Predictive Analytics and Futurism

June 2017
Issue 15

Understanding Autoregressive Model for Time Series as a Deterministic Dynamic System

By Dihui Lai and Bingfeng Lu

The autoregressive (AR) model is commonly used to model time-varying processes and solve problems in the fields of natural science, economics and finance, and others.[1] The models have always been discussed in the context of random process and are often perceived as statistical tools for time series data. However, randomness is only part of the story. The rich deterministic dynamics that an AR model produces is perhaps also worth some attention.

In this article, we are going to discuss the AR model by making connections to time-dependent ordinary differential equations. The goal is to understand the essential dynamics underlying the AR model and provide guidance on model usage in addition to statistical diagnostic tools.

AUTOREGRESSIVE MODEL

In general, the autoregressive model describes a system whose status (dependent variable) depends linearly on its own status in the past. The system can be mathematically described by a stochastic difference equation such as the following:

$$y_t = \beta_0 + \sum_{i=1}^{p} \beta_i y_{t-i} + \varepsilon_t .$$

Here, the βs describe how much the system's status *i* steps ago will impact current values. Normally, one would expect βs to decrease as *i* increases, that is, the events that happen further in the past have less impact on current events. Anything that happens earlier than *p* time steps ago will have no impact, and the model is noted as AR(*p*), where ε_t is a "noise" term that describes some random events that affect the status of the system. The "noise" term is often required to be stationary to make lots of statistical estimators valid (least-square estimation, maximum-likelihood estimation etc.).

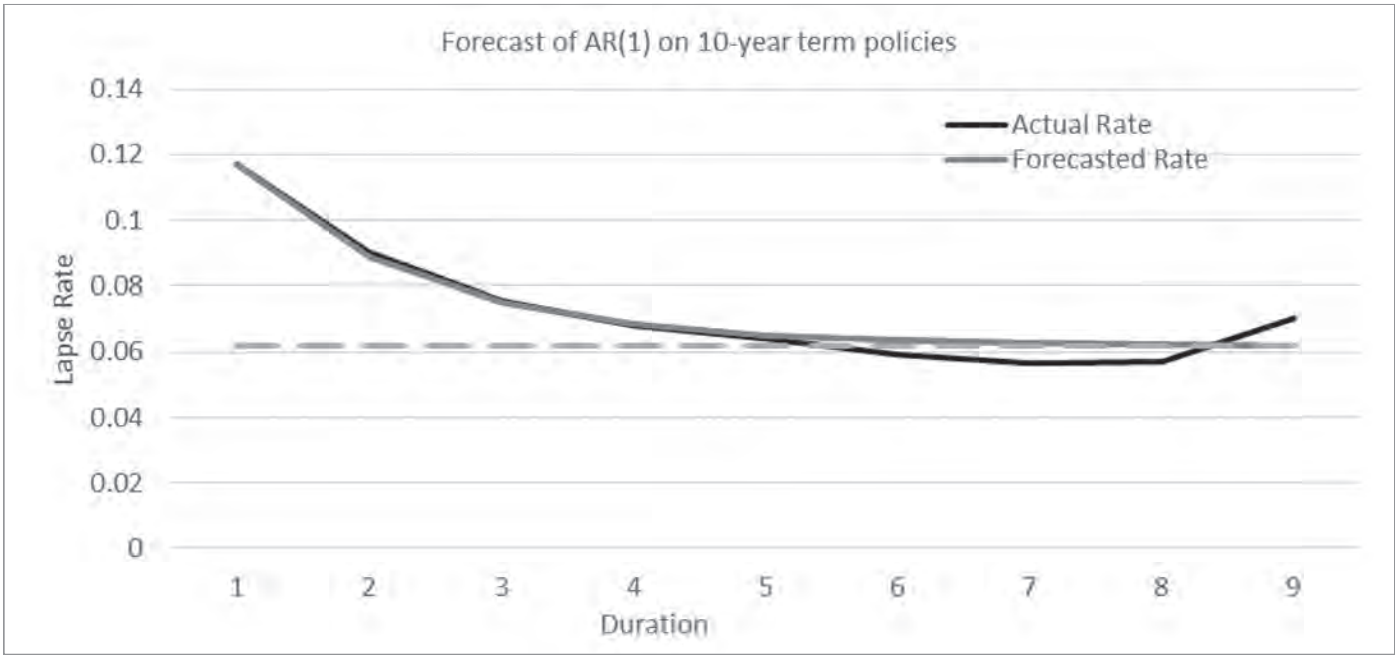
AR(1) MODEL AND FIRST ORDER TIME-DEPENDENT ORDINARY DIFFERENTIAL EQUATION (ODE) SYSTEM

In a very simple scenario where $p = 1$, we have an AR(1) model where the system's current status is dependent only on the system's status one time step ago: $y_t = \beta_0 + \beta_1 y_{t-1} + \varepsilon_t$. The continuous version of the system can be represented as a first-order time-dependent ODE with a noise term: $\frac{dy}{dt} = \beta_0 + (\beta_1 - 1) y + \varepsilon_t$ (see the appendix). Without considering the noise, the closed formula solution of the ODE is an exponential function: $y = constant * e^{(\beta_1 - 1)t} + \frac{\beta_0}{1-\beta_1}$. It follows immediately that the status of the system will reach an equilibrium point $\frac{\beta_0}{1-\beta_1}$, if $\beta_1 < 1$, as the exponential term vanishes in the long term. Not surprisingly, this is also the expected behavior of an AR(1) model in equilibrium status when $y_t = y_{t-1}$.

Now that we have made the connection between the two systems, it becomes clear that the parameter $(1-\beta_1)$ could be interpreted as a decay constant that describes how fast the system will reach a steady value as time elapses. When $\beta_1 < 1$, the AR(1) model is nothing more than a system that exponentially decays to a steady state from a certain initial value noted as *constant* in the close formula solution. On the other hand, when $\beta_1 > 1$, the dependent variable will exponentially increase to a very large value.

In another words, an AR(1) model can be used to describe the evolvement of systems that have decay-like behavior with a long-term equilibrium point. As an example, we modeled the lapse behavior of a 10-year term life policy over the level period with an AR(1) model. The model uses the lapse rate at each policy year as the target variable. To make a forecast, we provide the model with an initial lapse rate at duration 1, and the lapse rate evolves as an exponential decay toward a stable point (see Figure 1). The model forecast did quite well at early duration but underestimated the rate after duration 5, indicating that extra factors need to be considered beyond the dynamics described by AR(1).

Figure 1
Forecasting the lapse rate of a 10-year term life policy over a level period by the AR(1) model. The black line is the actual lapse rate, and the red line is the forecasted rate. The forecasted lapse rate quickly decays and reaches a stable point (green dashed line)



AR(p) MODEL AND pTH-ORDER TIME-DEPENDENT ODE SYSTEM

In general, an AR(p) model is a pth-order linear difference equation with a noise term. It can be proven with some linear algebra techniques that a pth-order linear difference equation can be reorganized into a set of p first-order ODEs. Thus, it is expected that an AR(p) model will inherent some dynamic properties of a pth-order ODE set. In the following section, we use an AR(2) model to reproduce the behavior of an oscillatory system.

SEASONALITY OR HARMONIC OSCILLATOR?

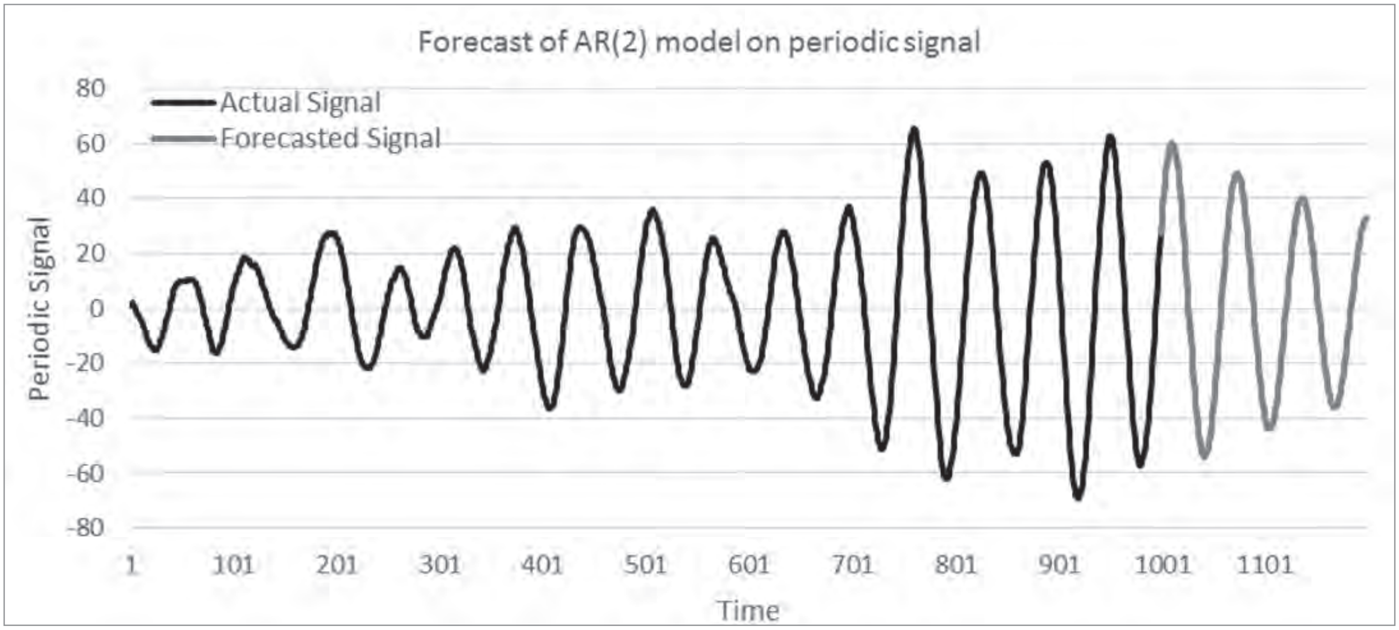
When studying time series, the periodic behavior is commonly modeled by constructing a new seasonal difference variable $\Delta y_t = y_t - y_{t-T_{period}}$. The evolvement of the system over time is then described by the new variable Δy_t. This clever approach avoids modeling the periodic behavior by removing the gross seasonal feature and considering only the change over seasons.[2] However, to make a forecast, this approach needs to have n initial condition parameters where $n = T_{period}$ and some prior knowledge for T_{period} are needed.

Alternatively, we know that a second-order ODE system will lead to oscillatory behavior (a harmonic oscillator can be described by a second-order ODE) given the right parameter sets, and therefore we expect the time series version of the system will produce periodic behaviors. As a demonstration, we build an AR(2) model on a sinusoidal time series signals (see Figure 2). Without explicitly modeling the seasonal activity, the model captures the essence of the oscillatory behavior (period) with only three parameters.

SUMMARY

When building an autoregressive model, it is often more of art than science to decide the value for p—that is, how far do we have to trace the system's past to make a reliable forecast? Some tools are available to help the decision-making process, such as an autocorrelation function (ACF) or a partial autocorrelation function (PACF).[3] Although the diagnostic tools provide convenient guidance on choosing the lag parameter, it is not always easy to find a clear-cut value. The judgment becomes even harder for a noisy data set.

Figure 2
Forecast of an AR(2) model on a periodic signal. The black line is the original signal, and the red line is the forecasted behavior of the system. The forecasted part reproduces the periods of the signal quite well.



In this article, we demonstrate the dynamic feature of AR models. By borrowing concepts and closed formula solutions from time-dependent ODEs, we gain some intuition for the parameters in AR models (βs and p) and relate them to the dynamic properties of continuous systems. We use some examples to demonstrate that an AR(1) model can be used to model a dynamic system showing decay-like behaviors. Besides the commonly used seasonality model, an AR(2) model could be used to model periodic oscillatory (seasonal) behaviors. ■

Dihui Lai, Ph.D., is a senior data scientist at RGA Reinsurance Company in Chesterfield, Mo. He can be reached at *dlai@rgare.com*.

Bingfeng Lu is an assistant data scientist at RGA Reinsurance Company in Chesterfield, Mo. He can be reached at *blu@rgare.com*.

ENDNOTES

1. Zhang, Yuanjie Michael, Jeffrey R. Russell and Ruey S. Tsay. 2001. "A nonlinear autoregressive conditional duration model with applications to financial transaction data," *Journal of Econometrics* 104, no. 1, pp. 179–207.
2. Hyndman, R.J. and G. Athanasopoulos. 2014. *Forecasting: Principles and Practice.* Melbourne: OTexts.
3. Box, G. E. P., G. M. Jenkins, G. C. Reinsel and G.M. Ljung. 2015. *Time Series Analysis: Forecasting and Control.* 5th ed. Hoboken, NJ: John Wiley & Sons.

APPENDIX

First-Order Difference Equation and First-Order ODE

A first-order difference equation can be written as $y_t = \beta_0 + \beta_1 y_{t-1}$. Here we have assumed a change over one time unit in the formula. In general, the time step can be of any unit, and by changing the unit of time, we can replace unit time with Δt, and the equation can be rewritten as $\frac{y_t - y_{t-\Delta t}}{\Delta t} = \beta_0 + (\beta_1 - 1) y_{t-\Delta t}$. When $\Delta t \to 0$, the difference equation becomes a first-order time-dependent ODE $\frac{dy}{dt} = \beta_0 + (\beta_1 - 1) y$.