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Managing Investment Risks of Insurance Contractual Designs

May 2021

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MANAGING INVESTMENT RISKS OF INSURANCE CONTRACTUAL DESIGNS

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Section 1: Executive Summary

Life and annuity markets around the world have seen increasingly complex investment-combined products, including universal life, fixed indexed annuities, variable annuities with a wide variety of guaranteed benefits and indexed variable annuities. Insurance companies are facing unprecedented exposure to financial risks in addition to traditional mortality and longevity risks. The risk assessment and modeling for such complex designs by conventional methods often focus heavily on their market competitiveness and profitability. However, there seems to be a lack of quantifiable and systematic approaches to assess investment risks associated with investment combined products. The objective of this study is to develop a framework for quantifying and analyzing various forms of contractual designs and showcase different techniques to assess the trade-off between competitive contractual designs and their practical implications with regard to the difficulty of risk management.

There are four commonly used risk management techniques—risk avoidance, reduction, sharing and retention. The latter three are often combined in various ways in contractual designs. Adding to the complexity is the interplay of various types of risks such as equity risk, inflation risk and mortality risk.

In Section 3, we establish a procedure to understand how risks are transferred among various stakeholders including policyholders, insurers, reinsurers and capital markets. For each type of product design discussed in this report, we aim to understand how each type of risk is managed through a cascade of risk management actions. Then we quantify and assess the extent to which the risk is transferred or mitigated through a particular risk management action. By dissecting each contractual design into a cascade of risk management actions, we aim to quantify the linkage between contractual design and the effect of risk management.

We introduce in Section 4 a series of variable annuity guaranteed benefits as examples of common contractual designs offered in the market. In contrast with current market practice of modeling them through spreadsheet calculations or their equivalents in specialized software packages, we take a minimalist approach to quantify structural components of these guaranteed benefits through mathematical formulation. In each formulation, we can clearly identify all risk factors and how they interact with each other.

An application of the cascade risk management model is offered in Section 5. Each product design is reviewed and analyzed from the lens of both a policyholder and an insurer. The decomposition of the risk-neutral value of an investment with each product design into those between a policyholder and an insurer enables us to understand the split of profit or deficiency between two parties, in agreement with the no-arbitrage theory. We then considerate the risk formulations after each step of risk management. By comparing various risk measures, we analyze the financial impact of each risk management action. As a result of this standardized measurement, we can compare how risks are absorbed or reduced on the cascade of risk management actions. From a practitioner’s point of view, the cascade model can offer a clear view of profitability of each product design through a series of actions.

While the previous sections focus on the interaction of equity risk and mortality risk, Section 6 offers another example of interactions between inflation risk and mortality risk. This section shows a variety of common product designs known in the industry.

We provide a framework of modeling inflation risk and mortality risk embedded in inflation-linked insurance in Section 7. In this section, we take one of the product designs as an example and focus on developing a delta-hedging strategy to transfer undiversifiable inflation risk to the capital market. Details of a hedging strategy is offered for practitioners interested in implementation.

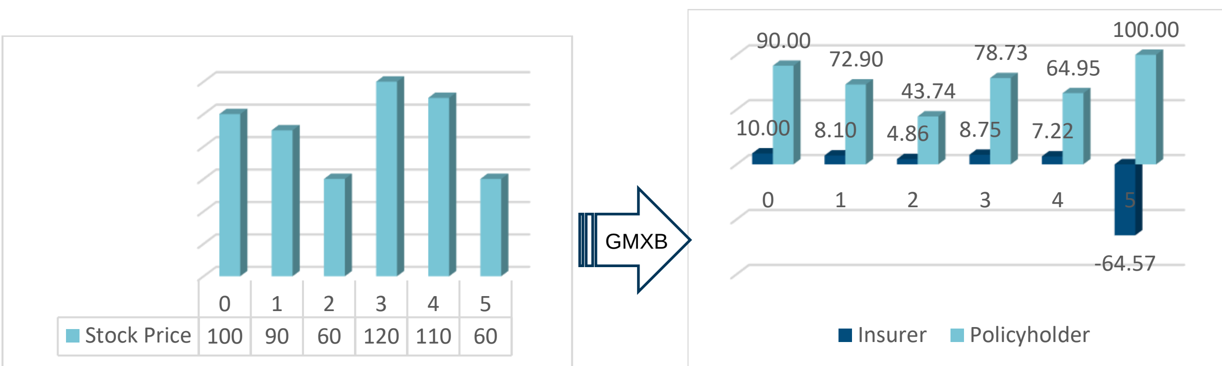
Section 2: Introduction

Over the last few decades, life insurers have introduced a wide variety of insurance and annuity contractual designs to appeal to consumers' changing needs and compete for market share. As these products are often combined with investments, different types of risks including equity, interest rate and inflation are embedded in insurers' product portfolios. Many contractual designs offer investment optionalities, often used to protect customers from the downside risk of investment. With the growing complexity of contractual designs, it is increasingly important for actuaries to understand the impact of these designs and the ease of managing underlying risks.

The purpose of this report is to develop a quantitative framework in which we can quantify and assess the connection between various contractual designs and associated risks.

As a general principle, we think of a contractual design as a mapping from the dynamics of the underlying risk(s) to an insurer's financial obligations. Suppose an investor has $100 to invest either directly in a stock or in a variable annuity contract linked to the same stock. See Figure 1, where GMXB represents various types of guaranteed minimum benefits, including guaranteed minimum death benefit, guaranteed minimum maturity benefit and guaranteed minimum withdrawal benefit.

Figure 1
A MAPPING FROM THE DYNAMICS OF EQUITY RISK TO AN INVESTMENT GUARANTEE



The left panel shows the evolution of stock prices over a period of five years. If left unprotected, the investor is fully exposed to equity risk and her investment varies exactly by the same amount as stock price. The VA contract offers a guaranteed minimum maturity benefit, which guarantees a minimum balance of $100 after five years. The cash flow is now split between the policyholder and the insurer who underwrites the variable annuity contract. The right panel shows the new set of cash flows between the policyholder and the insurer. In compensation for the guaranteed benefit, the policyholder pays 10% of her account value on an annual basis to the insurer. For example, in the first year, the policyholder has a total of $100 in her account prior to any fee deduction. The insurer receives $10. The remaining $90 is linked to stock prices on the left panel. Since the stock price drops about 10% at the end of first year, the policyholder ends up with $81. The fee charge at time 1 (the beginning of the second year) is given by 10% of the then-current balance, $81. The calculation can carry on as such. Even though the investor's investment still varies in proportion to the stock price, she receives a boost to her investment due to a benefit payment from the GMMB at maturity. Therefore, the equity risk is partially transferred from the policyholder to the insurer, who receives annual fee incomes and carries the financial obligation to absorb some of the policyholder's loss.

While this is well known, we take a theoretical viewpoint that each contractual design can be viewed as a set of two mappings, one of which maps to an insurer's cash flow and the other to a policyholder's cash flow.

Section 3: Risk Management Techniques

The purpose of a contractual design is to split the risks involved between the insurer and the insured so that the financial impacts of losses are acceptable to both parties. Without any investment guarantee, an insurer merely acts as a steward of its policyholders' investment funds and hence equity risk lies entirely with policyholders. When a certain investment guarantee is offered, a portion of equity risk is transferred from policyholders to the insurer. Therefore, how to manage risk is a critical component of an insurance business. In this report, we investigate contractual designs from the viewpoint of an insurer. We first break down contractual design into mechanical components by various risk management techniques. Then we gauge how much risk the insurer retains with each type of contract design by different measures.

In general, all techniques of risk management in the financial industry fall into one or more of the four categories.

3.1 RISK AVOIDANCE

An insurer may establish a risk management strategy to avoid a policy design that carries risk beyond a certain level of tolerance. For example, an inflation-linked insurance product may offer annuity payments increased by a fixed percentage each period, which is intended to keep up with rising cost of living. An alternative is to link the payments to the consumer price index, which measures the changes in consumer goods. While the CPI adjustment offers the best protection for policyholders, it involves significant inflation risk for the insurer. Assessing the difficulty with dealing with the uncertainty, the insurer may choose only to offer a fixed percentage increase to avoid inflation risk.

3.2 RISK REDUCTION

This may include a variety of approaches to reduce risk, such as providing policyholders with coverage or incentives for preventive health care in expectation of lowering the likelihood of health problems for policyholders and expensive bills for the insurer. However, in this report, we do not consider any external factor that may influence the probability distribution of risks involved. Instead, we are interested in the reduction of uncertainty due to the aggregation of contracts. The diversification of mortality risk among a large pool of contracts makes the average loss much more predictable and manageable.

3.3 RISK SHARING

This includes any design of insurance or financial product where certain risk is packaged and transferred from one party to another. An example would be the dynamic hedging of variable annuity contracts with a guaranteed minimum death benefit rider. While the mortality risk can be diversified in a large pool of policies, the equity risk cannot as all policies are linked to the same set of equity indices. The insurer may choose to transfer the undiversified risk to the capital market by developing a hedging program. When the equity indices perform poorly and the GMDB is in-the-money, the insurer can offset losses with payoffs from a hedging portfolio. More details will be offered later in this section.

3.4 RISK RETENTION

This means to accept a risk and absorb any loss from the risk with one's own resources. In the insurance industry, a common approach of risk retention is to set aside some resources, known as reserve or capital, to cover potential losses in the future.

3.5 CASADE MODEL OF RISK MANAGEMENT

We want to understand how the three most important tools of risk management—risk reduction, risk sharing and risk retention—are integrated in product design and development. Here we take a minimalist approach to only consider essential elements so that we can focus on the core principle of risk management. See Figure 2.

Figure 2

RISK MANAGEMENT TECHNIQUES EMBEDDED IN EQUITY-LINKED INSURANCE

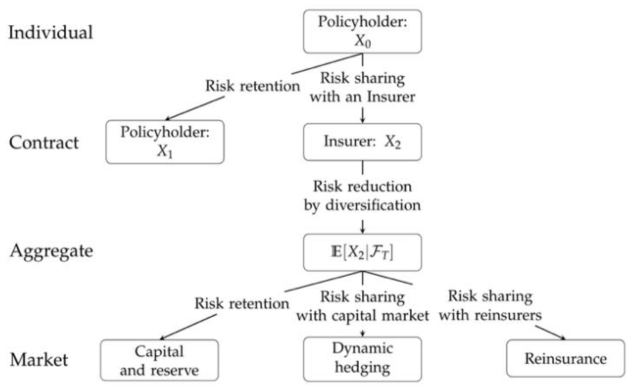


Figure 2 shows the various steps in which a product is engineered to deal with risks facing a policyholder. The first step of risk transfer from a policyholder to an insurer is of most interest to the policyholder, as it measures how much uncertainty is taken away when buying a product. One may also argue that such a quantitative analysis is useful for marketing purpose, as it reveals the degree to which a product improves a policyholder's risk profile. The second and third steps are of utmost interest to an insurer, as they show the insurer's capacity of coping with risks ceded by the policyholder. Hence, the report tends to focus on metrics of the last two steps from an insurer's viewpoint. In the figure, while X_0 and X_1 represents the policyholder's financial position by purchasing the product without and with the guarantee respectively and X_2 represents the insurer's financial position by selling the product with the guarantee. The conditional expectation $E[X_2|F_T]$ gives the insurer's financial position when the mortality risk is absorbed. The exact mathematical formulation of these quantities shall be introduced in Section 5 for each of three types of investment guarantees. The dissection of these quantities is used to understand how investment risk and mortality risk are re-distributed and absorbed by various stakeholders of each contract.

Section 4: Equity-linked Insurance and Annuity

To facilitate the discussion of cash flow models, we introduce the following notations to be used in this report.

- $\boldsymbol{S_t}$**.** The market value of the underlying equity index or fund at t. If more than one fund is involved, this is considered to be the portfolio value of all funds.
- $\{\boldsymbol{\mathcal{F}_t}: \boldsymbol{0 \leq t \leq T}\}$**.** The filtration with which a stochastic process $\{S_t: 0 \leq t \leq T\}$ is defined. The filtration represents the accumulation of market information over time.
- $\boldsymbol{F_t}$**.** The market value of the policyholder's sub-accounts at $\mathrm{t} \geq 0$. F_0 is considered to be the initial premium invested at the start of the contract.
- $\boldsymbol{G_t}$**.** The guaranteed base used to determine the amount of payment to the policyholder from various riders at time $\mathrm{t} \geq 0$. Examples will follow in the next section.
- $\mathbf{n}$. The number of valuations per year.
- $\mathbf{m}$. The nominal annualized rate at which asset-value-based fees are deducted from sub-accounts. Although we do not explicitly indicate the frequency of fee payments, we typically assume that the payment period matches the valuation period. We shall first consider m to be a nominal rate compounded

n times per annum in the discrete time model and then a nominal rate compounded continuously in the continuous time model.

- **m_e and m_d.** Rider charges for GMMB and GMDB, respectively. The portion available for funding the guarantee cost is called margin offset or rider charge and is usually split by benefit. We denote the annualized rate of charges allocated to the GMMB by m_e and that of the charges allocated to the GMDB by m_d. Note that in general, $\mathrm{m} > m_e + m_d$ to allow for overheads, commissions and other expenses.
- **r.** The continuously compounding annual risk-free rate. This typically reflects the overall yield rate of assets in the insurer's general account backing up guaranteed benefits.
- **T.** The target value date (or called maturity date), typically a rider anniversary on which the insurer is liable for guarantee payments.
- **T_x.** The random variable representing the future lifetime of the policyholder of age x at inception.
- K_x. The random variable representing the curtate future lifetime of the policyholder of age x at inception.
- **L.** The net present value of future liabilities at the start of the contract. The techniques to be introduced in the notes can be used to analyze liabilities at any other valuation date. We shall focus on the start of the contract for simplicity.

Let us first consider the cash flows of a stand-alone variable annuity contract. Consider a discrete time model with a valuation period of $1/\mathrm{n}$ of a time unit, where n represents the frequency of payments each period, such as quarterly ($\mathrm{n} = 4$), monthly ($\mathrm{n} = 12$) or daily ($\mathrm{n} = 252$), the number of trading days each year. In other words, cash flows are expected to occur only on time points $\mathrm{t} = 1/\mathrm{n},\ 2/\mathrm{n},\ \cdots,\ \mathrm{k}/\mathrm{n},\ \cdots,\ \mathrm{T}$. The fees and charges by annuity writers are typically taken as a fixed percentage of the-then-current account values for each period. The equity-linked mechanism for variable annuity dictates that at the end of each trading day, the account value fluctuates in proportion to the value of the equity fund in which it invests and is deducted by account-value-based fees:

$$F_{k/n} = F_0 \frac{S_{k/n}}{S_0}\left(1 - \frac{m}{n}\right)^k, k = 1,2,\ \cdots, nT, \tag{1}$$

where m is the annual rate of total charge compounded n times per year, and all charges are made at the beginning of each valuation period.

Observe that the income from the insurer's perspective is generated by a stream of account-value-based payments. To understand the effect of particular contractual design, we usually split fee incomes by benefit. Here we take for example the fee income collected to fund a particular rider; its present value up to the k-th valuation period is denoted and given by

$$M_{k/n} = \sum_{j=1}^{k} e^{-r(j-1)/n}\left(\frac{m_x}{n}\right) F_{(j-1)/n}, \tag{2}$$

where m_x represents the rate of fee charged by the insurer and there are three common riders, namely guaranteed minimum maturity benefit, guaranteed minimum death benefit and guaranteed minimum withdrawal benefit.

4.1 GUARANTEED MINIMUM MATURITY BENEFIT

The GMMB guarantees the policyholder a minimum monetary amount G at the maturity T. Recall the definition that T_x and K_x are the random variables representing the future and curtate lifetime of the policyholder, where the relationship between them is given by

$$K_x^{(n)} = \frac{1}{n}\lfloor nT_x \rfloor.$$

As the insurer is only possibly liable for the amount by which the guarantee exceeds the policyholder's account balance at maturity, the present value of the gross liability to the insurer is

$$e^{-rT}(G-F_T)^+ I(K_x > T), \tag{3}$$

where $(x)^+ = \max\{x, 0\}$ and $I(K_x > T) = \begin{cases} 1, & if K_x > T \\ 0, & if K_x \leq T \end{cases}$.

Consider the *individual net liability* of the guaranteed benefits from the insurer's perspective, which is the gross liability of guaranteed benefits less the fee incomes. The present value of the GMMB net liability is given by

$$L_e^{(n)}(T_x) := e^{-rT}(G-F_T)^+ I(K_x > T) - M_{T\wedge K_x},$$

where $x \wedge y = \min\{x, y\}$ and the margin offset is given by equation (2).

In practice, the insurer's liability is often determined under each economic scenario by recursive calculation of surplus or deficiency from period to period, whether carried out in an Excel spreadsheet or by other software packages. The expression above is a stochastic representation of a simplified version of such a liability under the GMMB rider. Under such a representation, one can generate a sample of net liabilities under different scenarios by running Monte Carlo simulations of equity returns and death events.

We shrink the valuation period to zero by taking n to ∞, reaching the limiting continuous-time model. Recall that

$$\lim_{n\to\infty} \left(1-\frac{m}{n}\right)^n = e^{-m}, \tag{4}$$

where m in this case should be interpreted as the continuously compounded annual rate of total charges. As a result, for each sample path, the continuous-time analogue of equation (1) is given by

$$F_t = \lim_{n\to\infty} F_{\frac{[nt]}{n}} = \frac{F_0}{S_0} \lim_{n\to\infty} S_{\frac{[nt]}{n}} \left[\left(1-\frac{m}{n}\right)^n\right]^{\frac{[nt]}{n}} = F_0 \frac{S_t}{S_0} e^{-mt}. \tag{5}$$

Using the definition of Riemann integral, we observe that the limit of the margin offset is given by

$$M_t = \lim_{n\to\infty} M_{\frac{[nt]}{n}} = \lim_{n\to\infty} \sum_{j=1}^{[nt]} \frac{1}{n} e^{-r(j-1)/n}\, m_e F_{(j-1)/n} = \int_0^t e^{-rs} m_e F_s ds,$$

where m_e is interpreted as the continuously compounded annual rate of rider charge allocated to the GMMB rider.

The limit of L leads to a continuous time model. In the case of the GMMB,

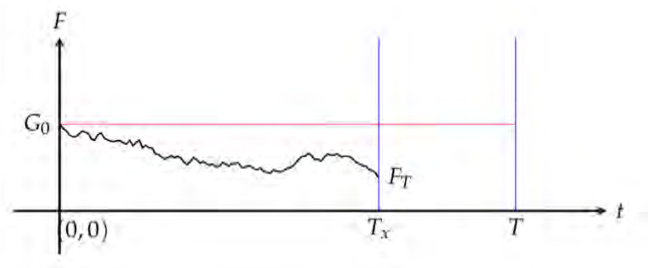
$$L_e^{(\infty)}(T_x) = e^{-rT}(G-F_T)^+ I(T_x > T) - \int_0^{T\wedge T_x} e^{-rs} m_e F_s ds. \tag{6}$$

The net liabilities L should be negative with a sufficiently high probability, as the products are designed to be profitable. However, in adverse scenarios, the net liabilities can become positive. Note that L depends on the path of equity values as well as mortality.

4.2 GUARANTEED MINIMUM DEATH BENEFIT

The GMDB guarantees the policyholder a minimum monetary amount G upon death payable at the end of the $1/n$-th period following his/her death. In other words, the insurer is responsible the difference between the guaranteed amount G and the actual account value at the time of payment, should the former exceed the latter. Figure 3 shows a particular scenario of account value falling below a guaranteed level at the time of death T_x, which represents a liability for the insurer.

Figure 3
GMDB GROSS LIABILITY



It is fairly common that the guarantee amount accumulates interest at a fixed rate $\rho > 0$, which is known as a roll-up option. The present value of the insurer's gross liability is

$$e^{-r\left(K_x^{(n)}+1/n\right)}\left(G\left(1+\frac{\rho}{n}\right)^{nK_x^{(n)}} - F_{K_x^{(n)}+1/n}\right)^{+} I\left(K_x^{(n)} < T\right).$$

The present value of the GMDB net liability is given by

$$L_d^{(n)}(T_x) := e^{-r\left(K_x^{(n)}+1/n\right)}\left(G\left(1+\frac{\rho}{n}\right)^{nK_x^{(n)}} - F_{K_x^{(n)}+1/n}\right)^{+} I\left(K_x^{(n)} < T\right) - \sum_{j=1}^{n\left(T\wedge K_x^{(n)}\right)} e^{-r(j-1)/n}\left(\frac{m_d}{n}\right)F_{(j-1)/n},$$

where m_d is the rate of fees per period allocated to fund the GMDB rider. Bear in mind that the rider fees are not charged separately, but rather as part of the mortality and expense, or M&E, fees, that is, $m > m_d$. Similarly, using the same argument as above, it is easy to show that the continuous-time version of the insurer's GMDB net liability is given by

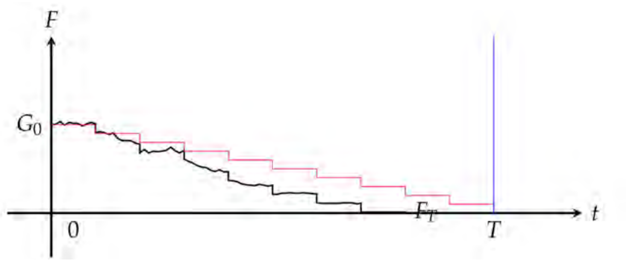
$$L_d^{(\infty)}(T_x) = e^{-rT_x}\left(Ge^{\rho T_x} - F_{T_x}\right)^{+} I(T_x < T) - \int_0^{T\wedge T_x} e^{-rs} m_d F_s ds\,. \quad (7)$$

Note that there are two sources of randomness in the formulation, namely the future lifetime T_x and the path of equity returns $\{F_s, 0 \le s \le T\}$. The interaction of both risks affects the severity of potential losses.

4.3 GUARANTEED MINIMUM WITHDRAWAL BENEFIT

The GMWB guarantees the policyholder the return of initial premium, typically through systematic withdrawals, regardless of the performance of the underlying equity funds. The policyholder is allowed to withdraw up to a maximal percentage of the initial premium per year out of the sub-account without penalty. The withdrawals are guaranteed to last until the initial premium is fully refunded, at which point any remaining balance of the investment account would be returned to the policyholder. Figure 4 shows a scenario with systematic withdrawals where the wiggly line at the bottom represents the balance of a policyholder's investment account and the staircase line at the top shows the remaining part of the initial premium to be withdrawn. While both lines drop by the same amount on each policy anniversary due to the systematic withdrawal, they can be far apart because the account value moves with the equity investment. As shown in the figure, when the market performs poorly, the investment account is depleted before the premium is fully returned. The GMWB rider enables the policyholder to continue receiving payments of the pre-specified withdrawal amount until the premium is refunded.

Figure 4
GMWB GROSS LIABILITY



Let us examine the cash flows for both the insurer and a policyholder. In practice, most policyholders choose to withdraw the maximum amount without penalty, denoted by w, which is typically a fixed percentage applied to the initial premium. Then it takes $\mathrm{T} = F_0/\mathrm{w}$ periods before the initial premium is fully refunded, at which point the GMWB rider expires. Note that w is represented as the absolute amount. For example, if the maximum is 10% per year and the initial deposit is $F_0 = 1{,}000$, then the withdrawal amount is $\mathrm{w} = 100$ and the maturity is in $1{,}000/100 = 10$ years. The margin offset per time unit used to fund the GMWB rider is denoted by m_w.

Although withdrawal activities can vary due to policyholders' behaviors, SOA and LIMRA's 2015 experience study (Drinkwater, Iqbal and Montiminy 2014) shows that "the majority of owner stake withdrawals through systematic withdrawal plans (SWPs). ... When owners use SWPs, they are likely to make withdrawals [for lifetime] within the maximum amount allowed in their contracts." This phenomenon has been observed consistently for many years by SOA and LIMRA's annual experience studies. The 2015 experience study shows that "79% of owners who took withdrawals in 2013 withdrew income that was below or close to the maximum amount calculated—up to 100%" and roughly 55% of policyholders who took withdrawals withdrew income between 90% and 110% of the maximum amount. The same report also states that "owners rarely add premium after the second year of owning a [guaranteed lifetime withdrawal benefit]." Therefore, in this paper, we make simplifying assumptions that are satisfied in the majority of cases in practice: 1) The policyholder under consideration starts taking withdrawals immediately at the time of valuation, although the results in this work can be easily extended to address the accumulation of an investment account in a waiting period until the first withdrawal; 2) the policyholder takes withdrawals at the penalty-free maximum amount every year; and 3) there is only an initial purchase payment (premium) at the start of the contract and no additional deposits are considered.

The pricing and hedging of the GMWB rider are studied in the literature by Milevsky and Salisbury (2006); Kling, Ruez and Russ (2011); and Feng and Jing (2016). Risk measures of the GMWB are considered in Feng and Vecer (2017) using partial differential equation methods.

To build up a model for the GMWB rider, we consider the incremental change of the policyholder's investment account over each period,

$$F_{(k+1)/n} - F_{k/n} = \frac{S_{(k+1)/n} - S_{k/n}}{S_{k/n}} F_{k/n} - \frac{m}{n} F_{k/n} - \frac{w}{n}, \tag{8}$$

which consists of the financial return from the equity-linking mechanism less the total management fee and the withdrawal for that period. Note that this recursive relation does not seem to have a simple explicit solution. To be more precise, however, one needs to make sure that the account does not go below zero and hence

$$F_{(k+1)/n} = \max\left\{\frac{S_{(k+1)/n}}{S_{k/n}} F_{k/n} - \frac{m}{n} F_{k/n} - \frac{w}{n}, 0\right\}. \tag{9}$$

Let us first consider the GMWB liability from a policyholder's point of view. The systematic withdrawals in the initial periods are typically taken directly out of the policyholder's account. When the equity performs badly, there is a chance that the account is depleted prior to time T. In that case, the insurer would have to continue to fund the systematic withdrawals out of its own general account. Therefore, the insurer's liability starts when the policyholder's sub-account is exhausted, called *ruin time*, that is,

$$\tau := \min\left\{\frac{k}{n} > 0 \colon F_{k/n} = 0\right\}.$$

Note that the ruin time is a random variable and can be greater than T, which means the fund in the sub-account will never be exhausted at the expiration time of the GMWB rider. For simplicity, we consider the case where the policyholder starts to withdraw immediately after the purchase of the contract. There would possibly be two sources of income for the policyholder:

1. The collection of all guaranteed withdrawals until the initial premium is returned or the time of death. The present value of all withdrawals is given by

$$\sum_{k=1}^{\lceil n(T\wedge T_x)\rceil} e^{-rk/n}\frac{w}{n}.$$

2. The balance of the policyholder's investment account, if there is any remaining at the maturity or the time of death $\mathrm{T} \wedge T_x$. This can be easily represented by $e^{-r(T\wedge T_x)} F_{T\wedge T_x}$. Therefore, the total sum of financial returns to the policyholder from the investment in the variable annuity contract can be written as

$$\sum_{k=1}^{\lceil n(T\wedge T_x)\rceil} e^{-rk/n}\frac{w}{n} + e^{-r(T\wedge T_x)} F_{T\wedge T_x} I\left(F_{T\wedge T_x} > 0\right). \tag{10}$$

From an insurer's point of view, the cash flows would look somewhat different. First of all, one has to keep in mind that the policyholder withdraws from his/her own account as long as it remains sufficient, and that the insurer only picks up the bill after the account plunges to zero prior to $\mathrm{T} \wedge T_x$.

On one hand, the total present value of the insurer's liabilities (guaranteed withdrawals) would be the sum of present values of all withdrawals after ruin time and before the earlier of the maturity T and the time of death T_x,

$$\sum_{k=n\tau}^{(n\tau-1)\vee\lceil n(T\wedge T_x)\rceil} e^{-rk/n}\frac{w}{n},$$

with the convention that $\sum_{k=m}^{m-1} = 0$ for any integer m. The symbol $\vee$ means the greater of two numbers, that is, $\mathrm{x} \vee y = \max\{\mathrm{x}, \mathrm{y}\}$. The random upper limit is used to make the sum zero when ruin does not occur prior to maturity. On the other hand, the insurer collects M&E fees from the start of the contract until the policyholder's account is exhausted and hence the present value of its income stream is given by

$$\sum_{k=1}^{(n\tau-1)\wedge\lceil n(T\wedge T_x)\rceil} e^{-r(k-1)/n} F_{(k-1)/n}\frac{m_w}{n}.$$

Piecing together both sides of the benefit outgo and the fee income, we obtain the individual net liability of a GMWB rider

$$L_w^{(n)} := \sum_{k=n\tau}^{(n\tau-1)\vee[n(T\wedge T_x)]} e^{-rk/n}\frac{w}{n} - \sum_{k=1}^{(n\tau-1)\wedge[n(T\wedge T_x)]} e^{-r(k-1)/n}F_{(k-1)/n}\frac{m_w}{n}. \tag{11}$$

The analogue of equation (9) is to impose the restriction that in the continuous-time model the process F is absorbed at zero once it reaches zero.

From a policyholder's point of view, we obtain the total worth of investment with the variable annuity contract with the GMWB rider in continuous-time by taking the limit of equation (10) as $\mathrm{n} \to \infty$,

$$\int_0^{T\wedge T_x} e^{-rt}w\mathrm{dt} + e^{-r(T\wedge T_x)}F_{T\wedge T_x}\mathrm{I}\big(F_{T\wedge T_x} > 0\big).$$

Similarly, the continuous-time individual net liability of a GMWB rider from an insurer's point of view is given by the limit of equation (11) as $\mathrm{n} \to \infty$,

$$L_w^{(\infty)} := \int_\tau^{\tau\vee(T\wedge T_x)} e^{-rt}w\,dt - \int_0^{\tau\wedge(T\wedge T_x)} e^{-rt}m_w F_t dt,$$

where the ruin time is defined by $\tau := \inf\{\, t > 0 : F_t \le 0\}$.

Section 5: Risk Management of Equity-linked Insurance

In this section, we shall apply the cascade model to the quantification and assessment of equity and mortality risks embedded in various variable annuity guaranteed benefits. After all benefits are analyzed under the same framework, we shall make a comparison of different product designs.

5.1 RISK MEASURES

Risk measures map a risk/loss random variable to a real number. There are several risk measures that are most commonly used in practice, including value-at-risk (VaR),

$$\mathrm{VaR_p}(X) = \inf\{\,\mathrm{x}: \mathrm{P}(X \le x) \ge p\},$$

the threshold exceeded by the loss random variable with the probability of at least p; tail value-at-risk (TVaR),

$$\mathrm{TVaR_p}(X) = \frac{1}{1-p}\int_p^1 \mathrm{VaR_q}(X)dq,$$

the average value of extreme VaRs with the confidence level $[p, 1]$; and conditional tail expectation (CTE),

$$\mathrm{CTE_p}(X) = \mathrm{E}\left[X \middle| X \ge \mathrm{VaR_p}[X]\right],$$

the average amount of loss when it exceeds the VaR at the confidence level p. When X is a continuous random variable, $\mathrm{TVaR_p}(X)$ and $\mathrm{CTE_p}(X)$ are equivalent. Risk measures have been used by the insurance industry for setting up reserves and capitals.

5.2 GUARANTEED MINIMUM MATURITY BENEFIT

There are two parties involved at the individual contract level—a policyholder and an insurer underwriting the variable annuity contract. Two types of risks are considered for this design—equity risk and mortality risk. The lapse risk is not considered here. Let us now consider the split of risks between the policyholder and the insurer at the contract level.

5.2.1 RISK RETENTION AND RISK SHARING AT CONTRACTUAL LEVEL

The investment with combined risks is split into two components – the policyholder's account with retained risks and the insurer's account with risks transferred from the policyholder. In the following, we examine financial positions of both the policyholder and the insurer.

1. **Without investment guarantee.** Had a policyholder not purchased a variable annuity, he or she would use the initial purchase payment F_0 to buy equity directly and hence be completely exposed to the equity risk with the uncertainty of financial returns. To make an easy comparison of the cases with and without investment guarantees, we look at economic outcomes at the time of termination $\mathrm{T} \wedge T_x$, which is the earlier of some maturity date T or the time of the policyholder's death. With an analogy to equation (5), an investment without any guarantee ends up with

$$F_0 \frac{S_{T \wedge T_x}}{S_0} = F_{T \wedge T_x} e^{m(T \wedge T_x)},$$

or written in terms of time-0 value as the risk $X_0 := F_{T \wedge T_x} e^{-(r-m)(T \wedge T_x)}$, where F refers to the account value process in equation (5).

2. **With investment guarantee.** In contrast, the equity risk is shared between the policyholder and the insurer once a contract with the GMMB rider is in force. The policyholder or beneficiary (heir) is now entitled to the balance in the sub-account and additional guarantee benefit in time-0 value,

$$X_1 := e^{-rT} \max\{F_T, G\} I(T_x \geq T) + e^{-rT_x} F_{T_x} I(T_x < T), \tag{12}$$

whereas the insurer collects equity-linked fees and covers the downside of equity risk,

$$X_2 := \int_0^{T \wedge T_x} e^{-rs} m_e F_s ds - e^{-rT} (G - F_T)^+ I(T_x \geq T). \tag{13}$$

Note that with the variable annuity contract the split of risk is complicated by the additional uncertainty with the policyholder's mortality risk.

3. **Equivalence of risk-neutral values.** Suppose that there is no friction cost in the product design. In other words, all fees and charges are used exactly to fund the GMMB rider, that is, $\mathrm{m} = m_e$. We show below that the split of equity and mortality risks through contractual design is justifiable in the sense that the combination of the risk retained by the policyholder and the risk ceded to the insurer is equivalent to the original equity risk in risk-neutral value. In other words, under a risk-neutral measure $\tilde{P}$, the expected value of X_0 is equal to the sum of expected value of X_1 and X_2, that is,

$$\tilde{E}[X_0] = \tilde{E}[X_1] + \tilde{E}[X_2]. \tag{14}$$

The calculation in equation (14) confirms the intuition that a VA contract is in essence a re-distribution of cash flows generated from the policyholder's original investment between himself and the insurer. The insurer offers some "smoothing" (downside risk protection) on the policyholder's cash flows in exchange for service fees. Regardless of contractual designs, the true value of investment, gauged by a risk-neutral measure, would remain the same before and after entering the contract. This is consistent with the no-arbitrage theory.

It is easy to prove the equivalence under the Black-Scholes model (independent log-normal model). On the left side of equation (14), it follows from martingale property and optional stopping theorem that

$$\tilde{E}[X_0] = F_0. \tag{15}$$

In view of the fact that $\max\{F_T, G\} - (G - F_T)^+ = F_T$, we must have that

$$X_1 + X_2 = e^{-r(T \wedge T_x)} F_{T \wedge T_x} + \int_0^{T \wedge T_x} e^{-rs} m_e F_s ds,$$

where F_t follows the following geometric Brownian motion

$$\mathrm{d}F_t = (r - m) F_t \mathrm{dt} + \sigma F_t \mathrm{d}W_t,$$

W_t is a Brownian motion. Then by Itô's calculus, we know that

$$F_t = F_0 \exp\left(\left(r - m - \frac{\sigma^2}{2}\right) t + \sigma W_t\right),$$

and $\tilde{E}(F_t) = F_0 e^{(r-m)t}$. Then, by assuming $m_e = \mathrm{m}$, we have

$$\tilde{E}\left[\int_0^{T \wedge T_x} e^{-rs} m_e F_s ds\right] = \int_0^{T \wedge T_x} e^{-rs} m \tilde{E}(F_s) ds = \int_0^{T \wedge T_x} e^{-rs} m F_0 e^{(r-m)s} ds$$
$$= F_0 - F_0 e^{-m(T \wedge T_x)} = F_0 - \tilde{E}\left[e^{-r(T \wedge T_x)} F_{T \wedge T_x}\right].$$

Therefore, on the right-hand side of equation (14), we also have

$$\tilde{E}[X_1 + X_2] = F_0,$$

which proves the identity equation (14) for the decomposition of risks on an individual contract. In practice, the total fee rate m is always larger than the rider charge rate m_e. Hence, the left-hand side of equation (14) is greater than the right-hand side due to the loss of friction cost (overheads, commission, compliance, etc.).

No-arbitrage pricing is well studied in the literature for variable annuity guaranteed benefits. See, for example, Cui Feng and MacKay (2017); Feng and Volkmer (2016); and Marshall, Hardy and Saunders (2010). These techniques can be applied to offer a breakdown of investment values between a policyholder and an insurer.

Let us consider a numerical example to better understand the engineering process of risk management. The model assumptions for the variable annuity product under consideration are provided in the Appendix B. The underlying equity process is driven by a geometric Brown motion, aka an independent lognormal model. Even though we presented the mathematical formulation in a continuous time model, all cash flows are projected on an annual basis for simplicity, that is, account balances are evaluated at the end of each year and fee incomes are deducted from the account at the beginning of each year. We only consider a single cohort of 60-year-olds. There is no friction cost in the model, that is, $\mathrm{m} = m_e$.

When the investor chooses to buy the variable annuity contract with the GMMB rider, her original financial position is in essence split into two, one part is retained by the investor and the other ceded to the insurer who underwrites the contract. As shown in equation (14), we can quantify the amount of risk-neutral value transfer that occurs with such a contract. The calculation is summarized below with parameters given in Table 23 in the Appendix B.

1. Simulate curtate lifetime K_x by Table 28 in the Appendix B and stock prices S_t under risk-neutral measures at discrete times $\mathrm{t} = 1, \dots, \mathrm{T}$ for n scenarios.
2. Determine financial positions of the policyholder X_1 and the insurer X_2 by the discretized version of equations (12) and (13) under each scenario.
3. Take sample mean of financial positions at time T for X_1 and X_2 and obtain $\tilde{E}[X_1]$ and $\tilde{E}[X_2]$.

Table 1 shows that the investor retains about 89% of her original financial position while transferring about 11% to the insurer. In other words, the investor is expected to give up 11% of the risk-neutral value of her original position because she is better off with a protection on the downside risk.

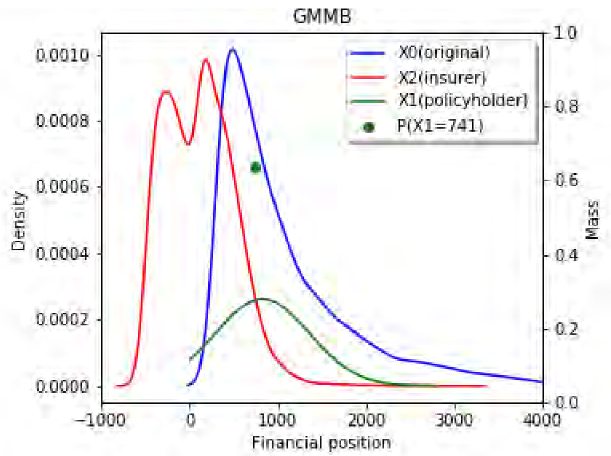
Table 1
VALUE DECOMPOSITION BY CONTRACTUAL DESIGN

	$\widetilde{E}[X_0]$	$\widetilde{E}[X_1]$	$\widetilde{E}[X_2]$
Risk-neutral value	1,000	889	111

While the risk-neutral values add up, that is, $\mathrm{E}[X_0] = \mathrm{E}[X_1] + \mathrm{E}[X_2]$, it does not mean that $X_0 = X_1 + X_2$. The two parties are willing to split the risky investment because both can gain some advantages with tail risk distributions. We can visualize the net effect of contractual design at the individual contract level in Figure 5. Keep in mind that the investor starts with an initial investment of $1,000. The horizontal axis shows the possible range of her financial positions with and without the GMMB rider. The blue line represents the probability density function of the policyholder's original financial position without any investment guarantee X_0 at the termination of the contract. The other two lines represent probability functions of the two parties after the investor enters into the contract with investment guarantee. The green line represents the probability density function of the policyholder's financial position X_1, whereas the red line represents that of the insurer's financial position X_2. The green dot represents the probability mass of X_1 at the level $e^{-rT}\mathrm{G}\ (= 741)$, which is the present value of minimum guarantee should the policyholder survive to maturity. Readers should refer to the right side of the vertical scale for the size of the probability mass, which is around 0.63. In other words, the policyholder has the chance of 63% receiving the minimum guaranteed balance. The policyholder's position has higher probability of extremely low financial returns and lower probability of extremely high financial returns with investment guarantee than that without guarantee, due to fees and charges. However, in general, the probability density of the policyholder's position with guarantee is significantly less than that of the position without guarantee. It shows that the policyholder's financial position is subject to less variability due to the protection from the investment guarantee. It is consistent with the observation that each policyholder gives up some potential of upside risk in exchange for a protection from the downside risk.

Figure 5

EFFECT OF RISK TRANSFER AT THE CONTRACT LEVEL



The red line shows the probability density function of the insurer's financial position. As we shall see throughout the report, the feature of two peaks is prevalent in the probability density function of all insurers' positions. The peak on the positive side points to the most likely range of fee incomes contributed by policyholders who survive to maturity, while the other peak on the negative side indicates the most likely range of insurer's net loss due to benefit payments. It is clear that the insurer carries the majority of downside risk as its probability density function centers around near zero levels. The split of the investor's original risk profile (blue line) into that of an insurer (red line) and that of a policyholder (green line) is a reflection of the risk-sharing effect of the GMMB contractual design.

To better quantify risk sharing at the contractual level, we can demonstrate the reduction in risk measures from the viewpoint of a policyholder. As shown in Table 2, we can measure the policyholder's financial returns by two risk measures, namely, value-at-risk and tail-value-at-risk. Since the policyholder gives up some upside profit, we can observe about 36% to 39% reduction in her extreme total returns. Since the policyholders receives protection on the downside risk of her investment, we also apply the two risk measures to the left tail of the investor's financial position in Table 3. In all cases, the left quantiles of policyholder's financial position doubled or tripled. This result indicates that the policyholder can earn much more under severe adverse scenarios with the GMMB rider than without the rider.

Table 2

REDUCTION IN VALUE-AT-RISK OF POLICYHOLDER'S RIGHT-TAIL DISTRIBUTION BY RISK SHARING

p	$\mathrm{VaR_p}(X_0)$	$\mathrm{VaR_p}(X_1)$	Reduction	$\mathrm{TvaR_p}(X_0)$	$\mathrm{TvaR_p}(X_1)$	Reduction
0.975	3,930.4	2,433.8	38.1%	5,946.0	3,652.8	38.6%
0.95	2,924.8	1,824.1	37.6%	4,652.4	2,868.7	38.3%
0.9	2,092.3	1,314.3	37.2%	3,551.1	2,201.7	38.0%
0.8	1,401.6	889.4	36.5%	2,622.6	1,635.8	37.6%

Table 3
INCREASE IN VALUE-AT-RISK OF POLICYHOLDER'S LEFT-TAIL DISTRIBUTION BY RISK SHARING

p	$\mathrm{VaR_p}(X_0)$	$\mathrm{VaR_p}(X_1)$	Increase	$\mathrm{TVaR_p}(X_0)$	$\mathrm{TVaR_p}(X_1)$	Increase
0.025	102.5	238.2	132.4%	74.5	156.9	110.6%
0.05	139.2	384.8	176.4%	98.1	234.1	138.6%
0.1	197.1	727.5	269.1%	133.7	388.5	190.6%
0.2	298.6	740.8	148.1%	191.2	564.6	195.3%

5.2.2 RISK REDUCTION AT AGGREGATE LEVEL

Recall from Section 4.1 that the individual net liability of the GMMB is defined to be the present value of future outgo less the present value of future income on a stand-alone contract basis. Unlike standardized exchange-traded financial derivatives with unit contract sizes, variable annuities are sold to individual investors as a retirement planning vehicle. Individuals may choose to purchase annuities with different payments. To introduce an aggregate model, let us re-formulate the individual net liability so as to distinguish different policies in a large pool.

We can write the individual net liability of the GMMB rider for the i-th policyholder where $\mathrm{i} = 1, 2, \cdots, \mathrm{n}$ as

$$L\big(T_x^{(i)}\big) \coloneqq e^{-rT}\big(G^{(i)} - F_T^{(i)}\big)^+ I\big(T_x^{(i)} > T\big) - \int_0^{T\wedge T_x^{(i)}} e^{-rs} m_e F_s^{(i)} ds\,, \tag{16}$$

where

1. $T_x^{(i)}$ is the future lifetime of the i-th policyholder of age x at issue;

2. $F_0^{(i)}$ is the initial purchase payment of the i-th policyholder;

3. $F_t^{(i)} \coloneqq e^{-mt} F_0^{(i)} S_t / S_0$ is the evolution of the i-th policyholder's investment account; and

4. $G^{(i)} \coloneqq \gamma F_0^{(i)}$ is the guaranteed minimum amount at maturity for the i-th policyholder, where γ determines the guaranteed amount $G^{(i)}$ as a percentage of the i-th policyholder's initial purchase payment.

Then the *aggregate net liability* of the GMMB rider for all n policies is determined by

$$\sum_{i=1}^{n} L\big(T_x^{(i)}\big) = \sum_{i=1}^{n} e^{-rT}\big(G^{(i)} - F_T^{(i)}\big)^+ I\big(T_x^{(i)} > T\big) - \sum_{i=1}^{n} \int_0^T e^{-rt} m_e F_t^{(i)}\, I\big(T_x^{(i)} > t\big) dt. \tag{17}$$

While we are interested in the tail risk of the aggregate net liability, it is clear that the total liability scales up with the size of the policy pool. To consider the diversification effect, we need to "normalize" the total liability by introducing the *average net liability*

$$\overline{L}^{(n)} \coloneqq \frac{1}{n}\sum_{i=1}^{n} L\big(T_x^{(i)}\big). \tag{18}$$

If the individual net liabilities ($L(T_x^{(1)}),\ L(T_x^{(2)}),\ \cdots,\ L(T_x^{(n)})$) are exchangeable,[1] then we have

$$\mathrm{TVaR_p}\left(\overline{L}^{(n+1)}\right) \leq \mathrm{TVaR_p}\left(\overline{L}^{(n)}\right).$$

The proof can be found in Theorem 5.41 and Corollary 5.42 of Feng (2018). The interpretation of this result is very similar to that of central limit theorem. When the insurer has a large set of policies, the losses and profits average out. The larger the portfolio, the less likely to observe extreme aggregate loss. The TVaR measures the severity of tail risk. The TVaR of the average net liability is a decreasing function of the sample size n. In other words, the tail risk of average net liability can always be reduced by diversification through a large pool of policies. Furthermore, if $\left(L(T_x^{(1)}), L(T_x^{(2)}), \cdots, L(T_x^{(n)})\right)$ are independent, then the strong law of large numbers implies that as $\mathrm{n} \rightarrow \infty$,

$$\mathrm{TVaR_p}\left[\overline{L}^{(n)}\right] \rightarrow E\left[L(T_x^{(i)})\right],$$

for all $\mathrm{p} \in [0,1]$.

However, due to the equity linking mechanism, individual net liabilities $\left(L(T_x^{(1)}), L(T_x^{(2)}), \cdots, L(T_x^{(n)})\right)$ are not mutually independent. Nonetheless, it is shown in Feng and Shimizu (2016) that there is a limit, denoted by $\overline{L}^{(\infty)}$, such that as $\mathrm{n} \rightarrow \infty$,

$$\overline{L}^{(n)} \longrightarrow \overline{L}^{(\infty)}. \tag{19}$$

The convergence above is almost sure, that is, the convergence holds with probability one. Subsequently, one can show that tail risks of averages decrease with the sample size and eventually converge to that of an average model, that is, as $\mathrm{n} \rightarrow \infty$,

$$\mathrm{TVaR_p}\left(\overline{L}^{(n)}\right) \downarrow \mathrm{TVaR_p}\left(\overline{L}^{(\infty)}\right).$$

The proof can be found in Corollary 5.47 in Feng (2018). In other words, the uncertainty of L^{∞} is entirely attributable to financial risk whereas the mortality risk is fully diversified.

As the insurance product is designed to transfer a certain amount of equity and mortality risks from individual policyholders to the insurer, the risks from individual contracts are pooled together in the insurer's "melting pot"—a line of business. In view of definitions and equations (17), (18) and (19), one can show that the insurer's average liability at the corporate level is in essence a conditional expectation of its individual liability at the contract level.

$$\tilde{E}[-X_2|\mathcal{F}_T] := e^{-rT}\ {}_Tp_x(G - F_T)^+ - \int_0^T e^{-rs} m_e\ \ {}_sp_x F_s ds, \tag{20}$$

where $\mathcal{F}_T$ represents the filtration at time T. As defined earlier, X_2 denotes the insurer's financial position. To avoid introducing another symbol, we think of $-X_2$ as the insurer's net liability. Positive net liability represents loss whereas a negative one means profit.

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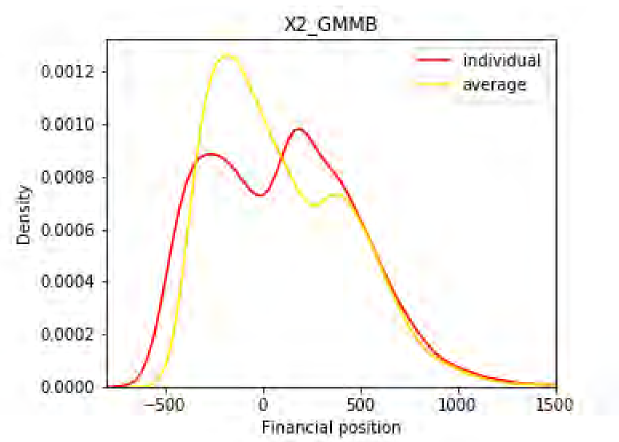
[1] The random variable $(X_1,\ \cdots,\ X_n)$ is exchangeable if it has the same joint distribution as $\left(X_{j_1}, \cdots,\ X_{j_n}\right)$ for all permutations of $(1,\ 2, \cdots,\ n)$. It means that all variables have the same marginal distribution and there is a symmetry in their dependence. Such a situation is common in practice even if the dependence of random variables is unknown or difficult to model.

It is also known that the tail risk of the average liability is reduced compared with that of the individual liability due to the diversification of mortality risk, that is, for any $p \in (0,1)$,

$$\mathrm{TVaR_p}\left[\tilde{E}[-X_2|\mathcal{F}_T]\right] \leq \mathrm{TVaR_p}[-X_2].$$

Let us now return to the numerical example of the 60-year-old cohort. Suppose there is a sufficiently large number of policyholders so that the law of large numbers applies and the mortality risk is completely diversified. Figure 6 shows that the probability density function of the average liability at the aggregate level (yellow line) is more concentrated than that of the insurer's individual liability at the contract level (red line). In a manner similar to the individual model, the two peaks in both density functions for the aggregate model roughly represent the case of no benefit payment and that of benefit payment. Note that there are two sources of randomness in the expression of individual liability $-X_2$ in equation (13), whereas the mortality risk is fully diversified in the expression of the average liability $\tilde{E}[-X_2|\mathcal{F}_T]$ in equation (20). The reason for their difference is that extreme scenarios are more likely to occur with the interaction of equity and mortality risk. For example, an adverse scenario may happen in the case of individual liability when a policyholder dies after a prolonged period of the underlying equity's persistently poor performance. But this may not happen in the case of average liability because the contract size declines in a deterministic fashion due to the survival rate. The distribution function of $\tilde{E}[-X_2|\mathcal{F}_T]$ (red line) represents the remaining uncertainty due to the net effect of diversified mortality risk and undiversifiable equity risk.

Figure 6
RISK REDUCTION AT AGGREGATE LEVEL



Left tails of probability density functions for $-X_2$ and $\tilde{E}[-X_2|\mathcal{F}_T]$ are shown in Figure 7. Note that we focus on the negative part of the liability distribution, which represent the insurer's possible losses. It is clear from the figure that the conditional expectation (yellow line) has a lighter tail than the unconditioned financial position (red line). Another angle to look at the same issue is through survival functions in Figure 8. Note that the survival function of the insurer's loss $-X_2$ (red line) dominates that of $\tilde{E}[-X_2|\mathcal{F}_T]$ (yellow line) at the right tail. It indicates that for a given level of loss, the probability of losses exceeding the level is greater for the insurer's loss for each contract than that for the insurer's average liability. Both graphs are consistent with the observation that an insurer can reduce the risk in its liability by diversifying its portfolio of policies.

Figure 7

RISK REDUCTION AT AGGREGATE LEVEL

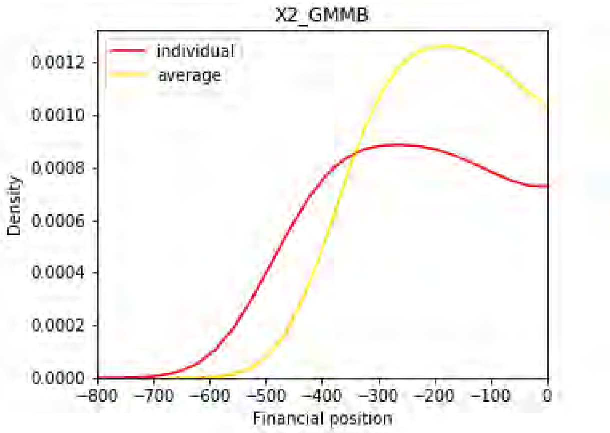
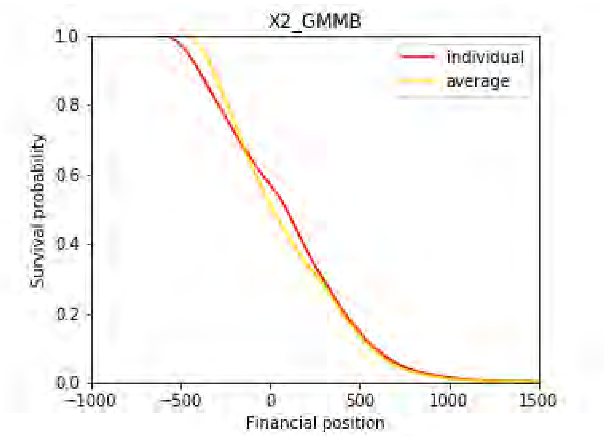


Figure 8

RISK REDUCTION AT AGGREGATE LEVEL



Unlike the case of classical insurance where mortality risk can be completely diversified, the equity risk is undiversifiable, which explains the remaining uncertainty in the insurer's financial position in equation (20). The significance of pooling effect to diversify mortality effect lies in the left tail of the insurer's position. Even though the two graphs of survival functions do not appear far apart in Figure 8, we can quantify the reduction in riskiness by calculating risk measures. Table 4 shows that two risk measures of the insurer's losses at extreme levels (80%, 90%, 95% and 97.5% quantiles of losses) have been reduced by 18% to 22%. In other words, this numerical experiment shows that an insurer can offset about 20% of its tail risk by the pooling effect and has yet to absorb the remaining 80% by other means.

Table 4

REDUCTION IN RISK MEASURES OF INSURER'S LOSS BY DIVERSIFICATION

p	$\mathrm{VaR}_p(X_2)$	$\mathrm{VaR}_p(E[X_2\|\mathcal{F}_T])$	Reduction	$\mathrm{TVaR}_p(X_2)$	$\mathrm{TVaR}_p(E[X_2\|\mathcal{F}_T])$	Reduction
0.025	−486.7	−382.6	21.4%	−523.5	−412.5	21.2%
0.05	−443.2	−349.7	21.1%	−493.7	−388.8	21.2%
0.1	−377.5	−299.3	20.7%	−451.3	−356.0	21.1%
0.2	−269.8	−220.7	18.2%	−387.1	−307.4	20.6%

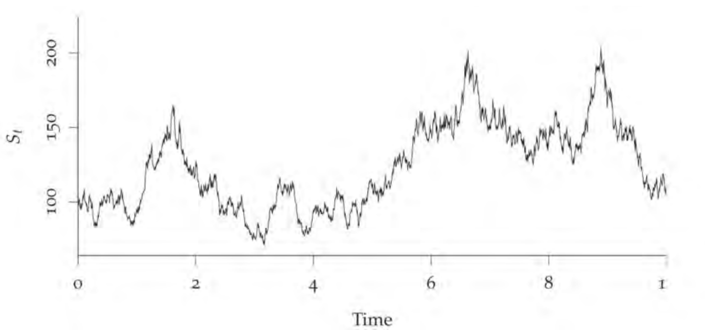
The pricing and hedging of the GMDB rider are first introduced in Milevsky and Posner (2001). Risk measures of insurer's net liability with the GMMB and the GMDB are studied in greater details in Feng and Volkmer (2012), Feng and Volkmer (2014), Feng (2014), Feng and Huang (2016), Feng and Yi (2019), and Feng, Kuznetsov and Yang (2019) under various stochastic models. The impact of policyholder behavior on risk measure is also discussed in Feng, Jing and Dhaene (2017).

5.2.3 RISK SHARING WITH FINANCIAL MARKET

The remaining undiversifiable aggregate risk is then processed in one of three ways. The insurer may choose to retain the aggregate risk by the traditional approach of setting up reserves and capitals. Under adverse circumstances, the insurer can use reserves and capitals to absorb losses from underwriting the risks. Keep in mind, however, reserves and capitals do not prevent insurers from suffering losses. They are meant for insurers to provide a temporary relief for the insurer. Another common approach is to set up a hedging program to offset losses. Under this approach, the aggregate risk is effectively transferred to the capital market. The third approach is for the insurer to enter into a reinsurance contract, which further splits the aggregate risk between the insurer and its reinsurer.

Here we consider the effect of dynamic hedging. The following numerical example is based on model assumptions and parameters of the GMMB rider listed in the appendix. We establish a discrete hedging program and shall illustrate the effectiveness of the program under a particular scenario of equity returns, as shown in Figure 9.

Figure 9
SAMPLE PATH OF EQUITY INDEX VALUE

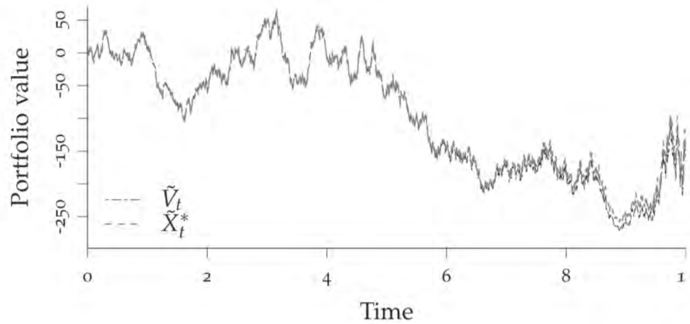


There are two methods of hedging against remaining risks in investment guarantees. The commonly used approach in practice is what we call gross liability hedging, which views the insurer's gross liability from investment guarantee as an embedded financial option and focuses on eliminating uncertainty from the gross liability. We also consider another approach, which we call net liability hedging. The approach also takes into account the interaction of equity risk and mortality risk from the income side. While the mortality risk is full diversified, it still affects the cash flows owing to the survivorship model. Hence the hedging strategy is developed to remove uncertainty of net liability, which includes both incomes and guaranteed liability outgoes.

1. **Gross liability hedging.** We can now inspect the result of a delta-hedging strategy in Figure 10. Details on the implementation of such a hedging program can be found in Feng (2018). The gray line represents the evolution of the hedging portfolio while the black line shows the path of time-t risk-neutral value of the GMMB gross liability less the accumulated value of its initial value. If the GMMB were to be compensated by a single up-front fee, then the value V_t would represent the net profit or loss of the GMMB rider at time t. One can hardly distinguish one line from the other in Figure 10, which implies that the daily rebalanced hedging strategy closely offsets the GMMB gross liability. However, one should be reminded that the GMMB rider is funded by a stream of equity-based fees. When only gross liability is considered, the equity risk on the asset generated by fee incomes is not taken into account. Such a hedging portfolio cannot eliminate the uncertainty with equity risk remaining on the income side.

Figure 10

SAMPLE PATH OF GROSS LIABILITY HEDGING PORTFOLIO

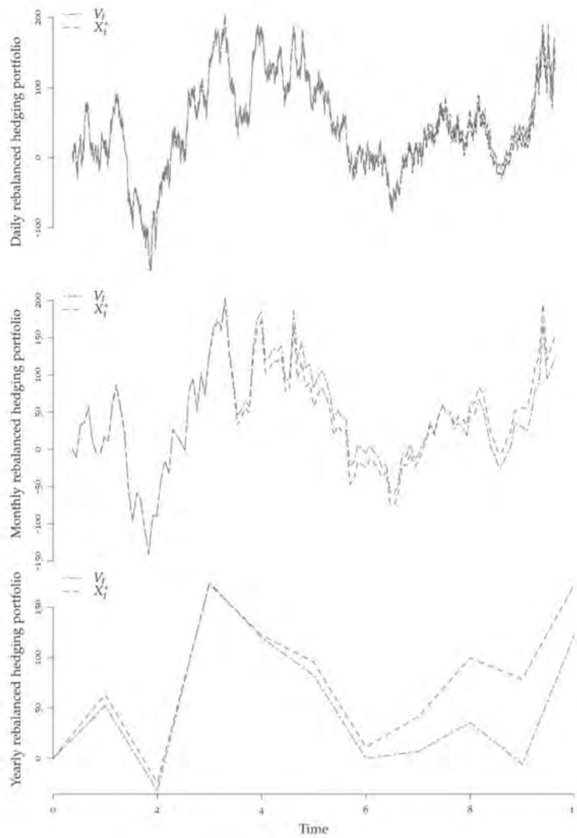


2. **Net liability hedging.** Consider a GMMB contract with the product specification listed in the appendix. The survivorship is based on the Illustrative Life Table in the Appendix B, Table 28. Survival probabilities for fractional ages are estimated using the uniform distribution of deaths assumption, that is, ${}_sq_x = s\ {}_1q_x$ for any integer x and $0 \leq s \leq 1$. To avoid any friction cost, we set $m_e = m$ in the model. The fee rate m is determined by the equivalence principle under the risk-neutral measure. In other words, the no-arbitrage value of the insurer's gross liability would exactly match that of the insurer's fee incomes.

   All illustrations will be based on the sample path of the equity index/fund $\{S_t: 0 \leq t \leq T\}$ in Figure 9. Following this particular scenario, we develop three hedging portfolios based on the above-described method, which are rebalanced on yearly, monthly and daily bases respectively, that is, $\Delta t = 1, 1/12, 1/252$. Path-wise comparison of portfolio values and GMMB net liability values at all time points for all three hedging portfolios can be seen in Figure 11. In each graph, the dashed line shows the fluctuation of hedging portfolio whereas the dash-dotted line represents the fluctuation of GMMB net liability. Portfolio values of a least active hedging program, which is rebalanced on an annual basis and depicted in the bottom graph, generally follow the pattern of fluctuation with net liability values. The path of the hedging portfolio and that of the net liability are almost indistinguishable for the most active hedging portfolio, which is rebalanced on a daily basis and depicted in the top graph. It is not surprising that hedging errors reduce with the frequency of rebalancing. However, a word of caution should be made. There is a trade-off between hedging error and transaction cost of hedging program. A prudent insurer strives to strike a balance and develop an affordable and effective hedging strategy.

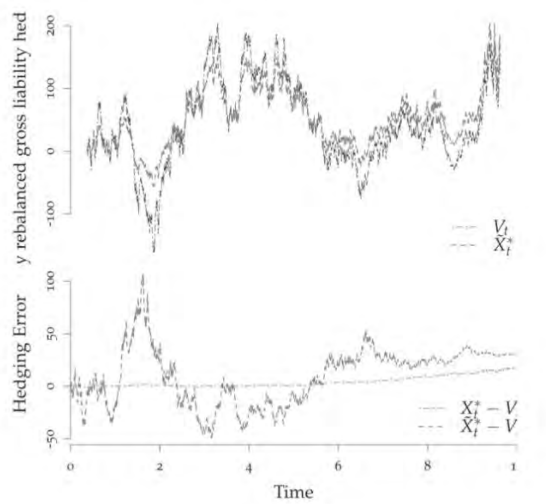
Figure 11

EFFECTIVENESS OF HEDGING PORTFOLIOS WITH VARIOUS REBALANCING FREQUENCIES



We can further compare the net liability hedging strategy with the gross liability hedging strategy. In the top graph of Figure 12, the dark line represents the evolution of the GMMB net liability whereas the light gray line illustrates the evolution of gross liability hedging portfolio. This graph should be compared with the top graph in Figure 11. The bottom graph of Figure 12 shows hedging errors from the two portfolios. The maximum absolute hedging error of the net liability hedging portfolio X_t^* is around \$17 while that for the gross liability hedging portfolio $\widehat{X_t^*}$ is clearly greater than \$100. It is not surprising that the gross liability hedging portfolio does not perform as well as the net liability hedging portfolio to hedge against the net liability. The reason is that the equity risk lingers with fee incomes and is not accounted for in the gross liability hedging portfolio.

Figure 12
COMPARISON OF GROSS AND NET LIABILITY HEDGING STRATEGIES



This example shows that fee incomes, if not hedged properly, can cause large fluctuation with an insurer's cash flows. Although gross liability hedging is most common in practice, the best practice for managing investment guarantees is to conduct the net liability hedging portfolio.

While the hedging example is only developed for the GMMB, the same technique can be applied to other riders. For example, Feng and Yi (2019) offers detailed analysis of hedging strategies for the guaranteed minimum accumulation benefit.

5.3 GUARANTEED MINIMUM DEATH BENEFIT

We use similar analysis to understand the contractual design of the GMDB. For comparison, we consider the investor who has a choice to make between direct investment in a stock or buying variable annuity written on the same equity with a GMDB rider. As shown in the previous section, the financial position of the investor without any investment guarantee is known to be

$$X_0 := F_0 \frac{S_{T \wedge T_x}}{S_0}.$$

5.3.1 RISK RETENTION AND RISK SHARING AT CONTRACTUAL LEVEL

When the investor enters the variable annuity contract, she retains some portion of the upside risk while ceding the downside risk to the insurer upon death. For simplicity, we consider a discrete time model with the death benefit payable on an annual basis. Therefore, the investor's financial position can be written as

$$\boldsymbol{X_1 = e^{-rT} F_T I(K_x > T) + e^{-rK_x} \max\{F_{K_x}, G\} I(K_x \le T)},$$

where the fund value is determined by

$$\boldsymbol{F_{k+1} = F_k \left(\frac{S_{k+1}}{S_k} - m\right)}.$$

The insurer provides a minimum guarantee at the time of the policyholder's death. Note however we consider only a term contract here. In other words, the investment guarantee expires on a fixed maturity date T.

$$X_2 = \sum_{k=0}^{(T \wedge K_x)-1} e^{-rk} m_d F_k - e^{-rK_x}\left(G - F_{K_x}\right)_+ I(K_x \leq T). \quad (21)$$

With analogy to equation (14), it is not difficult to show that the investor ends up with the same risk-neutral value with or without the investment guarantee.

$$\tilde{E}[X_0] = \tilde{E}[X_1] + \tilde{E}[X_2].$$

We consider a numerical example where a variable annuity contract with a GMDB rider is sold to a cohort of 60-year-olds. The model parameters are provided in the appendix. As shown in Table 5, by entering this contract, the investor gives up 33.8% of the risk-neutral value of her initial investment while maintaining 66.2% on her own. In exchange for the value transfer, the insurer is committed to offer protection on the investor's downside risk upon her death.

Table 5

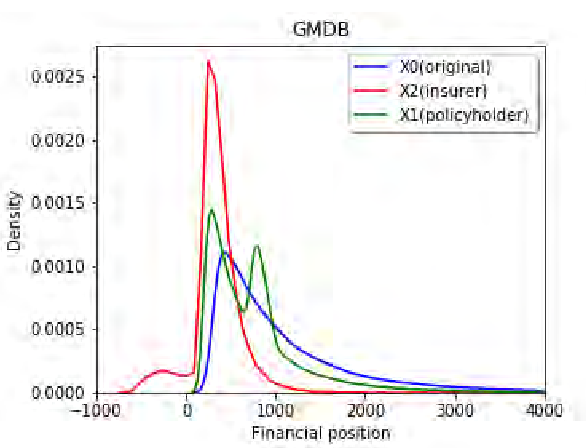
VALUE DECOMPOSITION BY CONTRACTUAL DESIGN

	$\tilde{E}[X_0]$	$\tilde{E}[X_1]$	$\tilde{E}[X_2]$
Risk-neutral value	1,000	662	338

While the risk-neutral value of the investor's investment is split between the insurer and the investor herself, the two parties end up with different risk profiles. As shown in Figure 13, the probability density function of the policyholder's financial position has thinner tails on both ends than that of her original position without any investment guarantee. In other words, she gives up some portion of upside risk in exchange for protection from the insurer for her downside risk. Note that the probability distributions of both insurer and policyholder have two peaks, which correspond to two cases depending on whether policyholders survive to maturity. When policyholders die prior to maturity, the rider makes pure profit for the insurer. Otherwise, the insurer may suffer loss when account values are persistently low.

Figure 13

EFFECT OF RISK TRANSFER AT THE CONTRACT LEVEL



Note that in Figure 13 the probability distribution of the policyholder's financial position after entering the VA contract has a thinner tail than that before entering the contract. Even though in this graph their differences do not appear quite visible, their tail distributions are in fact quite different. Table 6 shows that there is close to 35% to 39% reduction in profit measured by value-at-risk and tail-value-at-risk.

Table 6
REDUCTION IN RISK MEASURES OF POLICYHOLDER'S RIGHT-TAIL DISTRIBUTION

p	$\mathrm{VaR_p}(\boldsymbol{X_0})$	$\mathrm{VaR_p}(\boldsymbol{X_1})$	Reduction	$\mathrm{TVaR_p}(\boldsymbol{X_0})$	$\mathrm{TVaR_p}(\boldsymbol{X_1})$	Reduction
0.975	3,930.4	2,433.8	38.1%	5,946.0	3,652.8	38.6%
0.95	2,924.8	1,824.1	37.6%	4,652.4	2,868.7	38.3%
0.9	2,092.3	1,314.3	37.2%	3,551.1	2,201.7	38.0%
0.8	1,401.6	913.9	34.8%	2,622.6	1,637.3	37.6%

Similarly we can examine changes in the left tail of the policyholder's financial position due to entering the VA contract with the GMDB rider. It is interesting to observe in Table 7 that most of left-tail quantiles for the policyholder's position with the GMDB do not improve compared with those without the GMDB. It indicates that the policyholder would earn less with the GMDB than otherwise under severe adverse scenarios. This may sound contradictory with the fact that the policyholder buys the GMDB rider for the protection of the downside risk. However, this result is in fact not surprising because the severe adverse scenarios occur when the policyholder survives to maturity and the invest account performs poorly throughout the term. Under such scenarios, the policyholder's position is dominated by $e^{-rT}F_T$ with the GMDB rider. In contrast, the policyholder's position is given by $F_0 S_T/S_0$. The former is expected to be lower than the latter, due to the deduction of rider charges. In other words, the policyholder cannot take advantage of the death benefit but loses the value of rider charges.

Table 7
LEFT-TAIL RISK MEASURES OF POLICYHOLDER'S POSITION BY RISK SHARING

p	$\mathrm{VaR_p}(\boldsymbol{X_0})$	$\mathrm{VaR_p}(\boldsymbol{X_1})$	$\mathrm{TVaR_p}(\boldsymbol{X_0})$	$\mathrm{TVaR_p}(\boldsymbol{X_1})$
0.975	102.5	63.5	74.5	46.0
0.95	139.2	87.5	98.1	61.0
0.9	197.1	125.4	133.7	84.0
0.8	298.6	194.2	191.2	122.0

The real benefit of the GMDB rider is materialized when the policyholder dies before maturity. Therefore, we can look at the policyholder's financial position given that the death occurs during the term of the rider, that is, the distributions of X_0 and X_1 given that $T_x < T$. These density functions are more informative of the advantages from the GMDB than those in Figure 13 because it filters out cases where the policyholder survives to maturity. It is clear from Figure 14 that $X_1 | T_x < \mathrm{T}$ has thinner left tail and right tail than $X_0 | T_x < T$. Tables 8 and 9 show the increase of left-tail quantiles with the GMDB rider. As expected, the policyholder under the protection of the GMDB rider is more likely to receive higher financial returns than otherwise in the case of death prior to maturity.

Figure 14

PROBABILITY DENSITY FUNCTIONS OF X_1 AND X_2 GIVEN THAT $T_x < T$

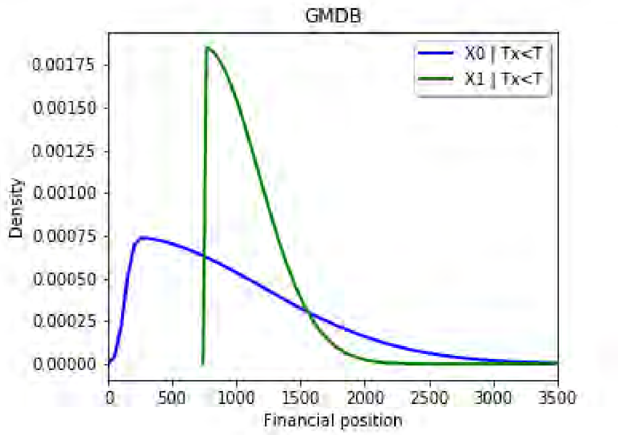


Table 8

INCREASE IN VALUE-AT-RISK OF POLICYHOLDER'S POSITION BY RISK SHARING

p	$\mathrm{VaR}_p(X_0 \vert T_x < T)$	$\mathrm{VaR}_p(X_1 \vert T_x < T)$	Increase
0.025	125.3	740.8	491.22%
0.05	153.3	740.8	383.24%
0.1	246.8	740.8	200.16%
0.2	377.2	763.4	102.39%

Table 9

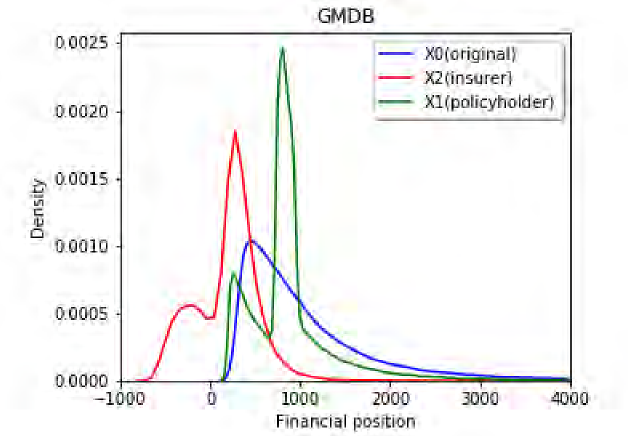
INCREASE IN TAIL-VALUE-AT-RISK OF POLICYHOLDER'S POSITION BY RISK SHARING

p	$\mathrm{TVaR}_p(X_0 \vert T_x < T$	$\mathrm{TVaR}_p(X_1 \vert T_x < T$	Increase
0.025	98.7	740.8	650.56%
0.05	120.9	740.8	512.74%
0.1	154.0	740.8	381.04%
0.2	231.8	751.0	223.99%

To show the effect of mortality risk, we plot the same set of risk profiles in Figure 15 when the contract is issued to a life of age 75. Keep in mind that it is much more likely for a 75-year-old to make a benefit claim than a 60-year-old. Observe that the difference between her financial position with and without the GMDB rider in Figure 13 is smaller than that in Figure 15, which suggests the improvement of her risk profile with the investment guarantee is less significant than that without the investment guarantee. The insurer's probability distribution of profit/loss in Figure 13 is also more concentrated in Figure 15 due to the fact that most 60-year-olds would survive to maturity and do not make benefit claims. Note also that the density functions of both the policyholder's and the insurer's positions are bimodal in the case of 75-year-olds in Figure 15 whereas they appear more like unimodal in the case of 60-year-olds in Figure 13. The bimodal shape results from the fact that the 75-year-old cohort breaks into two significant groups, one of which receive benefits and the other of which does not. But most 60-year-olds survive to maturity without any claim, which explains the unimodal nature of financial positions.

Figure 15

EFFECT OF RISK TRANSFER AT THE CONTRACT LEVEL



5.3.2 RISK REDUCTION AT AGGREGATE LEVEL

In a similar manner to the previous section, we investigate the effect of risk reduction by diversification. It is also known in Feng and Shimizu (2016) that when the insurer sells the identical contract to a large number of policyholders, the limit of its average liability can be represented by

$$\tilde{E}[X_2|\mathcal{F}_T] = \sum_{k=0}^{T-1} e^{-rk}\, m_d\; {}_k p_x F_k - \sum_{k=1}^{T} e^{-rk}\; {}_{k-1}p_x q_{x+k-1}(G - F_k)_+. \tag{22}$$

Starting from equation (21), K_x can take any value among $k = 1, \dots, T$ with probability $P(K_x = k) =_{k-1} p_x q_{x+k-1}$. By averaging the value in equation (21) with reference to $\mathrm{K_x}$ as well as the probability, the mortality risk can be fully diversified and we get equation (22).

It is easy to show by comparing equations (21) and (22) that the mortality risk is fully diversified and the equity risk remains. However, it is not to say that the diversified mortality risk has no role in the riskiness of the contract. Instead, the equity risk interacts with the mortality risk as the mortality risk imposes a temporal functional form of the accumulation of equity risk.

Figures 16 and 17 represent the probability density function and the survival function of the insurer's net liability prior to and post diversification. It is clear in Figure 16 that the insurer's liability is more concentrated after the diversification (yellow lines) than it is before the procedure (red lines). One can also observe in Figure 17 that the survival function of the insurer's liability after diversification lies below that of the insurer's liability before diversification after the two intersect. It indicates that probabilities of extremely large loss or large profit are both smaller after diversification. Both graphs confirm that the insurer can effectively reduce tail risk by pooling policies.

Figure 16

RISK REDUCTION AT AGGREGATE LEVEL

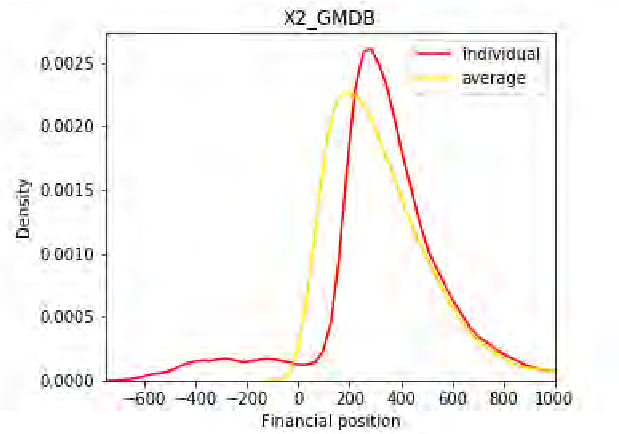
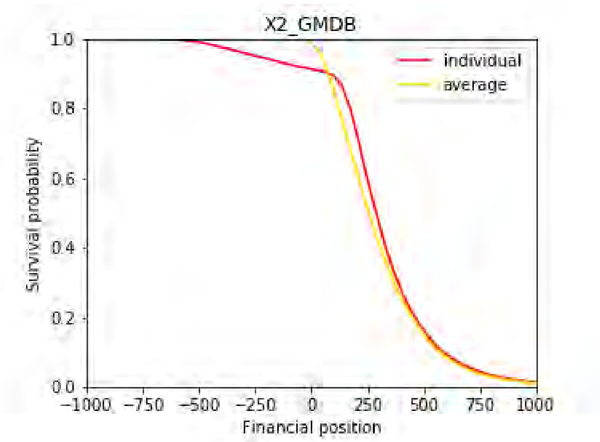


Figure 17

RISK REDUCTION AT AGGREGATE LEVEL



We can assess the extent to which tail distributions have changed by looking at risk measures in Table 10. Both results show that the diversification at the aggregate level can reduce the tail risk of the insurer's net liability by 106% to 136% at extreme quantile levels (e.g., $p = 0.025, 0.05$). Note that the reduction of riskiness in the case of GMDB by risk measure is more significant than that in the case of GMMB. This is likely due to the fact that mortality risk interacts with equity risk in all policy years during the term of the GMDB rider whereas the mortality risk only affects equity risk at maturity with the GMMB rider.

Table 10

CHANGES IN RISK MEASURES OF INSURER'S NET PROFIT BY DIVERSIFICATION

p	$\mathrm{VaR_p}(\boldsymbol{X_2})$	$\mathrm{VaR}_{\boldsymbol{p}}(\boldsymbol{E}[\boldsymbol{X_2}\vert\boldsymbol{\mathcal{F}_T}])$	Change	$\mathrm{TvaR_p}(\boldsymbol{X_2})$	$\mathrm{TvaR}_{\boldsymbol{p}}(\boldsymbol{E}[\boldsymbol{X_2}\vert\boldsymbol{\mathcal{F}_T}])$	Change
0.025	–364.8	50.7	113.90%	–460.5	28.8	106.25%
0.05	–204.7	74.2	136.25%	–373.2	45.8	112.27%
0.1	118.0	105.5	–10.59%	–209.3	68.0	132.49%
0.2	211.5	151.2	–28.51%	–14.4	98.2	781.94%

The remaining aggregate risk is undiversifiable and hence is left for other risk management techniques to deal with.

5.4 GUARANTEED MINIMUM WITHDRAWAL BENEFIT

The GMWB is a type of living benefit that offers income payments on a periodic basis.

5.4.1 RISK RETENTION AND RISK SHARING AT CONTRACTUAL LEVEL

To understand the split of equity risk between the policyholder and the insurer, we consider an investor's financial positions with and without the investment guarantee.

1. **Without investment guarantee.** If the investor chooses to invest in a stock on his own and draw down its principle at the rate of w each year, then his account value grows in the same fashion. For $k = 0, 1, \cdots$,

$$\widehat{F_{k+1}} = \max\left\{\widehat{F_k}\frac{S_{k+1}}{S_k} - w, 0\right\},$$

with the initial investment $\widehat{F_0} = F_0$. Note that the investor's account may run out of money before he can recoup his initial investment by time $T = F_0/w$. To facilitate the discussion, we consider the ruin time

$$\hat{\tau} = min\,\{t|\,\widehat{F_t}\ = 0\}.$$

Therefore, the investor's financial position without any investment guarantee is made up of his guaranteed annual withdrawals and the value of the remaining balance at the earlier of maturity and the time of death:

$$X_0 = \sum_{k=1}^{(\hat{\tau}-1)\wedge K_x} e^{-rk}w + e^{-r(T\wedge K_x)}\widehat{F_{T\wedge K_x}}.$$

2. **With investment guarantee.** As shown in an earlier section, the GMWB guarantees that the policyholder can take withdrawals until his initial investment is fully refunded regardless of the equity performance. In other words, even if the account is exhausted before T the policyholder is still entitled to annual withdrawal of amount w. The rider offers some protection on the downside risk of the investment and in exchange the investor gives up some portion of his financial returns as fees. The rider expires at time T. In this case, the account value grows in a different manner than the previous case. For $k = 0, 1, \cdots$,

$$\boldsymbol{F_{k+1} = \max\left\{F_k\left(\frac{S_{k+1}}{S_k} - m\right) - w, 0\right\}.}$$

The policyholder is expected to receive two sources of incomes, namely the guaranteed stream of withdrawals and the balance of his account at the end of the guaranteed period, $\boldsymbol{T \wedge K_x}$. Therefore, the present value of the policyholder's income with investment guarantee is given by

$$X_1 = \sum_{k=1}^{T\wedge K_x} e^{-rk}\,w + e^{-r(T\wedge K_x)}F_{T\wedge K_x}.$$

The insurer is responsible for the policyholder's withdrawals and also loses future fee incomes if his account runs out of money prior to the end of the guaranteed period, $T \wedge K_x$. In other words, the insurer is only compensated by fee incomes before $\tau \wedge T$.

$$\boldsymbol{X_2 = \sum_{k=0}^{(\tau-1)\wedge(T-1)\wedge K_x} e^{-rk}\,m_w\,F_k - \sum_{k=\tau}^{(\tau-1)\vee((T-1)\wedge K_x)} e^{-rk}\,w,}$$

where τ is the ruin time,

$$\tau = \min\{t: F_t = 0\}.$$

Keep in mind that all fees are collected at the beginning of each period.

We consider a numerical example to see how the risk is split between the policyholder and the insurer. The model parameters are provided in the appendix. Because most policyholders survive to maturity, we consider the simple case without the consideration of mortality risk.

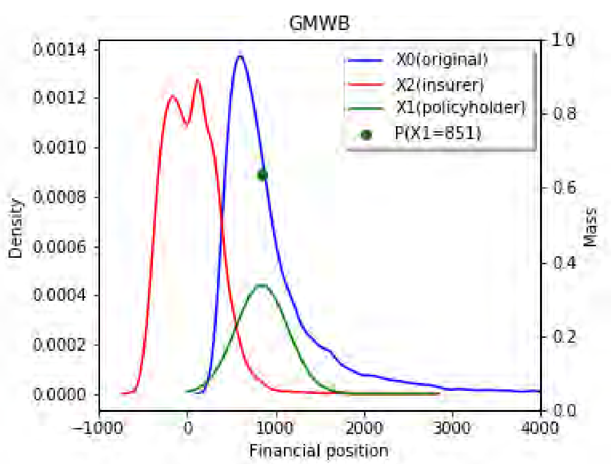
Table 11 shows that the policyholder retains about 94% of the risk-neutral value of his initial investment while ceding 6% to the insurer. In exchange, the insurer provides the guarantee on systemic withdrawals.

Table 11
VALUE DECOMPOSITION BY CONTRACTUAL DESIGN

	$\widetilde{E}[X_0]$	$\widetilde{E}[X_1]$	$\widetilde{E}[X_2]$
Risk-neutral value	1,000	940	60

Figure 18 shows probability distributions of the insurer's financial position and the policyholder's financial position with and without the investment guarantee. Note that the probability distribution of the policyholder's original position without the VA contract (the blue line) has two peaks because of two cases. The first case corresponds to the situation where the principal is drawn down before time T while the other corresponds to the circumstance where the account remains sufficient by time T and there is a random amount of terminal balance. The probability distribution of the policyholder's financial position with the VA contract (the green line) has a probability mass, which corresponds to the probability that the policyholders only receives the full refund. Note that the scale on the left side of the figure shows probability density, whereas the scale on the right side indicates probability mass. The value 851 is given by the present value of the guaranteed annuity $\sum_{k=1}^{T} e^{-rk}w$. It is clear from Figure 18 that the policyholder gives up some portion of his upside financial returns in exchange for the removal of downside returns. In fact, the present value of his total returns is guaranteed to be greater than 851.

Figure 18
EFFECT OF RISK TRANSFER AT THE CONTRACT LEVEL



The extent of reduction in upside financial returns is measured by risk measures in Table 12. Keep in mind that, as in previous product designs, there is a trade-off between left-tail return and right-tail return. The policyholder gives up financial returns in between 25% and 40% of upper quantiles in exchange for improvement in lower quantiles.

Table 12

REDUCTION IN VALUE-AT-RISK OF POLICYHOLDER'S POSITION BY RISK SHARING

p	$\mathrm{VaR_p}(\boldsymbol{X_0})$	$\mathrm{VaR_p}(\boldsymbol{X_1})$	Reduction	$\mathrm{TvaR_p}(\boldsymbol{X_0})$	$\mathrm{TvaR_p}(\boldsymbol{X_1})$	Reduction
0.975	3,142.1	1,968.2	37.4%	4,840.5	2,912.4	39.8%
0.95	2,353.8	1,527.8	35.1%	3,764.3	2,312.6	38.6%
0.9	1,723.7	1,184.0	31.3%	2,878.6	1,821.8	36.7%
0.8	1,228.0	910.4	25.9%	2,158.1	1,423.9	34.0%

We can observe the benefit of entering the GMWB rider at the left-tail quantiles in Table 13. Recall that the policyholder's financial position under the GMWB rider is bounded from below by the present value of guaranteed withdrawals 851. The policyholder's financial returns under the naked position can be significantly smaller under severe adverse scenarios. Therefore, the policyholder can be well protected under the scenarios included in the left tail of the probability distribution.

Table 13

INCREASE IN VALUE-AT-RISK OF POLICYHOLDER'S LEFT-TAIL DISTRIBUTION BY RISK SHARING

p	$\mathrm{VaR_p}(\boldsymbol{X_0})$	$\mathrm{VaR_p}(\boldsymbol{X_1})$	Increase	$\mathrm{TVaR_p}(\boldsymbol{X_0})$	$\mathrm{TVaR_p}(\boldsymbol{X_1})$	Increase
0.025	375.4	522.7	39.2%	338.1	422.6	25.0%
0.05	413.1	622.0	50.6%	366.8	495.1	35.0%
0.1	466.5	777.0	66.6%	404.1	595.3	47.3%
0.2	546.7	851.0	55.7%	455.9	717.6	57.4%

The mortality risk of the GMWB rider can be fully diversified by pooling together a large sample of policies issued to the same age cohort. As done in previous models, we can obtain the average liability per contract by

$$E[X_2|\mathcal{F}_\mathcal{T}] = \sum_{k=0}^{(\tau-1)\wedge(T-1)} e^{-rk} m_w \, {}_kp_x \, F_k - \sum_{k=\tau}^{(\tau-1)\vee(T-1)} e^{-rk} \, {}_kp_x \, w.$$

Note that negative financial positions are net losses for the insurer in the extreme cases. The diversification of mortality risk accounts for about 12% reduction in the absolute value of VaR and TVaR by the GMWB rider as shown in Table 14.

Table 14

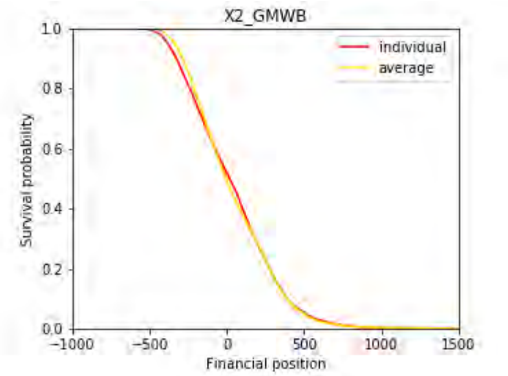
REDUCTION IN RISK MEASURES OF INSURER'S LOSS BY DIVERSIFICATION

p	$\mathrm{VaR_p}(\boldsymbol{X_2})$	$\mathrm{VaR_p}(\boldsymbol{E[X_2|\mathcal{F}_T\}})$	Reduction	$\mathrm{TVaR_p}(\boldsymbol{X_2})$	$\mathrm{TVaR_p}(\boldsymbol{E[X_2|\mathcal{F}_T\}})$	Reduction
0.025	–405.9	–358.9	11.6%	–450.9	–399.9	11.3%
0.05	–362.5	–320.6	11.6%	–417.3	–369.3	11.5%
0.1	–306.8	–269.1	12.3%	–375.1	–331.2	11.7%
0.2	–219.5	–195.0	11.2%	–318.8	–281.0	11.9%

Losses are reduced by 12%. The effect of risk control can also be observed in Figure 19. The probability density of the post-pooling average liability $E[X_2|\mathcal{F}_\mathcal{T}]$ has a lighter left tail than that of the pre-pooling individual liability X_2. In other words, it is less likely to observe severe losses after mortality risk pooling.

Figure 19

RISK REDUCTION AT AGGREGATE LEVEL



5.5 COMPARISON OF CONTRACTUAL DESIGNS

While there are many ways to compare contractual designs, we consider here a few metrics to understand their different risk profiles. Keep in mind that all three riders are offered to exactly the same age cohort.

The left tail of an insurer's liability represents the insurer's severe losses, whereas the right tail shows the insurer's profits. We compare the 90% confidence intervals of a policyholder's and an insurer's financial positions in Table 15. These values are calculated under real world measure. A policyholder has the most secured financial position with the GMWB rider among all three product designs based on the comparison of the first row. Note that X_0 is the same for GMMB and GMDB and hence the first two confidence intervals are identical. The policyholder's financial position without any guarantee X_0 is different for the GMWB because the policyholder is expected to draw down his principal gradually in the same way as taking withdrawals from the GMWB for fair comparison. Since the withdrawals are taken out of the account, the policyholder's asset is less exposed to equity risk and hence the confidence interval is narrower for the GMWB than those for GMMB and GMDB from the perspective of the policyholder. Observe that the upper bound of the confidence interval for the policyholder's position with the GMMB is the same as that with the GMDB. The reason is that the chance the policyholder dies within 10 years is rather low and hence the policyholder is most likely to end up with account value without any guarantee payment when the GMDB expires. The right tail captures favorable scenarios in which the account values outperform the guarantee base under the GMMB. Hence the GMMB and the GMDB have the same right tail. The lower bound of the confidence interval for the policyholder's position with the GMMB is higher than that with the GMDB. Again this is because the policyholder is likely to survive the 10-year period and is hence exposed to the equity risk after the GMDB expires, whereas the policyholder's account is floored by the GMMB after the 10-year period. The GMDB rider leads to narrowest confidence interval of the insurer's profit. If confidence intervals before and after risk pooling are compared, we observe that the improvement is the smallest with the GMWB. It has to do with the fact that the majority of individuals survive before the rider expires and the impact of mortality risk is rather small. The fact that the GMDB rider appears to be more profitable than the other two and the GMWB the least is that the 60-year policyholder has a relatively high probability of survival to maturity, 0.82, and hence the GMDB payments are not always needed.

Table 15

CONFIDENCE INTERVALS WITH AGE = 60

Risk metrics	GMMB		GMDB		GMWB	
90% confidence interval of policyholder's position without guarantee	139.2	2,924.8	139.2	2,924.8	413.1	2,353.8
90% confidence interval of policyholder's position with guarantee	384.8	1,824.1	87.5	1,824.1	622.0	1,527.8
90% confidence interval of insurer's position before mortality risk pooling	–443.2	758.8	–204.7	760.2	–362.5	536.3
90% confidence interval of insurer's position after mortality risk pooling	–349.7	724.3	74.2	727.4	–320.6	517.2

Table 16 offers a breakdown of the variance of the insurer's liability for each product type. As we explained earlier, the insurer's liability after the pooling of mortality risk, $E[X_2|\mathcal{F}_\mathcal{T}]$, is less than or equal to that prior to the pooling, X_2, and hence the variance of the former is clearly smaller than the variance of the latter. It is also well known that $E[X_2|\mathcal{F}_\mathcal{T}]$ is uncorrelated with $X_2 - E[X_2|\mathcal{F}_\mathcal{T}]$. The mortality risk is completely diversified in $E[X_2|\mathcal{F}_\mathcal{T}]$. Therefore, it is natural to separate the variances, which are attributable to different sources. It is clear from Table 16 that mortality risk contributes the most to the variability of the GMDB liability and the least to that of the GMWB liability.

Table 16

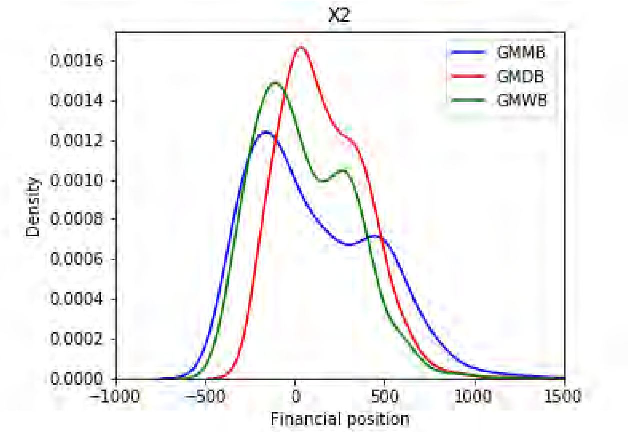
RISK ANALYSIS FOR DIFFERENT GUARANTEED-BENEFIT

Product type	GMMB	GMDB	GMWB
Variance of insurer's liability $(\mathrm{VaR}(X_2))$	153,417	76,330	84,607
attributable to equity risk $(\mathrm{VaR}(E[X_2|\mathcal{F}_\mathcal{T}])$	126,960 (82.75%)	48,823 (63.97%)	75,068 (88.75%)
attributable to mortality risk $\mathrm{Va}R(X_2 - E[X_2|\mathcal{F}_\mathcal{T}])$	26,457 (17.25%)	27,498 (36.03%)	9,539 (11.17%)

Figure 20 shows the full picture of their risk profiles. Note that the GMDB rider has the most concentrated probability density function, the GMWB has the heaviest right tail and the GMMB rider has the heaviest left tail. In other words, the GMDB tends to generate the most stable incomes for the insurer. The GMWB is less likely to suffer severe losses than the GMMB and more likely to generate large profit. The bimodal shape of distribution for GMMB and GMDB suggests the financial condition is likely to swing from profit to loss or vice versa.

Figure 20

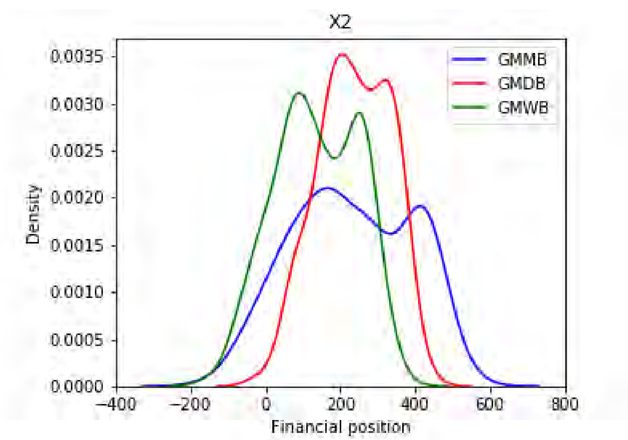
COMPARISON OF CONTRACTUAL DESIGNS AT THE AGGREGATE LEVEL ($\sigma = 0.3$)



One should keep in mind that the comparison should be done in relative terms. The fact that the density functions in Figure 20 have a relatively wide range is due to the volatility coefficient $\sigma = 0.3$. The general shape of the density functions stays the same as the volatility coefficient is reduced to $\sigma = 0.1$, as shown in Figure 21. Comparing Figures 20 and 21, we observe that the density functions become more concentrated and skewed toward the right in all cases. While all contractual designs are profitable with high probability, an insurer may need to pay particular attention to the left tails of liabilities.

Figure 21

COMPARISON OF CONTRACTUAL DESIGNS AT THE AGGREGATE LEVEL ($\sigma = 0.1$)



Another approach to compare contractual designs is to investigate their sensitivities with respect to changes in volatility of equity risk. For example, we can increase the volatility coefficient of the equity model and observe how the change affects risk measures of tail probabilities for all contractual designs.

Table 17 shows the left-tail TVaR at the 5% quantile level with various volatilities for the age cohort (60). The numbers in parentheses underneath each risk measure indicate the percentage changes from the base case of $\sigma = 0.2$. In terms of changes from the base case, the GMDB rider is most sensitive to changes in equity volatility, followed by the GMMB rider and the GMWB rider. It has to do with the fact that the GMDB product is in general more concentrated on its profitability than the other two designs. The 5% left tail represents rather extreme cases for the GMDB rider and hence is more sensitive to changes in volatility.

Table 17

LEFT-TAIL $\text{TVAR}_{0.05}(X_2|\mathcal{F}_{\mathcal{T}})$ UNDER DIFFERENT VOLATILITIES FOR LIFE-AGE 60

σ	0.1	0.15	0.2	0.25	0.3
GMMB	−69.5 (74.02%)	−181.7 (32.07%)	−267.5	−332.9 (−24.45%)	−382.7 (−43.07%)
GMDB	214.8 (87.59%)	159.9 (39.68%	114.5	76.9 (−32.84%)	45.9 (−59.96%)
GMWB	−86.8 (65.77%)	−177.7 (29.93%	−253.6	−316.9 (−24.96%)	−369.3 (−45.62%)

Table 18 shows the same risk measure for the age cohort (75). It is interesting to note that in this case, the GMMB rider tends to profit more or lose less than the other two riders. It also means it is more sensitive to changes in volatility than others.

Table 18

LEFT-TAIL $\text{TVaR}_{0.05}(X_2|\mathcal{F}_{\mathcal{T}})$ UNDER DIFFERENT VOLATILITIES FOR LIFE-AGE 75

σ	0.1	0.15	0.2	0.25	0.3
GMMB	56.2 (189.494%)	−10.7 (82.96%)	−62.8	−103.4 (−64.65%)	−135.1 (−115.134%)
GMDB	39.7 (132.48%)	−43.3 (64.57%)	−112.2	−169.1 (−38.38%)	−215.9 (−76.68%)
GMWB	1.3 (101.05%)	−64.8 (46.69%)	−121.4	−170.9 (−40.74%)	−213.5 (−75.79%)

5.6 IMPACT OF STOCHASTIC VOLATILITY AND JUMPS

We can extend the original Black-Scholes model to more sophisticated models such as a stochastic volatility jump model. According to Andersen, Benzoni and Lund (2002), the equity process is driven by the following dynamics,

$$\frac{dS_t}{S_t} = (\mu - \lambda(t)\bar{\kappa})dt + \sqrt{V_t}dW_{1,t} + \kappa_t dq_t$$
$$dV_t = (\alpha - \beta V_t)dt + \eta\sqrt{V_t}dW_{2,t}$$

where

1. (W_1, W_2) is a Brownian motion with correlations ρ.
2. q is a Poisson process with the intensity rate $\lambda(t)$, which is independent of W_1 and W_2, and, that is, $P(dq_t = 1) = \lambda(t)dt$.
3. The intensity rate is given by $\lambda(t) = \lambda_0 + \lambda_1 V_t$.
4. The magnitude of the jump κ_t is assumed to be log-normally distributed,
$$\text{Ln}(1+\kappa_t) \sim N\left(\text{Ln}(1+\bar{\kappa}) - \frac{1}{2}\delta^2, \delta^2\right),$$
and $\bar{\kappa}$ represents the average size of jump.
5. V represents the volatility or instantaneous variance. The starting point of volatility V_0 is given by its long term mean α/β.

The model parameters used in the following numerical examples can be found in Table 27 in Appendix B, which are estimated using data on S&P 500 from Jan. 2, 1953, to Dec. 31, 1996, in Andersen, Benzoni and Lund (2002).

It is worth pointing out that, when the average size of jump is set at $\bar{\kappa} = 0$ and the jump intensity is set at $\lambda(t) = 0$, the volatility becomes constant and the stochastic volatility jump model reduces to the BS model, which has constant volatility and no jump.

Figure 22(a) shows two sample paths generated from the BS and SVJ models. The sample path generated from the SVJ model is visibly different from that of the BS model due to the presence of jumps. Figure 22(b) shows sample paths of the corresponding volatility processes. The constant volatility in the BS model appears to dominate stochastic volatilities in the SVJ model.

Jumps can be observed in Figure 22(a). The probability densities corresponding to jump models do not have any jump because the financial positions are still continuous random variables.

Figure 22(a)
STOCK PRICE FOR BS AND SVJ MODEL

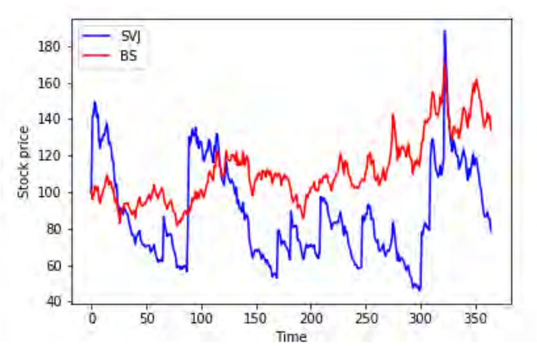
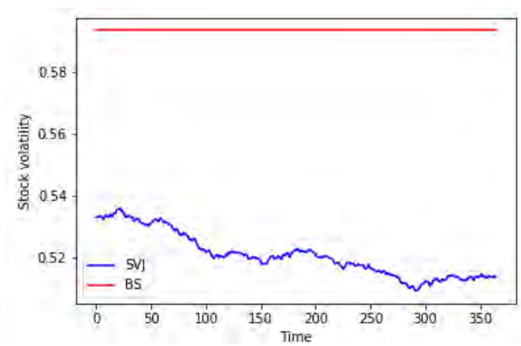


Figure 22(b)
VOLATILITY FOR BS AND SVJ MODEL



We present in Figures 23 to 25 the impact of stochastic volatility and jumps on the insurer's GMMB, GMDB and GMWB net liabilities respectively. For better visual contrast, we always show the density functions in the BS model with cool colors and those in the SVJ model with warm colors. In the case of the GMMB liability, the density functions in the SVJ model tend to shift left from those in the BS model, indicating that the product is less profitable in general when stochastic volatility and jumps are considered. We know that volatility and jump can increase the probability of extreme results. However, it is interesting to observe that the GMMB average liability in the SVJ model is more concentrated than that in the BS model. This is likely due to the fact that the probability of paying out maturity benefits is increased in the SVJ model, which leads to a higher chance of making a loss.

Figure 23

COMPARISON OF BS AND SVJ FOR INSURER'S FINANCIAL POSITION IN GMMB

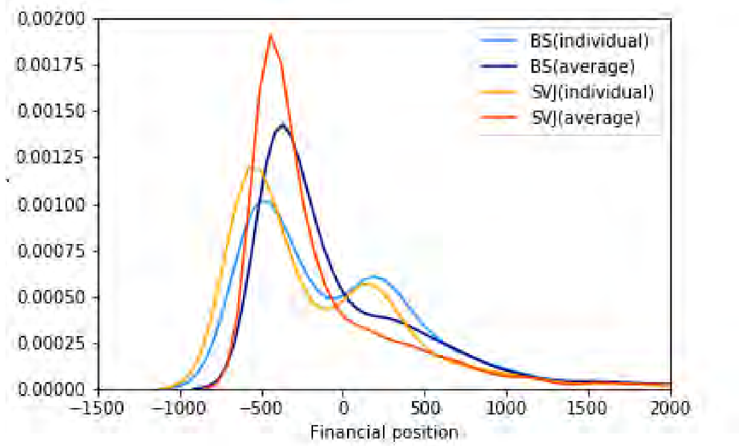


Figure 24

COMPARISON OF BS AND SVJ FOR INSURER'S FINANCIAL POSITION IN GMDB

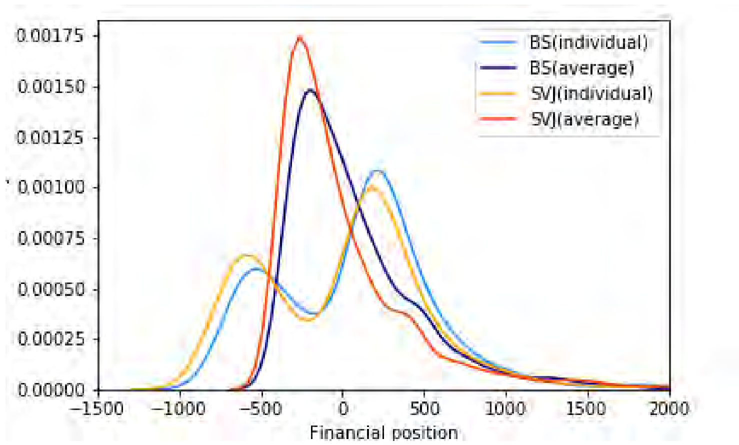
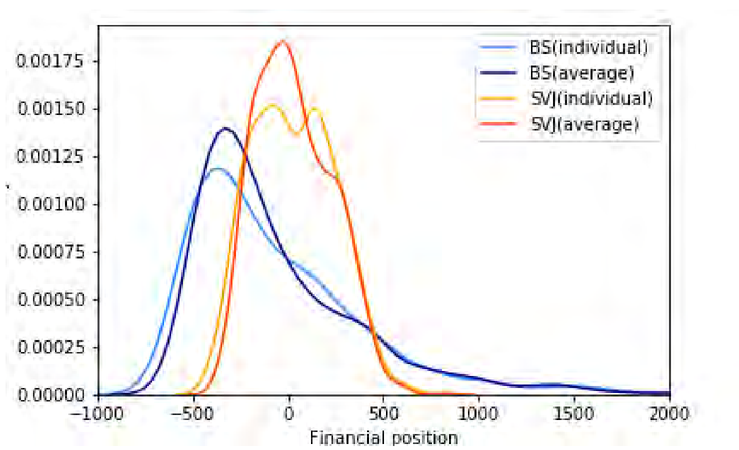


Figure 25

COMPARISON OF BS AND SVJ FOR INSURER'S FINANCIAL POSITION IN GMWB



It should also be pointed out here that these density functions are more spread out than those in Figure 6. This is because the model parameters in this section are estimated from empirical data whereas those in previous sections are set by authors. The volatility estimate from the empirical data in Andersen, Benzoni and Lund (2002) is close to 0.6, which appears to be higher than what is typically used in the insurance industry.

Stochastic volatility and jumps have a similar effect on GMDB net liabilities, including both individual and average liabilities. However, they have the opposite effect on GMWB liabilities. This is likely due to the fact that stochastic volatility and jump increase the chance and the duration of policyholder's account staying positive and hence reduces the amount of guaranteed withdrawal payments from the insurer's general account.

Section 6: Inflation-linked Insurance

Many insurance products offer inflation protection features, under which benefit payoffs increase periodically to keep up with inflation. Inflation protection features are common for products that offer benefits in the form of annuities, such as a cost-of-living adjustment on life annuity and disability insurance. Because these products are typically bought many years before the first annuity payment is drawn, the cost of living or care may have far exceeded its original estimate. Therefore, inflation protection is critical for policyholders to retain the real value of its insurance protection. While these features are considered desirable for policyholders, they can be difficult to manage for insurers.

6.1 COMMON INFLATION PROTECTION DESIGNS

Here we consider a variety of inflation indexation features that are used in practice.

- **Fixed percentage indexation.** The design offers fixed percentage increase of the benefit on an annual basis. For a 55-year-old, a $200 per day benefit with 5% compound inflation protection will be worth $677 per day at age 80. There is no inflation risk for the insurer, as the growth of benefit is pegged to a fixed rate rather than the actual inflation in the market. The fixed 3% compounded percentage is the most popular inflation protection option elected by policyholders. Suppose that the initial benefit is given by B_0. We denote the benefits in subsequent years by $\{B_k, k = 1,2,\cdots\}$. Let ρ be the fixed percentage increase per annum. Then the benefits can represented by

$$B_k = (1 + \rho)B_{k-1} = (1 + \rho)^k B_0.$$

- **CPI indexation or a percentage of the CPI increase.** With this option, the inflation increase is tied to the annual increase of the consumer price index. The CPI averaged about 2.69% between 1982 and 2019. Let us denote the CPI at time k by I_k. Then the benefits are adjusted according to changes in the CPI,

$$B_k = B_{k-1}\frac{I_k}{I_{k-1}} = B_0\frac{I_k}{I_0}.$$

  Such a product design exposes the insurer to inflation risk. One way to hedge against the inflation risk embedded in such a product design is to use inflation-linked securities. For example, Treasury inflation-protected securities, or TIPS, are linked to the CPI and the principal amount appreciates in accordance with changes in the index. However, such government-issued securities may be either unavailable or illiquid in other countries. Another option is to invest in inflation-related assets such as real estate investment trusts, or REIT, which invests in commercial, residential and industrial real estate. However, the changes in real estate may not be exactly in sync with those in the CPI, which creates basis risk for the insurer.

- **CPI indexation with a floor.** This design offers a floor on the CPI indexation, which can be viewed as offering the policyholder a call option on the CPI. Suppose the floor is set at f. Then the benefit rises according to

$$B_k = B_{k-1} \max\left\{1+f, \frac{I_k}{I_{k-1}}\right\} = B_0 \prod_{j=1}^{k} \max\left\{1+f, \frac{I_j}{I_{j-1}}\right\}.$$

- **CPI indexation with a cap.** This design imposes a cap on the CPI indexation. Suppose that the cap is set at c. Then the benefit rises according to

$$B_k = B_{k-1} \min\left\{1+c, \frac{I_k}{I_{k-1}}\right\} = B_0 \prod_{j=1}^{k} \min\left\{1+c, \frac{I_j}{I_{j-1}}\right\}.$$

- **CPI indexation with a cap and carry forward.** This design sits somewhere between the full CPI indexation and the capped one. While the indexation cannot exceed the cap, the excess is not forfeited but rather carried forward year by year. The year-to-year increase on CPI is always credited to the policyholder but with some time lag due to the yearly cap. Table 19 shows how the indexation works with a 3% cap and carry forward.

Table 19
EXAMPLE OF CPI INDEXATION WITH A CAP AND CARRY FORWARD

Year	CPI increase	Indexation	Cap & forward
1	2%	2%	Less than cap
2	5%	3%	2% carry forward
3	4%	3%	1% + 2% carry forward
4	1%	3%	1% carry forward
5	0%	1%	Less than cap; credit clear

We can formulate this product design as follows. Let a_k be the carry-forward credit and define the quantity by the following recursion:

$$a_k = \left(a_{k-1} + \frac{I_k}{I_{k-1}} - (1+c)\right)^+.$$

The benefit changes according to

$$B_k = B_{k-1}\left(a_{k-1} - a_k + \frac{I_k}{I_{k-1}}\right).$$

Section 7: Risk Management of Inflation-linked Insurance

In this section, we consider specifically one of the four risk management techniques—risk transfer—in the context of inflation-linked insurance. In particular, we are interested in how the combination of inflation risk and mortality risk can be transferred to the capital market by developing a hedging program using inflation-linked financial instruments.

7.1 RISK MANAGEMENT OF INFLATION-ADJUSTED ANNUITY

Consider an annuity contract that offers annual payments starting from an advanced age of the policyholder and lasting for the rest of his lifetime. Such an annuity arrangement is common for pension payout. The contract also offers various options for inflation protection on the annuity payments.

Suppose a policyholder is at age x when entering the contract and the first annuity payment starts at age $x+T$. To formulate the insurer's liability, we consider the random variable representing biometric risk, the curtate future lifetime, denoted by K_x.

Consider the insurer's liability, which is the collection of all inflation-protected benefits. The present value of the insurer's liability at the inception of the contract is hence given by

$$L := \sum_{k=0}^{K_x \vee (T-1)-T} e^{-r(T+k)} B_k,$$

where r is the average rate of return on assets backing up the liability. For example, say the benefit is available after $T=10$ years and the policyholder dies after 16 years and five months. Then $K_x=16$ and $K_x \vee (T-1)=16$. There are a total of $K_x \vee (T-1)-T=6$ payments in total if $K_x^r < K^x \wedge T$. If the policyholder dies after eight years, which is before he is eligible for annuity payments, then $K_x \vee (T-1)-1=-1$ and the summation yields 0 by convention that $\sum_{k=0}^{-1} \quad = 0$ and the insurer has no liability payment.

In general, we consider mortality risk to be diversifiable. When the pool of policyholders in the same cohort is large enough, the law of large numbers implies there is a fixed percentage of survivors at each time point. Let $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$ be the natural filtration generated by the process underlying the dynamics of inflation rates. In other words, $\mathcal{F}$ represents the accumulation of information with regard to inflation risk.

For simplicity, we first consider the conditional expectation of the insurer's liability at the time of first benefit payment given the filtration

$$\begin{aligned} E[L|\mathcal{F}_T] &:= E\left[\sum_{k=0}^{\infty} e^{-r(T+k)} B_k I(K_x > T-1, K_x - T \geq k) \middle| \mathcal{F}_T\right] \\ &= \sum_{k=0}^{T} e^{-r(T+k)} \quad {}_T p_x \quad {}_k p_{x+T} B_k, \end{aligned}$$

where $\quad {}_T p_x \quad {}_k p_{x+T} B_k$ represents the probability that the life-age (x) survives k additional years after reaching age $x+T$. Note that the only source of randomness is the dynamics of inflation risk $\{I_0, I_1, \cdots, I_T\}$.

7.2 CPI MODEL FRAMEWORK

As a critical component of inflation protection features, inflation risk is modeled by a well-studied stochastic process, which was introduced by Jarrow and Yildirim (2003). For convenience, we introduce the following notations to be used in the rest of this section.

- $\boldsymbol{r(t)}$**.** Real short rate at t.
- $\boldsymbol{n(t)}$**.** Nominal rate at t.
- $\boldsymbol{f_n(t,T)}$**.** Nominal forward rate at t for date T.
- $\boldsymbol{f_r(t,T)}$**.** Real forward rate at t for date T.
- $\boldsymbol{I_t}$**.** The consumer purchase index at t.
- $\boldsymbol{P_n(t,T)}$**.** Price of nominal zero-coupon bond at t with maturity at T in dollars.
- $\boldsymbol{P_r(t,T)}$**.** Price of real zero-coupon bond at t with maturity at T in CPI units.
- $\boldsymbol{B_n(t)}$**.** Time t money market account value in dollars.
- $\boldsymbol{B_r(t)}$**.** Time t money market account value in CPI units.

- $\boldsymbol{V_0^i}$. Price of the product at 0 with maturity at i.
- $\boldsymbol{V_0}$. Total price of the product at 0 with maturity at i for $i = T_1, \dots, T_k$.

Because the model considers term structure of interest rates, a key concept is the forward rate with index (t, T), which is an interest rate specified at t to be used at time T. The price of a nominal zero-coupon bond at time t with maturity time T and that of a real zero-coupon bond are given by

$$\mathrm{P_n(t,T)} = \exp\left\{-\int_{\mathrm{t}}^{\mathrm{T}} \mathrm{f_n(t,s)ds}\right\}, \tag{23a}$$

$$P_r(t,T) = \exp\left\{-\int_t^T f_r(t,s)ds\right\}. \tag{23b}$$

The real spot rate is the interest rate applied immediately to investment in CPI units, that is, $r(t) = f_r(t,t)$. Similarly, the nominal spot rate is the interest rate applied immediately to investment in dollars, that is, $n(t) = f_n(t,t)$. Hence, the money market account grows with spot rates,

$$B_r(t) = \exp\left(\int_0^t r(u)du\right),$$
$$B_n(t) = \exp\left(\int_0^t n(u)du\right).$$

In the Jarrow and Yildirim model, it is assumed that under the real world measure, the dynamics of forward rates and CPI are given by

$$df_n(t,T) = \mu_n(t,T)dt + \sigma_n(t,T)dW_n^P(t), \tag{24a}$$
$$df_r(t,T) = \mu_r(t,T)dt + \sigma_r(t,T)dW_r^P(t), \tag{24b}$$
$$dI(t) = I(t)\mu_I(t)dt + I(t)\sigma_I dW_I^P(t), \tag{24c}$$

with the boundary conditions $I(0) = I_0 > 0$ and

$$f_n(0,T) = f_n^M(0,T),$$
$$f_r(0,T) = f_r^M(0,T),$$

where

1. (W_n^P, W_r^P, W_I^P) is a Brownian motion with correlations $\rho_{n,r}$, $\rho_{n,I}$ and $\rho_{r,I}$.
2. μ_n, μ_r and μ_I are adapted processes.
3. σ_n and σ_r are deterministic functions; σ_I is positive constant.
4. $f_n^M(0,T)$ and $f_r^M(0,T)$ are nominal and real forward rates observed in the market at time 0 for maturity T.

It is known that the financial market would be complete and arbitrage free if there exists a unique equivalent probability measure Q such that

$$\frac{P_n(t,T)}{B_n(t)}, \frac{I(t)P_r(t,T)}{B_n(t)}, \frac{I(t)B_r(t)}{B_n(t)}$$

are Q-martingales. It means that all assets discounted in respective units should keep the same risk-neutral value at all times. This is consistent with the assumption of no arbitrage in the market.

It is shown in Jarrow and Yildirim (2003) that the dynamics under the martingale measure are given by

$$df_n(t,T) = \sigma_n(t,T)\left(\int_t^T \sigma_n(t,s)ds\right)dt + \sigma_n(t,T)d\widetilde{W_n^P}(t), \tag{25a}$$

$$df_r(t,T) = \sigma_r(t,T)\left(\int_t^T \sigma_r(t,s)ds - \rho_{rI}\sigma_I(t)\right)dt + \sigma_r(t,T)d\widetilde{W_r^P}(t), \tag{25b}$$

$$dI(t) = I(t)[n(t) - r(t)]dt + I(t)\sigma_I d\widetilde{W_I}^P(t). \tag{25c}$$

If we further assume that the volatility functions $\sigma_n(t,T)$ and $\sigma_r(t,T)$ are given by

$$\sigma_n(t,T) = \sigma_n e^{-\theta_n(T-t)},$$
$$\sigma_r(t,T) = \sigma_r e^{-\theta_r(T-t)},$$

Then stochastic processes in equation (25) reduce to the model framework that depicts the evolution for the nominal and real short rates and for the CPI as

$$dn(t) = [\alpha_n(t) - \theta_n n(t)]dt + \sigma_n dW_n(t), \tag{26a}$$
$$dr(t) = [\alpha_r(t) - \rho_{r,I}\sigma_I\sigma_r - \theta_r r(t)]dt + \sigma_r dW_r(t), \tag{26b}$$
$$dI(t) = I(t)[n(t) - r(t)]dt + I(t)\sigma_I dW_I(t), \tag{26c}$$

where

1. (W_n, W_r, W_I) is a Brownian motion with correlations $\rho_{n,r}$, $\rho_{n,I}$ and $\rho_{r,I}$.

2. σ_n, σ_r, θ_n and θ_r are positive constant.

3. $\alpha_n(t)$ and $\alpha_r(t)$ are functions given by

$$\alpha_n(t) = \frac{\partial f_n(0,t)}{\partial T} + \theta_n f_n(0,t) + \frac{\sigma_n^2}{2\theta_n}\left(1 - e^{-2\theta_n t}\right),$$
$$\alpha_r(t) = \frac{\partial f_r(0,t)}{\partial T} + \theta_r f_r(0,t) + \frac{\sigma_r^2}{2\theta_r}\left(1 - e^{-2\theta_r t}\right).$$

Thus, under martingale measure Q, $I(t)$ is lognormally distributed and for $t < T$,

$$I(T) = I(t)\exp\left\{\int_t^T [n(s) - r(s)]ds - \frac{1}{2}\sigma_I^2(T-t) + \sigma_I\big(W_I(T) - W_I(t)\big)\right\}.$$

Therefore, the CPI ratio follows a lognormal distribution given by

$$\frac{I_{T_i}}{I_{T_{i-1}}} = \exp\left\{\int_{T_{i-1}}^{T_i} [n(s) - r(s)]ds - \frac{1}{2}\sigma_I^2(T_i - T_{i-1}) + \sigma_I\big(W_I(T_i) - W_I(T_{i-1})\big)\right\}. \tag{27}$$

7.3 PRICING

As far as the evolution of the CPI ratio is known, we are ready to present the pricing procedure of the inflation-adjusted annuity. We take the CPI indexation with a floor as an example.

Recall from Section 6.1, the annuity payout for each time T_i are given for $i = 0, \ldots, k$ by

$$B_{T_i} = B_0 \prod_{j=1}^{i} \max\left\{1 + f, \frac{I_{T_j}}{I_{T_{j-1}}}\right\},$$

and the time-0 no-arbitrage value of the benefit B_{T_i} is defined by $E\left(e^{-\int_0^{T_i} n(s)ds} B_{T_i}\right)$, which is calculated from the following Theorem 7.1.

Theorem 7.1. Suppose E^T denotes the expectation taken under the T-forward measure; using results from the Black-Scholes formula, the price at time 0 for benefit B_{T_i} is calculated by the following explicit formula,

$$V_0^{T_i} = P_n(0, T_i) B_0 \prod_{j=1}^{i} \left[M(0, T_{j-1}, T_j) \Phi\left(d_1^{T_{j-1}, T_j}\right) + (1 + f) \Phi\left(-d_2^{T_{j-1}, T_j}\right) \right], \tag{28}$$

where

$$d_1^{T_{j-1}, T_j} = \left(\frac{\ln \frac{M(0, T_{j-1}, T_j)}{(1+f)} + \frac{1}{2} V^2(0, T_{j-1}, T_j)}{V(0, T_{j-1}, T_j)} \right),$$

$$d_2^{T_{j-1}, T_j} = d_1^{T_{j-1}, T_j} - V(0, T_{j-1}, T_j).$$

Note that the explicit formula for $M(0, T_{j-1}, T_j)$ and $V^2(0, T_{j-1}, T_j)$ will be addressed in equations (36) and (37) in Appendix A.

Now, for $j = 1, \ldots, k$, the value at time 0 for the benefit B_{T_j} is explicitly determined by equation (28). Adding back all the time interval, we obtain the no-arbitrage value of the inflation-adjusted annuity at time 0:

$$V_0 = \sum_{i=1}^{k} V_0^{T_i} = \sum_{i=1}^{k} \{ P_n(0, T_i) B_0 \prod_{j=1}^{i} [M(0, T_{j-1}, T_j) \Phi(d_1^j) + (1 + f) \Phi(-d_2^j)] \}.$$

7.4 HEDGING

While we have developed the no-arbitrage pricing of CPI indexed annuity with a floor, we have yet to address another important problem—how to hedge against inflation risk and provide a guaranteed rate of return embedded in this product should we underwrite such a product. Here we provide details in an example of delta hedging.

7.4.1 PRODUCT DESCRIPTION

Consider a compound ratchet inflation-linked contract with the following product features:

1. The contract lasts for 10 years and all benefits are paid at the end of each year.

2. Benefit base is $1,000.

3. The inflation adjustment is set at the floor rate 2%.

4. The dynamics of CPIs are modeled by the Jarrow-Yildirim model.

5. TIPS are freely traded in the market in units of $1,000 with the coupon rate of 1%.

7.4.2 PARAMETER ESTIMATION

Readers are referred to the estimation of model parameters in Jarrow and Yildirim (2003). In this example, we take advantage of empirical studies in the literature and estimated parameters in Table 20 are taken from Table 3 of Jarrow and Yildirim (2003).

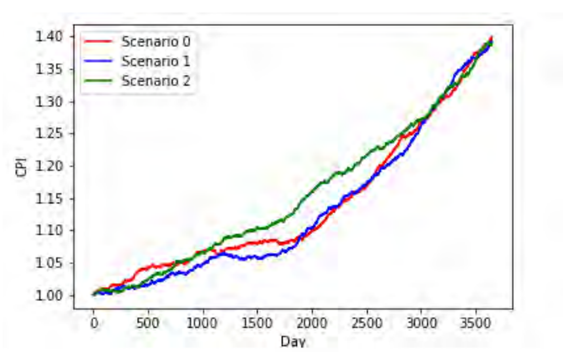
Table 20
PARAMETERS OF THE JY MODEL, TABLE 3 OF JARROW AND YILDIRIM (2003)

Item	Symbol	Value
Volatility for CPI	σ_I	0.00874
Volatility for real rates	σ_r	0.00299
	a_r	0.04339
Volatility for real rates	σ_n	0.00566
	a_n	0.03398
Correlation Coefficient	ρ_{rI}	−0.32127
	ρ_{rI}	0.06084
	ρ_{rI}	0.01482

Example of Simulation CPI

Suppose that the CPI at time 0 is 1. Figure 26 shows three different scenarios of daily CPI trajectories. Observe that the CPI in the Jarrow-Yildirim model exhibits an increasing trend. The 40% increase in the CPI after 10 years shows the importance of the inflation protection with the annuity product.

Figure 26
DAILY CPI TREND FOR 10 YEARS



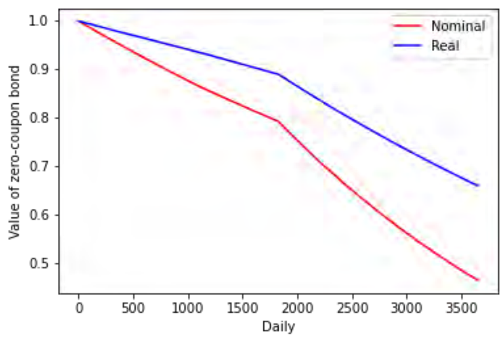
Calculation of $P_n(0,t)$ and $P_r(0,t)$

Suppose the nominal and real forward rate are given by Table 21. $P_n(0,t)$ and $P_r(0,t)$ can be determined by equation (23) and Figure 27 shows the corresponding daily values of zero-coupon bond.

Table 21
NOMINAL AND REAL FORWARD RATES

Item	$T \leq 3$	$3 < T \leq 5$	$5 < T \leq 10$
$f_n(0,T)$	0.048	0.044	0.107
$f_r(0,T)$	0.022	0.025	0.06

Figure 27
VALUE OF ZERO COUPON BOND



7.4.3 HEDGING STRATEGY

By definition, the delta δ_t of a given product at any given time t is measured by the first partial derivative of the price of the product V_t with respect to the CPI I_t, that is

$$\delta_t = \frac{\partial V_t}{\partial I_t}.$$

Define a delta-hedging strategy that consists of investment in Treasury inflation-protected securities and a money market account. TIPS provides inflation-adjusted coupon payments periodically where the coupon is determined by multiplying the principle amount by the ratio of inflation between two periods and the pre-determined coupon rate. When inflation goes up, coupon payment goes up by the same proportion. For example, suppose someone invests in TIPS with a face amount of $1,000 and coupon rate of 2%, and that the current CPI is 210. If the CPI changes to 220 after one year, then the coupon amount for the first year would be

$$1{,}000 \times 0.02 \times \frac{220}{210} = 20.95.$$

The number of shares Δ_t invested in TIPS at time t is determined by the relative ratio of the inflation-linked product's delta δ_t^P and the TIPS's delta δ_t^{TIPS}. The rest of money will be invested in the money market, which is denoted as M_t. Therefore, define $\Omega_t = \{\Delta_t, M_t\}$ to be the delta-hedging portfolio at time t, the value of the replicating portfolio V_t^Δ is given by

$$V_t^\Delta = \Delta_t V_t^{\text{TIPS}} + M_t = \frac{\delta_t^P}{\delta_t^{\text{TIPS}}} V_t^{\text{TIPS}} + M_t. \quad (29)$$

7.4.4 CALCULATION OF DELTAS

Firstly, we assume without the loss of generality that $T_j = j$ for $j = 1, \ldots, k$ and that the hedging portfolio is re-balanced m times each year.

Determination of δ_t^P

By the same argument in Section 7.3, denote the rounded down integer of t as $\lfloor t \rfloor$, the value of the product V_t^P at time t for $t = \frac{1}{m}, \dots, k - \frac{1}{m}, k$ is determined by,

$$V_t^P = \sum_{i=\lfloor t \rfloor+1}^{k} F(t,i) \left[\frac{I_t}{I_{\lfloor t \rfloor}} M(0,t,\lfloor t \rfloor+1) \Phi\left(d_1^{\widetilde{t,\lfloor t \rfloor+1}}\right) + (1+f) \Phi\left(-d_2^{\widetilde{t,\lfloor t \rfloor+1}}\right) \right], \tag{30}$$

with the δ_t^P determined by

$$\delta_t^P = \sum_{i=\lfloor t \rfloor+1}^{k} F(t,i) \frac{1}{I_{\lfloor t \rfloor}} M(0,t,\lfloor t \rfloor+1) \Phi\left(d_1^{\widetilde{t,\lfloor t \rfloor+1}}\right), \tag{31}$$

where

$$F(t,i) = P_n(t,i) B_0 \prod_{j=1}^{\lfloor t \rfloor} \max\left\{1+f, \frac{I_{T_j}}{I_{T_{j-1}}}\right\} \prod_{j=\lfloor t \rfloor+1}^{i-1} \left[M(0,j,j+1)\Phi\left(d_1^{j,j+1}\right) + (1+f)\Phi\left(-d_2^{j,j+1}\right)\right], \tag{32}$$

and

$$d_1^{\widetilde{t,\lfloor t \rfloor+1}} = \left(\frac{\ln \frac{\frac{I_t}{I_{\lfloor t \rfloor}} M(0,t,\lfloor t \rfloor+1)}{(1+f)} + (1/2) V^2(0,t,\lfloor t \rfloor+1)}{V(0,t,\lfloor t \rfloor+1)} \right),$$

$$d_2^{\widetilde{t,\lfloor t \rfloor+1}} = d_1^{\widetilde{t,\lfloor t \rfloor+1}} - V(0,t,\lfloor t \rfloor+1).$$

To implement the delta hedging, we need to simulate the path for CPI and get the value of I_t for $t = \frac{1}{m}, \dots, k - \frac{1}{m}, k$. Moreover, $P_n(0,t)$ and $P_r(0,t)$ are calculated by equation (23) and Table 21. Then for each re-balanced time t, using parameters in Table 20, we can calculate each function M and V by equations (36) and (37) in Appendix A. Then d_1, d_2, $\Phi(d_1)$ and $\Phi(-d_2)$ can be found. Consequently, V_t^P in equation (30), δ_t^P in equation (31) and function $F(t,i)$ in equation (32) can be calculated.

Determination of δ_t^{TIPS}

To find the value of δ_t^{TIPS}, we need to find the value of the TIPS on each re-balanced time t. TIPS pay coupons at coupon rate c twice a year, and the face amount H can be redeemed at the end of the term. The coupon is adjusted by the ratio of CPI $\frac{I_t}{I_0}$ on each coupon payment day t. On the redemption date k, the redemption amount is also adjusted by $\frac{I_k}{I_0}$ but should be no less than the face amount $H \max\left\{1, \frac{I_k}{I_0}\right\}$. Then the value of TIPS at time 0 would be given by

$$V_0^{\text{TIPS}} = E\left(Hc \sum_{i=1}^{2k} \frac{I_{\frac{i}{2}}}{I_0} e^{-\int_0^{\frac{i}{2}} n(s)ds} + H \max\left\{1, \frac{I_k}{I_0}\right\} e^{-\int_0^k n(s)ds} \right)$$

$$= Hc \sum_{i=1}^{2k} P_n\left(0, \frac{i}{2}\right) M\left(0,0,\frac{i}{2}\right) + H P_n(0,k)\left[M(0,0,k)\Phi\left(d_1^{0,k}\right) + \Phi\left(-d_2^{0,k}\right)\right].$$

Moreover, at any time $t = \frac{1}{m}, \dots, k - \frac{1}{m}, k$, the value of TIPS would be given by

$$V_t^{\text{TIPS}} = Hc \sum_{i=\lfloor 2t \rfloor+1}^{2k} \frac{I_t}{I_0} P_n\left(t, \frac{i}{2}\right) M\left(0, t, \frac{i}{2}\right) + HP_n(t,k)\left[\frac{I_t}{I_0} M(0,t,k)\Phi\left(\widehat{d_1^{t,k}}\right) + \Phi\left(-\widehat{d_2^{t,k}}\right)\right], \tag{33}$$

with the delta δ_t^{TIPS} given by

$$\delta_t^{\text{TIPS}} = \frac{1}{I_0}\left(Hc \sum_{i=\lfloor 2t \rfloor+1}^{2k} P_n\left(t, \frac{i}{2}\right) M\left(0, t, \frac{i}{2}\right) + HP_n(t,k)M(0,t,k)\Phi\left(\widehat{d_1^{t,k}}\right)\right), \tag{34}$$

where

$$\widehat{d_1^{t,k}} = \left(\frac{\ln \frac{I_t}{I_0} M(0,t,k) + (1/2)V^2(0,t,k)}{V(0,t,k)}\right),$$

$$\widehat{d_2^{t,k}} = \widehat{d_1^{t,k}} - V(0,t,k).$$

Again, to implement the delta hedging, we need to simulate the path for CPI and get the value of I_t for $t = \frac{1}{m}, \dots, k - \frac{1}{m}, k$. Moreover, $P_n(0,t)$ and $P_r(0,t)$ are calculated by equation (23) and Table 21. Then for each re-balanced time t, using the parameters in Table 20, we can calculate each function M and V by equations (36) and (37). Then d_1, d_2, $\Phi(d_1)$ and $\Phi(-d_2)$ can be found. Consequently, V_t^{TIPS} in equation (33) and δ_t^{TIPS} in equation (34) can be calculated.

For example, suppose we have a $k = 3$ year inflation-linked annuity product and the re-balance is implemented at each quarter, that is, $m = \frac{1}{4}$. The I_t, $P_n(0,t)$, $P_r(0,t)$, $\delta_{t-\frac{1}{m}}^P$ and $\delta_{t-\frac{1}{m}}^{\text{TIPS}}$ for $t = \frac{1}{4}, \dots, 3$ are summarized in Table 22.

Table 22

CALCULATION OF δ_t^P AND δ_t^{TIPS}

Time t		I_t	$P_n(0,t)$	$P_r(0,t)$	$\delta_{t-\frac{1}{4}}^P$	$\delta_{t-\frac{1}{4}}^{\text{TIPS}}$
Year 1	$t = 0.25$	1.007	0.988	0.995	1,639.41	992.77
	$t = 0.5$	1.014	0.976	0.989	1,740.60	998.64
	$t = 0.75$	1.022	0.965	0.984	1,798.51	994.41
	$t = 1$	1.030	0.953	0.978	1,972.40	1,000.15
Year 2	$t = 1.25$	1.035	0.942	0.973	1,132.07	995.80
	$t = 1.5$	1.049	0.931	0.968	1,056.32	1,001.32
	$t = 1.75$	1.055	0.919	0.962	1,492.22	996.86
	$t = 2$	1.066	0.908	0.957	1,534.88	1,002.35
Year 3	$t = 2.25$	1.067	0.898	0.952	667.62	997.88
	$t = 2.5$	1.073	0.887	0.946	502.49	1,003.39
	$t = 2.75$	1.080	0.876	0.941	505.41	998.93
	$t = 3$	1.087	0.866	0.936	444.62	1,004.45

7.4.5 EXAMPLE OF HEDGING STRATEGY

In this section, we would discuss the implementation of the delta-hedging strategy proposed in Section 7.4.3. For any re-balanced time $t = \frac{1}{m}, \dots, k - \frac{1}{m}, k$, the dynamic hedging program involves the following transactions:

1. Hold $\frac{\delta_t^P}{\delta_t^{\text{TIPS}}}$ units of TIPS.

2. Finance the change in units of TIPS by the money market account. In other words, the money market account changes by $\left(\frac{\delta^P_{t-\frac{1}{m}}}{\delta^{\text{TIPS}}_{t-\frac{1}{m}}} - \frac{\delta^P_t}{\delta^{\text{TIPS}}_t}\right) V_t^{\text{TIPS}}$. Moreover, at each coupon date (respectively, benefit payment date), the coupon payment (respectively, benefit payment) is deposited into (respectively, paid out of) the money market account.

Before moving on to the example, we should make some assumptions throughout the section. The hedging program is carried out on a monthly basis. The initial CPI at time 0 is set to be 1, that is, $I_0 = 1$. Recall that the hedging portfolio is composed of TIPS and a money market account and that TIPS coupons are paid twice a year.

At time $t = 0$, an insurer sells the inflation-protected annuity product for the price V_0^P. The insurer invests in $\frac{\delta_0^P}{\delta_0^{\text{TIPS}}}$ units of TIPS. The rest of sales proceeds $V_0^P - \frac{\delta_0^P}{\delta_0^{\text{TIPS}}} V_0^{\text{TIPS}}$ is deposited in the money market account. At the end of each month, time t, where $t = \frac{1}{12}, \dots, 10$, the insurer re-balances the hedging portfolio by adjusting the units invested in the TIPS bond according to the difference between the ratio of delta at time $t - \frac{1}{m}$ and t, that is, $\frac{\delta^P_{t-\frac{1}{12}}}{\delta^{\text{TIPS}}_{t-\frac{1}{12}}} - \frac{\delta^P_t}{\delta^{\text{TIPS}}_t}$. The cost of adjustment is financed entirely by the money market account. Moreover, coupon payments (respectively, benefit payments) are accumulated to (respectively, paid out of) the money market account at each coupon date (respectively, benefit payment date).

For example, at time $t_1 = \frac{1}{12}, if \frac{\delta_0^P}{\delta_0^{\text{TIPS}}} = 10, \frac{\delta_{t_1}^P}{\delta_{t_1}^{\text{TIPS}}} = 8, V_{t_1}^{\text{TIPS}} = 800$. Then at time t_1, we should sell $\frac{\delta_0^P}{\delta_0^{\text{TIPS}}} - \frac{\delta_{t_1}}{\delta_{t_1}^{\text{TIPS}}} = 2$ units of TIPS and accumulate $2 \times V_{t_1}^{\text{TIPS}} = 1{,}600$ in the money market account. Since it is neither a coupon payment date nor a benefit payment date, there is no other transaction in the money account.

Determination of the Hedging Error

In theory, if the hedging program can be carried out continuously, the value of the hedging portfolio should exactly match that of the inflation-adjusted annuity. In other words, the hedging portfolio should have just enough to pay for the inflation-adjusted annuity payments throughout the term of the contract. In practice, however, such a dynamic-hedging program can only be done on a discrete basis. We can assess the effectiveness of the hedging program by investigating the discrete hedging error. The accumulated hedging error is defined by

$$\epsilon^\Delta = \frac{V_k^\Delta - V_k^P}{V_0^P}. \tag{35}$$

Keep in mind that the smaller this ratio is, the more effective the hedging strategy.

7.4.6 VISUALIZATION AND ANALYSIS

Figure 28(a) shows a particular sample of the monthly delta-hedging program. We can make a comparison of the value of the replicating portfolio (red line) and that of the actual annuity product (blue line). It is obvious from the comparison that the value of the hedging portfolio matches that of the annuity reasonably well. Both sample paths show steep drops at the end of each year due to the outgoing benefit payments. The ending values of both the annuity and the hedging portfolio are larger than $1,000. That is because of the inflation adjustment to the benefit base, which is consistent with the result shown in Figure 26. To analyze the effectiveness of the hedging program, we compute the total hedging error defined in equation (35) for each sample path generated from the Jarrow-

Yildirim model. We generate a total of 1,000 sample path of CPIs. The distribution of accumulated hedging errors is shown in Figure 28(b). The vast majority of cases fall in the range between –3% and 3%, which indicates that the dynamic delta-hedging strategy is reasonably effective.

Figure 28(a)
REPLICATING PORTFOLIO VS. TRUE PRODUCT

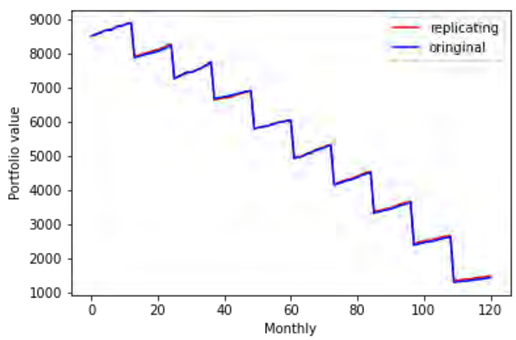
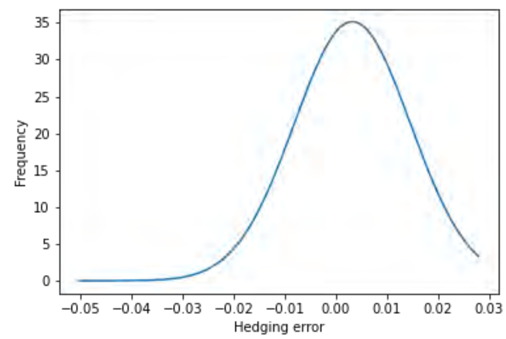


Figure 28(b)
MONTHLY DELTA-HEDGING ERROR DISTRIBUTION



Section 8: Acknowledgments

The researchers' deepest gratitude goes to those without whose efforts this project could not have come to fruition: The Project Oversight Group and others for their diligent work overseeing project development and reviewing and editing this report for accuracy and relevance.

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Appendix A: Proof of Theorem 7.1

Firstly, we can calculate the time-0 no-arbitrage value of the benefit B_{T_i} as follows:

$$
\begin{aligned}
E\left(e^{-\int_0^{T_i} n(s)ds} B_{T_i}\right) &= E\left(e^{-\int_0^{T_i} n(s)ds} B_0 \prod_{j=1}^{i} \max\left\{1+f, \frac{I_{T_j}}{I_{T_{j-1}}}\right\}\right) \\
&= B_0 E\left(\prod_{j=1}^{i} E\left[e^{-\int_{T_{j-1}}^{T_j} n(s)ds} \max\left\{1+f, \frac{I_{T_j}}{I_{T_{j-1}}}\right\} |\mathcal{F}_{T_{j-1}}\right]\right) \\
&= P_n(0,T_i) B_0 \prod_{j=1}^{i} \left[(1+f) + E_n^{T_j}\left(\left[\frac{I_{T_j}}{I_{T_{j-1}}} - (1+f)\right]_+ |\mathcal{F}_{T_{j-1}}\right)\right] \\
&= P_n(0,T_i) B_0 \prod_{j=1}^{i} \left[M(0,T_{j-1},T_j)\Phi\left(d_1^{T_{j-1},T_j}\right) + (1+f)\Phi\left(-d_2^{T_{j-1},T_j}\right)\right],
\end{aligned}
$$

where

$$
\begin{aligned}
d_1^{T_{j-1},T_j} &= \left(\frac{\ln\frac{M(0,T_{j-1},T_j)}{(1+f)} + (1/2)V^2(0,T_{j-1},T_j)}{V(0,T_{j-1},T_j)}\right), \\
d_2^{T_{j-1},T_j} &= d_1^{T_{j-1},T_j} - V(0,T_{j-1},T_j), \\
M(0,T_{j-1},T_j) &= E_n^{T_j}\left\{\frac{I_{T_j}}{I_{T_{j-1}}}\middle| \mathcal{F}_{T_{j-1}}\right\}, \\
V^2(0,T_{j-1},T_j) &= \mathrm{Var}_n^{T_j}\left\{\mathrm{Ln}\frac{I_{T_j}}{I_{T_{j-1}}}\middle| \mathcal{F}_{T_{j-1}}\right\}.
\end{aligned}
$$

Secondly, the explicit expressions for $M(0,T_{j-1},T_j)$ and $V^2(0,T_{j-1},T_j)$ can be determined as follows for $j = 1, \ldots, k$. For $M(0,T_{j-1},T_j)$, it follows from equation (27) that

$$
\begin{aligned}
E_n^{T_j}\left\{\frac{I_{T_j}}{I_{T_{j-1}}}\middle| \mathcal{F}_{T_{j-1}}\right\} &= E_n^{T_j}\left\{e^{\int_{T_{i-1}}^{T_i}[n(s)-r(s)]ds - (1/2)\sigma_I^2(T_i - T_{i-1}) + \sigma_I(W_I(T_i) - W_I(T_{i-1}))}\middle| \mathcal{F}_{T_{j-1}}\right\} \\
&= \frac{P_n(0,T_{j-1})}{P_n(0,T_j)} E_n^{T_j}\left(e^{-\int_{T_{i-1}}^{T_i} r(s)ds}\middle| \mathcal{F}_{T_{j-1}}\right),
\end{aligned}
$$

where $r(t)$ follows the stochastic process defined in equation (26). The calculation of the expectation is a classic problem for interest rate models. In the Hull-White model, the short real rate $r(t)$ is driven by

$$
dr(t) = [\alpha_r(t) - \theta_r r(t)]dt + \sigma_r dW_r(t).
$$

It is known that $\int_t^T r(s)ds\,|\mathcal{F}_t \sim N(m(t,T), v(t,T))$, where

$$
\begin{aligned}
m(t,T) &= B_r(t,T)(r(t) - \alpha(t)) + Ln\frac{P_r^M(0,t)}{P_r^M(0,T)} + \frac{1}{2}(v(0,T) - v(0,t)), \\
v(t,T) &= \frac{\sigma_r^2}{\theta_r^2}\left[(T-t) + \frac{2}{\theta_r}e^{-\theta_r(T-t)} - \frac{1}{2\theta_r}e^{-2\theta_r(T-t)} - \frac{3}{2\theta_r}\right],
\end{aligned}
$$

$$B_r(t,T) = \frac{1}{\theta_r}\left(1 - e^{-\theta_r(T-t)}\right),$$

$$\alpha(t) = f_r^M(0,t) + \frac{\sigma_r^2}{2\theta_r^2}\left(1 - e^{-\theta_r t}\right)^2.$$

Then the time-t price of a zero-coupon bond maturing at time T is known to be

$$P_r(t,T) = E\left(e^{-\int_t^T r(s)ds}\middle|\mathcal{F}_t\right) = A_r(t,T)e^{-B_r(t,T)r(t)},$$

with

$$A_r(t,T) = \frac{P_r^M(0,T)}{P_r^M(0,t)} e^{\left(B_r(t,T)f_r^M(0,t) - \frac{\sigma_r^2}{4\theta_r}\left[1-e^{-2\theta_r t}B_r^2(t,T)\right]\right)}.$$

Then, we obtain

$$E_n^{T_j}\left(e^{-\int_{T_{i-1}}^{T_i} r(s)ds}\middle|\mathcal{F}_{T_{j-1}}\right) = \frac{P_r(0,T_j)}{P_r(0,T_{j-1})}\exp\{C(0,T_{j-1},T_j)\},$$

where

$$C(0,T_{j-1},T_j) = \sigma_r B_r(T_{j-1},T_j)\Bigg\{B_r(0,T_{j-1})\left(\rho_{r,I}\sigma_I - 0.5\sigma_r B_r(0,T_{j-1}) + \frac{\rho_{n,r}\sigma_n}{\theta_n+\theta_r}\left(1+\theta_r B_r(0,T_{j-1})\right)\right) - \frac{\rho_{n,r}\sigma_n}{\theta_n+\theta_r}B_n(0,T_{j-1})\Bigg\}.$$

Next, $M(0,T_{j-1},T_j)$ is given by

$$M(0,T_{j-1},T_j) = \frac{P_n(0,T_{j-1})}{P_n(0,T_j)}\frac{P_r(0,T_j)}{P_r(0,T_{j-1})}\exp\{C(0,T_{j-1},T_j)\}. \tag{36}$$

Using the same argument, we can also determine $V^2(0,T_{j-1},T_j)$ by

$$\begin{aligned}
&V^2(0,T_{j-1},T_j)\\
&= \frac{\sigma_n^2}{2\theta_n^2}\left(1-e^{-\theta_n(T_j-T_{j-1})}\right)^2\left[1-e^{-2\theta_n(T_{j-1}-t)}\right] + \frac{\sigma_r^2}{2\theta_r^2}\left(1-e^{-\theta_r(T_j-T_{j-1})}\right)^2\left[1-e^{-2\theta_r(T_{j-1}-t)}\right]\\
&-2\rho_{n,r}\frac{\sigma_n\sigma_r}{\theta_n\theta_r(\theta_n+\theta_r)}\left(1-e^{-\theta_n(T_j-T_{j-1})}\right)\left(1-e^{-\theta_r(T_j-T_{j-1})}\right)\left[1-e^{-(\theta_n+\theta_r)(T_{j-1}-t)}\right]\\
&+\sigma_I^2(T_j-T_{j-1}) + \frac{\sigma_n^2}{\theta_n^2}\left[(T_j-T_{j-1}) + \frac{2}{\theta_n}e^{-\theta_n(T_j-T_{j-1})} - \frac{1}{2\theta_n}e^{-2\theta_n(T_j-T_{j-1})} - \frac{3}{2\theta_n}\right]\\
&+\frac{\sigma_r^2}{\theta_r^2}\left[(T_j-T_{j-1}) + \frac{2}{\theta_r}e^{-\theta_r(T_j-T_{j-1})} - \frac{1}{2\theta_r}e^{-2\theta_r(T_j-T_{j-1})} - \frac{3}{2\theta_r}\right]\\
&-2\rho_{n,r}\frac{\sigma_n\sigma_r}{\theta_n\theta_r}\left[(T_j-T_{j-1}) - \frac{1-e^{-\theta_n(T_j-T_{j-1})}}{\theta_n} - \frac{1-e^{-\theta_r(T_j-T_{j-1})}}{\theta_r} + \frac{1-e^{-(\theta_n+\theta_r)(T_j-T_{j-1})}}{\theta_n+\theta_r}\right]\\
&\quad -2\rho_{n,I}\frac{\sigma_n\sigma_I}{\theta_n}\left[(T_j-T_{j-1}) - \frac{1-e^{-\theta_n(T_j-T_{j-1})}}{\theta_n}\right] - 2\rho_{r,I}\frac{\sigma_r\sigma_I}{\theta_r}\left[(T_j-T_{j-1}) - \frac{1-e^{-\theta_r(T_j-T_{j-1})}}{\theta_r}\right].
\end{aligned} \tag{37}$$

More details of the derivation can be found in Brigo and Mercurio (2007).

Appendix B: Model Assumptions

For numerical examples in the previous sections, here is the summary of all parameters for GMMB, GMDB, GMWB, in the Black-Scholes model and the stochastic volatility jump model as well as an illustrative life table.

Table 23

GMMB PRODUCT SPECIFICATION

Item	Symbol	Value
Guarantee level	G	$1,000
Initial purchase payment	F_0	$1,000
Initial equity value	S_0	$100
Annualized rate of total fee	m	5%
Annualized risk-free interest rate	r	3%
Annualized rate of return	μ	3%
Annualized equity volatility rate	σ	30%
GMMB maturity date	T	10 years
Age of policyholder at issue	x	60

Table 24

GMDB PRODUCT SPECIFICATION

Item	Symbol	Value
Guarantee level	G	$1,000
Initial purchase payment	F_0	$1,000
Initial equity value	S_0	$100
Annualized rate of total fee	m	5%
Annualized risk-free interest rate	r	3%
Annualized rate of return	μ	3%
Annualized equity volatility rate	σ	30%
GMDB maturity date	T	10 years
Age of policyholder at issue	x	60

Table 25

GMWB PRODUCT SPECIFICATION

Item	Symbol	Value
Guarantee level	G	$1,000
Initial purchase payment	F_0	$1,000
Initial equity value	S_0	$100
Annualized amount of withdrawal	w	$100
Annualized rate of total fee	m	5%
Annualized risk-free interest rate	r	3%
Annualized rate of return	μ	3%
Annualized equity volatility rate	σ	30%
GMWB maturity date	T	10 years
Age of policyholder at issue	x	60

Table 26

PARAMETERS OF THE BS, TABLE III OF ANDERSEN, BENZONI AND LUND (2002)

Item	Symbol	Value
Drift mean of stock price	μ	0.0398
Volatility of stock price	σ	0.5933

Table 27

PARAMETERS OF THE SVJ MODEL, TABLE III OF ANDERSEN, BENZONI AND LUND (2002)

Item	Symbol	Value
Drift mean of stock price	μ	0.0304
Drift term of stock volatility	α	0.0064
	β	0.012
Volatility rate	η	0.0711
Correlation coefficient	ρ	–0.6219
Standard deviation	δ	0.0134
Jump intensity	λ_1	0.0202
	λ_2	0.00002
Average size of jump	$\bar{\kappa}$	0

Table 28

ILLUSTRATIVE LIFE TABLE

x	$1{,}000q_x$	${}_xp_0$	x	$1{,}000q_x$	${}_xp_0$	x	$1{,}000q_x$	${}_xp_0$
1	0.587	0.99941	40	1.317	0.97381	79	69.595	0.47012
2	0.433	0.99898	41	1.424	0.97242	80	77.114	0.43387
3	0.350	0.99863	42	1.540	0.97093	81	85.075	0.39696
4	0.293	0.99834	43	1.662	0.96931	82	93.273	0.35993
5	0.274	0.99806	44	1.796	0.96757	83	101.578	0.32337
6	0.263	0.99780	45	1.952	0.96568	84	110.252	0.28772
7	0.248	0.99755	46	2.141	0.96362	85	119.764	0.25326
8	0.234	0.99782	47	2.366	0.96134	86	130.583	0.22019
9	0.231	0.99709	48	2.618	0.95882	87	143.012	0.18870
10	0.239	0.99685	49	2.900	0.95604	88	156.969	0.15908
11	0.256	0.99660	50	3.223	0.95296	89	172.199	0.13169
12	0.284	0.99631	51	3.598	0.94953	90	188.517	0.10686
13	0.371	0.99599	52	4.019	0.94571	91	205.742	0.08488
14	0.380	0.99561	53	4.472	0.94148	92	223.978	0.06587
15	0.435	0.99518	54	4.969	0.93681	93	243.533	0.04982
16	0.486	0.99469	55	5.543	0.93161	94	264.171	0.03666
17	0.526	0.99417	56	6.226	0.92581	95	285.199	0.02621
18	0.558	0.99362	57	7.025	0.91931	96	305.931	0.01819
19	0.586	0.99303	58	7.916	0.91203	97	325.849	0.01226
20	0.613	0.99242	59	8.907	0.90391	98	344.977	0.00803
21	0.642	0.99179	60	10.029	0.89484	99	363.757	0.00511
22	0.677	0.99112	61	11.312	0.88472	100	382.606	0.00316
23	0.717	0.99041	62	12.781	0.87341	101	401.942	0.00189
24	0.760	0.98965	63	14.431	0.86081	102	422.569	0.00109
25	0.803	0.98886	64	16.241	0.84683	103	445.282	0.00060
26	0.842	0.98802	65	18.191	0.83142	104	469.115	0.00032
27	0.876	0.98716	66	20.259	0.81458	105	491.923	0.00016
28	0.807	0.98626	67	22.398	0.79633	106	511.560	0.00008
29	0.935	0.98534	68	24.581	0.77676	107	526.441	0.00004
30	0.959	0.98440	69	26.869	0.75589	108	536.732	0.00002
31	0.981	0.98343	70	29.363	0.73369	109	543.602	0.00001
32	0.997	0.98245	71	32.169	0.71009	110	547.664	0
33	1.003	0.98147	72	35.168	0.68505	111	549.540	0
34	1.005	0.98048	73	38.558	0.65863	112	550.000	0
35	1.013	0.97949	74	42.106	0.63090	113	550.000	0
36	1.037	0.97847	75	46.121	0.60180	114	550.000	0
37	1.082	0.97741	76	50.813	0.57122	115	1000.000	0
38	1.146	0.97629	77	56.327	0.53905			
39	1.225	0.97510	78	62.629	0.50529			

References

Andersen, Torben G., Luca Benzoni and Jesper Lund. 2002. An Empirical Investigation of Continuous-time Equity Return Models. *The Journal of Finance* 57, no. 3:1239–84.

Brigo, Damiano, and Fabio Mercurio. 2007. *Interest Rate Models—Theory and Practice: With Smile, Inflation and Credit.* New York: Springer Science & Business Media.

Cui, Zhenyu, Runhuan Feng and Anne MacKay. 2017. Variable Annuities with VIX-linked Fee Structure Under a Heston-type Stochastic Volatility Model. *North American Actuarial Journal* 21, no. 3:458–83.

Drinkwater, Matthew, Jafor Iqbal and Joseph E. Montiminy. 2014. Variable Annuity Guaranteed Living Benefits Utilization: 2012 Experience. Technical report, Society of Actuaries and LIMRA.

Feng, Runhuan. 2014. A Comparative Study of Risk Measures for Guaranteed Minimum Maturity Benefits by a PDE Method. *North American Actuarial Journal* 18, no. 4:445–61.

———. 2018. *An Introduction to Computational Risk Management of Equity-linked Insurance*. Boca Raton, Fla.: CRC Press.

Feng, Runhuan, and Huaxiong Huang. 2016. Statutory Financial Reporting for Variable Annuity Guaranteed Death Benefits: Market Practice, Mathematical Modeling and Computation. *Insurance: Mathematics and Economics* 67:54–64.

Feng, Runhuan, Alexay Kuznetsov and Fenghao Yang. 2019. Exponential Functionals and Variable Annuity Guaranteed Benefits. *Stochastic Processes and Their Applications* 22, no. 2:604–25.

Feng, Runhuan, and Xiaochen Jing. 2016. Analytical Valuation and Hedging of Variable Annuity Guaranteed Lifetime Withdrawal Benefits. *Insurance: Mathematics and Economics* 72:36–48.

Feng, Runhuan, Xiaochen Jing and Jan Dhaene. 2017. Comonotonic Approximations of Risk Measures for Variable Annuity Guaranteed Benefits with Dynamic Policyholder Behavior. *Journal of Computational and Applied Mathematics* 311:272–92.

Feng, Runhuan, and Yasutaka Shimizu. 2016. Applications of Central Limit Theorems for Equity-linked Insurance. *Insurance: Mathematics and Economics* 69:138–48.

Feng, Runhuan, and Hans W. Volkmer. 2012. Analytical Calculation of Risk Measures for Variable Annuity Guaranteed Benefits. *Insurance: Mathematics and Economics* 51, no. 3:636–48.

———. 2014. Spectral Methods for the Calculation of Risk Measures for Variable Annuity Guaranteed Benefits. *ASTIN Bulletin* 44, no. 3:653–81.

———. 2016. An Identity of Hitting Times and its Application to the Valuation of Guaranteed Minimum Withdrawal Benefit. *Mathematics and Financial Economics* 10, no. 2:127–49.

Feng, Runhuan, and Jan Vecer. 2017. Risk-based Capital Requirements for Guaranteed Minimum Withdrawal Benefit. *Quantitative Finance* 17, no. 3:471–78.

Feng, Runhuan, and Bingji Yi. 2019. Quantitative Modeling of Risk Management Strategies: Stochastic Reserving and Dynamic Hedging of Variable Annuity Guaranteed Benefits. *Insurance: Mathematics and Economics* 14:50–73.

Jarrow, Robert, and Yildiray Yildirim. 2003. Pricing Treasury Inflation Protected Securities and Related Derivatives Using an HJM Model. *Journal of Financial and Quantitative Analysis* 38, no. 2:337–58.

Kling, Alexander, Frederick Ruez and Jochen Russ. 2011. The Impact of Stochastic Volatility on Pricing, Hedging and Hedge Efficiency of Withdrawal Benefit Guarantees in Variable Annuities. *ASTIN Bulletin* 41, no. 2:511–45.

Marshall, Claymore, Mary Hardy and David Saunders. 2010. Valuation of a Guaranteed Minimum Income Benefit. *North American Actuarial Journal* 14, no. 1:38–58.

Milevsky, Moshe A., and Steven E. Posner. 2001. The Titanic Option: Valuation of the Guaranteed Minimum Death Benefit in Variable Annuities and Mutual Funds*. Journal of Risk and Insurance* 68, no. 1:93–128.

Milevsky, Moshe A., and Thomas S. Salisbury. 2006. Financial Valuation of Guaranteed Minimum Withdrawal Benefits. *Insurance: Mathematics and Economics* 38, no. 1:21–38.

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