a) were generally successful here as well. The problem again required use of the chain rule in applying the derivative operator.

Let $C = C\left(\frac{B^2}{S},K,t\right)$.

$$\frac{\partial^2 V}{\partial S^2} = \frac{\partial}{\partial S}\left[\frac{\partial V}{\partial S}\right]$$
$$= \frac{\partial}{\partial S}\left[\alpha S^{\alpha-1}C - B^2 S^{\alpha-2}\frac{\partial C}{\partial S}\right]$$
$$= \frac{\partial}{\partial S}[\alpha S^{\alpha-1}C] - B^2\frac{\partial}{\partial S}\left[S^{\alpha-2}\frac{\partial C}{\partial S}\right]$$
$$= \alpha(\alpha-1)S^{\alpha-2}C + \alpha S^{\alpha-1}\frac{\partial}{\partial S}[C] - B^2\left\{(\alpha-2)S^{\alpha-3}\frac{\partial C}{\partial S} + S^{\alpha-2}\frac{\partial}{\partial S}\left[\frac{\partial C}{\partial S}\right]\right\}$$
$$= \alpha(\alpha-1)S^{\alpha-2}C + \alpha S^{\alpha-1}\left(\frac{-B^2}{S^2}\right)\frac{\partial C}{\partial S} - B^2\left\{(\alpha-2)S^{\alpha-3}\frac{\partial C}{\partial S} + S^{\alpha-2}\left(\frac{-B^2}{S^2}\right)\frac{\partial^2 C}{\partial S^2}\right\}$$
$$= \alpha(\alpha-1)S^{\alpha-2}C - \alpha B^2 S^{\alpha-3}\frac{\partial C}{\partial S} - B^2(\alpha-2)S^{\alpha-3}\frac{\partial C}{\partial S} + B^4 S^{\alpha-4}\frac{\partial^2 C}{\partial S^2}$$
$$= \alpha(\alpha-1)S^{\alpha-2}C - B^2(2\alpha-2)S^{\alpha-3}\frac{\partial C}{\partial S} + B^4 S^{\alpha-4}\frac{\partial^2 C}{\partial S^2}$$

1. Continued

(c) Determine the value of α such that $V(S,t)$ satisfies the Black-Scholes PDE.

Commentary on Question:
Candidates performed below expectation on this part. Most did not get beyond substituting prior results for the partial derivatives of $\mathrm{V(S,t)}$ *into the Black-Scholes PDE. Some candidates attempted to guess a simple value for* α, *e.g.* α = 1. *A key step was to recognize that after substituting for the partials of* $\mathrm{V(S,t)}$ *to establish a PDE in terms of a call option, we could leverage another PDE we know that call satisfies, i.e. the Black-Scholes PDE. From there, the rest of the problem is largely aligning terms and performing some algebra.*

For $V(S,t)$ to satisfy the Black-Scholes PDE, we must have

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV=0$$

Substituting results from prior parts, we find

$$\begin{aligned}\frac{\partial V}{\partial t}&+\frac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV\\
&= S^{\alpha}\frac{\partial C}{\partial t}\\
&+\frac{1}{2}\sigma^2S^2\left[\alpha(\alpha-1)S^{\alpha-2}C-B^2(2\alpha-2)S^{\alpha-3}\frac{\partial C}{\partial S}\right.\\
&\left.+B^4S^{\alpha-4}\frac{\partial^2 C}{\partial S^2}\right]+rS\left[\alpha S^{\alpha-1}C-\ B^2S^{\alpha-2}\frac{\partial C}{\partial S}\right]-rS^{\alpha}C\\
&=S^{\alpha}\left\{\frac{\partial C}{\partial t}+\frac{1}{2}\sigma^2(B^4S^{-2})\frac{\partial^2 C}{\partial S^2}+(B^2S^{-1})\big((-\alpha+1)\sigma^2-r\big)\frac{\partial C}{\partial S}\right.\\
&\left.+\left(\frac{1}{2}\sigma^2\alpha(\alpha-1)+r\alpha-r\right)C\right\}\end{aligned}$$

All terms include a factor of S^{α}, so we may factor it out and simplify the expression in terms of partial derivatives with respect to the vanilla call option, C. For the Black-Scholes PDE to be satisfied, the expression within the curly brackets must be equal to 0.

We can leverage the fact that as a vanilla call option, C, also satisfies the Black-Scholes PDE, but where the underlying is $\frac{B^2}{S}$, rather than S directly, requiring the modified form of the Black-Scholes for C to be:

$$\frac{\partial C}{\partial t}+\frac{1}{2}\sigma^2\left(\frac{B^2}{S}\right)^2\frac{\partial^2 C}{\partial S^2}+r\left(\frac{B^2}{S}\right)\frac{\partial C}{\partial S}-rC=0$$

Equating the coefficients, of $\frac{\partial C}{\partial S}$ for example, in each PDE implies:

$$(-\alpha+1)\sigma^2-r=r \iff 2r=(1-\alpha)\sigma^2 \iff \frac{2r}{\sigma^2}=(1-\alpha)$$
$$\iff \alpha=1-\frac{2r}{\sigma^2}$$

(d)

(i) Describe the payoff for both a down-and-in call and a down-and-out call, each with no rebate.

(ii) Derive the formula for a down-and-out call option with respect to a vanilla call and down-and-in call with the same parameters.

(iii) Explain why the response in part (ii) implies that the down-and-out call option price also satisfies the Black-Scholes PDE.

Commentary on Question:
Candidates performed reasonably well on parts (i) and (ii), but below expectation on (iii).

(i) Down-and-in call: If the underlying stock price hits the barrier below during life of option, the payoff is identical to that of a vanilla call. Else there is no payoff.

Down-and-out call: If the underlying stock price hits the barrier below during life of option, the option has no payoff. Else the payoff is the same as a vanilla call at maturity.

(ii) A portfolio that holds both a down-and-in call and down-and-out call option with the same parameters will yield the same payoff as vanilla call. So, to avoid arbitrage, the value of such a portfolio should be equivalent to a vanilla call.

$$C_{vanilla}=C_{down-and-in}+C_{down-and-out}$$

$$\Rightarrow C_{down-and-out}=C_{vanilla}-C_{down-and-in}$$

1. Continued

(iii) Given α from part (c), we know $V(S,t)$ is a solution to the Black-Scholes PDE. Since the price of a down-and-in call is given as $\left(\frac{S}{B}\right)^{\alpha} C\left(\frac{B^2}{S},K,t\right) = \frac{1}{B^{\alpha}}V(S,t)$, it must also be a solution.

The formula of a down-and-out call option is a linear combination of a vanilla call and a down-and-in call. Linear combinations of solutions to the Black-Scholes PDE are also solutions.

2. **Learning Objectives:**

1. The candidate will understand the foundations of quantitative finance.

Learning Outcomes:

(1c) Understand Ito integral and stochastic differential equations.

(1d) Understand and apply Ito's Lemma.

(1h) Define and apply the concepts of martingale, market price of risk and measures in single and multiple state variable contexts.

Sources:
Chin Chapter 2

Commentary on Question:
This question tests candidates' understanding of the properties of Brownian motions, Ito's lemma, and martingales.

Solution:

(a) Evaluate the following expressions for $0 < s < t < u$:

(i) $E^{\mathbb{Q}}(W(s)\,W(t)W(u))$

(ii) $E^{\mathbb{Q}}(W(t)W(u) \mid \mathcal{F}_s)$

Commentary on Question:
Many candidates did well on this part by applying the independence properties of increments of Brownian motions.

This question is straight bookwork from Neftci and Chin. For both of these parts, use independent increments to simplify the expressions.

Part (i)

$$E^{\mathbb{Q}}(W(t) \times W(u) \times W(s))$$
$$= E^{\mathbb{Q}}\big((W(u) + W(t) - W(t)) \times W(t) \times W(s)\big)$$
$$= E^{\mathbb{Q}}\big((W(u) - W(t)) \times W(t) \times W(s)\big) + E^{\mathbb{Q}}(W^2(t) \times W(s)) = \boldsymbol{A} + \boldsymbol{B}$$

Since $\big(W(u) - W(t)\big)$ is independent of $W(t)$ and $W(s)$ we may go with

$$\boldsymbol{A} = E^{\mathbb{Q}}\Big(\big(W(u) - W(t)\big) \times W(t) \times W(s)\Big)$$
$$= E^{\mathbb{Q}}\big(W(u) - W(t)\big)E^{\mathbb{Q}}\big(W(t) \times W(s)\big) = 0$$

2. Continued

$\boldsymbol{B} = E^{\mathbb{Q}}(W^2(t) \times W(s))$

$= E^{\mathbb{Q}}((W(t) - W(s) + W(s))^2 \times W(s))$
$= E^{\mathbb{Q}}((W(t) - W(s))^2 \times W(s)) + 2E^{\mathbb{Q}}\big((W(t) - W(s)) \times W^2(s)\big) + E^{\mathbb{Q}}(W^3(s))$
$= E^{\mathbb{Q}}((W(t) - W(s))^2) \times E^{\mathbb{Q}}(W(s)) + 2E^{\mathbb{Q}}(W(t) - W(s)) \times E^{\mathbb{Q}}(W^2(s))$
$+ E^{\mathbb{Q}}(W^3(s))$
$= (t - s) \times 0 + 0 \times s + 0$

Hence $E^{\mathbb{Q}}(W(t) \times W(u) \times W(s)) = 0$

Part (ii)

Use the expressions obtained in part (ii) before the split of independent increments. For notation, denote $E^{\mathbb{Q}}(X|\mathcal{F}_s) = E_s^{\mathbb{Q}}(X)$

We may simply go with:

$$E_s^{\mathbb{Q}}\big(W(t) \times W(u)\big) = E_s^{\mathbb{Q}}\big(W(t) \times W(u)\big)$$
$$= E_s^{\mathbb{Q}}\Big(W(t) \times \big(W(u) - W(t) + W(t)\big)\Big)$$
$$= E_s^{\mathbb{Q}}\Big(W(t) \times \big(W(u) - W(t)\big)\Big) + E_s^{\mathbb{Q}}\big(W^2(t)\big)$$

$= 0 + E_s^{\mathbb{Q}}\Big(\big(W(t) - W(s)\big)^2\Big) + \big(W(s)\big)^2$ where in the last equality we use the fact that $E_s^{\mathbb{Q}}\big(W(t)\big) = W(s)$ and $E(X^2) = E[(X - \mu)^2] + \mu^2$
ence, $E_s^{\mathbb{Q}}\big(W(t) \times W(u)\big) = (t - s) + W^2(s)$

(b) Determine whether $X(t)$ is a martingale under $\mathbb{Q}$ using Ito's lemma.

Commentary on Question:
Many candidates were able to derive the correct formula by applying Ito's lemma and got the right answer.

This question is an application of the multivariate Ito's Lemma:

$$dX(t) = d\left((V(t))^2 \times W(t) - \int_0^t W(p)dp\right)$$

Let $\boldsymbol{A} *= d((V(t))^2 \times W(t))$
$= 2V(t)W(t)dV(t) + (V(t))^2 dW(t) + .5 * 2W(t)dt$

2. **Continued**

Let $\boldsymbol{B}* = d\left(-\int_0^t W(s)ds\right)$

$= -(W(t)dt)$

$\boldsymbol{A}* + \boldsymbol{B}* = 2V(t)W(t)dV(t) + ((V(t))^2)dW(t)$

As V and W are independent and the SDE is driftless, X(t) is a martingale.

(c) Determine whether $X(t)$ is a martingale under $\mathbb{Q}$ using the definition of a martingale.

Commentary on Question:
Most candidates did poorly in this part. Many candidates were able to list the three conditions of martingales. However, few were able to prove the second property.

To obtain full marks, candidates need to show $X(t)$ satisfies the full definition of a martingale and all 3 parts.

Criteria 1 – adaptability

Clearly $X(t)$ is adapted to $\mathcal{F}_t$

Criteria 2 – $E^{\mathbb{Q}}(|X(t)|) < \infty$. Note, there is more than one way to demonstrate this.

$$E^{\mathbb{Q}}(|X(t)|) = E^{\mathbb{Q}}\left(\left|(V(t))^2 \times W(t) - \int_0^t W(p)dp\right|\right)$$

By the triangle inequality, we have:

$$\leq E^{\mathbb{Q}}(|(V(t))^2 \times W(t)|) + E^{\mathbb{Q}}\left(\left|\int_0^t W(p)dp\right|\right)$$

By independence of V and W, and abs(integral) ≤ integral(abs)

$$\leq E^{\mathbb{Q}}(|(V(t))^2|) \times E^{\mathbb{Q}}(|W(t)|) + \left|\int_0^t E^{\mathbb{Q}}|W(p)|dp\right|$$

Evaluate $E^{\mathbb{Q}}(|W(t)|) = 2\sqrt{t}\int_0^\infty W(1)\frac{1}{\sqrt{2\pi}}e^{-\frac{W^2(1)}{2}}dW(1)$

$$= -\sqrt{\frac{2t}{\pi}}\int_0^\infty \frac{\partial e^{-\frac{W^2(1)}{2}}}{\partial W(1)}dW(1) = \sqrt{\frac{2t}{\pi}}$$

2. **Continued**

So, $E^{\mathbb{Q}}(|X(t)|) \le t\sqrt{\frac{2t}{\pi}} + \int_0^t \sqrt{\frac{2p}{\pi}}dp = t\sqrt{\frac{2t}{\pi}} + \frac{2}{3}\sqrt{\frac{2}{\pi}}\ t^{3/2} = \frac{5}{3}\sqrt{\frac{2}{\pi}}t^{3/2} < \infty$

This is unnecessarily complicated.

From $E(X^2) = \left(E(X)\right)^2 + E\left(\left(X - E(X)\right)^2\right)$ we know $E^{\mathbb{Q}}(|W(t)|) \le \sqrt{E^{\mathbb{Q}}\left(W^2(t)\right)} = \sqrt{t}$. Thus

$$E^{\mathbb{Q}}\left(\left|\left(V(t)\right)^2\right|\right) \times E^{\mathbb{Q}}(|W(t)|) + \left|\int_0^t E^{\mathbb{Q}}|W(p)|dp\right|$$

$$\le t \times \sqrt{t} + \left|\int_0^t \sqrt{p}dp\right| = \left(1 + \frac{2}{3}\right)t^{\frac{3}{2}} < \infty$$

Criteria 3 – martingale property

$$E_s^{\mathbb{Q}}(X(t)) = E_s^{\mathbb{Q}}\left((V(t))^2 \times W(t) - \int_0^t W(p)dp\right)$$

By the independence of V and W, and by splitting the integral we have

$$= E_s^{\mathbb{Q}}((V(t))^2) \times E_s^{\mathbb{Q}}(W(t)) - E_s^{\mathbb{Q}}\left(\int_0^s W(p)dp + \int_s^t W(p)dp\right)$$

Consider the first part of the equation. $W(t)$ *is a Brownian motion and a martingale. i.e.,*
$E_s^{\mathbb{Q}}\left(W(t) - W(s) + W(s)\right) = W(s)$. $E_s^{\mathbb{Q}}\left(\left(V(t)\right)^2\right) = E_s^{\mathbb{Q}}\left(\left(V(t) - V(s) + V(s)\right)^2\right) =$
$E_s^{\mathbb{Q}}((V(t) - V(s))^2) + 2E_s^{\mathbb{Q}}\left(V(s)(V(t) - V(s))\right) + E_s^{\mathbb{Q}}\left(\left(V(s)\right)^2\right) = t - s + \left(V(s)\right)^2$

So, the first part of the equation is $W(s)\left(t - s + \left(V(s)\right)^2\right)$

Next, consider the second part of the equation and use the measurability of the first integral and indepence of the integrand on (s,t) of $\mathcal{F}_s$

$$E_s^{\mathbb{Q}}\left(\int_0^s W(p)dp + \int_s^t W(p)dp\right)$$

$$= \int_0^s W(p)dp + \int_s^t E_s^{\mathbb{Q}}\left(W(p) - W(s) + W(s)\right)dp$$

$$= \int_0^s W(p)dp + W(s)\int_s^t dp = \int_0^s W(p)dp + (t - s)W(s)$$

2. Continued

Putting the two parts together, we have

$$E_s^{\mathbb{Q}}(X(t)) = W(s)\left(t - s + \left(V(s)\right)^2\right) - \left(\int_0^s W(p)dp + (t-s)W(s)\right)$$

$$= W(s)\left(V(s)\right)^2 - \int_0^s W(p)dp = X(s)$$

Hence, $X(t)$ *is a martingale*

3. **Learning Objectives:**

1. The candidate will understand the foundations of quantitative finance.

Learning Outcomes:

(1c) Understand Ito integral and stochastic differential equations.

(1d) Understand and apply Ito's Lemma.

(1f) Understand and apply Jensen's Inequality.

(1i) Demonstrate understanding of the differences and implications of real-world versus risk-neutral probability measures, and when the use of each is appropriate.

(1j) Understand and apply Girsanov's theorem in changing measures.

Solution:

(a) Show that the solution to the stochastic differential equation above is:

$$S_t = S_0 e^{-\theta t} + \mu\left(1 - e^{-\theta t}\right) + \sigma \int_0^t e^{-\theta(t-u)} dW_u.$$

Commentary on Question:

Candidates performed well on part(a). Most candidates were able to demonstrate that S_t as shown below satisfies the given SDE by way of Ito's Lemma.

Let $X_t = S_t e^{\theta t}$. Ito Lemma gives:

$$dX_t = e^{\theta t} dS_t + \theta S_t e^{\theta t} dt = e^{\theta t}[\mu\theta\, dt + \sigma\, dW_t]$$

therefore

$$X_t = X_0 + \mu\theta \int_0^t e^{\theta u}\, du + \sigma \int_0^t e^{\theta u}\, dW_u$$

and

$$S_t e^{\theta t} = S_0 + \mu(e^{\theta t} - 1) + \sigma \int_0^t e^{\theta u}\, dW_u$$

from which we obtain the desired solution.

(b) Derive

(i) a lower bound for $E^{\mathbb{P}}[S_t^2]$ using Jensen's inequality.

(ii) the exact value of $E^{\mathbb{P}}[S_t^2]$ using Ito isometry.

3. **Continued**

Commentary on Question:
Most candidates were able to recall Jensen's inequality, and the majority of them successfully applied the inequality to derive the lower bound for the expectation.

(i)
Jensen's theorem states that for f convex:

$$E\big(f(X)\big) \geq f\big(E(X)\big).$$

Let $f(x) = x^2$, convex.

Then

$$E(S_t^2) \geq [E(S_t)]^2 = \left[S_0 e^{-\theta t} + \mu\left(1 - e^{-\theta t}\right)\right]^2.$$

(ii)
First compute

$$Var(S_t) = \sigma^2 e^{-2\theta t} Var\left[\int_0^t e^{\theta u}\, dW_u\right] = \sigma^2 e^{-2\theta t} E\left[\int_0^t e^{\theta u}\, dW_u\right]^2$$

$$= \sigma^2 e^{-2\theta t}\left[\int_0^t E(e^{2\theta u})\, du\right]$$

by Ito isometry.

Evaluate the Riemann integral to obtain

$$Var(S_t) = \sigma^2 e^{-2\theta t}\,\frac{e^{2\theta t} - 1}{2\theta} = \frac{\sigma^2}{2\theta}\left(1 - e^{-2\theta t}\right).$$

Therefore:

$$E(S_t^2) = Var(S_t) + [E(S_t)]^2 = \frac{\sigma^2}{2\theta}\left(1 - e^{-2\theta t}\right) + \left[S_0 e^{-\theta t} + \mu\left(1 - e^{-\theta t}\right)\right]^2.$$

(c) Determine the Radon-Nikodym derivative $\frac{d\mathbb{Q}}{d\mathbb{P}}$ that validates your colleague's claim.

Commentary on Question:
Few candidates earned full credit in this part. While many could recall Girsanov's Theorem and the definition of the Radon-Nikodym derivative, quite a few candidates were not able to apply both correctly to validate the colleague's claim.

3. Continued

Let $\lambda_t = \frac{(r+\theta)S_t - \mu\theta}{\sigma}$; then define the measure $\mathbb{Q}$ by means of its RN derivative with respect to $\mathbb{P}$:

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = e^{\int_0^t \lambda_u dW_u - 0.5\int_0^t \lambda_u^2 du}$$

Girsanov's Theorem states that $W_t^{\mathbb{Q}}$, defined by

$$W_t^{\mathbb{Q}} = W_t - \int_0^t \lambda_u \, du$$

is a $\mathbb{Q}$-standard Wiener process with respect to the same filtration. This can be written as

$$dW_t^{\mathbb{Q}} = dW_t - \lambda_t \, dt$$

and substituted into the SDE of S_t to obtain

$$\begin{aligned} dS_t &= \theta(\mu - S_t)\, dt + \sigma\left(dW_t^{\mathbb{Q}} + \lambda_t \, dt\right) \\ &= \theta(\mu - S_t)\, dt + (r+\theta)S_t \, dt - \mu\theta \, dt + \sigma dW_t^{\mathbb{Q}} = rS_t \, dt + \sigma \, dW_t^{\mathbb{Q}} \end{aligned}$$

therefore

$$d(S_t e^{-rt}) = e^{-rt} dS_t - re^{-rt} S_t \, dt = \sigma e^{-rt} \, dW_t^{\mathbb{Q}},$$

confirming the colleague's claim.

(d) Compute the fair value of this option using risk-neutral valuation.

Commentary on Question:
Candidates performed poorly in this part. While many were able to identify the fair value of the option as the present value of the expected payoff under a risk-neutral measure, only a few were able to provide a complete solution.

Let V_0 be the option price at time $t = 0$.

Then:

$$V_0 = e^{-r} E^{\mathbb{Q}}\left[\mathbb{I}_{S_1 > K}\right] = e^{-r} \, \mathbb{Q}(S_1 > K).$$

Use part (a) with obvious substitutions to obtain

$$S_1 = S_0 e^r + \sigma \int_0^1 e^{-r(1-u)} \, dW_u^{\mathbb{Q}}$$

3. Continued

and observe that it is normally distributed, with

$$E^{\mathbb{Q}}(S_1) = S_0 e^r$$
$$Var^{\mathbb{Q}}(S_1) = \frac{\sigma^2}{2r}(e^{2r} - 1)$$

obtained from part (c), therefore

$$e^{-r}\,\mathbb{Q}(S_1 > K) = e^{-r}\left[1 - \mathrm{N}\left(\frac{K - S_0 e^r}{\sqrt{\frac{\sigma^2}{2r}(e^{2r} - 1)}}\right)\right]$$
$$= e^{-r}\,\mathrm{N}\left(\frac{S_0 e^r - K}{\sqrt{\frac{\sigma^2}{2r}(e^{2r} - 1)}}\right).$$

4. **Learning Objectives:**

2. The candidate will understand:
   - The Quantitative tools and techniques for modeling the term structure of interest rates.
   - The standard yield curve models.
   - The tools and techniques for managing interest rate risk.

Learning Outcomes:

(2b) Understand and apply various one-factor interest rate models.

(2c) Calibrate a model to observed prices of traded securities.

(2f) Apply the models to price common interest sensitive instruments including: callable bonds, bond options, caps, floors, and swaptions.

Sources:

1) Fixed Income Securities: Valuation, Risk, and Risk Management, Veronesi, Pietro, 2010 (Ch. 19)

2) An Introduction to the Mathematics of Financial Derivatives, Hirsa, Ali and Neftci, Salih N., 3rd Edition 2nd Printing, 2014 (Ch. 3)

Commentary on Question:

This question tests candidates' understanding of Hull-White model and applies the model to price call option. Most candidates were able to earn partial credits for this question.

Solution:

(a) Demonstrate that $\theta_t = m + \gamma^* c + \gamma^* m t + \frac{\sigma^2}{2\gamma^*}(1 - \mathrm{e}^{-2\gamma^* t})$

Commentary on Question:

Candidates performed well in this part. Candidates generally were able to derive θ_t and received full credit.

$$\theta_t = \frac{\partial f(0,t)}{\partial t} + \gamma^* f(0,t) + \frac{\sigma^2}{2\gamma^*}(1 - e^{-2\gamma^* t})$$

$f(0,t) = c + mt,\ \frac{\partial f(0,t)}{\partial t} = \mathrm{m}$

Hence,

$$\theta_t = m + \gamma^* c + \gamma^* m t + \frac{\sigma^2}{2\gamma^*}(1 - e^{-2\gamma^* t})$$

4. **Continued**

(b) Show that:

(i) $A(0;T) = \frac{c}{\gamma^*}(1 - \mathrm{e}^{-\gamma^* T}) - cT - \frac{mT^2}{2}$

(ii) $Z(r_0, 0; T) = e^{\frac{-mT^2}{2} - r_0 T}$

Commentary on Question:
Candidates performed below average for part (i). Most candidates who took solution 1 or solution 2 approach were able to derive A(0,T) successfully. Candidates who took solution 3 approach were mostly stopped at the first step.

Candidates performed as expected for part (ii). Candidates who failed to mention $r_0 = c$ *received partial credit.*

(i)
Solution 1:

$$f(t,T) = -\frac{\partial}{\partial T}(\ln Z(r,t;T))$$

$$f(0,T) = -\frac{\partial}{\partial T}(lnZ(r_0, 0; T))$$

Therefore,

$$Z(r_0, 0; T) = e^{-\int_0^T f(0,t)dt} = e^{-\int_0^T (c+mt)dt} = e^{-(cT + \frac{1}{2}mT^2)} = e^{A(0;T) - B(0;T)r_0}$$

$$\Rightarrow -\left(cT + \frac{1}{2}mT^2\right) = A(0;T) - B(0;T)r_0$$

And $r_0 = f(0{,}0) = c + m * 0 = c$

$$A(0;T) = B(0;T)r_0 - \left(cT + \frac{1}{2}mT^2\right) = \frac{C}{\gamma^*}(1 - e^{-\gamma^* T}) - cT - \frac{1}{2}mT^2$$

Solution 2:

$$f(t,T) = -\frac{\partial}{\partial T}(\ln Z(r,t;T))$$

$$f(0,T) = c + mT = -\frac{\partial}{\partial T}\ln Z(r_0, 0; T) == -\frac{\partial}{\partial T}\big(A(0;T) - r_0 B(0;T)\big)$$

But $B(0;T) = \frac{1 - e^{-\gamma^* T}}{\gamma^*} \Rightarrow \frac{\partial}{\partial T}B(0;T) = e^{-\gamma^* T}$

4. **Continued**

Therefore, $\frac{\partial}{\partial T}A(0;T) = \frac{\partial}{\partial T}r_0 B(0;T) - c - mT = r_0 e^{-\gamma^* T} - c - mT$

By integrating,

$A(0;T) - A(0;0) = r_0 * \frac{1-e^{-\gamma^* T}}{\gamma^*} - cT - \frac{1}{2}mT^2$

And $A(0;0) = 0$, (this implies from the fact that $Z(r,0;0) = 1$)

$$r_0 = f(0,0) = c + m * 0 = c$$

Therefore, $A(0;T) = \frac{c}{\gamma^*}(1 - \mathrm{e}^{-\gamma^* T}) - cT - \frac{mT^2}{2}$

Solution 3:

$$A(0;T) = \int_0^T -\theta_t B(0;T-t)dt + \frac{\sigma^2}{2(\gamma^*)^2}[T + \frac{1-e^{-2\gamma^*(T)}}{2\gamma^*}$$

$$= \int_0^T [-m - \gamma^* c - \gamma^* mt - \frac{\sigma^2}{2\gamma^*}(1 - e^{-2\gamma^* t})]\frac{\left(1 - e^{-\gamma^*(T-t)}\right)}{\gamma^*}dt$$

$$+ \frac{\sigma^2}{2(\gamma^*)^2}[T + \frac{1-e^{-2\gamma^*(T)}}{2\gamma^*} - 2B(0;T)]$$

$$= -\frac{mT}{\gamma^*} - cT - \frac{mT^2}{2} - \int_0^T \frac{\sigma^2}{2\gamma^{*2}}(1 - e^{-2\gamma^*(t)})dt$$

$$+ \frac{m}{\gamma^*}\int_0^T \frac{(e^{-\gamma^*(T-t)}}{1}dt + c\int_0^T e^{-\gamma^*(T-t)}\,dt + m\int_0^T t\,e^{-\gamma^*(T-t)}dt$$

$$+ \int_0^T \frac{\sigma^2}{2\gamma^{*2}}\left(e^{-\gamma^*(T-t)} - e^{-\gamma^*(T+t)}\right)dt$$

$$+ \frac{\sigma^2}{2(\gamma^*)^2}[T + \frac{1-e^{-2\gamma^*(T)}}{2\gamma^*} - 2B(0;T)]$$

$$= -\frac{mT}{\gamma^*} - cT - \frac{mT^2}{2} - \frac{\sigma^2}{2\gamma^{*2}}T - \frac{\sigma^2}{4\gamma^{*3}}e^{-2\gamma^* T} + \frac{\sigma^2}{4\gamma^{*3}}$$

$$+ m\frac{\left(1 - e^{-\gamma^*(T)}\right)}{\gamma^{*1}} + c\frac{\left(1 - e^{-\gamma^*(T)}\right)}{\gamma^*} + m\frac{(te^{-\gamma^*(T-t)}}{\gamma^*}|_0^T - \frac{m}{\gamma^*}\int_0^T e^{-\gamma^*(T-t)}\,dt$$

$$+ \frac{\sigma^2}{2\gamma^{*3}}\left(1 - e^{-\gamma^* T}\right) + \frac{\sigma^2}{2\gamma^{*3}}e^{-\gamma^*(2T)} - \frac{\sigma^2}{2\gamma^{*3}}e^{-\gamma^*(T)}$$

4. Continued

$$+\frac{\sigma^2}{2(\gamma^*)^2}[T+\frac{1-e^{-2\gamma^*(T)}}{2\gamma^*}-2\frac{1-e^{-\gamma^*(T)}}{\gamma^*}]$$

$$=-\frac{mT}{\gamma^*}-cT-\frac{mT^2}{2}-\frac{\sigma^2}{2\gamma^{*2}}T-\frac{\sigma^2}{4\gamma^{*3}}e^{-2\gamma^*T}+\frac{\sigma^2}{4\gamma^{*3}}$$

$$m\frac{(1-e^{-\gamma^*(T)})}{\gamma^{*2}}+c\frac{(1-e^{-\gamma^*(T)})}{\gamma^*}+m\frac{\left(te^{-\gamma^*(T-t)}\right)}{\gamma^*}|_0^T-\ m\frac{\left(1-e^{-\gamma^*(T)}\right)}{\gamma^{*2}}+\frac{\sigma^2}{2\gamma^{*3}}(1-e^{-\gamma^*T})$$

$$+\frac{\sigma^2}{2\gamma^{*3}}e^{-\gamma^*(2T)}-\frac{\sigma^2}{2\gamma^{*3}}e^{-\gamma^*(T)}+\frac{\sigma^2}{2(\gamma^*)^2}[T+\frac{1-e^{-2\gamma^*(T)}}{2\gamma^*}-2\frac{1-e^{-\gamma^*(T)}}{\gamma^*}]$$

$$=\frac{c}{\gamma^*}(1-e^{-\gamma^*T})-cT-\frac{mT^2}{2}$$

(ii)

$Z(r_0,t;T)=e^{A(t;T)-B(t;T)r_0}$

$Since\ r_0=f(0,0)=c+m*0=c$

$$Z(r_0,0;T)=e^{-cT-\frac{mT^2}{2}+\frac{c}{\gamma^*}(1-e^{-\gamma^*T})-\left(\frac{1-e^{-\gamma^*(T)}}{\gamma^*}\right)r_0}=e^{-r_0T-\frac{mT^2}{2}+\frac{r_0}{\gamma^*}(1-e^{-\gamma^*T})-\left(\frac{1-e^{-\gamma^*(T)}}{\gamma^*}\right)r_0}$$

$$Z(r_0,0;T)=e^{-\frac{mT^2}{2}-r_oT}$$

(c) Calculate γ^*.

Commentary on Question:

Most Candidates were able to answer the question correctly. For those who did not receive full marks was mostly due to calculation error.

$$\theta_t=m+\gamma^*c+\gamma^*mt+\frac{\sigma^2}{2\gamma^*}(1-e^{-2\gamma^*t})$$

$$\theta_{\left(\frac{1}{2\gamma^*}\right)}-\theta_0$$

$$=\gamma^*m(\frac{1}{2\gamma^*})+\frac{\sigma^2}{2\gamma^*}(1-e^{-2\gamma^*\frac{1}{2\gamma^*}})=\frac{m}{2}+\frac{\sigma^2}{2\gamma^*}(1-e^{-1})=\frac{0.0862}{2}+\frac{0.2^2}{2\gamma^*}(1-e^{-1})=0.084$$

$=>\ \gamma^*=0.309105$

4. Continued

(d) Calculate the price at t=0 of the1 year and 2 months European call with strike price of $90, written on zero-coupon bond with face value $100 and maturity of 2.5 years.

Commentary on Question:
Candidates performed below average in this part. Most candidates were able to show the formulas to calculate the call option but failed to modify the formulas for face amount $100. Some candidates did not have the correct formula for volatility.

Principal=100, K=90

$B(7/6;\ 2.5) = \frac{1}{0.309105}\left(1 - e^{-0.309105(2.5-7/6)}\right)$ = *1.09273*

$$S_Z\,(T_o; T_1) = B(T_o; T_B) * \sqrt{\frac{\sigma^2}{2\gamma}(1 - e^{-2\gamma T_O})}$$

$$S_Z\,(7/6; 2.5) = B(7/6; 2.5) * \sqrt{\frac{0.2^2}{2 * 0.309105}(1 - e^{-2*0.309105*7/6})}$$

$S_Z\,(7/6; 2.5) = 0.199248$

$$Z(0, r_0; 7/6) = e^{\frac{-0.0862\left(\frac{7}{6}\right)^2}{2} - 0.04*\frac{7}{6} = 0.900027}$$
=

$$Z(0, r_0; 2.5) = e^{\frac{-0.0862(2.5)^2}{2} - 0.04*2.5 = 0.691166}.$$

$$d_1 = \frac{1}{S_Z\,(T_o; T_B)} ln(Z(0, r_0; T_B)/(K_i Z(0, r_0; T_o)\,)) + \frac{S_Z\,(T_o; T_B)}{2}$$

$$d_1 = \frac{1}{0.199248} ln\left(\frac{100 * 0.691166}{90 * 0.900027}\right) + \frac{0.199248}{2}$$
$= -0.69679$

$d_2 = d_1 - S_Z\,(T_o; T_B)$ =*-0.89604*

$N(d_1)$ =*0.242967*, $N(d_2)$ =*0.185116*

The price of the call option
$V(r_0, 0) = 100 * Z(0, r_0; T_B)N(d_1) - 90Z(0, r_0; T_O)N(d_2)$
$= 100 * 0.691166 * N(d_1) - 90 * 0.900027 * N(d_2)$
=1.798

5. **Learning Objectives:**

2. The candidate will understand:
   - The Quantitative tools and techniques for modeling the term structure of interest rates.
   - The standard yield curve models.
   - The tools and techniques for managing interest rate risk.

Learning Outcomes:

(2b) Understand and apply various one-factor interest rate models.

(2c) Calibrate a model to observed prices of traded securities.

Commentary on Question:

This question tests candidates' understanding of interest rate calibrations. Most of the candidates earned full or partial credits from part (a) and (b), but only a few candidates earned partial credits from part (c).

Solution:

(a) Calculate the probability of simulating a negative interest rate for the next trading day.

Commentary on Question:

Most of the candidates were able to use the correct formula for this question. Candidates earned partial credits if they used the correct formula but failed to calculate the final numbers. Full credit will be given to candidates who calculated the correct value.

$P[r_{t+s} < 0|r_t] = \Phi(-\frac{\bar{r}+(r_t-\bar{r})e^{-\gamma s}}{\sigma\sqrt{\frac{(1-e^{-2\gamma s})}{2\gamma}}})$

Substituting values for paramteres

$$z = \frac{\bar{r} + (r_t - \bar{r})e^{-\gamma s}}{\sigma\sqrt{\frac{(1 - e^{-2\gamma s})}{2\gamma}}} = 2.66113$$

Then the probability is $\Phi(-2.66113) = 0.0039$

(b) Calculate the simulated rate for the next trading day using

(i) the Euler-Maruyama discretization method.

(ii) the transition density method.

5. Continued

Commentary on Question:
Some of candidates earned partial credits for using the correct formulas and parameters but only a few candidates calculated the correct values.

(i) Under Euler-Maruyama discretization

$$r(i) = \alpha + \beta r(i-1) + \epsilon_i, i = 1,2, \ldots$$

Where $\alpha = \gamma\bar{r}\Delta, \beta = 1 - \gamma\Delta$ and $\epsilon_i \sim N(0, {\sigma^*}^2)$ with $\sigma^* = \sigma\sqrt{\Delta}$ With given parameters

$$\alpha = 0.3 * 0.05 * \frac{1}{252} = 5.952 * 10^{-5}$$
$$\beta = 1 - 0.05 * \frac{1}{252} = 0.9980$$
$$\sigma^* = 0.06 * \sqrt{\frac{1}{252}} = 0.0037796$$
$$r(i-1) = 0.01$$
$$r(i) = 5.952 * 10^{-5} + 0.9980 * 0.01 - 1.96 * 0.0037796$$
$$r(i) = 0.002631$$

(ii) With the transition density method next random number from the Vasicek is given by

$$r_{t+s} = \bar{r} + (r_t - \bar{r})e^{-\gamma s} + \left(\frac{\sigma^2}{2\gamma}(1 - e^{-2\gamma s})\right)^{\frac{1}{2}} Z$$

With the given parameters

$$(r_t - \bar{r})e^{-\gamma s} = (0.01 - 0.05)e^{-\frac{0.3}{252}} = -0.03995241$$
$$\left(\frac{\sigma^2}{2\gamma}(1 - e^{-2\gamma s})\right)^{\frac{1}{2}} = 0.03777$$
$$r_{t+s} = 0.05 - 0.03995241 + 0.03777 * (-1.96)$$
$$r_{t+s} = 0.002644$$

(c) Compare and contrast the Euler-Maruyama discretization method and the transition density method for simulating interest rate paths in general and in this particular case for Vasicek model.

Commentary on Question:
Not many candidates attempted this question and some of them successfully identified Euler-Maruyama method is an approximating method and Transition density method is an exact method. However, few candidates pointed out that the differences between those two methods are minimal when s is small in the Vasicek model.

5. Continued

The Euler-Maruyama method is based on the first order discretization of a stochastic differential equation (or simiple discretization)
i.e.

$$dr_t = a(r_t)dt + b(r_t)dX_t$$

is approximated using

$$r_{t+\Delta} - r_t \approx a(r_t)\Delta + b(r_t)\sqrt{\Delta}Z$$

Where Δ is a small time step and Z is a standard normal random variable with mean 0 and variance 1.
Essentially in simulation we are assuming $r_{t+\Delta}|r_t$ is normally distributed with mean $r_t + a(r_t)\Delta$ and variance $b(r_t)^2\Delta$. So even if the original process doesn't take negative values, the approximation may give negative values.

Transition density method relies on the exact distribution of $r_{t+\Delta}|r_t$. So it is an exact method not an approximation. The disadvantage of this method is the exact distribution may not be available for many cases.

In the Vasicek method as we saw in part (a) and (b) the difference is minimal. That is because the exact distribution of $r_{t+s}|r_t$ is normal and since for small values of s

$$e^{-\gamma s} \approx 1 - \gamma s$$

With that

$$\bar{r} + (r_t - \bar{r})e^{-\gamma s} \approx \bar{r} + (r_t - \bar{r})(1 - \gamma s)$$
$$\bar{r} + (r_t - \bar{r})e^{-\gamma s} \approx \bar{r}\gamma s + r_t(1 - \gamma s)$$

and $\left(\frac{\sigma^2}{2\gamma}(1 - e^{-2\gamma s})\right) \approx \sigma^2 s$. These are the mean and variance in the Euler-Maruyama discretization method.

6. Learning Objectives:

2. The candidate will understand:
   - The Quantitative tools and techniques for modeling the term structure of interest rates.
   - The standard yield curve models.
   - The tools and techniques for managing interest rate risk.

Learning Outcomes:

(2b) Understand and apply various one-factor interest rate models.

(2c) Calibrate a model to observed prices of traded securities.

(2d) Describe the practical issues related to calibration, including yield curve fitting.

(2g) Understand and apply the techniques of interest rate risk hedging.

Sources:

Fixed Income Securities: Valuation, Risk, and Risk Management, Veronesi, Pietro, 2010, Chapter 15-16

Commentary on Question:

Commentary listed underneath question component.

Solution:

(a) Derive the expression for Z_t by solving the above ODEs.
(Hint: The solutions of an ODE $f'(x) = f(x)\gamma - 1$ are $f(x) = \frac{1}{\gamma} - Ce^{\gamma x}$ where C is a constant.)

Commentary on Question:
This was a challenging part. About half of the candidates attempted to solve it and of those very few went beyond applying the hint to solve for B. Some candidates did not realize that one needs a boundary condition to find the constants.

At time T Z(T,T) = 1, therefore

$$A(T,T) - B(T,T)r_T = 0$$

And

$$A(T,T) = 0$$
$$B(T,T) = 0$$

Since the exponent of Z must be 0 for any value of r_T.

From the hint:
$B(t,T) = C - \frac{1}{\gamma} e^{-\gamma(t-T)}$, from B(T,T) = 0 C =$\frac{1}{\gamma}$, and therefore

$$B(t,T) = \frac{1}{\gamma} (1 - e^{-\gamma(t-T)})$$

6. Continued

To solve for A, we integrate:

$\frac{dA(t,T)}{dt} = B(t,T)\gamma\bar{r} - \frac{1}{2}B(t,T)^2\sigma^2 = \left(1 - e^{-\gamma(T-t)}\right)\bar{r} - \frac{\sigma^2}{2}\left(\frac{1-e^{-\gamma(T-t)}}{\gamma}\right)^2$ to get:

$A(t,T) = \left(t - \frac{1}{\gamma}e^{-\gamma(T-t)} + C\right)\bar{r} - \frac{\sigma^2}{2\gamma^2}\left(t - \frac{2}{\gamma}e^{-\gamma(T-t)} + \frac{1}{2\gamma}e^{-2\gamma(T-t)} + D\right)$, where C and D are constants. From the boundary condition

$$A(T,T) = 0$$

$A(T,T) = (T - \frac{1}{\gamma} + C)\bar{r} - \frac{\sigma^2}{2\gamma^2}\left(T - \frac{2}{\gamma} + \frac{1}{2\gamma} + D\right) = 0$, *therefore* $C = \frac{1}{\gamma} - T$ *and* $D = \frac{3}{2\gamma} - T$

Plugging into the formula for A and rearranging we get:

$$A(t,T) = [B(t,T) - (T-t)]\left(\bar{r} - \frac{\sigma^2}{2\gamma^2}\right) - \frac{\sigma^2}{4\gamma}B(t,T)^2$$

(b) Choose the best parametrization γ and $\bar{r}$. Show your work to support the choice.

Commentary on Question:
Most candidates worked on this part and in general they did well, common mistakes were typos when entering the formulae in Excel. Candidates who had such errors got partial credit for this part

To parametrize γ and $\bar{r}$, the values of the parameters should produce bond prices that are closest to the observed market values. Minimum least square of difference between modeled vs. observed values can be used as the criteria.

Formulae for A and B are in the Formula Sheet provided to all candidates during the exam.

Option 3 is the correct answer, for calculations see the Excel spreadsheet.

(c) Determine the replicating portfolio at time 0, for the zero-coupon bond $Z_0(2\ year)$ using the zero-coupon bond $Z_0(4\ year)$ and cash.

Commentary on Question:
Most candidates attempted this part. Common errors were wrong formula for Δ or the replicating portfolio. Partial credit was given for the correct set up of the problem but wrong or no calculations.

The replicating portfolio for $Z_0(2)$ *using* $Z_0(4)$ *should consist of* Δ *unit of* $Z_0(4)$ *and cash:*

$$P_0 = \Delta Z_0(4) + C_0 = Z_0(2)$$

Choose Δ *such that*

$$\Delta = \frac{\partial Z_0(2)/\partial r}{\partial Z_0(4)/\partial r} = \frac{B(0,2)Z_0(2)}{B(0,4)Z_0(4)}$$

Then the Cash position is

$$C_0 = Z_0(2) - \Delta Z_0(4)$$

For calculations see Excel.

6. Continued

(d) Calculate, $P_{0.5}$, the value of the rebalanced replicating portfolio immediately after the rebalancing at time 0.5.

Commentary on Question:
Candidates did not do well in this part. Common errors were wrong Δ, wrong rollforward of the cash position, etc. Partial credits were given for calculating the correct Δ, or for the correct problem set up but no calculations.

At time 0.5, re-calculate the Δ as

$$\Delta = \frac{\partial Z_{0.5}(2)/\partial r}{\partial Z_{0.5}(4)/\partial r} = \frac{B(0.5,2)Z_{0.5}(2)}{B(0.5,4)Z_{0.5}(4)}$$

New Cash position at t=0.5 required is

$$C_{0+0.5} = C_0 + C_0 r_0 dt - Cash\ needed/generated\ for\ rebalancing$$

Cash needed/generated for rebalancing is

$$(\Delta_{0.5} - \Delta_0)Z_{0.5}(4)$$

Therefore the portfolio value at t=0.5 is

$$P_{0.5} = \Delta_{t+dt} Z_{4,t+dt} + C_{t+dt}$$

For Calculations see Excel.

(e)

(i) Illustrate a relative value trade strategy using $Z_0(2\ year)$ and $Z_0(4\ year)$.

(ii) Calculate the profit of the strategy at time 0.5.

Commentary on Question:
About half of the candidates attempted this part (i). Common mistakes were not identifying the correct arbitrage or wrong calculations. Candidates who identified the correct strategy only got partial credit.

(i) Based on the parameters and given bond prices, bond $Z_0(2)$ is over-priced (market price is higher than the modeled price) and $Z_0(4)$ under-priced according to the Vasicek model (from the question (b) above). Thus a relative value trade strategy can be set up to short $Z_0(2)$ and long replicating portfolio of $Z_0(2)$ using $Z_0(4)$.

Execute the strategy as follow:

1. At time 0:

- Short sell $Z_0(2)$ for 9,248.49
- Long $\Delta = 0.8059$ unit of $Z_0(4)$
- Invest cash position of 2,487.3 at overnight deposit yielding the short rate (initial yield is $r_0 = 3\%$)

6. Continued

2. After time 0, dynamically re-balance the replicating portfolio so that at time $T = 2$, the replication portfolio generates 10,000 to deliver the notional of the short sold bond. If the replicating portfolio is rebalanced in continuous time, the portfolio can replicate $Z_t(2)$ without gain/loss after time 0.

(ii) Profit from the strategy:

Based on observed bond prices, gain from setting up the strategy at time 0 is $9{,}400 - 0.8059 \times 8{,}300 - 2{,}487.3 = 224.01$. There is no future gain/loss after time 0 by dynamically rebalancing the replicating portfolio. Thus profit from the strategy is 224.01.

(f) List two considerations when executing a relative value trade strategy as in part (e)(i).

Commentary on Question:
About half of the candidates attempted this part. This part asks about the shortcomings of the Vasicek model. However, most of the attempts discussed general shortcomings of trading strategies, while such answers were correct in general, they do not answer the question asked, hence they got only partial credit.

The relative value trade strategy in part d) should work if Vasicek model is correct. However, the model is always imperfect, especially for a one factor model. The one-factor Vasicek model has known disadvantages such as:
Perfectly correlated short- and long-term interest rates, and thus inability to capture curvature of the yield curve properly.
Understating volatilities for long term rates

Thus, the strategy may not generate the profit as expected, as the replication strategy may not work as expected. Traders should only undertake such strategies when the apparent arbitrage is big enough to cover potential losses in the replication strategy.

7. **Learning Objectives:**

3. The candidate will understand:
   - How to apply the standard models for pricing financial derivatives.
   - The implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory.
   - How to evaluate risk exposures and the issues in hedging them.

Learning Outcomes:

(3d) Demonstrate an understanding of how to delta hedge, and the interplay between hedging assumptions and hedging outcomes.

(3g) Describe and explain some approaches for relaxing the assumptions used in the Black-Scholes-Merton formula.

(3h) Compare and contrast the various kinds of volatility, e.gl, actual, realized, implied and forward, etc.

(3i) Define and explain the concept of volatility smile and some arguments for its existence.

Sources:

QFIQ-115-17: Which Free Lunch Would You Like Today, Sir?: Delta Hedging, Volatility Arbitrage and Optimal Portfolios

Commentary on Question:

Commentary listed underneath question component.

Solution:

(a) List the pros and the cons of hedging with implied volatility and actual volatility.

Commentary on Question:
Candidates generally did well on this part of the question.

Pros of hedging with implied volatility:

- No local fluctuations in profit and loss (continually making a profit)
- Only need to be on the right side of the trade to profit (buy when actual is going to be higher than implied and sell if lower)
- The number that goes into the delta is implied volatility, which is easy to observe
- The profit each day is deterministic

Cons of hedging with implied volatility:

- You don’t know how much money you will make, only that it is positive. The present value of the total profit at expiration is path dependent

Pros of hedging with actual/realized volatility:

- Profit at expiration is known when hedging with actual volatility

7. Continued

Cons of hedging with actual/realized volatility:

- Subject to profit and loss fluctuations during the life of the option, which can be less appealing from a local risk management perspective
- Unlikely to be totally confident in your volatility forecast (the number put into the delta formula)

(b) Choose the most appropriate volatility for hedging under each of the following two constraints.

(i) Mark to model

(ii) Mark to market

Commentary on Question:
Candidates generally did well on this question.

Under the constraint of "Mark to model" where you are not concerned about the day-to-day fluctuations in the mark-to-market profit and loss, it is better to hedge with actual volatility if you are confident about estimating the actual volatilities. Its expected total profit is not far from the optimal payoff under hedging with implied vol and its standard deviation of final profit is zero.

Under the constraint of "Mark to Market" where you must worry about the short-term fluctuations of profit and loss, it is more appropriate to hedge with implied volatility under which you continuously make profit without much short-term fluctuation and annoyance from risk management despite the final profit is path dependent.

(c) Design a volatility arbitrage to make money assuming that your prediction is correct and that you hedge with actual volatility.

Commentary on Question:
Most candidates noted why you should buy the call option, but not all did (i.e., the call was undervalued since actual volatility is higher than implied). Most candidates correctly wrote to buy the call option and sell the stock, although not all mentioned that the number of shares is determined by delta (N(d1)). Many candidates missed the last piece of the volatility arbitrage strategy – to invest the cash earning the risk-free rate or borrow paying the risk-free rate – and many candidates missed the fact that the volatility arbitrage needs to be rebalanced frequently.

7. **Continued**

Because the predicted actual volatility is higher than the implied volatility, the call option is under-valued.

Thus, the volatility arbitrage strategy is to:
(a) Buy the call option
(b) Sell the stock XYZ by shares determined by the Delta N(d1) where d1 is calculated using actual volatility
(c) Invest the cash earning the risk-free rate or borrow paying the risk-free rate
The strategy needs to be executed and the delta hedge to be rebalanced as frequently as possible (e.g., daily)

(d) Calculate the final profit from the arbitrage executed in part (c).

Commentary on Question:
Many candidates did well here.

S=100, K=100, r=0%, T=1
σ(actual) = 30%; σ(implied) = 20%

$$F(S_t, t) = S_t N(d_1) - Ke^{-r(T-t)} N(d_2)$$

$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{\ln(S/K) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

Plug in all the values, the Black-Scholes formula for the call option can be simplified because of r=0, d=0, T=1, and S/K=1

7. **Continued**

c = 2*100*[Norm(0.15) – Norm(0.1)] = 3.958

$$F(S_t, t) = S_t N(d_1) - K e^{-r(T-t)} N(d_2)$$

$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{\ln(S/K) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

Value *x*	Standard Normal *N(x)*
0.10	0.5398
0.15	0.5596
0.20	0.5793
0.25	0.5987
0.30	0.6179

Simplied calculation

0.1500	0.1000
-0.1500	-0.1000
0.5596	0.5398
0.4404	0.4602

Call (actual vol)	Call (implied Vol)
11.9235	7.9656

Profit (hedging w/ actual vol)

3.9580 3.958

8. **Learning Objectives:**

3. The candidate will understand:
   - How to apply the standard models for pricing financial derivatives.
   - The implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory.
   - How to evaluate risk exposures and the issues in hedging them.

Learning Outcomes:

(3a) Demonstrate an understanding of option pricing techniques and theory for equity derivatives.

(3b) Identify limitations of the Black-Scholes-Merton pricing formula.

(3c) Demonstrate an understating of the different approaches to hedging – static and dynamic.

(3d) Demonstrate an understanding of how to delta hedge, and the interplay between hedging assumptions and hedging outcomes.

(3i) Define and explain the concept of volatility smile and some arguments for its existence.

Sources:

The Volatility Smile, Derman, Emanuel and Miller, Michael B., 2016

QFIQ-120-19: Chapters 6 and 7 of Pricing and Hedging Financial Derivatives, Marroni, Leonardo and Perdomo, Irene, 2014

QFIQ-115-17: Which Free Lunch Would You Like Today, Sir?: Delta Hedging, Volatility Arbitrage and Optimal Portfolios

Commentary on Question:

This question tested candidates' knowledge on profiles of the Greeks such as Delta, Vega, Theta and Gamma, etc. and its application to analyze the risk profiles on synthetic trading strategies. It also tested the Vanna approach to deal with Greek profile change due to volatility skew. Majority of the candidates performed reasonably well from (a) – (e) as they can get partial credits by providing some attributes for Greeks but unable to cover all the key characteristics. However, candidates performed below expectation from (f) – (g) as they show less familiarity with the topic.

Solution:

(a) Describe the key characteristics of its Delta profile.

8. Continued

Commentary on Question:
In general, candidates performed as expected. Most candidates listed a few characteristics of the Delta of Strategy A. However, only handful candidates provided all key characteristics.

Option Strategy A is a call spread strategy. The delta of the call spread is the sum of the delta of the two calls.
The delta of the call spread is positive (above x-axis) because it is long the ATM call (with a lower strike and thus higher delta) and short the OTM call (with a higher strike and thus lower delta)
The delta of the call spread approaches zero when it is far out of the money when stock price is far away from 100 on both sides (S<70 and S>160, for example).
The delta of the call spread is the highest when the stock price is in between of the two strikes (100 and 120).

(b) Draw the Delta profile in the Excel spreadsheet.

Commentary on Question:
Candidates performed well on this question. Majority of the candidates received full credits. Some candidates made minor formula mistakes and received partial credits.

See excel.

(c) Describe the key characteristics of its Gamma profile.

Commentary on Question:
Many candidates were able to identify the basic features of Gamma profiles of vanilla options and applied them to describe that of the call spread. However, many candidates simply provided the conclusion of the Gamma profile without providing details that support the conclusions. Some candidates misstate that strategy price approaches 0 when stock price is below 100 or above 120, failed to realize that strategy need to be far away from the strikes to approach 0.

The Gamma of the call spread is the sum of the Gamma of the two calls.

When the stock moves further and further away out-of-the-money, option value becomes very insensitive and thus Gamma (and Delta) should approach zero.

8. **Continued**

When the stock price is close to 100, the gamma of the ATM call, which the call spread is long, should dominate the gamma of the OTM call, which the call spread is short, and thus the net Gamma is positive. Conversely, when the stock price is close to 120, the gamma of the OTM call, which the call spread is short, should dominate the gamma of the ATM call, which the call spread is short, and thus the net Gamma is negative.

In between, the net Gamma should cross zero where the net Delta is maximized.

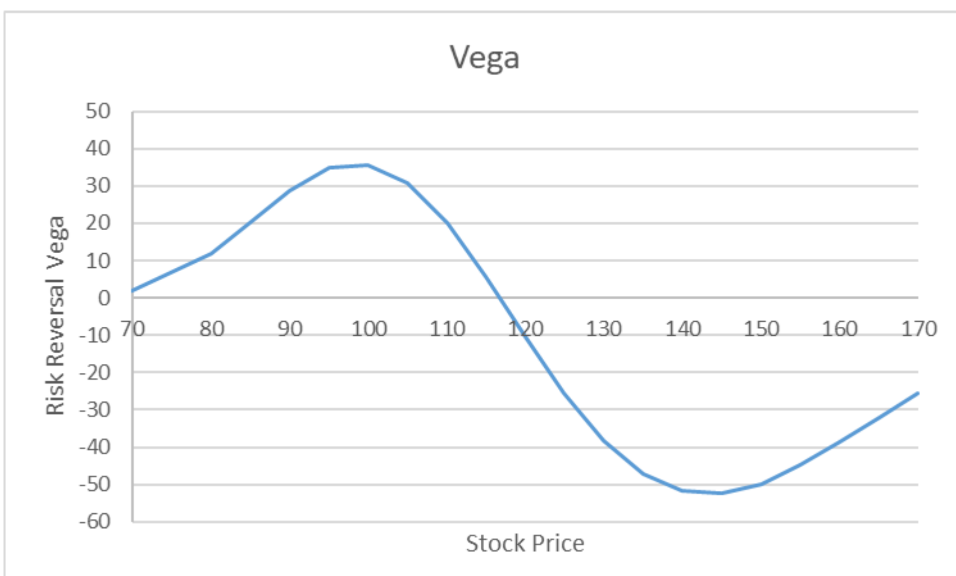
(d) Describe their key characteristics of its Vega and Theta profiles.

Commentary on Question:
Many candidates identified that Vega has a similar profile to Gamma and Theta is approximately a flip of the Gamma profile. However, many candidates did not provide sufficient reasoning on why the strategy of Vega and Theta has such a profile.

Vega has a similar profile as Gamma and flips from positive to negative as the stock price rises and flips from negative to positive as the stock price falls. Theta usually shows the opposite of the Gamma and thus its profile is approximately a flip of the Gamma profile.

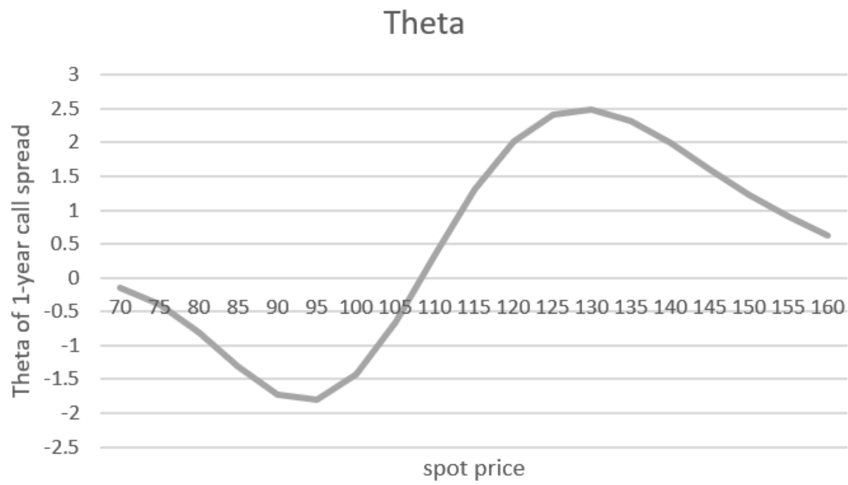
Both Vega and Theta will approach zero when the stock price moves sufficiently further away from the spot price.

Vega:



8. Continued

Theta:



(e) Sketch the Vega profile and describe the key characteristics.

Commentary on Question:
Many candidates identified that strategy B was a risk reversal strategy and were able to describe its Vega profile. Some candidates only sketched the Vega profile without sufficiently describing its characteristics therefore received partial credits.

Option Strategy B is a risk reversal strategy. The long position on the call gives a long Vega position, whereas the short position on the put gives a short Vega position. The overall Vega is the sum of the Vega of the long call and the short put.

Vega, Gamma, and Theta tend to reach their maximum values when the underlying is close to their respective strikes. Thus, as the underlying approaches 110, the Vega from the call option will dominate. Similarly, as the underlying approaches 90, the Vega from the put option will prevail.

Thus, Vega is positive around 110 area and negative around 90 area and crosses zero around 100.

8. **Continued**

Sketch the graph as follows:

(f) Evaluate the impact of the change to Option Strategy B.

Commentary on Question:
Candidates performed below expectations. Many candidates did not analyze the relationship between the option values and the corresponding volatility change and therefore, made the wrong conclusion. Some candidates did not realize that the price change for 90-strike option is larger than that of 110-strike due to the magnitude of the volatility change.

Option Strategy B takes a long position on a call option with a strike at 110 and a short position on a put option with a strike at 90.

This shows that the strategy takes a long position on the implied volatility at 110 and a short position on the implied volatility at 90.

The change in the volatility skew indicates both implied volatilities at 110 and at 90 have decreased but the implied vol at 110 has decreased less than the implied vol at 90 has decreased, so the long call loses less than the short put loses.

On the net basis, the option strategy benefits from the flattening of the volatility skew.

(g) Define its *Vanna ratio* and *Vanna contribution*.

Commentary on Question:
Candidates performed below expectations. While many candidates were able to give the definition of Vanna, they could not describe well what Vanna contribution is. Almost all candidates failed to identify that the Vanna ratio is 1 as the strategy B itself is a risk reversal.

8. Continued

Vanna ratio is the ratio of the Vanna of the exotic option, which itself is a Risk Reversal, over the Vanna of Risk Reversal, and this ratio is 1.

Vanna contribution is the difference between the market price of the Risk Reversal and the Risk Reversal priced with the 50 Delta volatility. It measures how "wrong" a risk reversal would be if the volatility smile is ignored and only the 50-delta volatility is used for pricing the option.

(h) Describe how to apply the Vanna adjustment.

Commentary on Question:
Candidates do not perform well in general in this question. Many candidates are able to list the formula for Vanna cost but missed one or two steps for the procedure for applying the Vanna adjustment. Very few candidates received full credits.

Calculated the Vanna ratio and the Vanna contribution defined in (g)

Calculate the Vanna cost = Vanna ratio X Vanna contribution

Obtain the option price using Black-Scholes framework, call it BSPrice for the risk reversal option in Option B.

The Vanna adjusted option price
Vanna-Adjusted Price = BSPrice + Vanna cost

9. Learning Objectives:

3. The candidate will understand:
   - How to apply the standard models for pricing financial derivatives.
   - The implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory.
   - How to evaluate risk exposures and the issues in hedging them.

4. The candidate will learn how to apply the techniques of quantitative finance to applied business contexts.

Learning Outcomes:

(3a) Demonstrate an understanding of option pricing techniques and theory for equity derivatives.

(3e) Analyze the Greeks of common option strategies.

(4a) Identify and evaluate embedded options in liabilities, specifically indexed annuity and variable annuity guarantee riders (GMAB, GMDB, GMWB and GMIB).

(4c) Demonstrate an understanding of dynamic and static hedging for embedded guarantees, including:
 (i) Risks that can be hedged, including equity, interest rate, volatility and cross Greeks.
 (ii) Risks that can only be partially hedged or cannot be hedged including policyholder behavior, mortality and lapse, basis risk, counterparty exposure, foreign bonds and equities, correlation and operation failures

Sources:
QFIQ-132-21, Volatility Smile-Derman-Miller-Ch 03

Commentary on Question:
The majority of candidates performed poorly on this question. Particularly for parts (b) and (c), many candidates either performed poorly on or entirely skipped those questions. For those that did attempt the question, the most common mistakes were not being able to derive the Greeks for the Asian call option.

Solution:
(a)
 (i) Identify the type of options which should be purchased.

 (ii) Calculate the values in the table below, (assuming a Black-Scholes framework):

Commentary on Question:
Candidates performed adequately for this part. Most were able to identify that a geometric mean Asian option needed to be purchased.

RB = 1,000,000*(1- exp(-0.03)) = 29,554.4664515

9. Continued

Amount of ZCBs to buy = (1,000,000 – RB)/1000 = 970.4455335 (assuming each bond has a notional of $1,000)

The risk budget should be invested in long ATM geometric mean Asian call options with maturity of 1 year.

Using the BS framework, the value of a vanilla European call option is:
C_0^E = N(d1) $S_0 e^{-qt}$ – N(d2)*K*e^{-rt} where q= continuous dividend rate and d1 = $\frac{\ln\left(\frac{S_t}{k}\right)+\left(r-q+\frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}}$

Substituting in q = $-\frac{1}{2}\left(r-\frac{\sigma_a^2}{2}\right)+r$ and $\sigma_a = \frac{\sigma}{\sqrt{3}}$ gives:

C_0^a = N(d1) $S_0 e^{\left(\frac{1}{2}\left(r-\frac{\sigma_a^2}{2}\right)-r\right)t}$ – N(d2)*K*e^{-rt} and

d1 = $\frac{\ln\left(\frac{S_t}{k}\right)+\left(r-\left(\frac{-1}{2}\left(r-\frac{\sigma_a^2}{2}\right)+r\right)+\frac{\sigma_a^2}{2}\right)t}{\sigma_a\sqrt{t}} = \frac{\ln\left(\frac{S_t}{k}\right)+\left(\frac{1}{2}\left(r-\frac{\sigma_a^2}{2}\right)+\frac{\sigma_a^2}{2}\right)t}{\sigma_a\sqrt{t}} = \frac{\ln(1)+\left(\frac{1}{2}\left(0.03-\frac{\frac{0.2^2}{3}}{2}\right)+\frac{\frac{0.2^2}{3}}{2}\right)}{\frac{0.2}{\sqrt{3}}}$ = 0.158771324

d2 = d1 - $\sigma_a\sqrt{t}$ = 0. 0.158771324 – $\frac{0.2}{\sqrt{3}}$ = 0.04330127

C_0^a = $N(0.158771324) * 100e^{\left(\frac{1}{2}\left(0.03-\frac{\frac{0.2^2}{3}}{2}\right)-0.03\right)}$ – N(0.04330127)*100*$e^{-0.03}$

= $0.563075 * 100e^{\left(\frac{1}{2}\left(0.03-\frac{\frac{0.2^2}{3}}{2}\right)-0.03\right)}$ – 0.517269*100$e^{-0.03}$

= 5.086478857

# of call options to purchase = RB/C = 29554.4664515/5.086478857 = 5810.398

$p_{dpa} = \frac{RB}{\frac{1{,}000{,}000 C_0^a}{100}}$ = 29554.4664515/(10^4*5.086478857) = 0.581039798

(b)

(i) Determine the Vega of the Asian call options above.

(ii) Explain the value of the above Vega in relation to the Vega of a European call option and why this relation intuitively makes sense.

Commentary on Question:
Candidates performed poorly in this part. A large majority of candidates did not correctly derive the expression for the Vega of an Asian call option necessary for part (i). In part (ii), most candidates were able to correctly explain that the Vega of an Asian call is less than that of a European call.

9. **Continued**

C_0^a = N(d1) $S_0 e^{\left(\frac{1}{2}\left(r-\frac{\sigma_a^2}{2}\right)-r\right)t}$ – N(d2)*K*e^{-rt} = $S_0 e^{-rt}$[N(d1)$e^{\frac{1}{2}\left(r-\frac{\sigma^2}{6}\right)} - N(d2)$]

By the product rule and chain rule:

$\frac{\partial C_0^a}{\partial \sigma} = S_0 e^{-rt}[\frac{-2\sigma}{12} N(d1) e^{\frac{1}{2}\left(r-\frac{\sigma^2}{6}\right)} + e^{\frac{1}{2}\left(r-\frac{\sigma^2}{6}\right)} n(d1) * \frac{\partial}{\partial \sigma} d1 - n(d2) * \frac{\partial}{\partial \sigma} d2]$

$$\frac{\partial}{\partial \sigma} d1 = \frac{\partial}{\partial \sigma} \frac{\left(\frac{1}{2}\left(0.03 - \frac{\frac{\sigma^2}{3}}{2}\right) + \frac{\frac{\sigma^2}{3}}{2}\right)}{\frac{\sigma}{\sqrt{3}}} = \frac{\partial}{\partial \sigma} \frac{\left(0.015 + \frac{\sigma^2}{12}\right)}{\frac{\sigma}{\sqrt{3}}} = \frac{-0.015 * \sqrt{3}}{\sigma^2} + \frac{\sqrt{3}}{12} = -0.50518$$

Note: d2 = d1 - $\frac{\sigma}{\sqrt{3}}$ -> $\frac{\partial}{\partial \sigma} d2 = \frac{\partial}{\partial \sigma} d1 - \frac{1}{\sqrt{3}} = -1.08253$

From Part A, N(d1) = N(0.158771324) = 0.563075

$\frac{\partial C_0^a}{\partial \sigma}$ = $100e^{-0.03}$($\frac{-0.2}{6}$*1.011735* 0.563075+($e^{\frac{1}{2}\left(r-\frac{\sigma^2}{6}\right)} n(d1) * -0.50518) - (n(d2) * -1.08253)$)

n(d1) = $\frac{e^{\frac{-(0.158771324^2)}{2}}}{\sqrt{2\pi}}$= 0.393946, n(d2) = $\frac{e^{\frac{-(0.04330127^2)}{2}}}{\sqrt{2\pi}}$= 0.398568

$\frac{\partial C_0^a}{\partial \sigma}$ = $100e^{-0.03}$(-0.018899 + 1.011735 * 0.393946 * -0.50518 - 0.398568*-1.08253) = 20.48845

➔ Total Option Vega = 20.48845* 1,000,000/100 = 204,884.5

European call option Vega = $S_0 n(d1)$, d1 = $\frac{(r+\frac{\sigma^2}{2})}{\sigma}$=$\frac{(0.03+\frac{0.04}{2})}{0.2}$ = 0.25

Vega = 100*$\frac{e^{\frac{-0.25^2}{2}}}{\sqrt{2\pi}}$=38.66681168

The Vega of the European call option is greater than the Vega of the Asian option.

This relation makes sense since Asian options sample the underlying asset price across the entire option period rather than simply the final price, resulting in a shorter average duration for the impact of the volatility. Since volatility and its impact on option prices scales with time this results in a lower Vega.

Note: award ½ point for recognizing the European option Vega is larger if appropriate value or logic is given. Award second half point as long as the candidate references stock prices being sampled across the period rather than just the final price, and this resulting in a lower sensitivity to volatility.

9. Continued

(c) Determine an initial Delta-Vega hedge position using an ATM 1-year European call option and the underlying stocks.

Commentary on Question:
Candidates performed very poorly in this part. Many candidates skipped this question. For those that attempted the question, they were not able to correctly calculate the Greeks for the Asian call option.

Need to solve for position such that delta and Vega = 0

Vega of European Call Option = S_0 *N'(d1) * Sqrt(T-t)

d1 = $\frac{\ln\left(\frac{S_t}{k}\right)+\left(r-q+\frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}} = \frac{0+\left(0.03+\frac{0.2^2}{2}\right)}{0.2}$ = 0.25
N'(d1) = 0.386668
Vega = 100 * 0.386668 =38.6668

- ➔ Need Vega of option to equal 204,884.5
- ➔ Need to buy 204,884.5/38.6668 = 5,298.718798 ATM 1 year euro call options

Delta of European Call Option = N(d1) = 0.598706

Delta of Asian option:

C_0^a = N(d1) $Se^{\left(\frac{1}{2}\left(r-\frac{\sigma_a^2}{2}\right)-r\right)t}$ – N(d2)*K*e^{-rt}

= $e^{-rt}[S\text{N}(\text{d}1)e^{\frac{1}{2}\left(r-\frac{\sigma^2}{6}\right)} - KN(d2)]$

By the product rule and chain rule:

$$\frac{\partial C_0^a}{\partial S} = e^{-rt}[\text{N}(\text{d}1)e^{\frac{1}{2}\left(r-\frac{\sigma^2}{6}\right)} + Se^{\frac{1}{2}\left(r-\frac{\sigma^2}{6}\right)}n(d1) * \frac{\partial}{\partial S}d1 - K\,n(d2) * \frac{\partial}{\partial S}d2]$$

$$\frac{\partial}{\partial S}d1 = \frac{\ln(S) - \ln(k) - f(\text{r},\sigma)}{\sigma_a} = \frac{1}{\sigma_a S}$$

$$\frac{\partial}{\partial S}d2 = \frac{\partial}{\partial S}(d1 - \sigma) = \frac{1}{\sigma_a S}$$

Note: since k =S_0

$$\frac{C_0^a}{S} = e^{-rt}[\text{N}(\text{d}1)e^{\frac{1}{2}\left(r-\frac{\sigma^2}{6}\right)} + \frac{Se^{\frac{1}{2}\left(r-\frac{\sigma^2}{6}\right)}n(d1)-S_0 n(d2)}{\sigma_a S}]$$

$$\frac{\partial C_0^a}{\partial S} = e^{-0.03}\left[0.563075 * 1.011735 + \frac{1.011735 * n(d1) - n(d2)}{0.2}\right]$$

$$= e^{-0.03}\left[0.563075 * 1.011735 + \frac{1.011735 * 0.393946 - 0.398568}{0.2}\right]$$

= 0.970446*[0.569683163 + 0] = 0.552846

9. **Continued**

Portfolio Delta = Asían Option # *Asían Option Delta – European Option Delta
=0.552846 *1,000,000/100 - 5,298.718798 *0.598706= 2356.09007

Delta of Stock = 1
Need to sell 2356.09007 shares of stock.

10. **Learning Objectives:**

4. The candidate will learn how to apply the techniques of quantitative finance to applied business contexts.

Learning Outcomes:

(4a) Identify and evaluate embedded options in liabilities, specifically indexed annuity and variable annuity guarantee riders (GMAB, GMDB, GMWB and GMIB).

(4c) Demonstrate an understanding of dynamic and static hedging for embedded guarantees, including:
 (i) Risks that can be hedged, including equity, interest rate, volatility and cross Greeks.
 (ii) Risks that can only be partially hedged or cannot be hedged including policyholder behavior, mortality and lapse, basis risk, counterparty exposure, foreign bonds and equities, correlation and operation failures

(4e) Demonstrate an understanding of how differences between modeled and actual outcomes for guarantees affect financial results over time.

Sources:

QFIQ-134-22 An Introduction to Computational Risk Management

QFIQ-128-20_Mitigating Interest Rate Risk

Commentary on Question:

This question is testing the candidates' ability to recognize embedded option in a variable annuity contract with a GMDB rider and derive a delta-rho hedge for it. In addition, it tests the candidates' knowledge of how the difference between the model and actual outcomes affect the hedging results for this product. Overall, the attempt rate for this question was low, especially for parts a) and b) which involve calculations.

Solution:

(a) Derive the no-arbitrage value of the net liability L_t at time *t*.

Commentary on Question:

Most candidates did not attempt this part of the question. To earn points for this question, candidates needed to manipulate the given equation for $\mathrm{L_t}$ *and derive the equation for the expected value. Partial points were awarded to the candidates who successfully took the calculation further than copying down the given equation, mostly for recognizing that* $E\left[e^{-r(s-t)}\max(G-F_s,0)\right]$ *is a put option and writing down the value.*

10. Continued

Net Liability = Expected PV of benefits – Expected PV of fee income, which is given:

$$L_t = {}_tp_x(\Omega_t - \Upsilon_t) - {}_tp_x E^{\mathbb{Q}}\left[\int_t^T mF_s e^{-r(s-t)}{}_{s-t}p_{x+t}ds\right]$$

$$= {}_tp_x E^{\mathbb{Q}}\left[\int_t^T e^{-r(T-t)}\max(G - F_s, 0)_{s-t}p_{x+t}\mu_{x+s}ds\right]$$

$$- {}_tp_x E^{\mathbb{Q}}\left[\int_t^T mF_s e^{-r(s-t)}{}_{s-t}p_{x+t}ds\right]$$

The no-arbitrage value of the net liability is the expected value with respect to the risk neutral measure. Due to independence of mortality and equity return, the first term can be written as

$${}_tp_x E^{\mathbb{Q}}\left[\int_t^T e^{-r(T-t)}\max(G - F_s, 0)_{s-t}p_{x+t}\mu_{x+s}ds\right]$$

$$= \int_t^T E^{\mathbb{Q}}[e^{-r(T-t)}\max(G - F_s, 0)]\ {}_tp_{x\,s-t}p_{x+t}\mu_{x+s}ds$$

$$= \int_t^T E^{\mathbb{Q}}[e^{-r(T-t)}\max(G - F_s, 0)]\ {}_sp_x\mu_{x+s}ds$$

Since $F_t = S_t e^{-mt}$

$$dF_t = -me^{-mt}S_t dt + e^{-mt}dS_t = -me^{-mt}S_t dt + re^{-mt}S_t dt + \sigma e^{-mt}S_t dW_t$$

$$= e^{-mt}S_t(r - m)dt + \sigma e^{-mt}S_t dW_t = F_t(r - m)dt + \sigma F_t dW_t$$

$E\left[e^{-r(s-t)}\max(G - F_s, 0)\right]$ is the no-arbitrage price of a put option, thus

$$Value\ of\ benefits = \int_t^T \left[Ge^{-r(s-t)}N(-d_2) - S_t e^{-m(s-t)}N(-d_1)\right]{}_sp_x\mu_{x+s}ds$$

where $d_1 = \frac{ln\frac{S_t}{G} + (s-t)(r - m + \frac{\sigma^2}{2})}{\sigma\sqrt{s-t}}$ and $d_2 = d_1 - \sigma\sqrt{s - t}$.

The second term is the PV of fees:

$${}_tp_x E^{\mathbb{Q}}\left[\int_t^T mF_s e^{-r(s-t)}{}_{s-t}p_{x+t}ds\right] = E\left[\int_t^T e^{-r(s-t)}mF_s\ {}_sp_x ds\right] =$$
$$\int_t^T E\left[e^{-r(s-t)}mF_s\ {}_sp_x\right]ds = \int_t^T E\left[e^{-r(s-t)}S_s\right]e^{-ms}m\ {}_sp_x ds =$$
$$\int_t^T S_t e^{-ms}m\ {}_sp_x ds = mF_t\int_t^T e^{-m(s-t)}\ {}_sp_x ds$$

(b) Derive the positions of stock, zero-coupon bond and money market account for a portfolio Π_t that hedges the Delta and Rho of the net liability in part (a).

Commentary on Question:
A lot of candidates did not attempt this part of the question. To earn points for this part, candidates needed to correctly describe the positions in underlying, bond, and money market account to set up the hedge, and derive the equation for the positions, especially for ρ_t *and* B_t. *Partial points were awarded for describing the hedge, although most candidates did not finish the derivation of the positions.*

10. Continued

To hedge delta and rho of L_t, invest in

- Δ_t share of the underlying S_t and $\Delta_t = \frac{\partial L_t}{\partial S_t}$, which is given
- ρ_t unit in the zero-coupon bond and $\rho_t = \frac{\partial L_t / \partial r}{\partial P_t / \partial r}$
- B_t in the money market account
- $\Pi_t = \Delta_t S_t + \rho_t P_t + B_t = L_t$

$$\rho_t = \frac{\partial L_t / \partial r}{\partial P_t / \partial r} = \frac{-\int_t^T G(s-t)e^{-r(s-t)}N(-d_2)\ {}_sp_x\mu_{x+s}ds}{-(T-t)e^{-r(T-t)}} = \\ = \frac{\int_t^T G(s-t)e^{-r(s-t)}N(-d_2)\ {}_sp_x\mu_{x+s}ds}{(T-t)e^{-r(T-t)}}$$

And

$$\begin{aligned} B_t = L_t - \Delta_t S_t - \rho_t P_t \\ &= \int_t^T \left[Ge^{-r(s-t)}N(-d_2) - S_t e^{-m(s-t)}N(-d_1)\right]\ {}_sp_x\mu_{x+s}ds \\ &- mF_t\int_t^T e^{-m(s-t)}\ {}_sp_x ds - S_t\int_t^T Ge^{-m(s-t)}[N(d_1)-1]\ {}_sp_x\mu_{x+s}ds \\ &+ mS_t\int_t^T e^{-ms}\ {}_sp_x ds - \frac{\int_t^T G(s-t)e^{-r(s-t)}N(-d_2)\ {}_sp_x\mu_{x+s}ds}{(T-t)e^{-r(T-t)}}e^{-r(T-t)} \end{aligned}$$

$$mS_t\int_t^T e^{-ms}\ {}_sp_x ds = mF_t e^{mt}\int_t^T e^{-ms}\ {}_sp_x ds = mF_t\int_t^T e^{-m(s-t)}\ {}_sp_x ds$$

Thus

$$B_t = \int_t^T \Big[Ge^{-r(s-t)}N(-d_2) - S_t e^{-m(s-t)}N(-d_1) - S_t Ge^{-m(s-t)}[N(d_1)-1] \\ - \frac{s-t}{T-t}Ge^{-r(s-t)}N(-d_2)\Big]\ {}_sp_x\mu_{x+s}ds$$

$$B_t = \int_t^T \left[\frac{T-s}{T-t}Ge^{-r(s-t)}N(-d_2) - S_t e^{-m(s-t)}N(-d_1)(1-G)\right]\ {}_sp_x\mu_{x+s}ds$$

(c) Describe the hedging effectiveness you expect to observe under each of the 3 models of simulating interest rates (specified in the table above). Explain your reasoning.

Commentary on Question:
Many of the candidates who attempted this question did relatively well, as they were able to correctly order the three models for their hedging effectiveness and describe reasoning for their response. But some candidates did not properly understand the question and directly compared the pros and cons of the three models for interest rate hedging, which did not earn points.

10. Continued

For the control where interest rate is not simulated stochastically, and follows the deterministic path r_t, the hedging is expected to be effective, since the hedging model is the same as the simulation model used for the assessment. Hedging gain/loss at time T should be small.

For interest rate model option 1, interest rates are simulated stochastically, instead of the deterministic interest r_t which is used to develop the hedge. Higher hedging errors are expected at maturity T due the model risk that deterministic assumption r_t does not capture all the variabilities in the simulated interest scenarios.

For interest rate model option 2, additional difference between the simulation model and the hedging model is introduced due to the additional factors in the simulating model, which allows yield curve to take on different shapes. Thus, the hedging error is expected to be higher than option 1.

(d) Describe changes in hedging effectiveness in comparison to part (c) for the Interest rate model 1 and the Interest rate model 2.

Commentary on Question:
Some candidates did well on this part, though many did not properly understand the question and made general comparison of the two models for interest rate hedging, which did not earn points.

For interest rate model option 1, using the one-factor Vasicek model to develop the hedge should improve the hedging effectiveness when compared to using deterministic r_t. GMDB is more impacted by the long-term trend in the interest rate, which is relatively well captured by the one-factor Vasicek model compared to the simulation model of CIR. Thus, using a stochatic model for developing the hedge reduces the model risk vs. the simulation model and improves the hedging results.

For interest rate model option 2, using the one-factor Vasicek model to develop the hedge may not have significant improvement on the hedging results. As the simulation model has a lot more flexibility with three factors and can produce simulations with more variability in the shape of the term structure, using a one-factor stochastic model to develop the hedge does not significantly reduce the model risk vs. the simulation model, when compared to a deterministic r_t.