Appendix 3

Calculation of key-rate metrics and their cross impacts using calculus

Realizing that key-rate metrics are simply orders of differentiation of the value being calculated with respect to the part of rates on the curve, we can simply use calculus and the use of Taylor series expansion to approximate value change after a change in one of the variables (key rates).

In the below, f represents the discounted value of what is of interest (generally cash flows or profits), x and y are key rates, a and b are the starting positions of rates in x and y respectively. the superscript +/– means rates increase/decrease by h. The subscript x means the first order differentiation is used in variable x, xx refers to second order differentiation, and xy means cross impact of x and y is being calculated when change in x happens after change in y.

$f_x^+(a,b) = \frac{f(a+h,b)-f(a,b)}{h} = \frac{P_x}{h}$, this equals KRxDV01 when rates have increased (directional)

$f_x^-(a,b) = \frac{f(a,b)-f(a-h,b)}{h} = \frac{-M_x}{h}$, this equals KRxDV01 when rates have decreased (directional)

$f_x(a,b) = \frac{f(a+h,b)-f(a-h,b)}{2h} = \frac{[f(a+h,b)-f(a,b)]-[f(a-h,b)-f(a,b)]}{2h} = \frac{f_x^+(a,b)+f_x^-(a,b)}{2} = \frac{P_x-M_x}{2h}$, this equals KRxDV01 when up and down rates are incorporated (nondirectional)

$f_y^+(a,b) = \frac{f(a,b+h)-f(a,b)}{h} = \frac{P_y}{h}$

$f_y^-(a,b) = \frac{f(a,b)-f(a,b-h)}{h} = \frac{-M_y}{h}$

$f_y(a,b) = \frac{f(a,b+h)-f(a,b-h)}{2h} = \frac{[f(a,b+h)-f(a,b)]-[f(a,b-h)-f(a,b)]}{2h} = \frac{f_y^+(a,b)+f_y^-(a,b)}{2} = \frac{P_y-M_y}{2h}$

$f_{xx}^+(a,b) = \frac{f_x^+(a+h/2,b)-f_x^+(a,b)}{h/2} = \frac{[f(a+h,b)-f(a+h/2,b)]-[f(a+h/2,b)-f(a,b)]}{h^2/4} = \frac{f(a+h,b)-2f(a+h/2,b)+f(a,b)}{h^2/4} = \frac{P_x-2P_{x/2}}{h^2/4}$, this equals KRxCV01 when rates have increased (directional)

$f_{xx}^-(a,b) = \frac{f_x^-(a,b)-f_x^-(a-h/2,b)}{h/2} = \frac{[f(a,b)-f(a-h/2,b)]-[f(a-h/2,b)-f(a-h,b)]}{h^2/4} = \frac{f(a,b)-2f(a-h/2,b)+f(a-h,b)}{h^2/4} = \frac{M_x-2M_{x/2}}{h^2/4}$, this equals KRxCV01 when rates have decreased (directional)

$f_{xx}(a,b) = \frac{f_x(a+h/2,b)-f_x(a-h/2,b)}{2h/2} = \frac{[f(a+h,b)-f(a,b)]-[f(a,b)-f(a-h,b)]}{h^2} = \frac{P_x+M_x}{h^2}$, this equals KRxCV01 when up and down rates are incorporated (nondirectional)

$f_{yy}^+(a,b) = \frac{f_y^+(a,b+h/2)-f_y^+(a,b)}{h/2} = \frac{[f(a,b+h)-f(a,b+h/2)]-[f(a,b+h/2)-f(a,b)]}{h^2/4} = \frac{f(a,b+h)-2f(a,b+h/2)+f(a,b)}{h^2/4} = \frac{P_y-2P_{y/2}}{h^2/4}$

$f_{yy}^-(a,b) = \frac{f_y^-(a,b)-f_y^-(a,b-h/2)}{h/2} = \frac{[f(a,b)-f(a,b-h/2)]-[f(a,b-h/2)-f(a,b-h)]}{h^2/4} = \frac{f(a,b)-2f(a,b-\frac{h}{2})+f(a,b-h)}{h^2/4} = \frac{M_y-2M_{y/2}}{h^2/4}$

$f_{yy}(a,b) = \frac{f_x(a,b+h/2)-f_x(a,b-h/2)}{2h/2} = \frac{[f(a,b+h)-f(a,b)]-[f(a,b)-f(a,b-h)]}{h^2} = \frac{P_y+M_y}{h^2}$

$f_{xy}^{++}(a,b) = \frac{f_y^+(a+h,b)-f_y^+(a,b)}{h} = \frac{[f(a+h,b+h)-f(a+h,b)]-[f(a,b+h)-f(a,b)]}{h^2} = \frac{f(a+h,b+h)-f(a+h,b)-f(a,b+h)+f(a,b)}{h^2} = \frac{PP_{xy}-P_x-P_y}{h^2}$, this reflects cross impact of key-rate x against key-rate y when the both have increased

$f_{xy}^{+-}(a,b) = \frac{f_y^-(a+h,b)-f_y^-(a,b)}{h} = \frac{[f(a+h,b)-f(a+h,b-h)]-[f(a,b)-f(a,b-h)]}{h^2} = \frac{f(a+h,b)-f(a+h,b-h)-f(a,b)+f(a,b-h)}{h^2} = \frac{P_x-PM_{xy}+M_y}{h^2}$, this reflects cross impact of key-rate x against key-rate y when x has increased while y has decreased

$f_{xy}^{--}(a,b) = \frac{f_y^-(a,b)-f_y^-(a-h,b)}{h} = \frac{[f(a,b)-f(a,b-h)]-[f(a-h,b)-f(a-h,b-h)]}{h^2} = \frac{f(a,b)-f(a,b-h)-f(a-h,b)+f(a-h,b-h)}{h^2} = \frac{MM_{xy}-M_x-M_y}{h^2}$, this reflects cross impact of key-rate x against key-rate y when the both have decreased

$f_{xy}^{-+}(a,b) = \frac{f_y^+(a,b)-f_y^+(a-h,b)}{h} = \frac{[f(a,b+h)-f(a,b)]-[f(a-h,b+h)-f(a-h,b)]}{h^2} = \frac{f(a,b+h)-f(a,b)-f(a-h,b+h)+f(a-h,b)}{h^2} = \frac{P_y-MP_{xy}+M_x}{h^2}$

$f_{xy}(a,b) = \frac{f_y(a+h,b)-f_y(a-h,b)}{2h} = \frac{[f(a+h,b+h)-f(a+h,b-h)]-[f(a-h,b+h)-f(a-h,b-h)]}{4h^2} = \frac{f(a+h,b+h)-f(a+h,b-h)-f(a-h,b+h)+f(a-h,b-h)}{4h^2} = \frac{PP_{xy}-PM_{xy}-MP_{xy}+MM_{xy}}{4h^2} = \frac{f_{xy}^{++}(a,b)+f_{xy}^{+-}(a,b)+f_{xy}^{-+}(a,b)+f_{xy}^{--}(a,b)}{4}$, this reflects cross impact of key-rate x against key-rate y when nondirectional differentiation is incorporated

$f_{yx}^{++}(a,b) = \frac{f_x^+(a,b+h)-f_x^+(a,b)}{h} = \frac{[f(a+h,b+h)-f(a,b+h)]-[f(a+h,b)-f(a,b)]}{h^2} = \frac{f(a+h,b+h)-f(a,b+h)-f(a+h,b)+f(a,b)}{h^2} = \frac{PP_{xy}-P_y-P_x}{h^2} = f_{xy}^{++}(a,b)$

$f_{yx}^{+-}(a,b) = \frac{f_x^-(a,b+h)-f_x^-(a,b)}{h} = \frac{[f(a,b+h)-f(a-h,b+h)]-[f(a,b)-f(a-h,b)]}{h^2} = \frac{f(a,b+h)-f(a-h,b+h)-f(a,b)+f(a-h,b)}{h^2} = \frac{P_y-MP_{xy}+M_x}{h^2} = f_{xy}^{-+}(a,b)$

$f_{yx}^{--}(a,b) = \frac{f_x^-(a,b)-f_x^-(a,b-h)}{h} = \frac{[f(a,b)-f(a-h,b)]-[f(a,b-h)-f(a-h,b-h)]}{h^2} = \frac{f(a,b)-f(a-h,b)-f(a,b-h)+f(a-h,b-h)}{h^2} = \frac{MM_{xy}-M_x-M_y}{h^2} = f_{xy}^{--}(a,b)$

$f_{yx}^{-+}(a,b) = \frac{f_x^+(a,b)-f_x^+(a,b-h)}{h} = \frac{[f(a+h,b)-f(a,b)]-[f(a+h,b-h)-f(a,b-h)]}{h^2} = \frac{f(a+h,b)-f(a,b)-f(a+h,b-h)+f(a,b-h)}{h^2} = \frac{P_x-PM_{xy}+M_y}{h^2} = f_{xy}^{+-}(a,b)$

$f_{yx}(a,b) = \frac{f_x(a,b+h)-f_x(a,b-h)}{2h} = \frac{[f(a+h,b+h)-f(a-h,b+h)]-[f(a+h,b-h)-f(a-h,b-h)]}{4h^2} = \frac{f(a+h,b+h)-f(a-h,b+h)-f(a+h,b-h)+f(a-h,b-h)}{4h^2} = f_{xy}(a,b)$

$f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + \frac{1}{2}f_{xx}(a,b)(x-a)^2 + f_{xy}(a,b)(x-a)(y-b) + \frac{1}{2}f_{yy}(a,b)(y-b)^2$, this is Taylor series expansion of order 2 with two variables