QFI QF Model Solutions
Spring 2020

1.

Learning Objectives:

1. The candidate will understand the foundations of quantitative finance.

Learning Outcomes:

(1a) Understand and apply concepts of probability and statistics important in mathematical finance.

(1c) Understand Ito integral and stochastic differential equations.

(1d) Understand and apply Ito's Lemma.

(1h) Define and apply the concepts of martingale, market price of risk and measures in single and multiple state variable contexts.

Sources:
An Introduction to the Mathematics of Financial Derivatives, Hirsa, Ali and Neftci, Salih N., 3rd Edition 2nd Printing, 2014 – Chapters: 6, 8, 10, & 13.

Commentary on Question:
This question tests candidates' understanding of exotic options and Ito's Lemma and concept of martingale.

Solution:

(a) Describe three common differences between exotic options and standard options.

Commentary on Question:
Candidates performed below expectations. Candidates knew some of the differences between exotic and standard options but most failed to recognize the common differences.

Three common differences between exotic and standard options is that for exotic options:

- The expiration value of the option may depend on some event or the path of the asset over the life of the option;
- The option may have random expiration dates;
- The options may be written on more than one asset.

1. Continued

(b) Derive the stochastic differential equation of S_t.

Commentary on Question:
This is a standard application of Ito's Lemma. Most candidates demonstrated knowledge of how to apply Ito's Lemma. However, some candidates made mistakes in the application of the formula and/or in the calculations.

Let $F(x, y) = \sqrt{xy}$.

$S_t = F(X_t, Y_t)$.

By Ito's lemma, we have

$$dS_t = dF = F_t dt + F_x dX_t + F_y dY_t + \frac{1}{2}\left(F_{xx} dX_t^2 + F_{yy} dY_t^2\right) + F_{xy} dX_t dY_t$$

Note that

$F_t = 0$

$F_x = \frac{1}{2} X_t^{-\frac{1}{2}} Y_t^{\frac{1}{2}}$

$F_y = \frac{1}{2} X_t^{\frac{1}{2}} Y_t^{-\frac{1}{2}}$

$F_{xx} = -\frac{1}{4} X_t^{-\frac{3}{2}} Y_t^{\frac{1}{2}}$

$F_{yy} = -\frac{1}{4} X_t^{\frac{1}{2}} Y_t^{-\frac{3}{2}}$

$F_{xy} = \frac{1}{4} X_t^{-\frac{1}{2}} Y_t^{-\frac{1}{2}}$

$dX_t^2 = X_t^2(\alpha^2 + \beta^2)dt$

$dY_t^2 = Y_t^2(\beta^2 + \gamma^2)dt$

$dX_t dY_t = X_t Y_t(\alpha\beta + \beta\gamma)dt$

Plugging the above equations into the first equation, we get

$$\begin{aligned} dS_t = &\frac{1}{2}\sqrt{X_t Y_t}\left(\mu_1 dt + \alpha dW_t^{(1)} + \beta dW_t^{(2)}\right) + \frac{1}{2}\sqrt{X_t Y_t}\left(\mu_2 dt + \beta dW_t^{(1)} + \gamma dW_t^{(2)}\right) \\ &- \frac{1}{8}\sqrt{X_t Y_t}(\alpha^2 + 2\beta^2 + \gamma^2)dt + \frac{1}{4}\sqrt{X_t Y_t}(\alpha\beta + \beta\gamma)dt \end{aligned}$$

Rearranging the terms gives

$$\begin{aligned} dS_t = &\frac{1}{2} S_t\left(\mu_1 + \mu_2 - \frac{1}{4}(\alpha^2 + 2\beta^2 + \gamma^2 - 2\alpha\beta - 2\beta\gamma)\right)dt \\ &+ \frac{1}{2} S_t\left((\alpha + \beta)dW_t^{(1)} + (\beta + \gamma)dW_t^{(2)}\right) \end{aligned}$$

$$\begin{aligned} dS_t = &\frac{1}{2} S_t\left(\mu_1 + \mu_2 - \frac{1}{4}\{(\alpha - \beta)^2 + (\beta - \gamma)^2\}\right)dt \\ &+ \frac{1}{2} S_t\left((\alpha + \beta)dW_t^{(1)} + (\beta + \gamma)dW_t^{(2)}\right) \end{aligned}$$

1. Continued

(c) Show that B_t is a standard Wiener process.

Commentary on Question:
Candidates did relatively well on this part as most recognized the conditions required for B_t to be a *Wiener process.*

To show that B_t is a Wiener process, we need to verify four conditions.

1. $B_0 = 0$ → Since $W_t^{(1)}$ and $W_t^{(2)}$ are Wiener processes, we have $B_0 = 0$.

2. Stationary, Independent Increments → Since $W_t^{(1)}$ and $W_t^{(2)}$ are independent Wiener processes, B_t also has stationary, independent increments.

3. Continuous → Since $W_t^{(1)}$ and $W_t^{(2)}$ are Wiener processes, which are continuous in t, B_t is also continuous in t.

4. $E[B_t - B_s] = 0$ and $Var[B_t - B_s] = |t-s|$

Consider $B_t - B_s = \frac{(\alpha+\beta)\left(W_t^{(1)} - W_s^{(1)}\right) + (\beta+\gamma)\left(W_t^{(2)} - W_s^{(2)}\right)}{\sqrt{(\alpha+\beta)^2 + (\beta+\gamma)^2}}$.

Since $W_t^{(1)}$ and $W_t^{(2)}$ are Wiener processes, we know that $B_t - B_s$ follows a normal distribution.

$$E[B_t - B_s] = 0$$

and

$$Var[B_t - B_s] = \frac{(\alpha+\beta)^2|t-s| + (\beta+\gamma)^2|t-s|}{(\alpha+\beta)^2 + (\beta+\gamma)^2} = |t-s|$$

(d) Derive the condition on the parameters under which S_t is a martingale with respect to the filtration generated by B_t.

Commentary on Question:
Candidates did relatively well on this part as they recognized that the drift of S_t with respect to the filtration generated by B_t should be 0 to be a martingale.

From (b).:

$$dS_t = \frac{1}{2}S_t\left(\mu_1 + \mu_2 - \frac{1}{4}\{(\alpha-\beta)^2 + (\beta-\gamma)^2\}\right)dt + \frac{1}{2}S_t\left((\alpha+\beta)dW_t^{(1)} + (\beta + \gamma)dW_t^{(2)}\right)$$

1. **Continued**

$$\frac{dS_t}{S_t} = \frac{1}{2}\left(\mu_1 + \mu_2 - \frac{1}{4}\{(\alpha - \beta)^2 + (\beta - \gamma)^2\}\right) dt + \frac{1}{2}\sqrt{(\alpha + \beta)^2 + (\beta + \gamma)^2}\, dB_t$$

For $\frac{dS_t}{S_t}$ *to be a martingale, we must have the drift (dt) = 0*

$$\frac{1}{2}\left(\mu_1 + \mu_2 - \frac{1}{4}\{(\alpha - \beta)^2 + (\beta - \gamma)^2\}\right) = 0$$

$$\mu_1 + \mu_2 = \frac{(\alpha - \beta)^2 + (\beta - \gamma)^2}{4}$$

2. Learning Objectives:

1. The candidate will understand the foundations of quantitative finance.

Learning Outcomes:

(1a) Understand and apply concepts of probability and statistics important in mathematical finance.

(1c) Understand Ito integral and stochastic differential equations.

(1d) Understand and apply Ito's Lemma.

Sources:

Hirsa-Neftci Ch5, 9, 10, Chin-Olafsson Ch 2, 3

Commentary on Question:

The question is trying to test how to apply the concept of Ito Integral

Solution:

(a) Apply Ito's isometry to prove that

$$E\left[\left(\int_0^t f(u)\,dW(u)\right)\left(\int_0^t g(u)\,dW(u)\right)\right] = E\left[\int_0^t f(u)\,g(u)\,du\right].$$

Now, denote $X(t) = \int_0^t sgn\left(W(s)\right)dW(s)$, where $sgn(x) = \begin{cases} 1, & if\ x \geq 0 \\ -1, & if\ x < 0 \end{cases}$.

Commentary on Question:

Overall, candidates did poorly on this section. Most, who attempted it, stated isometry as a result, but did not apply it to complete the proof.

Ito's isometry says that

$$E\left[\left(\int_0^t (f(u) + g(u))dW(u)\right)^2\right] = E\left[\int_0^t (f(u) + g(u))^2 du\right].$$

For the expression on the left-hand side, expanding the squares followed by applying Ito isometry lead to

$$E\left[\left(\int_0^t f(u)dW(u)\right)^2 + \left(\int_0^t g(u)dW(u)\right)^2 + 2\left(\int_0^t f(u)dW(u)\right)\left(\int_0^t g(u)dW(s)\right)\right]$$
$$= E\left[\left(\int_0^t f^2(u)du\right) + \left(\int_0^t g^2(u)du\right) + 2\left(\int_0^t f(u)dW(s)\right)\left(\int_0^t g(u)dW(s)\right)\right]$$

For the expression on the right-hand side, expanding the squares gives

2. Continued

$$E\left[\int_0^t \left(f^2(u) + g^2(u) + 2f(u)g(u)\right)du\right]$$

Hence, subtracting like terms on both sides gives the desired results.

(b) Show using the result in part (a) or otherwise, that

(i) $E\left[X(t)W(t)\right] = 0$.

(ii) $E\left[X(t)W^2(t)\right] = \frac{4}{3}\sqrt{\frac{2t^3}{\pi}}$.

Hint: $E\left[\left|W(s)\right|\right] = \sqrt{\frac{2s}{\pi}}$.

Commentary on Question:
Of the two parts to this question, (b)(i) was attempted by most candidates. About half were able to attain full marks on part (b)(i). Fewer had success on part (b)(ii) and only the best candidates obtained full marks on part (b)(ii).

Part (i)

$$E[X(t)W(t)]$$
$$= E\left[\left(\int_0^t sgn\,[W(s)]dW(s)\right)\left(\int_0^t dW(s)\right)\right]$$
$$= E\left[\int_0^t sgn[W(s)] \cdot 1ds\right] \text{ by applying part (a)}$$
$$= \int_0^t E\big[sgn[W(s)]\big]ds$$
$$= 0$$

Alternative solution for part (i):

Appply Ito's Lemma, one has
$$d\big(X(t)W(t)\big)$$
$$= X(t)\,dW(t) + W(t)\,dX(t) + dX(t)dW(t)$$
$$= X(t)\,dW(t) + W(t)\,sgn[W(t)]\;dW(t) + sgn[W(t)]\,dt$$

Hence, integrating both sides followed by taking expectation gives
$$E[X(t)W(t)]$$
$$= E\left[\int_0^t [X(s) + W(s)\,sgn[W(s)]]\,dW(s) + \int_0^t sgn[W(s)]\,ds\right]$$

2. Continued

$= \int_0^t E[sgn[W(s)]]ds$
$= 0$

Part (ii)

Note that, by Ito formula,
$dW^2(t) = 2W(t)dW(t) + dt$
or equivalently,
$W^2(t) = 2\int_0^t W(s)dW(s) + t.$

Hence,
$E[X(t)W^2(t)]$
$$= E\left[\left(\int_0^t sgn\,[W(s)]dW(s)\right)\left(2\int_0^t W(s)dW(s) + t\right)\right]$$
$$= 2E\left[\int_0^t sgn[W(s)]W(s)ds\right] + tE\left[\int_0^t sgn[W(s)]dW(s)\right] \dots (*)$$
$$= 2\int_0^t E[|W(s)|]ds + 0$$
$$= 2\int_0^t \frac{2^{1/2}s^{1/2}}{\sqrt{\pi}} ds \dots (**)$$
$$= \frac{2^{5/2}t^{3/2}}{3\sqrt{\pi}}$$

where (**) is evaluated as follows: note that $W(s){\sim}N(0,s)$, which gives
$E[|W(s)|]$
$$= \sqrt{\frac{1}{2\pi s}}\int_{-\infty}^{\infty}|u|e^{-\frac{u^2}{2s}}du$$
$$= \sqrt{\frac{2}{\pi s}}\int_0^{\infty}|u|e^{-\frac{u^2}{2s}}du$$
$$= \sqrt{\frac{2}{\pi s}}\left[-se^{-\frac{u^2}{2s}}\right]_{u=0}^{u=\infty}$$
$$= \frac{2^{1/2}s^{1/2}}{\sqrt{\pi}}$$

An alternative way to arrive at (*) is given below.
Applying Ito formula, one has
$dX(t)W^2(t)$
$= X(t)\,dW^2(t) + W(t)\,dX(t) + \,dX(t)dW^2(t)$
$= X(t)[2W(t)dW(t) + dt] + W^2(t)[sgn[W(t)]\,dW(t)] + \,2W(t)\,sgn[W(t)]\,dt$
Hence, integrating both sides followed by taking expectation gives
$= E\left[\int_0^t[2X(s)W(s) + W^2(s)\,sgn[W(s)]]\,dW(s) + \int_0^t[X(s) + 2W(s)\,sgn[W(s)]]ds\right]$

2. Continued

(c) Show that $E[|X(t)W(t)|] \leq t$.

Commentary on Question:
Overall, candidates did poorly on this part, and very few obtained full marks. There were two approaches candidates could take, shown in the solutions below.

Solution 1

Note that By Cauchy-Schwarz inequality
$|X(t)W(t)| \leq \frac{X(t)^2+W(t)^2}{2}$
and
$E[X(t)^2] = E\left[\left(\int_0^t sgn\,[W(s)]dW(s)\right)^2\right]$
$= E\left[\left(\int_0^t (sgn[W(s)])^2 ds\right)\right] = E\left[\left(\int_0^t 1ds\right)\right] = t$

$E[W(t)^2] = t$

We have $[|X(t)W(t)|] \leq t$

Solution 2

$$E[|X(t)W(t)|] = E\left[\sqrt{\big(X(t)W(t)\big)^2}\right]$$

By isometry, we have $X^2(t) = \int_0^t sgn^2(W(s))ds = \int_0^t 1\,ds = t$

$$E\left[\sqrt{\big(X(t)W(t)\big)^2}\right] = E\left[\sqrt{t\,\big(W(t)\big)^2}\right] = \sqrt{t}\,E\left[\sqrt{\big(W(t)\big)^2}\right]$$

By Jensen's inequality, the concavity of square root function gives us that

$$\sqrt{t}\,E\left[\sqrt{\big(W(t)\big)^2}\right] \leq \sqrt{t}\sqrt{E[W^2(t)]} = t$$

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Sources:
Hirsa-Neftci Ch5, 6, 10; Chin-Olafsson Ch1, 2, 3

Commentary on Question:
In general, candidates did well on this question. Most lost points on parts (b) and (c).

Solution:
(a)

(i) State the three conditions for a process to be a martingale.

(ii) Show that, for a fixed u, $\{M_u(0,t)\}$ satisfies those conditions.

Commentary on Question:
Most candidates received full credit for part (i) as long as a clear, complete definition was stated and all conditions were correctly stated.

$\{M_u(0,t)\}$ *is* $\mathcal{F}_t$*-adapted.*
This is clear since $I(t)$ *is* $\mathcal{F}_t$ *measurable.*

$\mathbb{E}|M_u(0,t)|$ *is finite.*
Let $g = exp\left\{ux - \frac{1}{2}u^2 \int_0^t f^2(s)ds\right\}$*. Direct differentiation gives* $g_t = -\frac{1}{2}u^2 f^2(t)g$*,* $g_x = ug$ *and* $g_{xx} = u^2 g$*. Using Ito's formula,*

$$dM_u(0,t)$$
$$= g_t dt + g_x dI(t) + \frac{1}{2}g_{xx}[dI(t)]^2$$
$$= \left[udI(t) - \frac{1}{2}u^2 f^2(t)dt + \frac{1}{2}u^2 f^2(t)\right]g$$
$$= ugf(t)dW(t)$$

3. Continued

Writing in integral form and taking expectation gives

$$\mathbb{E}[M_u(0,t)] - M_u(0,0) = \mathbb{E}\left[\int_0^t uf(s)dW(s)\right]$$

so that

$$\mathbb{E}[M_u(0,t)] = \mathbb{E}|M_u(0,t)| = 1 < \infty$$

$\mathbb{E}\left[M_u(0,t_2)|\mathcal{F}_{t_1}\right] = M_u(0,t_1)$ *for* $0 \le t_1 < t_2$.

In evaluating the dynamic of $M_u(0,t)$ *above, observe that the drift term vanishes. Therefore* $\mathbb{E}\left[M_u(0,t_2)|\mathcal{F}_{t_1}\right] = M_u(0,t_1)$ *for* $0 \le t_1 < t_2$.

(b) Show that $\mathbb{E}\left[M_u(t_1,t_2)\middle|\mathcal{F}_{t_1}\right] = 1$.

Commentary on Question:

Candidates who referred to part (a) without dealing specifically with the more general question asked in this part received partial credit only.

Using martingale property and conditional expectation,

$$M_u(0,t_1) = \mathbb{E}\left[M_u(0,t_2)|\mathcal{F}_{t_1}\right]$$
$$= \mathbb{E}\left[exp\left\{uI(t_2) - \frac{1}{2}u^2\int_0^{t_2} f^2(s)ds\right\}\middle|\mathcal{F}_{t_1}\right]$$
$$= \mathbb{E}\left[exp\left\{uI(t_1) + u[I(t_2) - I(t_1)] - \frac{1}{2}u^2\int_0^{t_1} f^2(s)ds - \frac{1}{2}u^2\int_{t_1}^{t_2} f^2(s)ds\right\}\middle|\mathcal{F}_{t_1}\right]$$
$$= M_u(0,t_1)\mathbb{E}\left[exp\left\{u[I(t_2) - I(t_1)] - \tfrac{1}{2}u^2\textstyle\int_{t_1}^{t_2} f^2(s)ds\right\}\middle|\mathcal{F}_{t_1}\right].$$

Since $M_u(0,t_1)$ *is a non zero process, this gives*

$$1 = \mathbb{E}\left[M_u(t_1,t_2)|\mathcal{F}_{t_1}\right].$$

Alternatively, for $t \ge t_1$, *consider*

$$X_t = u\big(I(t) - I(t_1)\big) - \frac{u^2}{2}\int_{t_1}^{t} f^2(s)ds$$

Then

$$dX_t = uf(t)dW(t) - \frac{u^2}{2}f^2(t)dt$$

From Ito's formula

$$de^{X_t} = e^{X_t}\left(dX_t + \frac{(dX_t)^2}{2}\right)$$
$$= e^{X_t}\left(uf(t)dW(t) - \frac{u^2}{2}f^2(t)dt + \frac{\big(uf(t)\big)^2}{2}dt\right)$$

3. Continued

$= ue^{X_t} f(t) dW(t)$
Hence

$$e^{X_t} - e^{X_{t_1}} = u \int_{t_1}^{t} e^{X_s} f(s)\, dW(s)$$

Note that using martingale property,
$\mathbb{E}\left[\int_{t_1}^{t} e^{X_s} f(s)\, dW(s) \middle| \mathcal{F}_{t_1}\right] = \int_{t_1}^{t_1} e^{X_s} f(s)\, dW(s) = 0.$
Also, $X_{t_1} = u\big(I(t_1) - I(t_1)\big) - \frac{u^2}{2}\int_{t_1}^{t_1} f^2(s)ds = 0$. *Thus it follows that*
$\mathbb{E}\left[e^{X_t} | \mathcal{F}_{t_1}\right] = 1$. *The desired result follows by putting* $t = t_2$.

(c) Show that $I(t_1)$ and $I(t_2) - I(t_1)$ are independent normal random variables. Identify their means and variances.

Commentary on Question:
To receive full credit, candidates needed to prove independence and show the variables are normal random variables with the correct means and variances.

From part (b),

$$\mathbb{E}\left[M_{u_1}(0, t_1) M_{u_2}(t_1, t_2) | \mathcal{F}_{t_1}\right] = M_{u_1}(0, t_1)$$

Meanwhile, from part (b), $\mathbb{E}\left[M_{u_1}(0, t_1)\right] = 1$ *due to martingale property.*
Taking expectation on both sides and rearranging gives

$$\mathbb{E}[\exp\{u_1 I(t_1) + u_2[I(t_2) - I(t_1)]\}] = \exp\left\{\frac{1}{2}u_1^2 \int_0^{t_1} \mathrm{f}^2(s)ds + \frac{1}{2}u_2^2 \int_{t_1}^{t_2} \mathrm{f}^2(s)ds\right\}.$$

Note that the left-hand side corresponds to the joint moment generating function of $\big(I(t_1), I(t_2) - I(t_1)\big)$, whereas the right hand side corresponds to the product of moment generating functions of two (marginal) normal distributions. This implies $I(t_1)$ and $I(t_2) - I(t_1)$ are independent normal random variables.
From the joint moment generating function, means and variances can be identified as $I(t_1) \sim N\left(0, \int_0^{t_1} \mathrm{f}^2(s)ds\right)$ and $I(t_2) - I(t_1) \sim N\left(0, \int_{t_1}^{t_2} \mathrm{f}^2(s)ds\right)$.

Alternative solution:
From (b) $\mathbb{E}\left(\exp\left\{u[I(t_2) - I(t_1)] - \frac{1}{2}u^2 \int_{t_1}^{t_2} \mathrm{f}^2(s)ds\right\} \middle| \mathcal{F}_{t_1}\right) = 1$ for all $t_2 \geq t_1$
Thus, $\mathbb{E}\big(\exp\{u[I(t_2) - I(t_1)]\} \,| \mathcal{F}_{t_1}\big) = \exp\left\{\frac{1}{2}u^2 \int_{t_1}^{t_2} \mathrm{f}^2(s)ds\right\}$
Since RHS is the moment generating function of $N\left(0, \int_{t_1}^{t_2} \mathrm{f}^2(s)ds\right)$ we find that for all pairs t_1, t_2 with $0 \leq t_1 \leq t_2$

3. Continued

$$I(t_2) - I(t_1) \sim N\left(0, \int_{t_1}^{t_2} \mathrm{f}^2(s)ds\right)$$

In particular for pairs $0, t_1$ with $0 \le t_1$ we have ,

$$I(t_1) = I(t_1) - I(0) \sim N\left(0, \int_{0}^{t_1} \mathrm{f}^2(s)ds\right)$$

Finally since $W(s_4) - W(s_3)$ and $W(s_2) - W(s_1)$ are independent for $s_1 \le s_2 \le s_3 \le s_4$, we conclude that $I(t_1) = \int_0^{t_1} \mathrm{f}(s)\, dW(s)$ and $I(t_2) - I(t_1) = \int_{t_1}^{t_2} \mathrm{f}(s)\, dW(s)$ are also independent.

(d) $\mathrm{Cov}\left(I(t_1), I(t_2)\right)$

Commentary on Question:
Overall, candidates did very well on this part.

Using part (c),

$$Cov(I(t_1), I(t_2))$$
$$= Cov\big(I(t_1), I(t_2) - I(t_1)\big) + Var\big(I(t_1)\big)$$
$$= \int_0^{t_1} \mathrm{f}^2(s)\, ds$$

4. **Learning Objectives:**

1. The candidate will understand the foundations of quantitative finance.

Learning Outcomes:

(1i) Demonstrate understanding of the differences and implications of real-world versus risk-neutral probability measures, and when the use of each is appropriate.

(1j) Understand and apply Girsanov's theorem in changing measures.

Sources:

Hirsa–Neftci, Chapter 14

Commentary on Question:

This question tests candidates' knowledge of option pricing, Girsanov's theorem, the relationship between risk-neutral measures and real-world measures. Most candidates were able to complete some parts and earn partial credit. However, very few candidates were able to earn full credit.

Solution:

(a)

(i) Write the arbitrage-free option price C as an expectation with respect to the risk-neutral probability measure $\mathbb{Q}$.

(ii) Derive the formula $C = S_0 N(d_1)$ where $d_1 = \dfrac{\ln\left(\dfrac{S_0}{K}\right) + \left(r + \dfrac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$.

(iii) Calculate C using the above formula.

Commentary on Question:

Most candidates did well on parts (i) and (iii) by writing out the option pricing formula based on the risk-neutral pricing theory and calculating the option price. However, few candidates were able to derive the formula in part (ii).

(i) Let C denote the arbitrage-free option price. Then

$C = E^Q\left(S_1\, 1_{\{S_1 > K\}}\, e^{-r}\right)$ with Q the risk-neutral probability measure.

(ii) Continue from (i): Since $S_1 = S_0\, e^{(r - 0.5\sigma^2 + \sigma W_1{}^Q)}$, we obtain

$$S_1 > K \;\Rightarrow\; W_1{}^Q > -\frac{\ln\left(\frac{S_0}{K}\right) + (r - 0.5\sigma^2)}{\sigma}$$

and

4. **Continued**

$$E^Q\left(S_1\, 1_{\{S_1>K\}}\right) e^{-r} = S_0 \frac{1}{\sqrt{2\pi}} \int_{-\frac{\ln\left(\frac{S_0}{K}\right)+(r-0.5\sigma^2)}{\sigma}}^{\infty} e^{-0.5\sigma^2+\sigma x-0.5x^2} dx$$

$$= S_0 \frac{1}{\sqrt{2\pi}} \int_{-\frac{\ln\left(\frac{S_0}{K}\right)+(r-0.5\sigma^2)}{\sigma}}^{\infty} e^{-0.5(x-\sigma)^2} dx$$

$$= S_0 \frac{1}{\sqrt{2\pi}} \int_{-\frac{\ln\left(\frac{S_0}{K}\right)+(r+0.5\sigma^2)}{\sigma}}^{\infty} e^{-0.5y^2} dy$$

$$= S_0 \left[1 - N\left(-\frac{\ln\left(\frac{S_0}{K}\right)+(r+0.5\sigma^2)}{\sigma}\right)\right]$$

$$= S_0 N\left(\frac{\ln\left(\frac{S_0}{K}\right)+(r+0.5\sigma^2)}{\sigma}\right) = S_0 N(d_1).$$

(iii) Substituting the given values into the formula in (ii), we have $d_1 = -4.23$ and $C = 0$.

(b)

(i) State how Girsanov's Theorem can be used in this case.

(ii) Show that the adapted process on the probability space $(\Omega,\ I_t,\ \mathbb{Q})$ defined by $X_t = d_1$ satisfies the Novikov condition with d_1 being the constant calculated in part (a).

Commentary on Question:
Most candidates did well on this part.

(i) Girsanov Theorem can be applied to any two equivalent measures. Since we need to go from (Ω, I_t, Q) to (Ω, I_t, P), we can state it as follows:

Let W_t be a Wiener process defined on the probability space (Ω, I_t, Q) and X_t be an adapted process satisfying the Novikov condition. (A weaker hypothesis than Novikov, which implies the conclusion of the theorem is that the random process

$$\xi_t = e^{\left(\int_0^t X_u\, dW_u^{\mathrm{Q}} - 0.5\int_0^t {X_u}^2\, du\right)}, \qquad 0 \le t \le T$$

4. Continued

is a martingale with respect to the information sets I_t and the probability Q)
Then

$$W_t^{\mathrm{P}} = W_t^{\mathrm{Q}} - \int_0^t X_u \, du, \qquad 0 \le t \le T$$

is a Wiener process with respect to I_t and the probability P given by $P(A) = E^Q(1_A \, \xi_T)$ for any event A determined by I_T.

(ii) Novikov condition is satisfied since:

$$E^Q\left(e^{0.5\int_0^t |X_u|^2 \, du}\right) = E^Q\left(e^{0.5{d_1}^2 t}\right) < \infty$$

(c)

(i) Define the probability measure $\mathbb{P}$.

(ii) Express $N(d_1)$ as an expectation with respect to $\mathbb{P}$.

Commentary on Question:
Many candidates did well on part (i). However, few candidates were able to do part (ii).

(i) The application of Girsanov Theorem yields that $W_t^{\mathrm{P}} = W_t^{\mathrm{Q}} - d_1 t$ is a Wiener process with respect to (Ω, I_t, P) and $P(A) = E^Q\left(1_A \, e^{d_1 W_t{}^Q - 0.5 d_1^2 t}\right)$.

(ii) $N(d_1) = Q(\{W_1^{\mathrm{Q}} \le d_1\}) = E^Q\left(1_{\{W_1^{\mathrm{Q}} \le d_1\}}\right)$

$$= E^Q\left(1_{\{W_1^{\mathrm{Q}} \le d_1\}} e^{d_1 W_1^Q - 0.5 d_1^2} e^{-d_1 W_1^Q + 0.5 d_1^2}\right) = E^P\left(1_{\{W_1^{\mathrm{Q}} \le d_1\}} \, e^{-d_1 W_1^Q + 0.5 d_1^2}\right)$$

$$= E^P\left(1_{\{W_1^{\mathrm{P}} \le 0\}} \, e^{-d_1 W_1^P - 0.5 d_1^2}\right)$$

(d) Calculate y and z and estimate C using the simulation results and the table above.

Commentary on Question:
Most candidates did poorly on this part.

4. **Continued**

Using the simulation results:

$$E^P\left(1_{\{W_1^P \le 0\}}\, e^{-d_1 W_1^P - 0.5 d_1^2}\right)$$
$$\approx 0.003e^{(4.23)(-2.85)-0.5(4.23)^2} + 0.009e^{(4.23)(-2.55)-0.5(4.23)^2}$$
$$+ 0.008e^{(4.23)(-2.2)-0.5(4.23)^2} + 0.015e^{(4.23)(-1.85)-0.5(4.23)^2}$$
$$+ 0.027e^{(4.23)(-1.5)-0.5(4.23)^2} + 0.057e^{(4.23)(-1.15)-0.5(4.23)^2}$$
$$+ 0.098e^{(4.23)(-0.85)-0.5(4.23)^2} + 0.118e^{(4.23)(-0.5)-0.5(4.23)^2}$$
$$+ 0.149e^{(4.23)(-0.15)-0.5(4.23)^2} \approx 1.25 \times 10^{-5}$$

(disregarding all but the last three terms).

$x = 0.098e^{(4.23)(-0.85)-0.5(4.23)^2} = 3.50208E-07$ is given

$$y = 0.118e^{(4.23)(-0.5)-0.5(4.23)^2} = 1.85334E-06$$
$$z = 0.149e^{(4.23)(-0.15)-0.5(4.23)^2} = 1.02857E-05$$

Therefore, $C = S_0 N(d_1) \approx\ 1.25 \times 10^{-2} = 0.0125.$

5. **Learning Objectives:**

2. The candidate will understand the fundamentals of fixed income markets and traded securities.

Learning Outcomes:

(2c) Understand measures of interest rate risk including duration, convexity, slope, and curvature.

Sources:

Veronesi, Fixed Income Securities (Chapters 3, 4)

Commentary on Question:

The question was about the comprehension, the application and the calculations around the concepts of the duration and the convexity of a portfolio and their relation to the interest rate risk.

The global results were lower than expected because of the poor results of parts d), e) and f). Those three parts referred to the general understanding of the duration and convexity of Barbell-Bullet portfolio, and whether or not such a portfolio can always achieve positive return, and if it consists of an arbitrage opportunity.

It seems that the candidates have overlooked the section of the reading material explaining the situation. It is as if they wanted to concentrate more on difficult issues, involving formulae and calculations, but they should be remembered that more general concepts are often good concepts to know well, and that it is an occasion to earn exam points that may be important.

Part a) on the calculation of the duration and the convexity from the given data was well done.

Solution:

(a) Calculate the duration and the convexity of the GIC, assuming the term structure of interest rates currently is flat at a continuously compounded rate of 4%.

Commentary on Question:

The candidates were very successful in their calculations.

Duration $D_{GIC} = \sum_{i=1}^{n} W_i * T_i = 2.92$

Convexity $C_{GIC} = \sum_{i=1}^{n} W_i * T_i^2 = 10.523$

5. **Continued**

(b) Construct a hedging portfolio based on the CIO's suggestion.

Commentary on Question:
The results seemed to be of two categories. Many were not able to go through the question, did not remember the formulae, but other were more successful. They have usually used the development of the formula d V = d P + k1 × d P1 + k2 × d P2, leading to the equations:

k1 × D1 × P1 + k2 × D2 × P2 = −D × P (Delta Hedging)

k1 × C1 × P1 + k2 × C2 × P2 = −C × P (Convexity Hedging)

and they solved for k1 and k2 based on the data. However, some forgot the negative sign for the D and C, when they are at the right side of the equation.

Finally, in general, the candidates have interpreted the negative value of k1 and k2 as being a decision to short, but it was the contrary in this situation. The GIC sold by the company is a SHORT action. The negative signs of k1 and k2 mean the immunization actions are both opposite actions. So, we need to LONG two zero bonds to immunize the GIC.

The hedging portfolio would be :

$$V = GIC + k_1 * Z(0,2) + k_2 * Z(0,5)$$

Where

GIC : Present Value (PV) of the GIC (in millions)
$Z(0,2)$: PV of 2-year zero-coupon bond (in millions)
$Z(0,5)$: PV of 5-year zero-coupon bond (in millions)

$$k_1 = -{GIC}\big/{Z(0,2)} * ((D_{GIC} * C_{Z(0,5)} - C_{GIC} * D_{Z(0,5)}) / (D_{Z(0,2)} * C_{Z(0,5)} - C_{Z(0,2)} * D_{Z(0,5)}))$$

$$k_2 = -{GIC}\big/{Z(0,5)} * ((D_{GIC} * C_{Z(0,2)} - C_{GIC} * D_{Z(0,2)}) / (D_{Z(0,5)} * C_{Z(0,1)} - C_{Z(0,5)} * D_{Z(0,2)}))$$

5. Continued

Where

$D_{Z(0,T)} = T, C_{Z(0,T)} = T^2$: the durations and convexities of two zero-coupon bonds

PV of $GIC = \sum_{i=1}^{n} CF_i * Z(0, T_i) = 6.662$

$$Z(0,2) = 0.923, Z(0,5) = 0.819$$

$$D_{Z(0,2)} = 2, C_{Z(0,2)} = 2^2 = 4$$

$$D_{Z(0,5)} = 5, C_{Z(0,5)} = 5^2 = 25$$

$$k_1 = -\frac{6.662}{0.923} * \left(\frac{(2.92*25-10.523*5)}{(2*25-4*5)}\right) = -4.906$$

$$k_2 = -\frac{6.662}{0.819} * \left(\frac{(2.92*4-10.523*2)}{(5*4-25*2)}\right) = -2.54$$

So '***LONG***' 4.906*1,000,000=4,906,000 of the face value of the 2-year zero-coupon bond and 2.54*1,000,000=2,540,000 of the 5-year zero-coupon bond.

(c) Assess the gains or losses of the hedging portfolio constructed in (b), if interest rates were to decrease by 25 basis points from 4% to 3.75%.

Commentary on Question:

Some have used the Alternative Solution approach solving or dV, but they made errors in applying the formula, in particular with the second part of the equation where we have to consider dr^2.

$$GIC_{revised} = \sum_{i=1}^{n} CF_i * Z_{revised}(0, T_i) = 6.711$$

$$dGIC = GIC_{revised} - GIC = 6.711 - 6.662 = 0.05$$

$$dZ(0,2) = Z_{revised}(0,2) - Z(0,2) = 0.928 - 0.923 = 0.005$$

$$dZ(0,5) = Z_{revised}(0,5) - Z(0,5) = 0.829 - 0.819 = 0.01$$

So $dV = dGIC + k_1 * dZ(0,2) + k_2 * dZ(0,5)$

$dV = 0.05 - 4.906 * .005 - 2.54 * 0.01 \approx 0$

Alternative solution :

$$dV = -(D_{GIC} * GIC + k_1 * D_{Z(0,2)} * Z(0,2) + k_2 * D_{Z(0,5)} * Z(0,5)) * dr$$

$$+\frac{1}{2} * (C_{GIC} * GIC + k_1 * C_{Z(0,2)} * Z(0,2) + k_2 * C_{Z(0,5)} * Z(0,5)) * dr^2$$

5. Continued

$$dV = -(2.92 * 6.662 - 4.906 * 2 * 0.923 - 2.54 * 5 * 0.819) * 0.0025 + \frac{1}{2} * (10.523 * 6.662 - 4.906 * 4 * 0.923 - 2.54 * 25 * 0.819) * 0.0025^2 \approx 0$$

(d) Explain how to construct a barbell-bullet bond portfolio.

A barbell-bullet bond portfolio consists of a barbell bond portfolio which is long both long-dated bonds and short-dated bonds, and a bullet bond portfolio which is short medium-dated bonds.

The duration from the long-dated bonds and short-dated bonds is largely offset by the duration from the medium-dated bonds, resulting in an overall portfolio duration close to 0 (or zero). The overall convexity of the portfolio is positive.

(e) Explain whether the barbell-bullet bond portfolio can achieve a positive portfolio return when the moves of interest rates are small and in parallel.

By construct, this portfolio is duration hedged but not convexity hedged.

Pulling together the terms in dr and dr^2 we obtain

$$dV = -(D \times P + k \times D_z \times P_z) \times dr + \frac{1}{2} \times (P \times C + k \times P_z \times C) \times dr^2$$

The first parenthesis is zero while the second parenthesis is positive.

This shows that the portfolio can always achieve a positive portfolio return when the moves (dr) of interest rates are small and random. (Small is required because otherwise the formula for $d\,V$ would not valid).

(f) Critique the analyst's claim.

This shows that the portfolio can always achieve a positive portfolio return when the moves (dr) of interest rates are small and random. (Small is required because otherwise the formula for dV would not valid).
The barbell-bullet trading strategy does not represent an arbitrage opportunity.

The gain in value from higher convexity is offset by a lower gain due to passage of time.

The formula for $d\ V$ has one more component missing – the passage of time (Theta), which captures the changes in value in the long-dated, short-dated bonds and the medium-dated zero bonds due to passage of time exactly offset the convexity gain.

6. **Learning Objectives:**

3. The candidate will understand:
   - The Quantitative tools and techniques for modeling the term structure of interest rates.
   - The standard yield curve models.
   - The tools and techniques for managing interest rate risk.

Learning Outcomes:

(3a) Understand and apply the concepts of risk-neutral measure, forward measure, normalization, and the market price of risk, in the pricing of interest rate derivatives.

(3b) Understand and apply various one-factor interest rate models.

(3f) Apply the models to price common interest sensitive instruments including: callable bonds, bond options, caps, floors, and swaptions.

(3g) Understand and apply the techniques of interest rate risk hedging.

Sources:

Fixed Income Securities: Valuation, Risk, and Risk Management, Veronesi, Pietro, 2010

Commentary on Question:

Commentary listed underneath question component.

Solution:

(a) Show that these values are not consistent with a single factor arbitrage free model of the short rate.

Commentary on Question:

Candidates generally did poorly on this part of the question. A significant portion skipped it outright and an additional portion just wrote a bunch of formulas without trying to explain what they were trying to show. The answer below is illustrative of the approach needed but does not represent the only acceptable solution.

In a long-short portfolio that is insensitive to the interest rate movement, the following should hold under the no arbitrage condition for any two bonds:

$$\frac{\frac{\partial Z_1}{\partial t}+\frac{1}{2}\frac{\partial^2 Z_1}{\partial r^2}\sigma^2-r_t Z_1}{\frac{\partial Z_1}{\partial r}}=\frac{\frac{\partial Z_2}{\partial t}+\frac{1}{2}\frac{\partial^2 Z_2}{\partial r^2}\sigma^2-r_t Z_2}{\frac{\partial Z_2}{\partial r}}$$

Rearrange the above by dividing both the numerator and demonimator by the pricing of bond, and we get:

6. Continued

$$\frac{\frac{1}{Z_1}\frac{\partial Z_1}{\partial t}+\frac{1}{2}\left(\frac{1}{Z_1}\frac{\partial^2 Z_1}{\partial r^2}\right)\sigma^2-r_t}{\frac{1}{Z_1}\frac{\partial Z_1}{\partial r}}=\frac{\frac{1}{Z_2}\frac{\partial Z_2}{\partial t}+\frac{1}{2}\left(\frac{1}{Z_2}\frac{\partial^2 Z_2}{\partial r^2}\right)\sigma^2-r_t}{\frac{1}{Z_2}\frac{\partial Z_2}{\partial r}}$$

That is to say, if no arbitrage condition holds, the following ratio must be the same between any bonds

$$\frac{\frac{Theta}{Z}+\ 0.5*Convexity*Volatility^2-Current\ rate-Spot\ rate}{duration}$$

And thus we can assess the ratio for each of the two bonds, respectively.

1-year Bond ratio = $\frac{\frac{0.791}{0.9891}+\ 0.5*\ 0.821*0.022^2-0.01}{0.906}=0.002$

10-year Bond ratio = $\frac{\frac{0.178}{0.5917}+\ 0.5*\ 2.037*0.022^2-0.01}{1.427}=0.005$

The two ratios are signifiantly different, indicating that the non-arbitrage condition does not hold.

(b) Calculate the pricing errors of the bonds.

Commentary on Question:
Candidates performed well on this part. Most common mistakes tended to be calculation errors and the wrong sign for the pricing error.

Using the Vasicek model, For 1-year bond:

$$\mathrm{B}(1)=\frac{1}{\gamma^*}\left(1-e^{-\gamma^*}\right)=0.7192$$

$$\mathrm{A}(1)=(B(1)-1)*\left(\bar{r}^*-\frac{\sigma^2}{2(\gamma^*)^2}\right)-\ \frac{\sigma^2}{4\gamma^*}*B(1)^2$$
$$=(0.7192-1)*\ 0.0595-\ 0.0002*0.7192^2$$
$$=\ -0.01686$$

$$\mathrm{Z}_{1-year}=\ e^{(A(1)-B(1)*r_0)}=e^{(-0.01686-0.7192*0.01)}=0.9762$$

Pricing error is $P_{data}-P_{Vasicek}=98.91-0.9762*100=1.29$
Or 1.32% higher of the model prediction.

Similarily, for the 10-year bond:

$$\mathrm{B}(10)=\frac{1}{\gamma^*}\left(1-e^{-10\gamma^*}\right)=1.4273$$

6. Continued

$$A(10) = (B(10) - 10) * \left(\bar{r}^* - \frac{\sigma^2}{2(\gamma^*)^2}\right) - \frac{\sigma^2}{4\gamma^*} * B(10)^2$$
$$= (1.4273 - 10) * 0.0595 - 0.0002 * 1.4273^2$$
$$= -0.5105$$

$$Z_{10-year} = e^{(A(10)-B(10)*r_0)} = e^{(-0.5105-1.4273*0.01)} = 0.5917$$

Pricing error is $P_{data} - P_{Vasicek} = 59.17 - 0.5917 * 100 = 0$

(c) Determine the optimal hedge ratio, instrument(s) to trade and position (long or short).

Commentary on Question:
Candidates performed generally poorly on this part. Most recognized that the 1-year bond needed to be shorted. Difficulties arose in both calculating the hedge ratio and the development of the replicating portfolio and the need to use cash.

Since $P_{data} > P_{Vasicek}$ 1-year bond is over priced by the market. The 10-year bond is accurately priced.

To arbitrage, we should sell the overpriced 1-year bond and replicate the Vasicek model price of 1-year bond using the 10-year bond and cash position.

The optional hedge ratio is calculated as

$$\Delta = \frac{\partial Z_1}{\partial r} / \frac{\partial Z_2}{\partial r}$$

Under Vasicek model,

$$\frac{\partial Z}{\partial r} = e^{(A(t;T)-B(t;T)\, r_t)}(-B(t;T))$$

Therefore

$$\Delta = \frac{\frac{\partial Z_1}{\partial r}}{\frac{\partial Z_2}{\partial r}} = \frac{Z_{1-year}\, B(1)}{Z_{10-year} B(10)} = \frac{0.9762 * 0.7192}{0.5917 * 1.4273} = 0.8313$$

For each 1-year bond sold, we must purchase 0.8313 of the 10-year bond.

$$C_t = -P_{1-year}{}^{Model} + \Delta\, P_{10-year}{}^{Model}$$

The initial cash position is thus
$C_0 = -0.9762 * 100 + 0.8313 * 59.17 = -48.32198 \approx -48.32$
We lend at risk-free rate of 48.32 for each 100 1-year bond.

6. Continued

(d) Calculate the initial profit.

Commentary on Question:
Candidates generally answered this question satisfactorily. Most common error was not recognizing the initial cash position.

The replicating portfolio is given by
$P_0^{1-year} = -C_0 + \Delta\, P_{10-year}^{Model} = 48.32 + 0.8313 * 59.17 = 97.62$
Initial profit is thus $98.91 - 97.62 = 1.29$ for each 100 1-year bond sold.

It was also acceptable to note that the initial mispricing calculated in part b would also be the initial profit.

(e) Critique the following statement:

"By executing the arbitrage strategy, we are hedged and will reap a guaranteed profit from the position."

Commentary on Question:
Candidate generally did well if they attempted to answer this part. Additional responses besides the model solution were accepted as long as they make a reasonable argument relevant to the question such as parametrization, or appropriateness of the model.

The statement is not true.
The position is hedged only at time zero and we make a profit only today. The position needs to be dynamically rebalanced going forward.

7. **Learning Objectives:**

4. The candidate will understand:
   - How to apply the standard models for pricing financial derivatives.
   - The implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory.
   - How to evaluate risk exposures and the issues in hedging them.

Learning Outcomes:

(4a) Demonstrate an understanding of option pricing techniques and theory for equity derivatives.

(4d) Demonstrate an understanding of how to delta hedge, and the interplay between hedging assumptions and hedging outcomes.

(4f) Appreciate how hedge strategies may go awry.

(4h) Compare and contrast the various kinds of volatility, e.gl, actual, realized, implied and forward, etc.

Sources:

QFIQ-114-17: Chapter 2, 162-173 and 223-225 of Frequently Asked Questions in Quantitative Finance, Wilmott, Paul, 2nd Edision, 2009

The Volatility Smile, Derman, Miller, and Park, 2016, Ch 5

QFIQ-120-19: Chapters 6 and 7 of Pricing and Hedging Financial Derivatives, Marroni, Leonardo and Perdomo, Irene, 2014

Commentary on Question:

Commentary listed underneath question component.

Solution:

(a) Define actual volatility and realized volatility.

Commentary:

Candidates did well on this part of the question. Most candidates were able to provide a good description of actual and realized volatility.

Model Solution:

Actual volatility is the instantaneous amount of noise in a stock price return. Realized volatility is a backward-looking statistical measure of what volatility has been over a given period.

7. Continued

(b) Estimate ACL's realized volatility for the 5-day period expressed as annualized volatility assuming 252 trading days per year.

Commentary:
Most candidates did well on this question. Realized volatility should be calculated as the variance of the stock price returns, using the historical stock prices provided. Some candidates mistakenly calculated the variance of the stock prices.

Actual volatility = 16.*2%*

Day	0	1	2	3	4	5
Daily price return*	-	0.5%	2%	-2%	0%	-0.5%

*Log return is also accepted, LN(S(t)/S(t-1))

Mean = (0.5% + 2% - 2% + 0% -0.5%)/5 = 0%
**Daily var = (0.5%^2 + 2%^2 + 2%^2 + 0%^2 + 0.5%^2)/5 = 0.017%
Realized vol = (252* 0.017%)^0.5 = 20.7%.

**Also valid if daily variance used divisor of 4 rather than 5:
Daily var = (0.5%^2 + 2%^2 + 2%^2 + 0%^2 + 0.5%^2)/4 = 0.0213%
Realized vol = (252* 0.0213%)^0.5 = 23.1%%

(c) Determine if ACL's implied volatility is greater than, equal to, or less than its actual volatility.

Commentary:
Many candidates did well in this part of the question. To compare the implied vol to the actual volatility, the actual volatility can be used to calculate the price of call option, which is then compared to the market price that is determined by implied volatility. Points are awarded to candidates who demonstrated the understanding of the positive correlation between call option price and volatility, even if calculation errors were carried forward from previous parts.

Using the actual vol of 16.20% to calculate the option price.

Volatility	σ	16.20%
Stock price	S	999.58
Strike price	K	1000.00
Interest rate	r	2%
Maturity(year)	T	1

d1=[ln(S/K)+(r +0.5σ^2)]/(σ*T^0.5)=0.2019
d2=d1 - σ * T^0.5=0.0399
N(d1)=0.5800
N(d2)=0.5159
Call = S*N(d1) - K*exp(-r*T)*N(d2)=74.07

7. Continued

Since 74.07 < market price of 77, the implied vol > actual vol.

(d) Determine if ACL's implied volatility is greater than, equal to, or less than the volatility your firm used in its delta hedging strategy.

Commentary:
Many candidates did well on this question. Many reused the results from their answer to part (c). The key to this question is to realize that the delta of the option price calculated in part (c) is equivalent to the delta used in hedging.

Given that the firm uses delta hedging, the call option delta = N(d1). The number of shares the firm bought to hedge = 100*N(d1) = 58. This means N(d1) = 0.58. From part (c), we have N(d1) = 0.58 when actual volatility is used.

This means that the hedging volatility the firm used is equal to the actual volatility. Since, from part (c), we know that implied volatility > actual volatility, we can conclude that implied volatility > hedging vol.

OR, since $N(d_1) = \ 0.58$, therefore, $d_1 = 0.2019$

That is, $ln(0.99958) + (0.02 + 0.5\sigma^2) = 0.2019\sigma.$

$$\sigma = 16.2\% \ or \ 24.2\%$$

If candidates solved the equation, with two solutions, and concluded the implied volatility < hedging volatility, full credits were also given.

(e) Outline implications for your firm's hedging profit/loss given your firm's choice of the volatility in its delta hedging strategy.

Commentary:
Many candidates were able to identify at least one or two bullets listed below. Full points were awarded only to those candidates that identified all three bullets.

Characteristics:

- The firm's present value of hedging P&L is locked in at inception if the firm's view turns out to be correct.
- The firm's hedging P&L will exhibit large variations before reaching the maturity
- It's very difficult to foresee future realized volatility, thus, there is no assurance that the firm's view is actually correct.

7. Continued

(f) Determine the present value of your firm's hedging profit/loss.

Commentary:
Some candidates left this question blank. Many candidates appropriately utilized the call option price calculated in part (c). Common mistakes were either the candidate forgot to multiply the difference in option prices by the number of call options sold (100) or the sign was switched.

Because the hedging volatility is based on its view of 16.2% and its view turns out to be correct, the present value of hedging P&L is the difference between the option value the firm believes it should have been (which is 74.07), and the price the firm actually transacted at (which is 77). Thus, PV of P&L = 100 * (77 – 74.07) = 293.

(g) Determine the simulated Index value at the end of year 3.

Commentary:
Very few candidates calculated this part correctly. Many candidates left it blank. For candidates who performed the calculations, most of them did not convert the spot volatility to forward volatility for year 2 and 3, instead, spot volatilities were applied to the formulae directly. In these cases, partial credits were given.

S(t) = S(t-1)* e [r- σ(t)^2/2 + σ(t) * N(t)]
* σ(1)=.17
S(1) = 1000 * e[.02-(.17)^2/2+ (.17)(.1)] = 1022.8
* σ(2)^2 = 2 (.19)^2 - (.17)^2 => σ(2) = 20.81%
S(2) = 1022.8 * e[.02-(.2081)^2/2+(.2081)(-.1)]= 1000.08
* σ(3)^2 = 3(.20)^2 - 2(.19)^2 => σ(3) = 21.86%
S(3) = 1000.08 * e(.02-(.2186)^2/2 +(.2186)(0)) = 996.19

OR

For year 1: σ1 = 17%
For year 2: 2*19%^2 = 17%^2 + σ2^2
* σ2 = (2*19%^2 - 17%^2)^0.5 = 20.81%
For year 3: 3*20%^2 = 2*19%^2 + σ3^2
* σ3 = (3*20%^2 - 2*19%^2)^0.5 = 21.86%
S(3) = S0*e^((3*r -(σ1^2)/2-(σ2^2)/2-(σ3^2)/2 + σ1*N1 +σ2 *N2+σ3 *N3))
S(3)=1000*e^((3*2% -(17^2)/2-(20.81^2)/2-(21.86^2)/2 + 17%*0.1 +20.81%*(-0.1)+21.86%*0))
S(3) = 996.21

8. **Learning Objectives:**

4. The candidate will understand:
   - How to apply the standard models for pricing financial derivatives.
   - The implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory.
   - How to evaluate risk exposures and the issues in hedging them.

Learning Outcomes:

(4e) Analyze the Greeks of common option strategies.

(4i) Define and explain the concept of volatility smile and some arguments for its existence.

(4k) Describe and contrast several approaches for modeling smiles, including: stochastic volatility, local-volatility, jump-diffusions, variance-gamma, and mixture models.

Sources:

QFIQ 120-19 – *The Volatility Smile*, Derman, Emanuel and Miller, Michael B., 2016

Commentary on Question:

The purpose of this question is to find out the usefulness of straddle and strangle themselves, overall understanding of their Greeks, and valuation using local volatility models. Therefore, candidates were expected to include justification for problem-solving rather than simply filling out the answer.

Solution:

(a) Explain how a straddle strategy and a strangle strategy provide Vega protection with minimal exposure to other market risks.

Commentary on Question:

Most candidates understood what straddles and strangles were, and explained Vega protection well, but lacked explanations about why there was minimal exposure to other market risks.

- A long straddle position consists of the simultaneous purchase of an at-the-money call and an at-the-money put with the same maturity.
- A long strangle position consists of the simultaneous purchase of an out-of-money call and an out-of-the-money put with the different maturities, where the strike price of the call is higher than the strike of the put.

Straddles and strangles have positive, large Vega because Vega is positive for both calls and puts. Therefore, they show a strong sensitivity to the absolute level of volatility. They also have very low sensitivity to dV/dS and $dV/d\sigma$. Therefore, they can be used to trade and to express views on the absolute level of volatility.

8. **Continued**

Straddles and strangles have minimal exposure to other market risks because:

- Straddles and strangles are considered to be delta-neutral because the positive delta of the call option offsets the negative delta of the put option.
- Straddles and strangles have positive gamma, which benefits the investors when the underlying price moves in either direction.

(b)

(i) Calculate the cost of each of the two strategies in part (a).

(ii) Calculate Delta, Gamma, Vega, and Theta of each of the two strategies in part (a).

Commentary on Question:
Most candidates did well. Some candidates solved the question using formulas rather than calculating it using the information given in the problem.

(i)

$$Straddle = 6.371 + 4.882 = 11.253$$
$$Strangle = 1.767 + 0.977 = 2.744$$

(ii)

Delta

$$Straddle = 0.570 + (0.570 - 1) = 0.14$$
$$Strangle = 0.195 + (0.904 - 1) = 0.099$$

Gamma

$$Straddle = 2 \times 0.028 = 0.056$$
$$Strangle = 0.009 + 0.016 = 0.025$$

Vega

$$Straddle = 2 \times 27.772 = 55.544$$
$$Strangle = 12.089 + 19.521 = 31.610$$

Theta

$$Straddle = -7.074 - 4.035 = -11.109$$
$$Strangle = -5.413 - 1.340 = -6.753$$

(c) Estimate Delta and Vega of the two strategies in part (a) when

(i) Underlying price moves from 100 to 50;

(ii) Underlying price moves from 100 to 200.

8. Continued

Commentary on Question:
Most candidates struggled to approach the problem. Many candidates had shown attempts to compute Delta or Vega using their second derivatives (e.g. gamma). However, this can only be applied in the case of instantaneous underlying value change and cannot be used when there is a significant underlying value change, such as given in the exam problem.

$$\Delta^{Straddle} = N(d_1) + [N(d_1) - 1] = 2 \times N(d_1) - 1$$

$$\Delta^{Strangle} = N\left(d_1^{call}\right) + \left[N\left(d_1^{put}\right) - 1\right]$$

Therefore, $d_1 \to \infty$ $(i.e. \Delta \to 1 + (1-1) = 1 \ \ as \ \ S \to \infty$ and $d_1 \to -\infty$ $(i.e. \Delta \to 0 + (0-1) = -1 \ \ as \ \ S \to 0.$

$N(d_1)$ increases as S increases and $N(d_1)$ decreases as S decreases.

(i)

If the underlying price decreases from 100 to 50, Delta will decrease. It can be negative and will converge to -1
As the option gets more and more in-the-money or out-of-the-money, its Vega will stabilize at levels converging on 0. Hence, if S decreases from 100 to 50, Vega will decrease. Note that the call option will be more out-of-the-money and the put option will be more in-the-money.

(ii)

If the underlying price increases from 100 to 200, Delta will increase and approach 1.
As the option gets more and more in-the-money or out-of-the-money, its Vega will stabilize at levels converging on 0. Hence, if S increases from 100 to 200, Vega will decrease. Note that the call option will be more in-the-money and the put option will be more out-of-the-money.

(d) Describe how Vega changes with:

(i) The time to maturity;

(ii) The underlying value.

Commentary on Question:
Many candidates showed the right approach, but some candidates misunderstood the nature of Vega.

8. Continued

(i)

As the expiry approaches, the tails of the Vega function get thinner, and the dynamics that can be observed become increasingly stronger.

The effect of the reduction in the time to expiration on Vega is to reduce progressively the range of prices within which changes in volatility have meaningful effects.

For long-dated options, not only will the Vega tend to be bigger, but its effects are observable even for large deviations of the price of the underlying assets from the strike.

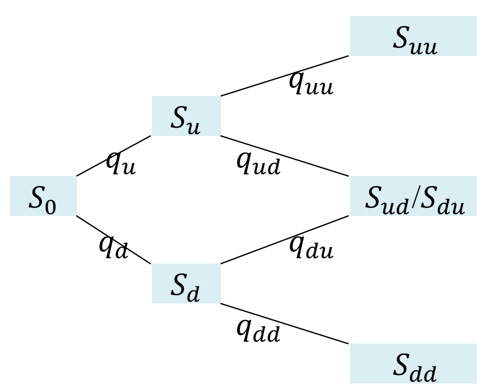
(ii)

When the option is not very far away from being at-the-money, the magnitude of Vega is not particularly affected by the level of volatility. When the option is closest to at-the-money, the option is most sensitive to underlying price changes, and therefore the Vega is maximized.

For very deep in-the-money or out-of-the-money options, the Vega exposure is close to zero, since it would take a significant change in stock price to impact the in or out of the moneyness of the options.

(e) Calculate the cost of the two strategies in part (a) using a tree assuming that $\Delta t = 0.25$ and the continuous risk-free interest rate is 3%.

Commentary on Question:
Most candidates had difficulty solving this problem. Very few completed the entire answer and many received no credit or very little credit.



$$S_u = S_0 \times e^{\sigma(S_0)\cdot\sqrt{\Delta t}} = 100 \times e^{0.2\times\sqrt{0.25}} = 110.52$$
$$S_d = S_0 \times e^{-\sigma(S_0)\cdot\sqrt{\Delta t}} = 100 \times e^{-0.2\times\sqrt{0.25}} = 90.48$$

8. Continued

$$q_u = \frac{F(= S_0 \times e^{r\Delta t}) - S_d}{S_u - S_d} = \frac{100.75 - 90.48}{110.52 - 90.48} = 51.26\%$$

$$q_d = 1 - q_u = 1 - 0.5126 = 48.74\%$$

$$\sigma_u = \frac{(110.52 - 100)^2}{110.52^2} + 0.2 = 20.91\%$$

$$\sigma_d = \frac{(90.48 - 100)^2}{90.48^2} + 0.2 = 21.11\%$$

$$S_{ud} = S_{du} = S_0 = 100$$

$$S_{uu} = S_u \times e^{r\Delta t} + \frac{(S_u)^2 \times \sigma(S_u)^2 \times \Delta t}{S_u \times e^{r\Delta t} - S_{ud}}$$

$$= 110.52 \times e^{0.03\times0.25} + \frac{110.52^2 \times 0.2091^2 \times 0.25}{110.52 \times e^{0.03\times0.25} - 100} = 123.11$$

$$S_{dd} = S_d \times e^{r\Delta t} - \frac{(S_d)^2 \times \sigma(S_d)^2 \times \Delta t}{S_{du} - S_d \times e^{r\Delta t}}$$

$$= 90.48 \times e^{0.03\times0.25} + \frac{90.48^2 \times 0.2111^2 \times 0.25}{100 - 90.48 \times e^{0.03\times0.25}} = 80.84$$

$$q_{uu} = \frac{S_u \times e^{r\Delta t} - S_{ud}}{S_{uu} - S_{ud}} = \frac{110.52 \times e^{0.03\times0.25} - 100}{123.11 - 100} = 49.11\%$$

$$q_{ud} = 1 - q_{uu} = 50.89\%$$

$$q_{du} = \frac{S_d \times e^{r\Delta t} - S_{dd}}{S_{du} - S_{dd}} = \frac{90.48 \times e^{0.03\times0.25} - 80.84}{100 - 80.84} = 53.88\%$$

$$q_{dd} = 1 - q_{du} = 46.12\%$$

$$\begin{aligned} Straddle = e^{-r\times2\Delta t}\{&[\max(S_{uu} - 100{,}0) \times q_{uu} \times q_u] \\ &+ [\max(S_{ud} - 100{,}0) \times q_{ud} \times q_u] \\ &+ [\max(S_{du} - 100{,}0) \times q_{du} \times q_d] \\ &+ [\max(S_{dd} - 100{,}0) \times q_{dd} \times q_d] \\ &+ [\max(100 - S_{uu}, 0) \times q_{uu} \times q_u] \\ &+ [\max(100 - S_{ud}, 0) \times q_{ud} \times q_u] \\ &+ [\max(100 - S_{du}, 0) \times q_{du} \times q_d] \\ &+ [\max(100 - S_{dd}, 0) \times q_{dd} \times q_d]\} = 9.97 \end{aligned}$$

8. Continued

$$
\begin{aligned}
Strangle = e^{-r\times 2\Delta t}\{&[\max(S_{uu}-120{,}0)\times q_{uu}\times q_u]\\
&+[\max(S_{ud}-120{,}0)\times q_{ud}\times q_u]\\
&+[\max(S_{du}-120{,}0)\times q_{du}\times q_d]\\
&+[\max(S_{dd}-120{,}0)\times q_{dd}\times q_d]\\
&+[\max(80-S_{uu},0)\times q_{uu}\times q_u]\\
&+[\max(80-S_{ud},0)\times q_{ud}\times q_u]\\
&+[\max(80-S_{du},0)\times q_{du}\times q_d]\\
&+[\max(80-S_{dd},0)\times q_{dd}\times q_d]\} = 0.77
\end{aligned}
$$

(f) Describe advantages and disadvantages of the local volatility model.

Commentary on Question:
Some candidates understood the local volatility model well, but many candidates showed incorrect analysis.

Advantages
A great advantage of the model is its closeness to the original Black-Scholes Model and its dynamics. The notion that the implied Black-Scholes Model volatility is the average of the local volatilities from the initial stock price to the strike leads to intuitive rules of thumb about how options values and hedge ratios differ from their Black-Scholes Model values in the presence of a skew.

Disadvantages
- The models need to be frequently recalibrated. (The necessity for periodic recalibration)
- The models tend to have difficulty matching the future short-term skew. (Inability to match the short-term skew)

9.

Learning Objectives:

4. The candidate will understand:
   - How to apply the standard models for pricing financial derivatives.
   - The implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory.
   - How to evaluate risk exposures and the issues in hedging them.

Learning Outcomes:

(4a) Demonstrate an understanding of option pricing techniques and theory for equity derivatives.

(4c) Demonstrate an understating of the different approaches to hedging – static and dynamic.

(4i) Define and explain the concept of volatility smile and some arguments for its existence.

Sources:

The Volatility Smile, Derman, Emanuel and Miller, Michael B., 2016 Ch. 2-3

Commentary on Question:

To receive full credit, candidates should have solid knowledge in understanding option payoffs, use options to replicate payoffs, understanding the difference of difference replication approaches and their limitations.

Solution:

(a) Compare and contrast static and dynamic replication.

Commentary on Question:
Candidates generally did well on this question.

Static replication:
- Reproduces the payoffs of the target security over its entire lifetime with an initial portfolio of securities whose weights will never need to be changed.
- No additional trading is required for the lifetime of the security once static replicating portfolio is created

Dynamic replication:
- the components and weights of the replicating portfolio must change over time.
- need to continually buy and sell securities as time passes and price changes to achieve replication

9. Continued

(b) List four limitations of replication.

Commentary on Question:
Candidates generally were able to identify some limitations. Candidates who did not list a) and d) below received partial credit.

a) It is challenging to choose a financial model with the right balance of complexity and accuracy
b) Have to adjust weights in the market where you are subject to bid-ask spreads, illiquidity, and other market impacts
c) Transaction cost
d) Estimation the future values of certain parameters that are difficult or impossible to observe in the market.

(c) Construct a structured product using the options above to meet ABC's goal.
Not understand call/put options

Commentary on Question:
Some candidates did not know how to use options to construct portfolios with desired payoff.

To construct the product:
Bank ABC can create a collar is created by buying a put $P_{90}(100,1)$, with a strike price of 90 (L) and stock price 100 (S) and selling a call $C_{110}(100,1)$, with a strike price of 110 (U) and stock price 100 (S), where $L < S < U$. Both options have the same expiration date T=1.

The put will limit our losses if the price of the stock falls below L=90, and the call will cap our profits if the stock rises above U=110.

(d) Calculate the price at time 0 of the structured product constructed in part (c).
Most candidates did well

Commentary on Question:
Most candidates did well on this part.

Value of a collar at time 1 $= S_1 + P_{90}(100,1) - C_{110}(100,1)$
$= S_1 + 2 - 1.9 = S_1 + 0.1$

9. Continued

(e) Replicate the payoff of the alternative product using only riskless bonds, the stock, and calls/puts on the stock.

Commentary on Question:
Similar to part (c), some candidates did not know how to use options to construct portfolios with desired payoff, particularly some failed to identify the ½ put option to protect 50% of the downside risk. Most candidates knew to use the call option with strike $115 to cap the gains.
Candidates received full credits using the alternative construction.

The payoff of a structured product is a piecewise-linear function of an underlying stock, S. The payoff has the following break points:
S = $0: payoff = $50
S = $100: payoff = $100
S = $115: payoff = $115
S= $140: payoff = $115

The portfolio's value at t is $V(t) = Ie^{-r(T-t)} + \lambda_0 S_t + (\lambda_1 - \lambda_0)\, C(K0) + (\lambda_2 - \lambda_1)\, C(K1) + \cdots$

Because the riskless rate is 0%, we do not need to worry about the $e^{-r(T-t)}$ term, but I is 0.5*B (y-axis intercept)

Slope between the first and second break points: ($100 – $50)/ ($100 − $0) =1/2.
Slope between the second and third: ($115−$100))/ ($115 − $100) = 1
Change in slope between them: 1 −1/2 = 1/2.

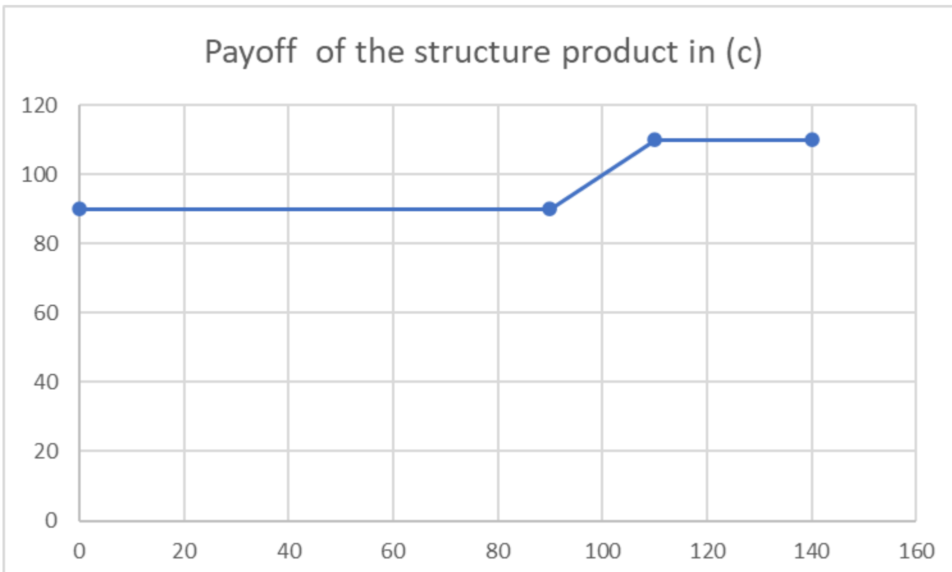
Slope between the third and fourth break points: ($115 – $115))/ ($140 − $115) =0.
Change in slope between them: 0 −1 = -1.
We need to buy 1 share of the underlying stock, buy ½ Put option at the strike of $100 and short one call option at the strike of $115.

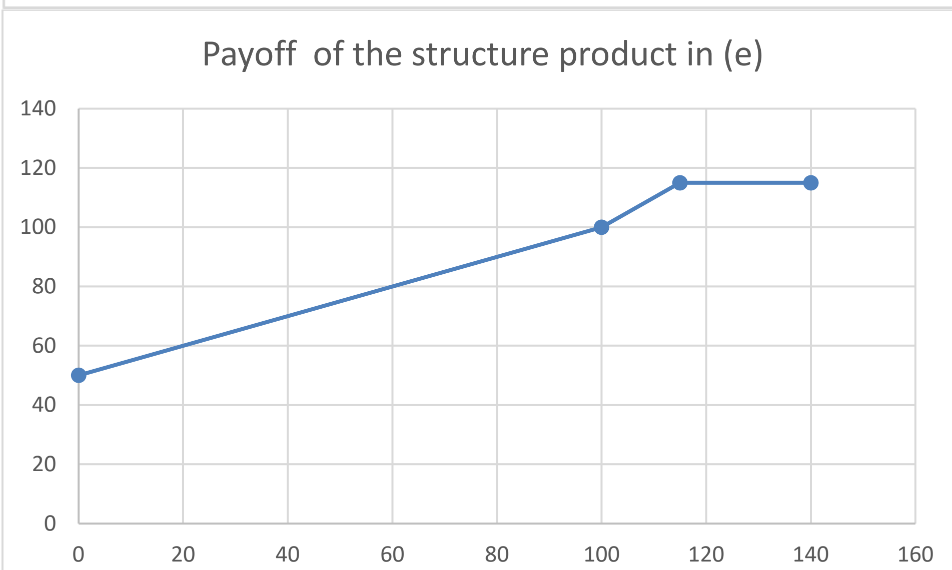
Alternatively, we buy $50 of riskless bonds, buy 1/2 share of the underlying stock, buy ½ call option with a strike price of $100 and short one call option of a strike price of $115 (this can be derived by applying the put-call parity on the 0.5 of put option with $100 strike).

9. **Continued**

(f) Sketch the payoffs of the strategies in part (c) and part (e) and compare their costs.

Commentary on Question:
Few candidates compared the cost between part (c) and part (e), some did not have labels in the payoff chart and thus earned partial credits.





9. **Continued**

Value of the structured product at time 1
$= 0.5*B + 0.5*S_1 + 0.5* C_{100}(100,1) - C_{115}(100,1)$
$= 0.5*B + 0.5* C_{100}(100,1) - C_{115}(100,1) +0.5*S_1$ (re-arranging)
$= 0.5*S_1 + 0.5*P_{100}(100,1) - C_{115}(100,1) +0.5*S_1$ (call put parity)
$= S_1 + 0.5*P_{100}(100,1) - C_{115}(100,1)$
$= S_1 + 0.5* 6 – 1.5 = S_1 + 1.5$

Value of the structured product at time 1 $= S_1 + 1.5$
comparing to collar $= S_1 + 0.1$
The portfolio in part (e) has richer optionality.

10. **Learning Objectives:**

2. The candidate will understand the fundamentals of fixed income markets and traded securities.

Learning Outcomes:

(2a) Understand the characteristics of fixed rate, floating rate, and zero-coupon bonds.

(2b) Bootstrap a yield curve.

(2c) Understand measures of interest rate risk including duration, convexity, slope, and curvature.

Sources:

Fixed Income Securities: Valuation, Risk, and Risk Management, Veronesi, Pietro, 2010 (Chapter 2, 3, and 4)

Commentary on Question:

This question tests candidates' understanding of the basics of fixed income securities and the application of interest risk management. Candidates perform as expected. Partial credits were given for each step that was completely correctly.

Solution:

(a) Calculate the discount factors Z(0,0.5) and Z(0,1) as of Dec 31, 2018.

Commentary on Question:

Candidates performed well on this part.

Pbill (0, 0.5) = \$97 = \$100 * Z(0, 0.5)
Therefore, Z (0,0.5) = 97 / 100 = 0.97

Pnote (0, 1) = \$95.85 = \$1.5 * Z(0, 0.5) + \$101.5 * Z(0,1)
Substituting Z(0,0.5) into the equation above, we have
\$95.85 = \$1.5 * 0.97 + \$101.5 * Z(0,1)
Z(0,1) = 0.93

(b) Describe two disadvantages of yield curve bootstrapping and how alternative approaches can be used to overcome each of the disadvantages.

Commentary on Question:

Candidates performed as expected on this part. Many were able to identify the disadvantages of bootstrapping correctly suggested the alternatives. Descriptions of how to apply the alternatives are required to receive full marks.

Disadvantage: For short-term maturities, there are too many bonds that mature on the same day to choose from. To perform the bootstrap methodology, we then must cherry pick the bonds that we deem have the highest liquidity.

10. Continued

Alternative: Regression
Estimates the yield curve based on the bond prices observed using Ordinary Least Squares regression

Disadvantage: For longer maturities, not all of the bonds may be available
Alternative: Curve fitting
We can postulate a parametric functional form for the discount factor Z(0,T) as a function of maturity T and use the current bond prices to estimate the parameters of this functional form.

(c)

(i) Calculate the Macaulay duration of the one-year Treasury note.

(ii) Explain why the Macaulay duration of the one-year Treasury note is shorter than one year.

Commentary on Question:
Candidates performed well on this part. For part (ii), full credit is awarded for correctly identifying the coupon at 6^th month reduces the weighted average of the CFs. Interest rate sensitivity is not a must-have to receive full credit for this part.

(i)

$$w_1 = \frac{0.03/2 \, * \, P_z(0{,}0.5)}{P_c(0,1)} = \frac{0.015 * 97}{95.85} = 0.01518$$

$$w_2 = \frac{(1+0.03/2) * P_z(0{,}1)}{P_c(0{,}1)} = \frac{1.015*(100*0.93)}{95.85} = 0.98482$$

$$D_c = \sum_{i=1}^{2} w_i T_i = (0.01518 * 0.5) + (0.98482 * 1) = 0.99241$$

(ii)

Since we are receiving intermediate coupons before its maturity, the average time to cash flow payments is lower

Cash flows that arrive sooner rather than later are less sensitive to changes in interest rate. Since we are receiving semi-annual coupons, it implies an overall lower sensitivity to changes in discount rates when compared to a one-year zero coupon bond.

10. Continued

(d) Explain the differences of cash flow matching and immunization in terms of hedging a stream of liability annuity cash flows.

Commentary on Question:
Candidates performed as expected on this part. Difference in practicality such as liquidity and cost are required to receive full credit.

Cash flow matching

- Purchase a set of securities that matches the cash outflows with the cash inflows.
- Not guaranteed to be able to find exactly the type of securities that are required for cash flow matching as these securities could be costly and illiquid.

Immunization

- Invest in a portfolio of securities with the same present value and duration of the cash flow commitments to pay
- Able to choose securities that have favorable properties in terms of liquidity and transaction costs

(e) Recommend a duration hedge strategy that uses the six-month Treasury bill to mitigate the interest rate risk.

Commentary on Question:
Candidates performed below expectation on this part. Many candidates simply calculated Ks as the ratio of the durations of the Treasury note and the 6-month Treasury bill.

The six-month Treasury bill does not pay any coupon and therefore its duration is 0.5 years.

The market price for the Treasury bill is \$97
The market price for the Treasury note is \$95.85

The Macaulay duration of the Treasury note is 0.99241

The recommended duration hedging can be achieved by taking a position Ks in the short-term T-bill

$$k_S = -\frac{D \times P}{D_S \times P_S} = - \frac{0.99241*95.85}{0.5*97} = - -1.96129$$

10. Continued

This means that you must short 1.96129* 10 = 19.6129 units of the Treasury bill to hedge the interest rate risk.

At the six-month maturity of the T-bill, the positions have to be rolled forward for another six months.

(f) Determine k_S and k_L in terms of $P, P^S, P^L, D_1, D_2, D_1^S, D_2^S, D_1^L,$ and D_2^L such that the portfolio P plus the short-dated and the long-dated bonds is immunized against changes in the level and slope factors.

Commentary on Question:
Candidates performed as expected on this part. Candidates didn't receive full credit when they do not follow the proof steps exactly. In fact, this part is directly coming from the source material.

the change in $V = P + k_S \times P_z^S + k_L \times P_z^L$ satisfies

$$dV = dP + k_F \times dP_z^S + k_L \times dP_z^L = 0$$

Substituting into dP, dP_z^S, dP_z^L

$$\frac{dP}{P} = -D_1 \times d\phi_1 - D_2 \times d\phi_2$$

We have

$$\begin{aligned} 0 \quad = \quad & -D_1 \times P \times d\phi_1 - D_2 \times P \times d\phi_2 \\ & + k_S \times (-D_{z1}^S \times P_z^S \times d\phi_1 - D_{z2}^S \times P_z^S \times d\phi_2) \\ & + k_L \times (-D_{z1}^L \times P_z^L \times d\phi_1 - D_{z2}^L \times P_z^L \times d\phi_2) \end{aligned}$$

Pool together all the elements containing the level and slope factors $d\phi_1$ and $d\phi_2$

$$\begin{aligned} 0 \quad = \quad & -(D_1 \times P + k_S \times D_{z1}^S \times P_z^S + k_L \times D_{z1}^L \times P_z^L) \times d\phi_1 \\ & -(D_2 \times P + k_S \times D_{z2}^S \times P_z^S + k_L \times D_{z2}^L \times P_z^L) \times d\phi_2 \end{aligned}$$

In order for the equation to be zero for all possible values of $d\phi_1$ and $d\phi_2$, each parenthesis on the right-hand side must be zero, thus we have,

$$k_S \times D_{z1}^S \times P_z^S + k_L \times D_{z1}^L \times P_z^L = -D_1 \times P$$
$$k_S \times D_{z2}^S \times P_z^S + k_L \times D_{z2}^L \times P_z^L = -D_2 \times P$$

10. **Continued**

The determination of k_s and k_L

$$
\begin{aligned}
k_S &= -\frac{P}{P_z^S}\left(\frac{D_1 \times D_{z2}^L - D_2 \times D_{z1}^L}{D_{z1}^S \times D_{z2}^L - D_{z2}^S \times D_{z1}^L}\right) \\
k_L &= -\frac{P}{P_z^L}\left(\frac{D_1 \times D_{z2}^S - D_2 \times D_{z1}^S}{D_{z1}^L \times D_{z2}^S - D_{z2}^L \times D_{z1}^S}\right)
\end{aligned}
$$

11. Learning Objectives:

1. The candidate will understand the foundations of quantitative finance.

3. The candidate will understand:
   - The Quantitative tools and techniques for modeling the term structure of interest rates.
   - The standard yield curve models.
   - The tools and techniques for managing interest rate risk.

Learning Outcomes:

(1c) Understand Ito integral and stochastic differential equations.

(3b) Understand and apply various one-factor interest rate models.

(3c) Calibrate a model to observed prices of traded securities.

(3f) Apply the models to price common interest sensitive instruments including: callable bonds, bond options, caps, floors, and swaptions.

(3i) Understand and apply the Heath-Jarrow-Morton approach including the Libor Market Model.

Sources:

Fixed Income Securities: Valuation, Risk, and Risk Management, Veronesi, Pietro, 2010 Chapters 14, 15, 19-21

An Introduction to the Mathematics of Financial Derivatives, Hirsa, Ali and Neftci, Salih N., 3rd Edition 2nd Printing, 2014, Chapters 9, 11

Problems and Solutions in Mathematical Finance: Stochastic Calculus, Chin, Eric, Nel, Dian and Olafsson, Sverrir, 2014, Chapter 3

Commentary on Question:

This question tested a candidate's understanding of one-factor interest models and the usage of these models in pricing interest rate sensitive instruments.

Solution:

(a) Compare and contrast the LIBOR market model, the Vasicek model, and the Black, Derman, and Toy (BDT) model in terms of:

   (i) The interest rates being modeled;

   (ii) Diffusion processes for interest rates.

11. Continued

Commentary on Question:
Most candidates did well and were awarded marks for applicable statements that were not necessarily in the model solution.

(i) LIBOR market model (or BGM model) uses forward LIBOR rates that are observable.

The Vasciek model and Black, Derman and Toy (i.e. BDT) model use spot rate inputs.

(ii) LIBOR market model is a model for the LIBOR-based forward rates, according to which each forward rate with maturity T follows a lognormal diffusion process under the dynamics implied by the zero-coupon bond with maturity T.

BDT assumes that the logarithm of interest rates is normally distributed, implying that interest rates are always positive.

Vasicek model assumes that interest rate is normally distributed.

(b) Describe briefly the common approach of calibrating the BDT results to observed prices of caplets.

Commentary on Question:
Most candidates did poorly on this question and did not describe a correct approach.

Common practice is to use directly the forward volatilities σ_i , computed from caps and floors from the Black formula.

Using this methodology, the BDT tree results are simpler to build, as at each step i, it is necessary to search only for one variable, $r_{i,1}$ that matches the term structure of interest rates.

(c) Prove that $r_t = r_0^{\frac{\sigma_t}{\sigma_0}} e^{\sigma_t \int_0^t \frac{\theta_s}{\sigma_s} ds} e^{\sigma_t X_t}$.

Hint: First find the SDE for $\dfrac{y_t}{\sigma_t}$.

Commentary on Question:
Most candidates were able to follow the hint to derive an appropriate proof.

11. Continued

$$dy_t = \left(\theta_t + \frac{\frac{\partial \sigma_t}{\partial t}}{\sigma_t} y_t\right) dt + \sigma_t dX_t$$

$$\frac{dy_t}{\sigma_t} = \left(\frac{\theta_t}{\sigma_t} + \frac{\frac{\partial \sigma_t}{\partial t}}{\sigma_t^2} y_t\right) dt + dX_t$$

$$\frac{dy_t}{\sigma_t} - \frac{\frac{\partial \sigma_t}{\partial t}}{\sigma_t^2} y_t \mathrm{dt} = \left(\frac{\theta_t}{\sigma_t}\right) dt + dX_t$$

$$d\left(\frac{y_t}{\sigma_t}\right) = \left(\frac{\theta_t}{\sigma_t}\right) dt + dX_t$$

Integrating on both sides from 0 to t and solving for $\mathrm{y_t}$ leads to

$$y_t = \frac{y_0}{\sigma_0}\sigma_t + \sigma_t \int_0^t \theta_s/\sigma_s ds + \sigma_t X_t$$

$$e^{y_t} = e^{\frac{y_0}{\sigma_0}\sigma_t + \sigma_t \int_0^t \theta_s/\sigma_s ds + \sigma_t X_t}$$

$$r_t = e^{y_t} = r_0^{\frac{\sigma_t}{\sigma_0}} e^{\sigma_t \int_0^t \frac{\theta_s}{\sigma_s} ds} e^{\sigma_t X_t} \qquad \text{as } e^{\frac{y_0}{\sigma_0}\sigma_t} = e^{\frac{\sigma_t}{\sigma_0}\ln r_0} = r_0^{\frac{\sigma_t}{\sigma_0}}$$

(d) Evaluate $E(r_t)$ and $Var(r_t)$, assuming $\sigma_t = \sigma$ for all $t \geq 0$ where σ is a constant.

Commentary on Question:
Most candidates were able to answer this question well, though some candidates missed a simplification step by not using the assumption regarding σ_t or failed to show intermediate steps in the calculations.

When $\sigma_t = \sigma,\ r_t = e^{y_t} = \mathrm{r_0}\ e^{\int_0^t \theta_s ds} e^{\sigma X_t}$

As X_t is normally distributed random variable with mean zero and variance t, by using the moment generating funtion property of a normal random variable

$$E(e^{\sigma X_t}) = e^{1(0)+\frac{1}{2}(1)^2\sigma^2 t} = e^{\frac{\sigma^2 t}{2}}$$

This also follows from the property $E(e^Y) = e^{E(Y)+\frac{Var(Y)}{2}}$:

$$E(e^{\sigma X_t}) = e^{0+\frac{\sigma^2 t}{2}} = e^{\frac{\sigma^2 t}{2}}$$

11. Continued

$E(r_t) = \mathrm{r}_0\, e^{\int_0^t \theta_s ds} e^{\frac{\sigma^2 t}{2}}$

$Var(e^{\sigma X_t}) = e^{\sigma^2 t}(e^{\sigma^2 t} - 1)$

Again this follows from the property $E(e^Y) = e^{E(Y)+\frac{Var(Y)}{2}}$:

$Var(e^{\sigma X_t}) = E(e^{2\sigma X_t}) - (E(e^{\sigma X_t}))^2 = e^{\frac{4\sigma^2 t}{2}} - e^{\frac{2\sigma^2 t}{2}} = e^{\sigma^2 t}(e^{\sigma^2 t} - 1)$

$Var(r_t) = \mathrm{r}_0^2\, e^{2\int_0^t \theta_s ds} e^{\sigma^2 t}(e^{\sigma^2 t} - 1)$

(e) Demonstrate that $r_K^* = 3.32325\%$.

Commentary on Question:
Most candidates did poorly, primarily due to incorrectly setting up the initial pricing equation. For candidates who had the correct pricing equation, common mistakes were not to fully calculate the final steps to numerically prove that $r_K^* = 3.32325\%$ *produced the desired price.*

The first step is to define the equation that solves the interest rate r_K^* at time 2.1 years such that the value of the underlying coupon bond at time 2.1 years is the strike price of 99.

Coupon of \$1 will be paid at time 2.5 years and 3 years. Principal of \$100 will be paid at time 3 years.

$$1 * e^{A(2.1,2.5)-B(2.1,2.5)r_K^*} + 101 * e^{A(2.1,3.0)-B(2.1,3.0)r_K^*} = 99$$

$$B(t,T) = B(0, T-t) = \frac{1}{\gamma}(1 - e^{-\gamma(T-t)})$$

$$A(t,T) = A(0, T-t) = \left(B(t,T) - (T-t)\right)\left(\bar{r} - \frac{\sigma^2}{2\gamma^2}\right) - \frac{\sigma^2 B(t,T)^2}{4\gamma}$$

$$\gamma = 0.05, \bar{r} = \frac{0.002}{0.05} = 0.04, \sigma = 0.015$$

$B(2.1, 2.5) = 0.39602653$
$A(2.1, 2.5) = -0.00015657$

$B(2.1, 3) = 0.88005036$
$A(2.1, 3) = -0.00077155$

$Define\ K_i = Z(r_K^*, 2.1, T_i)\ \ for\ each\ coupon\ date\ T_i$

$$100\ x 0.01 x\ K_1 = 1 * Z(r_K^*, 2.1, 2.5) = 1 * e^{A(2.1,2.5)-B(2.1,2.5)r_K^*}$$

$$100\ x 1.01 x\ K_2 = 101 * Z(r_K^*, 2.1, 3.0) = 101 * e^{A(2.1,3.0)-B(2.1,3.0)r_K^*}$$

11. Continued

When $r_K^* = 3.32325\%$,

$$1 * e^{A(2.1,2.5)-B(2.1,2.5)r_K^*} + 101 * e^{A(2.1,3.0)-B(2.1,3.0)r_K^*}$$

$$= e^{-0.00015657-0.39602653*0.0332325}$$
$$+101 * e^{-0.00077155-0.88005036*0.0332325}$$

$= 0.9867708 + 98.013252$
$=99.000023$ (difference due to rounding)

(f) Compute the value at time $t = 0$ of the above European call option on the coupon bond.

Commentary on Question:
Most candidates did poorly on this question as they were not able to set up appropriate pricing formulas and numerical calculation mistakes were common. Partial marks were awarded for stating the correct formulas (e.g. $s_Z\,(T_o, T_1)$).

Based on part (e), the call option on the coupon bond can be decomposed into:

(1) A call option with a strike price of 0.9867708 on a bond that pays off \$1 at time 2.5 years and
(2) A call option with a strike price of 98.013252 on a bond that pays off \$101 at time 3 years.

For the first option, c(1)= 0.01, principal=100

$$\sigma B(2.1, 2.5) = \frac{0.015}{0.05}\left(1 - e^{-0.05(2.5-2.1)}\right) = 0.005940398$$

$$s_Z\,(T_o, T_1) = B(T_o, T_B) * \sqrt{\frac{\sigma^2}{2\gamma}(1 - e^{-2\gamma T_o})}$$

$$s_Z\,(2.1, 2.5) = B(2.1, 2.5) * \sqrt{\frac{0.015^2}{2 * 0.05}(1 - e^{-2*0.05*2.1})}$$

$$s_Z\,(2.1, 2.5) = [B(2.1, 2.5) * 0.015] * \sqrt{\frac{1}{2 * 0.05}(1 - e^{-2*0.05*2.1})}$$

$s_Z\,(2.1, 2.5) = 0.005940398 * 1.376284$=0.00817567

$$d_1(1) = \frac{1}{s_Z\,(T_o, T_1)}\ln(Z(0, r_0; T_1)/(K_i Z(0, r_0 T_o)\,) + \frac{s_Z\,(T_o, T_1)}{2}$$
$$d_1(1) = \frac{1}{0.00817567}\ln\left(\frac{0.9268484}{0.9382455 * 0.9867708}\right) + \frac{0.00817567}{2}$$

$= 0.1381309$

$d_2(1) = d_1(1) - s_Z\,(T_o, T_1) = 0.1299552$

$N\big(d_1(1)\big) = 0.5549315,\ \ N\big(d_2(1)\big) = 0.5516991$

11. Continued

The price of the first call option
$V(r_0) = Z(0, r_0; T_B)N\big(d_1(1)\big) - K_I Z(0, r_0; T_O)N\big(d_2(1)\big)$
=0.9268484 * 0.5549315 -0.9867708 $*$ 0.9382455 $*$ 0.5516991
=0.0035561

Hence the value at time $t = 0$ of the European Call option on the coupon bond= 0.0035561+0.790097=0.7936534.

12. **Learning Objectives:**

3. The candidate will understand:
   - The Quantitative tools and techniques for modeling the term structure of interest rates.
   - The standard yield curve models.
   - The tools and techniques for managing interest rate risk.

Learning Outcomes:

(3a) Understand and apply the concepts of risk-neutral measure, forward measure, normalization, and the market price of risk, in the pricing of interest rate derivatives.

(3g) Understand and apply the techniques of interest rate risk hedging.

Sources:

Fixed Income Securities: Valuation, Risk, and Risk Management, Veronesi, Pietro, 2010 – Chapter 14: Page 515-520, Chapter 15: 535 -537

Commentary on Question:

Most of the candidates attempted part (a) to (c). Many of the candidates were not able to get full credits due to solving (b) and (c) using the same approach.

Solution:

(a) Show that $Z(r,t;T)$ follows the process:

$$\frac{dZ(r,t;T)}{Z(r,t;T)} = a(r,t;T)\,dt - q(r,t;T)\,dX_t$$

where

$$a(r,t;T) = \frac{1}{Z(r,t;T)}\left[a(r)\frac{\partial Z}{\partial r} + \frac{1}{2}\sigma^2(r)\frac{\partial^2 Z}{\partial r^2} + \frac{\partial Z}{\partial t}\right]$$

$$q(r,t;T) = -\frac{1}{Z(r,t;T)}\frac{\partial Z}{\partial r}\sigma(r)$$

Commentary on Question:

Most candidates were able to solve this question and receive full marks. Some common mistakes are:

- *Candidates started with equation* $dZ = (Z_r\,dr + 0.5Z_{rr}(dr)^2 + Z_t\,dt)*Z$
- *Solving the question specifically to a certain interest rate model*

12. **Continued**

Since the price of a bond is a function of r, using Ito's Lemma, we have:

$$dZ = \frac{\partial Z}{\partial r}dr + \frac{1}{2}\frac{\partial^2 Z}{\partial r^2}(dr)^2 + \frac{\partial Z}{\partial t}dt$$
$$= \frac{\partial Z}{\partial r}[a(r)dt + \sigma(r)\, dX_t] + \frac{1}{2}\frac{\partial^2 Z}{\partial r^2}[a(r)dt + \sigma(r)\, dX_t]^2 + \frac{\partial Z}{\partial t}dt$$
$$= a(r)\frac{\partial Z}{\partial r}dt + \sigma(r)\frac{\partial Z}{\partial r}dX_t + \frac{1}{2}\frac{\partial^2 Z}{\partial r^2}[\sigma(r)]^2 dt + \frac{\partial Z}{\partial t}dt$$
$$= \left[a(r)\frac{\partial Z}{\partial r} + \frac{1}{2}\frac{\partial^2 Z}{\partial r^2}[\sigma(r)]^2 + \frac{\partial Z}{\partial t}\right]dt + \sigma(r)\frac{\partial Z}{\partial r}dX_t$$
$$= \alpha(r,t,T)\cdot Z\, dt - q(r,t,T)\cdot Z\, dX_t$$

Dividing both sides by Z, we have:

$$\frac{dZ}{Z} = \alpha(r,t,T)\, dt - q(r,t,T)\, dX_t$$

(b) Show that for a delta-hedged portfolio

$$N = \frac{Z(r,t;T_1)\ q(r,t;T_1)}{Z(r,t;T_2)\ q(r,t;T_2)}$$

Commentary on Question:
Only small number of candidates were able to score full marks on this question. Many candidates used approach in question (c) to solve this question. To receive full marks, candidates have to show that they are using q(r,t,T) in (a) to get to the final equation for N.

The delta of the portfolio is:

$$\frac{\partial \pi}{\partial r} = \frac{\partial Z(r,t,T_1)}{\partial r} - N\cdot\frac{\partial Z(r,t,T_2)}{\partial r}$$

To delta hedge, set the delta of the portfolio to 0:

$$\frac{\partial Z(r,t,T_1)}{\partial r} - N\cdot\frac{\partial Z(r,t,T_2)}{\partial r} = 0$$

Rearranging, we have:

$$N = \frac{\frac{\partial Z(r,t,T_1)}{\partial r}}{\frac{\partial Z(r,t,T_2)}{\partial r}}$$

12. Continued

From (a), we have:

$$q(r,t,T) = -\frac{1}{Z(r,t,T)}\frac{\partial Z}{\partial r}\sigma(r) \Rightarrow \frac{\partial Z}{\partial r} = -\frac{q(r,t,T)Z(r,t,T)}{\sigma(r)}$$

$$\mathrm{N} = \frac{-\frac{q(r,t,T_1)Z(r,t,T_1)}{\sigma(r)}}{-\frac{q(r,t,T_2)Z(r,t,T_2)}{\sigma(r)}} = \frac{Z(r,t,T_1)\cdot q(r,t,T_1)}{Z(r,t,T_2)\cdot q(r,t,T_2)}$$

(c) Demonstrate that $\Pi(r,t)$ with a delta-hedged position has no volatility.

Commentary on Question:
Most of the candidates use the following to solve question (b). Marks were credited to question (b) in this case. Partial marks were given if candidate mention that the diffusion term vanish implies no volatility.

$$\begin{aligned}\mathrm{d}\Pi(r,t) &= dZ(r,t,T_1) - N\cdot dZ(r,t,T_2)\\ &= [\alpha(r,t,T_1)\,dt - q(r,t,T_1)\,dX_t]Z(r,t,T_1) - N\\ &\qquad \cdot[\alpha(r,t,T_2)\,dt - q(r,t,T_2)\,dX_t]Z(r,t,T_2)\\ &= [\alpha(r,t,T_1)Z(r,t,T_1) - N\cdot\alpha(r,t,T_2)Z(r,t,T_2)]dt\\ &\qquad + [-q(r,t,T_1)Z(r,t,T_1) + N\cdot q(r,t,T_2)Z(r,t,T_2)]dX_t\end{aligned}$$

Considering the coefficient of dX_t and substituting N:

$$\begin{aligned}&-q(r,t,T_1)Z(r,t,T_1) + N\cdot q(r,t,T_2)Z(r,t,T_2)\\ &= -q(r,t,T_1)Z(r,t,T_1) + \frac{Z(r,t,T_1)\cdot q(r,t,T_1)}{Z(r,t,T_2)\cdot q(r,t,T_2)}\cdot q(r,t,T_2)Z(r,t,T_2)\\ &= -q(r,t,T_1)Z(r,t,T_1) + Z(r,t,T_1)\cdot q(r,t,T_1)\\ &= 0\end{aligned}$$

Since the coefficient of dX_t is 0, it follows that the volatility of $\Pi(r,t)$ is 0.

<u>Approach 2</u>:

$$\mathrm{d}\Pi(r,t) = dZ(r,t,T_1) - N\cdot dZ(r,t,T_2)$$

$$= \left[\frac{\partial Z_1}{\partial r}dr + \frac{1}{2}\frac{\partial^2 Z_1}{\partial r^2}(dr)^2 + \frac{\partial Z_1}{\partial t}dt\right] - N\left[\frac{\partial Z_2}{\partial r}dr + \frac{1}{2}\frac{\partial^2 Z_2}{\partial r^2}(dr)^2 + \frac{\partial Z_2}{\partial t}dt\right]$$

12. Continued

From (b), we have

$$\frac{\partial Z_1}{\partial r} - N \cdot \frac{\partial Z_2}{\partial r} = 0$$

$$dr^2 = \sigma(r)^2 dt$$

$$\mathrm{d}\Pi(r,t) = \left[\frac{1}{2}\frac{\partial^2 Z_1}{\partial r^2}\sigma(r)^2 dt + \frac{\partial Z_1}{\partial t}dt\right] - N\left[\frac{1}{2}\frac{\partial^2 Z_2}{\partial r^2}\sigma(r)^2 dt + \frac{\partial Z_2}{\partial t}dt\right]$$

Since the coefficient of dX_t is 0, it follows that the volatility of $\Pi(r,t)$ is 0.

(d) Show that:

$$\frac{a(r,t;T_1)-r}{q(r,t;T_1)} = \frac{a(r,t;T_2)-r}{q(r,t;T_2)}$$

Commentary on Question:
Half of the candidates did not attempt this question. Partial marks were given if candidate mentioned riskless portfolio earn risk-free rate, or market price of risk is the same for no-arbitrage riskless portfolio.

Since the volatility of $\Pi(r,t)$ is 0, the portfolio is riskless. The no arbitrage principle that the portfolio Π must now earn the risk-free rate, and thus:

$$\frac{\mathrm{d}\Pi(r,t)}{\Pi(r,t)} = r dt$$

$$\mathrm{d}\Pi(r,t) = r\Pi(r,t)\, dt$$

Setting the coefficient of dt to $r\Pi(r,t)$:

$$\alpha(r,t,T_1)Z(r,t,T_1) - N \cdot \alpha(r,t,T_2)Z(r,t,T_2) = r\Pi(r,t)$$

Substituting $\Pi(r,t)$:

$$\alpha(r,t,T_1)Z(r,t,T_1) - N \cdot \alpha(r,t,T_2)Z(r,t,T_2) = r[Z(r,t,T_1) - NZ(r,t,T_2)]$$
$$\alpha(r,t,T_1)Z(r,t,T_1) - N \cdot [\alpha(r,t,T_2)Z(r,t,T_2) - rZ(r,t,T_2)] = rZ(r,t,T_1)$$

Substituting N:

$$\alpha(r,t,T_1)Z(r,t,T_1) - \frac{Z(r,t,T_1)\cdot q(r,t,T_1)}{Z(r,t,T_2)\cdot q(r,t,T_2)} \cdot [\alpha(r,t,T_2)Z(r,t,T_2) - rZ(r,t,T_2)]$$
$$= rZ(r,t,T_1)$$

12. Continued

$$\alpha(r,t,T_1) - \frac{q(r,t,T_1)}{q(r,t,T_2)} \cdot [\alpha(r,t,T_2) - r] = r$$

$$\frac{\alpha(r,t,T_1) - r}{q(r,t,T_1)} = \frac{\alpha(r,t,T_2) - r}{q(r,t,T_2)}$$

13. **Learning Objectives:**

3. The candidate will understand:
   - The Quantitative tools and techniques for modeling the term structure of interest rates.
   - The standard yield curve models.
   - The tools and techniques for managing interest rate risk.

Learning Outcomes:

(3a) Understand and apply the concepts of risk-neutral measure, forward measure, normalization, and the market price of risk, in the pricing of interest rate derivatives.

(3b) Understand and apply various one-factor interest rate models.

(3e) Demonstrate understanding of option pricing theory and techniques for interest rate derivatives.

(3f) Apply the models to price common interest sensitive instruments including: callable bonds, bond options, caps, floors, and swaptions.

(3h) Understand the application of Monte Carlo simulation to risk neutral pricing of interest rate securities.

Sources:

Fixed Income Securities: Valuation, Risk, and Risk Management, Veronesi, Pietro, 2010 Chapter 21 – Forward Risk Neutral Pricing and The LIBOR Market Model.

Commentary on Question:

This question is testing the understanding of the forward risk neutral pricing models (emphasis on the forward volatilities) and the ability to apply appropriate analytic formula to calculate option value. Most candidates demonstrated a good understanding of the basic concepts and the relationship among variances and the Monte Carlo method. Only a number of candidates applied the correct analytic formula to calculate the power option value. A significant portion of candidates skipped this question completely or partially.

Solution:

(a) Describe the distribution of $r_n(\tau, T)$ in the LIBOR market model.

Commentary on Question:

Most candidates answered that $r_n(\tau, T)$ *follows lognormal distribution and specified the mean and the variance of the distribution. Few of them mentioned the boundary condition.*

13. Continued

Under the T-forward risk neutral dynamics, the LIBOR spot rate $r_n(\tau,T)$ has a log-normal distribtution with mean $f_n(0,\tau,T)$ and variance $\int_0^\tau \sigma_f^2\,(t)dt$:

- $r_n(\tau,T)\sim LogN(f_n(0,\tau,T),\int_0^\tau \sigma_f^2\,(t)dt)$

  i.e.$E[r_n(\tau,T)] = f_n(0,\tau,T)$ and $Var[\ln(r_n(\tau,T)] = \int_0^\tau \sigma_f^2\,(t)dt$

The forward rate converges to the spot rate at maturity, $r_n(\tau,T) = f_n(\tau,\tau,T)$.

(b) Define caplet forward volatilities, $\sigma_f^{Fwd}\left(T_{i+1}\right),\ i=0,\ 1,\ ...,$ and identify their advantages in pricing caps.

Commentary on Question:
Most candidates demonstrated a good understanding of the concept by pointing out that caplet forward volatilities are implied volatilities and are constant over (t,T_i). A number of candidates mentioned that the caplet forward volatilities are independent of which cap the caplet belongs to and described the main advantages in pricing caps.

The caplet forward volatility $\sigma_f^{Fwd}(T_{i+1})$ is the volatility that characterizes particular caplet (i.e. implied volatility for a particular caplet)), independent of which cap the caplet belongs to. (Definition 20.2)

For each caplet expiry time T_{i+1}, the caplet forward volatility in $\sigma_f^{Fwd}(T_{i+1})$ is constant over (t,T_i). The forward rate volatility S_i is constant over each time period (T_{i-1},T_i) and may not be constant over (t,T_i). The variance derived from the caplet forward volatility and the variance derived from the forward rate volatility are equal for all T_{i+1}.

The forward risk neutral pricing methodology and information about caplet volatilities provide a straightforward way to value a fixed income secuity regardless of the payoff function.

(c) Calculate the value of the call option.

Commentary on Question:
Only couple of candidates recognized the power option and applied the correct Black's formula to solve this question.

Denote the (T-forward risk neutral) expected 6-month LIBOR raised up to cubic, and its variance, respectively by
$g(0{,}0.5{,}1)$ and σ_T^2. Then

$$g(0{,}0.5{,}1) = E_f^*[r_n(0.5{,}1)] = f_n(0{,}0.5{,}1)^{0.5} e^{0.5(0.5-1)\sigma_f^{Fwd^2}*0.5/2}$$
$$= 0.172773$$
$$\sigma_1^2 = Var[\ln\,(r_n(0.5{,}1)^{0.5})] = 0.5^2\sigma_f^{Fwd}(1)^2 * 0.5 = 0.005.$$
$$\sigma_1 = 0.0707.$$

From the Black's formula,

13. Continued

Power call = $NZ(0,1)[g(0,0.5,1)N(d_1) - KN(d_2)]$
Where

$$d_1 = \frac{1}{\sigma_1}\ln\left(\frac{g(0,0.5,1)}{K}\right) + 0.5 * \sigma_1 = -2.0342$$
$$d_2 = d_1 - \sigma_1 = -2.1049$$

Power call = \$87.77

(d) Outline an algorithm to calculate the option value using the Monte Carlo method.

Commentary on Question:
Most candidates knew the main steps of the Monte Carlo method. However, only a number of candidates received full marks. Many candidates lost partial marks due to missing formula in steps.

Since we know under the T-forward risk neutral dynmics, the 6-month LIBOR rate $r_n(\tau, T)$ has a log-normal distribution. Log $(r_n(\tau,T))\sim Normal(\log\left(f_n(0,\tau,T) - \frac{1}{2}\sigma_f^2\tau, \sigma_f^2\tau\right))$, here σ_f means the caplet forward volatility σ_f^{Fwd}. We can proceed as follows:
From $i = 1,2, \dots N$ (where N is large number such as 10,000)

1. Simulating $r_n(\tau, T)$
   $r_n{}^i = e^{\log(f(0,\tau,T) - \frac{1}{2}\sigma_f^2\tau + \sigma_f^2\sqrt{\tau}\varepsilon}$,
   where $\varepsilon \sim N(0,1)$
2. Compute the discount final payoffs
   $$V^i = Z(0,T)Nexp(-\lambda|r_n - K|)$$
3. Compute the call option price as the average of N values
   $$C = \frac{1}{N}\sum_{i}^{N} V^i$$

(e) Critique your colleague's suggestion in light of the relationship between $\sigma_f^{i+1}(t)$ and $\sigma_f^{Fwd}(T_{i+1})$ for $i = 0, 1, 2, \dots, M-1$.

Commentary on Question:
The purpose of this question is to test the candidates' understanding of the two variances: The variance derived from the Black formula and the variance implied by the forward rates. Many candidates shown the correct formula describing the relationship between the two variances. Few candidates compared the two variances and specified the main difference between the two variances.

13. Continued

Assuming σ_f^{i+1} constant for each forward rate is not reasonable.

In most situations caplet volatitlities shows a hump at around two years to maturity and this relatively stable over time. Standard alternate assumptions about $\sigma_f^{i+1}(t)$ is
$\sigma_f^{Fwd}(T_{i+1})^2 * (T_i - t) = S_i^2 * (T_1 - t) + S_{i-1}^2 * \Delta + \cdots + S_1^2 * \Delta,$
Where $\sigma_f^{Fwd}(T_{i+1})^2$ is the implied volatility of a caplet maturing at T_{i+1}

(f) Calculate the corresponding forward rate volatilities S_i in your plan.

Commentary on Question:
Most candidates did well in this question.

Based on the formula (21.39), S_i can be derived as follows:
$S_1 = 0.03$,
$S_2 = \sqrt{(0.045^2 * 1 - 0.03^2 * 0.5)/_{0.5}} = 0.056,$
$S_3 = \sqrt{(0.05^2 * 1.5 - 0.03^2 * 0.5 - 0.056^2 * 0.5)/_{0.5}} = 0.059,$
$S_4 = \sqrt{(0.045^2 * 2 - 0.03^2 * 0.5 - 0.056^2 * 0.5 - 0.059^2 * 0.5)/_{0.5}} = 0.024,$

Alternatively,
$S_1 = 0.03$,
$S_2 = \sqrt{(0.045^2 * 1 - 0.03^2 * 0.5)/_{0.5}} = 0.056,$
$S_3 = \sqrt{(0.05^2 * 1.5 - 0.045^2)/_{0.5}} = 0.059,$
$S_4 = \sqrt{(0.045^2 * 2 - 0.05^2 * 1.5)/_{0.5}} = 0.024,$

14. Learning Objectives:

5. The candidate will learn how to apply the techniques of quantitative finance to applied business contexts.

Learning Outcomes:

(5a) Identify and evaluate embedded options in liabilities, specifically indexed annuity and variable annuity guarantee riders (GMAB, GMDB, GMWB, and GMIB).

(5b) Demonstrate an understanding of embedded guarantee risk including: market, insurance, policyholder behavior, and basis risk.

(5c) Demonstrate an understanding of dynamic and static hedging for embedded guarantees, including:
   (i) Risks that can be hedged, including equity, interest rate, volatility and cross Greeks.
   (ii) Risks that can only be partially hedged or cannot be hedged including policyholder behavior, mortality and lapse, basis risk, counterparty exposure, foreign bonds and equities, correlation and operation failures

(5e) Demonstrate an understanding of how differences between modeled and actual outcomes for guarantees affect financial results over time.

Sources:

QFIQ-126-20: Malcolm Life Enhances its Variable Annuities

On the Importance of Hedging Dynamic Lapses in Variable Annuities, Risk and Rewards, 2015 issue 66

Commentary on Question:

This question tests candidates on their understanding of the embedded options in a VA product and the effectiveness of hedging VA guarantees in the presences of model risk.

Solution:

(a) Explain how these features impact the value of the embedded options in the riders.

Commentary on Question:

This question is relatively straightforward question but only half of the candidates provided the correct answers. Candidates are expected to demonstrate an understanding of how the product features impact the value of the embedded guarantees of VA.

14. Continued

- The higher the floor value, the more costly is the value of the embedded options to insurer. Thus, no floor value in the first 5 years would lower the embedded option value.
- An annual ratchet feature resets the guaranteed amount to the higher of contract value and initial/current guaranteed amount on an annual basis. This will cost more to insurer as this would increase the value of the embedded option.

(b)

(i) Explain how this assumption could be adjusted in order to make the products more competitive.

(ii) Suggest one way to manage the longevity risk after making this change.

Commentary on Question:
This question is relatively straightforward question but only half of the candidates provided the correct answers. Candidates are expected to demonstrate an understanding of the embedded guarantee risk associated with insurance risk.

- Increase the mortality rate assumption such that their expectation of how long they need to make the income payment is shorter than the industry average. So, the value of the embedded option in the riders is relatively lower, which make their products more competitive than competitors.
- Actually, it is the longevity risk that they need to manage. The insurer will incur a loss when making income payment for longer than expected. They can reinsure the longevity risk with a reinsurer.

(c)

(i) Identify the financial instruments that can be used to hedge against the following financial risks inherent in the GMWB:

- stock market volatility
- increase in stock market volatility.

(ii) State two reasons why the risks in the GMWB cannot be perfectly hedged.

Commentary on Question:
This is a relatively straight forward question but only half of the candidates provided the correct answers. Candidates are expected to demonstrate an understanding of dynamic and static hedging for embedded guarantees, including risks that can be hedged, risks that can only be partially hedged or cannot be hedged. For part (ii), most candidates were able to indicate that policyholder behavior cannot be perfectly hedged but failed to identify other risks.

14. Continued

(i)

- For stock market volatility: Long an at-the-money put option in stock market
- For increase in stock market volatility: Long a VIX futures / volatility swap to hedge against non-linear change in option value

(ii)

- Future behavior of customers who purchase the GMWB rider cannot be directly hedged
  - More owners than expected exercise to take the guaranteed minimum withdrawal
  - Customers may not hold the contracts as long as expected
  - How long the owner will keep the rider in force cannot be known in advance, so can't establish a perfectly hedged position
- If owner invested in mutual fund, the risk cannot be perfectly hedged with an index fund option
- Assets used to back the rider are not received up front, so the insurer doesn't receive the amount at which it has priced the stock market put from the customer on day one.
- Counterparty risk in the hedged position cannot be perfectly hedged. For example, derivatives markets might be close while the annuity contract values are changing drastically, the basis risk arising from difference between counterparty name and the reference entity of the CDS, etc.

(d)

(i) Justify your assertion.

(ii) Identify and explain the conclusions that can be drawn from comparing the above results after switching the results for Assumption Set III and IV.

(iii) Compare the above results with respect to the following aspects between the Black-Scholes and RS-GARCH models (assuming that insurer uses delta-hedging under the Black-Scholes model to manage the risk of the GMMBs):

- Risk measures (including standard deviation)
- Model risk
- Hedging error

(iv) Explain whether dynamic lapsation should be hedged, based on the comparison in part (iii).

14. Continued

Commentary on Question:
For part (i), most candidates were able to provide the justification with a direct comparison between scenario III and IV. But, most of them, didn't provide justification with a comparison with scenario II. Also, they failed to explicitly state that the scenarios are flipped.

For part (ii), most candidates were able to comment on the incorrect moneyness ratio. Other than that, most of them failed to comment with respect to hedging under the ideal conditions, standard deviation and risk measures, and the risk reduction associated with hedging dynamic lapses.

For part (iii), only a few candidates were able to clearly state the difference in the model risk between Black-Scholes and Regime Switching-GARCH, and to provide the correct answer on the model comparison with respect to mean, standard deviation and risk measures.

Most candidates answered part (iv) correctly.

(i)

- The outcome for scenario III and scenario IV appears to be flipped
- Regardless of model used, the mean, standard deviation, and risk measures of the net hedging error should be higher when dynamic lapsation is not hedged at all compared to when it is hedged at an incorrect moneyness ratio assumption
- Risk measures are similar between scenario III and scenario II under the Black-Scholes (BS) model, suggesting that scenario III does have a hedge in place for dynamic lapses; it is unlikely that the numbers are so close if no hedge is in place for dynamic lapses at all
- Risk measures in scenario IV are significantly higher compared to a perfectly hedged scenario II, suggesting that no hedge is in place for the dynamic lapses at all

(ii)

- By analyzing scenarios I and II, hedging under ideal conditions where there are no model or policyholder behavior risks yields an important risk reduction
- Even if the moneyness ratio is assumed incorrectly in the hedge, the risk measures are much lower than those obtained when dynamic lapsation risk is not hedged at all
- The standard deviation and risk measures under wrong moneyness ratio are approximately twice as large as in scenario II (perfect hedge) but when dynamic lapses are not hedged, they are about five times larger

14. Continued

- Even if the assumption on the moneyness ratio is set wrong in the hedge, it is still possible to achieve a very significant risk reduction by hedging dynamic lapses

(iii)

- The BS model assumes that the value of the reference portfolio follows a geometric Brownian motion. Since the hedging Greeks are computed under the BS model as well, there will be no model risk
- The RS-GARCH market model assumes the state of economy is driven by a latent Markov chain and captures jumps in returns and volatility dynamics.
- RS-GARCH model gives a lower mean of net hedging error compared to the BS model
- When hedging under RS-GARCH model, standard deviation is higher compared to the BS model
- Risk measures under RS-GARCH model are roughly twice compared to the BS model for all scenarios except when lapsation risk is not hedged at all

(iv)

- Even if the market model significantly deviates from the BS model, the analysis still points to hedging dynamic lapses with the wrong moneyness ratio as better off than not hedging at all

15. Learning Objectives:

5. The candidate will learn how to apply the techniques of quantitative finance to applied business contexts.

Learning Outcomes:

(5d) Demonstrate an understanding of target volatility funds and their effect on guarantee cost and risk control.

Sources:

QFIQ-124-20: Variable Annuity Volatility Management: An Era of Risk-Control

Commentary on Question:

This question is to test how well the candidates be able to apply the techniques of quantitative finance to applied business contexts, with the focus on the ability of the candidates to demonstrate an understanding of target volatility funds and their effect on guarantee cost and risk control. The question strikes a balance among Retrieval, Comprehension, Analysis, and/or Knowledge Utilization.

Solution:

(a) Describe principal objectives of volatility management strategies of equity-based guarantee products from the perspectives of manufacturer and client respectively.

Commentary on Question:

Q*uite a few candidates answer*ed *this question by listing the different perspective*s *without any description. The candidates are expected to give more explanations (rather than list) for each perspective.*

Manufacturer perspective:

- Write profitable business – product teams must satisfy bottom-line at-issue economic value / IRR thresholds
- Stabilize ALM & hedging performance – volatility management solutions stabilize ALM performance by narrowing dispersion in liability value changes; another concern specific to certain volatility solutions is "basis risk"
- Optimize capital requirements – reduce the liability performance "tails" that drive increased statutory reserves and capital levels

Client perspective:

- Maintain investment upside potential – clients invest in equity-based portfolios to harness potential market upside
- Minimize impact to guarantee value – clients invest in VAs to obtain income guarantee

15. Continued

(b) Determine the equity allocation of the portfolio after any changes driven by the capped volatility strategy.

Commentary on Question:
Less than half candidates were able to answer this question *well. The remaining were unable to either to recall or apply the* relevant *formula for the calculation*s.

formula for determining if volatility is too high; realized equity vol = SQRT(252 * 21-day sum of squared daily returns / 21)

realized equity vol = SQRT(252*.0081 / 21) = 31.177%
31.177% > 30% the trigger level, so adjustment needed

formula for determining new equity ratio; equity ratio = minimum (100%, 30%/realized equity volatility)

equity ratio = min (100%, 30%/31.177%) = 96.225%
current allocation is 60%, so new allocation is 96.225% * 60% = 57.735%

(c) Describe actions, if any, to take to achieve the changes in equity allocation in part (b).

Commentary on Question:
This question is directly related to part b). Credits were also awarded for candidates who were able to identify the correct actions, e.g., reduce the equity allocation. Candidates are expected to provide solutions/actions that are clearly effective.

- Action required is to de-risk using futures
- To reduce the exposure, sell futures

(d) Critique your coworker's thoughts on VIX-indexed volatility management strategies.

Commentary on Question:
Very few candidates can articulate the critique well. But most candidates could identify at least one or two of the properties below:

15. Continued

- VIX-indexed fee rider allows insurer to adjust charges periodically as needed (e.g., quarterly, etc.)
- Rider fee adjusts around a base fee level plus an increment amount for every unit the VIX exceeds a target value
- Rider fee is bounded overall by a floor and ceiling
- VIX-based solutions are
  - most effective in "body" of volatility distribution
  - least effective in "tail" of distribution
  - incremental fee in "spike" scenario are insufficient to offset the hedge losses driven by square of volatility (and/or) reduced fees are too great in periods of extremely low volatility

(e) A consulting firm recommended a joint VIX-indexed and capped volatility strategy as the volatility management strategy.

Evaluate the recommended strategy.

Commentary on Question:
A few candidates are able to correctly draw the right recommendation/conclusion based on the information provided in the question. Albeit, most candidates failed to perform well for this part.

- Reducing volatility and Vega is important from insurer (company) perspective while minimizing impact on returns is important from client perspective
- Joint strategy is more effective than using only capped volatility strategy
  - reduction in volatility (40% v 15%) and Vega (0.24% v 0.40%) (insurer perspective) with
  - negligible impact on returns and minimal increase in fees (client perspective)
- Target volatility can increase equity allocations beyond 60% in calm markets / heavy allocation to cash in high volatility times
  - loss of upside potential which would be negative from client perspective
- Capital preservation strategy has better reduction in volatility cost and Vega metrics but has significantly lower returns from 2010-17 which would be unfavorable from client perspective