CURATED PAST EXAM ITEMS
- Questions -

INV 201 – Quantitative Finance

Important Information:

- These curated past exam items are intended to allow candidates to focus on past SOA fellowship assessments. These items are organized by topic and learning objective with relevant learning outcomes, source materials, and candidate commentary identified. We have included items that are relevant in the new course structure, and where feasible we have made updates to questions to make them relevant.
- Where an item applies to multiple learning objectives, it has been placed under each applicable learning objective.
- Candidate solutions other than those presented in this material, if appropriate for the context, could receive full marks. For interpretation items, solutions presented in these documents are not necessarily the only valid solutions.
- Learning Outcome Statements and supporting syllabus materials may have changed since each exam was administered. New assessment items are developed from the current Learning Outcome Statements and syllabus materials. The inclusion in these curated past exam questions of material that is no longer current does not bring such material into scope for current assessments.
- Thus, while we have made our best effort and conducted multiple reviews, alignment with the current system or choice of classification may not be perfect. Candidates with questions or ideas for improvement may reach out to education@soa.org. We expect to make updates annually.

Course INV 201

Curated Past Exam Questions

All Learning Objectives

1. **Key Types of Derivatives**
2. **Valuation of Derivatives**
3. **Applications and Risks of Derivatives**

The following questions are taken from QFI Exams offered from 2020 to 2024. They have been mapped to the learning objectives and syllabus materials for the INV 201 2025-2026 course and, in some cases, modified to fit the 2025-2026 curriculum.

The related solutions and Excel spreadsheets are provided in separate files.

These questions have been carefully organized according to the relevant learning objectives, providing a structured approach to review and preparation.

This compilation aims to familiarize candidates with the format, scope, and level of difficulty they can expect, while also reinforcing key concepts and problem-solving techniques required for success on the INV 201 assessment.

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Learning Objective 1: The candidate will understand key types of derivatives

QFI QF Fall 2021 Question 5

Learning Outcomes:

a) Understand the payoffs of basic derivative instruments

b) Be able to identify the key differences between forwards and futures

d) Understand the mechanics of derivatives trading

Source References:

- *Options, Futures, and Other Derivatives*, Hull, John C., 11th Edition, 2021, Chapters 1, 7, pages 28-30, 39, 183-185

Question:

(a) Describe forward rate agreements, forward contracts, and interest rate swaps.

The response for this part is required on the paper provided to you.

All rates provided below are continuous compounding, swaps have semi-annual coupon payment.

t	T_0 (spot rate time 0)	$T_{0.5}$ (spot rate time 0.5)
0.5	2.00%	1.00%
1	2.50%	1.50%
1.5	3.00%	2.00%
2	3.50%	2.50%
2.5	3.50%	2.50%
3	4.00%	3.00%

ABC bank quotes 96.08 for a one-year forward contract on 1-year zero-coupon bond.

(b) Determine arbitrage strategy based on the above data.

The response for this part is required on the paper provided to you.

(c) Calculate the 2-year swap rate and the value of the swap at time 0.

The response for this part is to be provided in the Excel spreadsheet.

(d) Calculate the value of the 2-year swap in part (c) at time 0.5 (after cash payment).

The response for this part is to be provided in the Excel spreadsheet.

(e) Calculate the forward swap rate of the 2-year forward swap contract with expiry of 1 year.

The response for this part is to be provided in the Excel spreadsheet.

Learning Objective 2: The candidate will understand the principles and techniques for the valuation of derivatives

QFI QF Fall 2020 Question 1

Learning outcomes:

d) Understand Stochastic Calculus theory and technique used in pricing derivatives

Source references:

- *Options, Futures, and Other Derivatives*, Hull, John C., 11th Edition, 2021, Chapter 14, pages 318-321, 327
- *Problems and Solutions in Mathematical Finance: Stochastic Calculus*, Chin, Eric, Nel, Dian and Olafsson, Sverrir, 2014, pages 52, 57-58, 128-130, 132-137

Question:

Let $\{W_t : t \geq 0\}$ be a standard Wiener process and $M_t = \int_0^t f(s) dW_s$, where $f(s)$ is a square integrable deterministic function.

Let $X_t = e^{-\theta M_t - \frac{1}{2}\theta^2 \int_0^t f(s)^2 ds}$, where θ is a constant.

(a) Show that X_t satisfies the stochastic differential equation

$$dX_t = \theta f(t) X_t dW_t$$

(b) Show that M_t ~Normal(0, $\int_0^t f(s)^2 ds$) for any $t > 0$.

Suppose that Z_t satisfies the stochastic differential equation

$$dZ_t = \frac{y - Z_t}{1 - t} dt + dW_t \text{ for } 0 \leq t < 1,$$

with $Z_0 = z$ and $Z_1 = y$ where both y and z are constants.

(c) Show that $Z_t = yt + (1-t)(z + \int_0^t \frac{1}{1-s} dW_s)$ for $0 \le t < 1$.

(d) Find the mean and the variance of Z_t for $0 \le t < 1$.

(e) Show that Z_t follows a normal distribution for $0 < t < 1$.

QFI QF Fall 2020 Question 2

Learning outcomes:

d) Understand Stochastic Calculus theory and technique used in pricing derivatives

e) Understand and apply the concepts of risk-neutral measure, forward measure, normalization, and the market price of risk

Source references:

- *Options, Futures, and Other Derivatives, Hull, John C.,* 11th Edition, 2021, Chapter 28, pages 671-672
- *Problems and Solutions in Mathematical Finance: Stochastic Calculus*, Chin, Eric, Nel, Dian and Olafsson, Sverrir, 2014, pages 72-73, 221-227

Question:

You are an actuary working in the asset liability management department. Recently, you were assigned to examine nontraditional annuities, including equity indexed annuities.

Suppose that X_t and Y_t satisfy the following stochastic processes under the real-world measure $\mathbb{P}$:

$$dX_t = \mu_1 X_t dt + \sigma_1 X_t dW_t, \quad X_0 = 1$$

$$dY_t = \mu_2 Y_t dt + \sigma_2 Y_t dW_t, \quad Y_0 = 1$$

where μ_1, μ_2, $\sigma_1 > 0$, $\sigma_2 > 0$ are constants and W_t is a standard Wiener process under $\mathbb{P}$.

(a) Establish a condition on μ_1, μ_2, σ_1 and σ_2 such that both $X_t e^{-rt}$ and $Y_t e^{-rt}$ are martingales under the risk-neutral measure $\mathbb{Q}$.

(b) Derive the Radon-Nikodym derivative $\frac{d\mathbb{Q}}{d\mathbb{P}}$ by assuming that the condition in part (c) holds.

QFI QF Fall 2020 Question 3

Learning outcomes:

d) Understand Stochastic Calculus theory and technique used in pricing derivatives

Source references:

- *Options, Futures, and Other Derivatives, Hull, John C.,* 11th Edition, 2021, Chapter 14, page 327, 329

Question:

Let $\{W_t: t \geq 0\}$ be a standard Wiener process on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$.

Let $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T$ be the partitioning of the interval $[0, T]$ into n smaller subintervals all of size h for an arbitrary integer number $n > 0$ and an arbitrary real number $T > 0$.

Denote by

$$\Delta W_{t_i} = W_{t_{i+1}} - W_{t_i}, \qquad i = 0,1, \dots, n-1$$

(a) Show that for all $i, j = 0, 1, \dots, n-1$

(i) $E\left[\left(\Delta W_{t_i}\right)^4\right] = 3h^2$ using Ito's lemma.

(ii) $E\left[\left(\Delta W_{t_i}\right)^2 \left(\Delta W_{t_j}\right)^2\right] = h^2$ if $i < j$.

QFI QF Fall 2020 Question 7

Learning Outcomes:

f) Understand option pricing techniques

j) Define and explain the concept of volatility smiles and describe several approaches for modeling smiles, including stochastic volatility, local- volatility, jump-diffusions

Source References:

- *The Volatility Smile*, Derman, Emanuel and Miller, Michael, 2016, Chapters 8, 10, pages 144-145, 163-164
- *Options, Futures, and Other Derivatives*, Hull, John C., 11th Edition, 2021, Chapter 15, pages 352-353, 451

Question (Excel can be used to do the calculation):

You are a pricing actuary of an insurance company.

Your company is considering offering an index annuity product where interest rate credited is max (0%, min (the index return of T&T 400, a cap)).

Currently you are considering offering an annual cap of 5%. The index annuity is renewable annually.

Your company is planning to replicate the index credit using options.

You asked a junior analyst on your team to collect the information to price the replicating portfolio. He has provided the following:

- Risk free interest rate is 3.0% per annum.
- Dividend rate is 0.0% per annum.
- The current T&T 400 level is 100.
- Based on the implied volatility (IV) for options of 1year maturity, he has fitted an implied volatility function IV(K) = 15% + (K-100) *1.4%, where K is the strike level.
- Your company is budgeting 0.5% of the initial contract deposit for the replicating portfolio.

(a)

(i) Explain volatility smiles.

ANSWER:

(b) Describe the most salient characteristics of the equity volatility smile.

ANSWER:

(c) Identify the trades of the replicating portfolio.

ANSWER:

For part (d), identify the formula used (with reference to the formula # in the Formula Sheet as provided.)

(d) Calculate the price for the replicating portfolio and determine whether the budget is sufficient for the hedging, using the fitted implied volatility function IV(K) provided.

The response for this part is required on the paper provided to you.

(e) Explain the reasonableness of the implied volatility function IV(K) in the context of smile arbitrage.

ANSWER:

As an improvement to the current product offering, the product actuary would like to add a guarantee so that the index credit cap would be at least 3%. The cap is reset so that the total replication cost equals the 0.5% budget.

(f)

(i) Identify types of market conditions that would negatively affect the ability to manage the product with the added guarantee.

ANSWER:

(ii) Suggest a modeling approach to better measure the risk.

ANSWER:

QFI QF Fall 2020 Question 8

Learning Outcomes:

d) Understand Stochastic Calculus theory and technique used in pricing derivatives

e) Understand and apply the concepts of risk-neutral measure, forward measure, normalization, and the market price of risk

Source References:

- *Options, Futures, and Other Derivatives*, Hull, John C., 11th Edition, 2021, Chapter 31, pages 721-724
- *Problems and Solutions in Mathematical Finance: Stochastic Calculus*, Chin, Eric, Nel, Dian and Olafsson, Sverrir, 2014, pages 132-137

Question:

Consider the Cox, Ingersoll and Ross (CIR) model for the short-term interest rate r_t:

$$dr_t = \gamma(\bar{r} - r_t)dt + \sqrt{ar_t}\,dX_t \text{ for } t > 0 \text{ with } r_0 > 0$$

where $\gamma, \bar{r}$, and a are constants such that $\gamma\bar{r} > \frac{1}{2}a$, $a > 0$, and X_t is a standard Brownian motion.

(a) Explain why interest rates are always positive in this model.

(b) Show that $r_t = e^{-\gamma t} r_0 + \bar{r}(1 - e^{-\gamma t}) + \sqrt{a} e^{-\gamma t} \int_0^t e^{\gamma s} \sqrt{r_s}\, dX_s$.

(c) Determine $E[r_t]$ and $Var[r_t]$.

Recall that the price at time t of a zero-coupon bond with \$1 principal with maturity date T is given by

$$Z(r_t, t, T) = e^{A(t,T) - B(t,T) r_t},$$

where $A(t,T)$ and $B(t,T)$ are functions of t and T and $A(T,T) = B(T,T) = 0$.

(d) Express $\frac{\partial Z}{\partial t}$, $\frac{\partial Z}{\partial r}$ and $\frac{\partial^2 Z}{\partial r^2}$ in terms of $Z(r_t,t,T)$, $A(t,T)$ and $B(t,T)$.

(e) Show that

(i) $\frac{\partial A}{\partial t} = \gamma \bar{r} B(t,T)$

(ii) $\frac{\partial B}{\partial t} = \gamma B(t,T) + \frac{1}{2} a B(t,T)^2 - 1$

QFI QF Fall 2020 Question 12

Learning Outcomes:

a) Understand the principles of no-arbitrage and replication in asset pricing

d) Understand Stochastic Calculus theory and technique used in pricing derivatives

e) Understand and apply the concepts of risk-neutral measure, forward measure, normalization, and the market price of risk

Source References:

- *Options, Futures, and Other Derivatives*, Hull, John C., 11th Edition, 2021, Chapters 14, 28, pages 327, 675
- *Problems and Solutions in Mathematical Finance: Stochastic Calculus*, Chin, Eric, Nel, Dian and Olafsson, Sverrir, 2014, pages 97-98, 221-227

Question:

Consider a short rate process given by

$$dr = m^*(r,t)dt + s(r,t)dX_t,$$

where X_t is a standard Brownian motion under the risk-neutral measure $\mathbb{Q}$.

Consider pricing functions of two traded securities with maturity at time T, $V(r,t)$ and $Z(r,t)$, whose risk-neutral processes are given by

$$dZ(r,t) = rZ(r,t)dt + \sigma_z Z(r,t)dX_t,$$

$$dV(r,t) = rV(r,t)dt + \sigma_V V(r,t)dX_t,$$

where $\sigma_z = \sigma_z(r,t)$ and $\sigma_v = \sigma_v(r,t)$ are two volatility functions.

By no arbitrage conditions, both securities satisfy Fundamental Pricing Equations,

$$rV = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial r} m^*(r,t) + \frac{1}{2}\frac{\partial^2 V}{\partial r^2} s(r,t)^2$$

$$rZ = \frac{\partial Z}{\partial t} + \frac{\partial Z}{\partial r} m^*(r,t) + \frac{1}{2}\frac{\partial^2 Z}{\partial r^2} s(r,t)^2 .$$

Consider the following renormalization using the numeraire $Z(r,t)$

$$\widetilde{V}(r,t) = \frac{V(r,t)}{Z(r,t)}$$

(a) Show, using Ito's Lemma, that

$$\frac{d\widetilde{V}}{\widetilde{V}} = (\sigma_Z^2 - \sigma_V \sigma_Z)dt + (\sigma_V - \sigma_Z)dX_t$$

Using Girsanov's theorem we can construct a forward risk neutral measure $\mathbb{Q}^Z$ such that $\widetilde{X}_t = X_t - \int_0^t \sigma_Z(r,u)du$ is a standard Brownian motion under $\mathbb{Q}^Z$. You are also given that $\widetilde{V}$ satisfies a partial differential equation that is similar to the Fundamental Pricing Equation, namely

$$0 = \frac{\partial \widetilde{V}}{\partial t} + \frac{\partial \widetilde{V}}{\partial r}\Big(m^*(r,t) + \sigma_Z s(r,t)\Big) + \frac{1}{2}\frac{\partial^2 \widetilde{V}}{\partial r^2} s(r,t)^2 .$$

(b) Show that $\widetilde{V}$ is a martingale under $\mathbb{Q}^Z$ using:

(i) the result in part (a);

(ii) the Feynman-Kac theorem.

(c) Derive expressions for σ_z and σ_v in terms of $s(r,t)$, V, and Z.

QFI QF Spring 2021 Question 1

Learning Outcomes:

d) Understand Stochastic Calculus theory and technique used in pricing derivatives

Source References:

- *Options, Futures, and Other Derivatives*, Hull, John C., 11th Edition, 2021, Chapters 14, 28, pages 329, 671-675
- *Problems and Solutions in Mathematical Finance: Stochastic Calculus*, Chin, Eric, Nel, Dian and Olafsson, Sverrir, 2014, pages 72-73, 221-227

Question:

Under the real-world measure $\mathbb{P}$, the risk-free asset B_t and the stock price S_t satisfy the following processes:

$$dB_t = 0.01 B_t dt, \quad 0 \le t \le 1$$

$$dS_t = \begin{cases} 0.05 S_t dt + 0.2_t dW_t, & 0 \le t \le 0.5, \\ -0.05 S_t dt + 0.3 S_t dW_t, & 0.5 < t \le 1, \end{cases}$$

with $B_0 = S_0 = 1$, where $\{W_t : 0 \le t \le 1\}$ is a standard Brownian motion under $\mathbb{P}$. Let $\mathbb{Q}$ represent the risk-neutral measure.

(a) Determine the market price of risk for all $t \le 1$.

(b) Calculate $E^{\mathbb{P}}\left[S_1 | S_{0.5} \right]$.

(c) Derive the Radon-Nikodym derivative of the risk-neutral measure $\mathbb{Q}$ with respect to the real-world measure $\mathbb{P}$.

(d) Show that $\{S_t e^{-0.01t} : 0 \le t \le 1\}$ is a $\mathbb{Q}$-martingale.

QFI QF Spring 2021 Question 2

Learning Outcomes:

d) Understand Stochastic Calculus theory and technique used in pricing derivatives

Source References:

- *Options, Futures, and Other Derivatives*, Hull, John C., 11th Edition, 2021, Chapters 14, 28, pages 327, 675
- *Problems and Solutions in Mathematical Finance: Stochastic Calculus*, Chin, Eric, Nel, Dian and Olafsson, Sverrir, 2014, pages 128-130

Question:

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\{W_t\}_{t\geq 0}$ be a standard Brownian motion with respect to the filtration $\{\mathcal{F}_t\}_{t\geq o}$.

(a) Evaluate $Var^{\mathbb{P}}\left[|W_t|\right]$.

(b) Determine integer k that makes W_t^k a martingale.

QFI QF Spring 2021 Question 3

Learning Outcomes:

d) Understand Stochastic Calculus theory and technique used in pricing derivatives

Source References:

- *Options, Futures, and Other Derivatives*, Hull, John C., 11th Edition, 2021, Chapters 14, 28, pages 327, 675
- *Problems and Solutions in Mathematical Finance: Stochastic Calculus*, Chin, Eric, Nel, Dian and Olafsson, Sverrir, 2014, pages 128-130

Question:

Let W_t be a standard Wiener process defined on the interval $[0, T]$ with $\{\mathcal{F}_t\}_{0\leq t\leq T}$ its natural filtration.

You are given that for a standard normal random variable Z, we have $E\left[Z^4\right] = 3$.

(a) Derive $E\left[W_s^3 W_t\right]$ for $t > s$.

(b) Determine the value of c such that $W_t^3 - ctW_t$ is a martingale.

(c) Show that $X_t = \int_0^t W_u du$ is not a martingale.

Let $V = \int_0^1 e^{-s} dW_s$ and $Y = \int_0^2 e^{-s} dW_s$.

(d) Calculate

(i) $E[V^2]$

(ii) $E[VY]$

QFI QF Spring 2021 Question 4

Learning Outcomes:

a) Understand the principles of no-arbitrage and replication in asset pricing

d) Understand Stochastic Calculus theory and technique used in pricing derivatives

f) Understand option pricing techniques

Source References:

- *Options, Futures, and Other Derivatives*, Hull, John C., 11th Edition, 2021, Chapter 14, page 327
- *Problems and Solutions in Mathematical Finance: Stochastic Calculus*, Chin, Eric, Nel, Dian and Olafsson, Sverrir, 2014, pages 221-227
- INV201-101-25: Chapter 6 of Introduction to Stochastic Finance with Market Examples by Privault

Question:

Assume that the price S_t at time $t \geq 0$ of a non-dividend paying stock follows the stochastic differential equation below:

$$\frac{dS_t}{S_t} = 0.045\, dt + \sigma\, dW_t$$

where σ is a positive constant and W_t is a standard Wiener process under the risk-neutral probability measure.

You are given that there is a real number c such that:

$$\frac{d\left(S_t\right)^c}{\left(S_t\right)^c} = 0.18\, dt + 0.6\, dW_t$$

(a) Show, using Ito's lemma, that $\sigma = 0.3$.

Recall that for any real number k and a standard normal random variable Z:

$$E\left[Z^2 e^{kZ}\right] = \left(1 + k^2\right) e^{0.5k^2}$$

Consider a derivative security on the stock. The derivative security pays $S_3\left[\ln S_3\right]^2$ at time 3, and nothing at any other time. Assume $S_0 = 1$.

(b) Calculate the time-0 no-arbitrage price of this derivative security.

QFI QF Spring 2021 Question 7

Learning Outcomes:

d) Understand Stochastic Calculus theory and technique used in pricing derivatives

i) Calibrate a model to observed prices of traded securities including fitting to a given yield curve

Source References:

- *Problems and Solutions in Mathematical Finance: Stochastic Calculus*, Chin, Eric, Nel, Dian and Olafsson, Sverrir, 2014, pages 128-130, 132-137
- Calibrating Interest Rate Models (Section 1.1-4.3 excl 4.1.2)

Question:

The short rate *r(t)* follows the following process:

$$dr_t = [v - ar_t]dt + \sigma dX_t$$

where X_t is a standard Brownian motion and v, a, and σ are positive constants.

(a)

(i) Solve the stochastic differential equation.

(ii) Identify the distribution of r_t by providing its mean and variance.

(b) Show that the limiting distribution of r_t as t approaches infinity is $N\left(\frac{v}{a}, \frac{\sigma^2}{2a}\right)$

Assume that $r_m \sim N\left(\frac{v}{a}, \frac{\sigma^2}{2a}\right)$ for some $m > 0$.

(c) Demonstrate that the interest rate, r_{t+m}, follows the same distribution. Hint: Use time frame $(m, t+m)$ from solution of part (a).

(d)

(i) Estimate the parameters for interest rate process above.

(ii) Describe for the estimation of arbitrage free parameters using the table below observed in the market.

Hint: In a linear regression on time series $x_i = a + \beta x_{i-1}$, the coefficients are

$$\beta = \frac{n \cdot \sum_{i=1}^{n} x_{i-1} \cdot x_i - \sum_{i=1}^{n} x_i \cdot \sum_{i=1}^{n} x_{i-1}}{n \cdot \sum_{i=1}^{n} x_{i-1}^2 - \left(\sum_{i=1}^{n} x_{i-1}\right)^2}$$

$$a = \frac{\left(\sum_{i=1}^{n} x_i - \beta \sum_{i=1}^{n} x_{i-1}\right)}{n}$$

	t	r_i
0	0	0.03
1	0.25	0.034735

$\sum_{i=1}^{20} r_i = 0.821151$

$\sum_{i=1}^{20} r_{i-1} = 0.834376$

2	0.5	0.015108
3	0.75	0.049016
4	1	0.035745
5	1.25	0.03859
6	1.5	0.070412
7	1.75	0.037342
8	2	0.016796
9	2.25	0.057236
10	2.5	0.027911
11	2.75	0.044456
12	3	0.046928
13	3.25	0.06747
14	3.5	0.07093
15	3.75	0.040857
16	4	0.031479
17	4.25	0.042559
18	4.5	0.050222
19	4.75	0.026584
20	5	0.016775

$$\sum_{i=1}^{20} r_i^2 = 0.039037$$

$$\sum_{i=1}^{20} r_{i-1}^2 = 0.03965$$

$$\sum_{i=1}^{20} r_{i-1} \cdot r_i = 0.034693$$

$$Var(r) = \frac{1}{20}\sum_{i=1}^{20} r_i^2 - \left(\frac{1}{20}\sum_{i=1}^{20} r_i\right)^2$$

QFI QF Spring 2021 Question 10

Learning Outcomes:

d) Understand Stochastic Calculus theory and technique used in pricing derivatives

e) Understand and apply the concepts of risk-neutral measure, forward measure, normalization, and the market price of risk

Source References:

- *Options, Futures, and Other Derivatives*, Hull, John C., 11th Edition, 2021, Chapters 29, 31, pages 688-68, 722-723
- *Problems and Solutions in Mathematical Finance: Stochastic Calculus*, Chin, Eric, Nel, Dian and Olafsson, Sverrir, 2014, pages 132-137

Question:

Consider a Vasicek model of interest rates as follows:

$$dr_t = \gamma\left(\bar{r} - r_t\right)dt + \sigma dX_t,$$
$$m\left(r, t\right) = \gamma\left(\bar{r} - r_t\right)$$

And the arbitrage-free condition for long-short bond portfolio with Z_1, Z_2 is shown

$$\frac{\frac{\partial Z_1}{\partial t} + \frac{1}{2}\frac{\partial^2 Z_1}{\partial r_t^2} - r_t Z_1}{\partial Z_1 / \partial r_t} = \frac{\frac{\partial Z_2}{\partial t} + \frac{1}{2}\frac{1\partial^2 Z_2}{\partial r_t^2} - r_t Z_2}{\partial Z_2 / \partial r_t} = -m^*\left(r, t\right)$$

(a) Compare $m\left(r, t\right)$ with an arbitrage-free parameter $m^*\left(r, t\right)$ and explain the meaning of the parameters when $m^*\left(r, t\right) = \gamma^*\left(\bar{r}^* - r\right)$.

Consider the following parameters estimated as follows:

$$\gamma = 0.3262; \bar{r} = 5.09\%; \sigma = 2.21\%; \gamma^* = 0.4653; \bar{r}^* = 6.34\%$$

Assume today's short-term interest rate is $r_0 = 2\%$ and the bond pricing formula of Vasicek model

$$Z\left(r, t; T\right) = e^{A(t;T) - B(t;T)r}$$

where $B\left(t; T\right) = \frac{1}{\gamma^*}\left(1 - e^{-\gamma^*(T-t)}\right)$

and $A\left(t; T\right) = B\left(t; T\right) - \left(T - t\right)\left(\bar{r}^* - \frac{\sigma^2}{2\gamma^*}\right) - \frac{\sigma^2 B\left(t; T\right)^2}{4\gamma^*}$

(b) Show that $E\left[\frac{dZ / dt}{Z}\right] = E\left(r_t\right) + \frac{\sigma^2 B}{2\gamma^*}\left(1 - \gamma^*\right)$ using Ito's lemma.

(c) Compute $E\left[\frac{dZ / dt}{Z}\right]$ on zero-coupon bond with 10 year to maturity.

You are given $Z\left(r_0, 0; 1\right) = 0.975$ and $Z\left(r_0, 0; 5\right) = 0.898$.

(d) Calculate the value of a call option with 1 year to maturity $\left(T_0 = 1\right)$, strike price $K = 0.9$, written on a zero-coupon bond with 5 years to maturity.

QFI QF Fall 2021 Question 1

Learning Outcomes:

d) Understand Stochastic Calculus theory and technique used in pricing derivatives

Source References:

- *Options, Futures, and Other Derivatives*, Hull, John C., 11th Edition, 2021, Chapter 14, page 327
- *Problems and Solutions in Mathematical Finance: Stochastic Calculus*, Chin, Eric, Nel, Dian and Olafsson, Sverrir, 2014, pages 128-130

Question:

Let W_t be a standard Wiener process defined on the interval $[0, T]$. For $t \in [0, T]$, let X_t be defined as

$$X_t = \int_0^t W_u du.$$

(a) Explain why X_t is a normally distributed random variable for $t > 0$.

(b) Compute

(i) $E[X_t]$.

(ii) $Var[X_t]$.

Let Y_t be defined as

$$Y_t = \int_0^t \sqrt{|W_u|} dW_u.$$

(c) Compute $Var[Y_t]$.

QFI QF Fall 2021 Question 2

Learning Outcomes:

b) Understand Arrow-Debreau security and the distinction between complete and incomplete markets

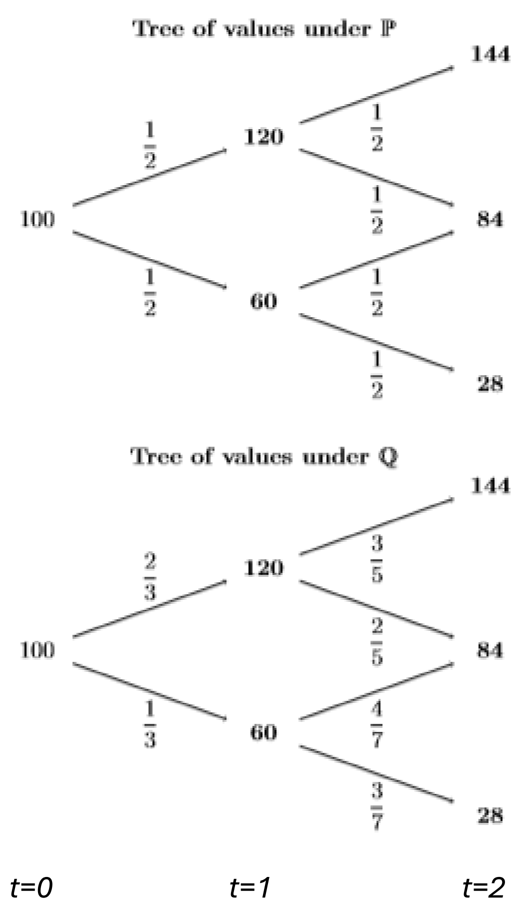
f) Understand option pricing techniques

Source References:

- INV201-100-25: Chapter 5 of Financial Mathematics – A Comprehensive Treatment, Campolieti
- INV201-101-25: Chapter 6 of Introduction to Stochastic Finance with Market Examples by Privault

Question:

Consider a price process A_t that evolves in discrete, integer time t according to a recombining binomial tree. The set of all possible values of the process are shown below for $t = \{0, 1, 2\}$ under two different probability measures $\mathbb{P}$ and $\mathbb{Q}$. Assume the risk-free rate is 0%.



(a) Show that $\mathbb{P}$ and $\mathbb{Q}$ are equivalent probability measures on the probability space implied by the price process A_t.

(b) Determine if the price process A_t is a:

(i) $\mathbb{Q}$-martingale.

(ii) $\mathbb{P}$-martingale.

(c) Calculate the values of the Radon-Nikodym derivative $\frac{d\mathbb{Q}}{d\mathbb{P}}$ for all paths through the tree, (i.e. up-up, up-down, down-up, down-down nodes).

(d) Evaluate the process $\xi_t = E^{\mathbb{P}}\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\middle|\mathcal{F}_t\right)$ at time $t = 1$ for both up and down nodes where $\mathcal{F}_t$ is the filtration history up to time t.

Consider a claim X that pays a fixed amount of \$20 at time $t = 2$ if the price process A_t attains a value of more than \$75, otherwise it pays nothing.

(e) Show numerically that $E^{\mathbb{Q}}\left[X\right] = E^{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}X\right]$ at time 0 by using the results in part (d).

QFI QF Fall 2021 Question 3

Learning Outcomes:

a) Understand the principles of no-arbitrage and replication in asset pricing.

c) Understand put-call parity and price bounds

d) Understand Stochastic Calculus theory and technique used in pricing derivatives

e) Understand and apply the concepts of risk-neutral measure, forward measure, normalization, and the market price of risk

f) Understand option pricing techniques

Source References:

- *Options, Futures, and Other Derivatives*, Hull, John C., 11th Edition, 2021, Chapters 11, 14, 28, pages 255-256, 327, 675

- *Problems and Solutions in Mathematical Finance: Stochastic Calculus*, Chin, Eric, Nel, Dian and Olafsson, Sverrir, 2014, pages 186-188, 221-227

Question:

Consider one risky asset S_t and one risk-free asset B_t with risk-neutral price dynamics:

$$dS_t = rS_t dt + \sigma S_t dW_t^{\mathbb{Q}}$$
$$dB_t = rB_t dt, B_0 = 1$$

where $W_t^{\mathbb{Q}}$ is a standard Wiener process under the risk-neutral measure $\mathbb{Q}$ and r, σ are constants.

Let X be a European claim with exercise time T, and price $\pi_t(X)$ at time $t < T$. Define $\pi_t^d(X) = B_t^{-1}\pi_t(X)$ to be its discounted price.

Suppose the claim can be replicated by a self-financing portfolio consisting of α_t units in the risky asset S_t and β_t units in the risk-free asset B_t, i.e.:

$$\pi_t(X) = \alpha_t S_t + \beta_t B_t,$$
$$d\pi_t(X) = \alpha_t dS_t + \beta_t dB_t.$$

(a)

(i) Determine the stochastic differential equation satisfied by the discounted price process $S_t^d = B_t^{-1}S_t$.

(ii) Explain why $\pi_T^d(X) = \pi_t^d(X) + \int_t^T \alpha_u \sigma S_u^d dW_u^{\mathbb{Q}}$.

(iii) Show that the discounted derivative prices $\pi_t^d(X), t < T$ form a $\mathbb{Q}$-martingale using part (a) (ii).

Let $C_t(K,T)$ be the price of the call option at time t with strike price K and maturity T and $P_t(K,T)$ be the price of the put option at time t with the same maturity and strike price.

(b) Prove that $C_t(K,T) - P_t(K,T) = S_t - Ke^{-r(T-t)}, t < T$.

A European chooser option, V, with exercise time $T_c < T$ is the option to choose, at time T_c, between a put and a call with identical maturity T and strike price K. Its payoff at T_c is $\max\left(P_{T_c}, C_{T_c}\right)$.

(c) Show that $\pi_t^d(V) = P_t(K,T) + C_t\left(Ke^{-r(T-T_c)}, T_c\right), t < T_c$.

QFI QF Fall 2021 Question 8

Learning Outcomes:

d) Understand Stochastic Calculus theory and technique used in pricing derivatives

e) Understand and apply the concepts of risk-neutral measure, forward measure, normalization, and the market price of risk

i) Calibrate a model to observed prices of traded securities including fitting to a given yield curve

Source References:

- *Options, Futures, and Other Derivatives*, Hull, John C., 11th Edition, 2021, Chapter 14, 31, pages 327, 721-723
- *Problems and Solutions in Mathematical Finance: Stochastic Calculus*, Chin, Eric, Nel, Dian and Olafsson, Sverrir, 2014, pages 132-137, 221-227
- Calibrating Interest Rate Models (Section 1.1-4.3 excl 4.1.2)

Question:

You were given the short rate model with the following stochastic differential equation:

$$dr_t = a(b - r_t)dt + \sigma dW_t$$

where *a, b,* σ are constant and W_t is a standard Wiener process with respect to the measure $\mathbb{P}$.

Let $B = B(r,t,T)$ be the price at time *t* of the default-free discount bond with maturity *T*.

Consider a self-financing portfolio, with a long position of η units of a short-term bond with maturity date T_1 and a short position of θ units of long-term bond with maturity date $T_2 > T_1$ chosen to eliminate interest rate risk.

$$P = \eta B_1(t, T_1) - \theta B_2(t, T_2)$$
$$dP = \eta B_1(t, T_1) - \theta dB_2(t, T_2)$$

(a) Show that

$$\frac{r_t B_1 - \frac{\partial B_1}{\partial t} - \frac{1}{2}\frac{\partial^2 B_1}{\partial r^2}\sigma^2}{\frac{\partial B_1}{\partial r}} = \frac{r_t B_2 - \frac{\partial B_2}{\partial t} - \frac{1}{2}\frac{\partial^2 B_2}{\partial r^2}\sigma^2}{\frac{\partial B_2}{\partial r}}$$

The response for this part is required on the paper provided to you.

Denote by $m(t) = \frac{r_t B - \frac{\partial B}{\partial t} - \frac{1}{2}\frac{\partial^2 B}{\partial r^2}\sigma^2}{\frac{\partial B}{\partial r}}$,which is independent of the maturity of the bond from the result in part (a), and define the market price of risk $\lambda(r_t, t) = \frac{a(b - r_t) - m(t)}{\sigma}$.

(b) Show that the price of a default-free discount bond satisfies the following partial differential equation

$$\frac{\partial B}{\partial r}\left[a(b - r_t) - \sigma\lambda\right] + \frac{\partial B}{\partial t} + \frac{1}{2}\frac{\partial^2 B}{\partial r^2}\sigma^2 - r_t B = 0$$

The response for this part is required on the paper provided to you.

(c) Describe the key features of this interest rate model.

The response for this part is required on the paper provided to you.

You are given data on one month T-bill rates from 1960 to 2010.

(d) Explain how to estimate the interest rate model parameters, using the given data. Identify the estimated parameters that can be used in pricing interest rate derivative.

The response for this part is required on the paper provided to you.

Given the following information:

- Long-run mean of the spot rate: $b = 0.05$
- Speed of mean reversion: $a = 0.25$
- Absolute interest rate volatility: $\sigma = 0.015$
- Market price of interest rate risk: $\lambda = -0.1$

(e) Calculate the default-free discount bond price with 30-year maturity with $r = 0.1\%$, 5%, and 10%, respectively.

The response for this part is to be provided in the Excel spreadsheet.

(f) Generate the yield curves for the same set of spot rates in part (e) with different maturities,1 through 30 years.

The response for this part is to be provided in the Excel spreadsheet.

QFI QF Fall 2022 Question 1

Learning Outcomes:

d) Understand Stochastic Calculus theory and technique used in pricing derivatives

Source References:

- *Options, Futures, and Other Derivatives*, Hull, John C., 11th Edition, 2021, Chapter 14, 28, pages 327, 675
- *Problems and Solutions in Mathematical Finance: Stochastic Calculus*, Chin, Eric, Nel, Dian and Olafsson, Sverrir, 2014, pages 52, 72-73

Question:

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\{W_t\}_{t\geq 0}$ be a standard Wiener process with respect to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$.
Also assume a constant risk-free rate $r > 0$.

(a) List the criteria for a stochastic process to be a martingale with respect to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$.

Let $X_t = \int_0^t {W_s}^2 ds + \alpha_t$, where $\alpha_t = f(W_t, t)$.

(b) Derive a stochastic differential equation for X_t using Ito's Lemma.

Assume from this point on that $\alpha_t = -t{W_t}^2 + \beta_t$, with β_t a deterministic function of time.

(c) Identify an appropriate β_t, if it exists, that makes X_t a martingale.

(d) Calculate $E({W_t}^4)$ using Ito's Lemma.

QFI QF Fall 2022 Question 2

Learning Outcomes:

a) Understand the principles of no-arbitrage and replication in asset pricing

d) Understand Stochastic Calculus theory and technique used in pricing derivatives

e) Understand and apply the concepts of risk-neutral measure, forward measure, normalization, and the market price of risk

f) Understand option pricing techniques

Source References:

- *Options, Futures, and Other Derivatives*, Hull, John C., 11th Edition, 2021, Chapters 13, pages 296-298
- *Problems and Solutions in Mathematical Finance: Stochastic Calculus*, Chin, Eric, Nel, Dian and Olafsson, Sverrir, 2014, pages 221-227
- Understanding the Connection Between Real-World and Risk-Neutral Generators, SOA Research

Question:

You are considering investing in an asset S that is known to attain the following discrete prices:

$$S_0 = 100,$$

$$S_t = \begin{cases} 1.3S_{t-1} \text{ with probability } 0.7 - t/20 \\ \alpha S_{t-1} \text{ with probability } 0.3 + t/20 \end{cases} \quad \text{(for } t = 1, 2, \dots, 10\text{)},$$

where α is a positive constant.

Assume that the above probabilities are under the real-world measure and the annual effective risk-free rate of interest r is 5%.

(a) Determine the range of α so that there is no arbitrage opportunity.

Let d be the number of annual down-movements for $t \leq 4$.

(b) Derive the price S_4 as a function of d and give the possible range of S_4.

Let $\alpha = 0.9$ and consider a double barrier option Z that pays 100 at $t = 5$ if S_t either exceeds 290 or falls below 70 at any time $t \leq 5$.

(c) Calculate the real-world probability that the double barrier option will be exercised.

(d) Calculate the risk-neutral probability of an up-movement in the price of the asset S.

(e) Calculate the price Z_0 of the double barrier option at $t = 0$ under the risk-neutral measure.

QFI QF Fall 2022 Question 4

Learning Outcomes:

d) Understand Stochastic Calculus theory and technique used in pricing derivatives

Source References:

- *Options, Futures, and Other Derivatives*, Hull, John C., 11th Edition, 2021, Chapter 14, page 327
- *Problems and Solutions in Mathematical Finance: Stochastic Calculus*, Chin, Eric, Nel, Dian and Olafsson, Sverrir, 2014, pages 52, 128-130

Question:

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\{W_t\}_{t\geq 0}$ be a standard Wiener process.

(a) Show that $Z_t = \int_0^t W_u du$ is a normally distributed random variable.

Define a stochastic process $\{Y_t\}_{t\geq 0}$ given by $Y_0 = 0,\ Y_t = \frac{\sqrt{3}}{t} Z_t$ for $t > 0$.

(b) Determine whether Y_t is a Wiener process with respect to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$.

Suppose S_t satisfies the Geometric Brownian Motion model:

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma\, dW_t,$$

with μ and σ both constant, and define a new stochastic process:

$$G_t = e^{\frac{1}{t}\int_0^t \ln(S_u)\,du}, G_0 = S_0.$$

(c) Derive an expression for dG_t in terms of dY_t.

QFI QF Spring 2023 Question 2

Learning Outcomes:

d) Understand Stochastic Calculus theory and technique used in pricing derivatives

e) Understand and apply the concepts of risk-neutral measure, forward measure, normalization, and the market price of risk

f) Understand option pricing techniques

Source References:

- *Options, Futures, and Other Derivatives*, Hull, John C., 11th Edition, 2021, Chapters 14, 28, pages 327, 675
- *Problems and Solutions in Mathematical Finance: Stochastic Calculus*, Chin, Eric, Nel, Dian and Olafsson, Sverrir, 2014, pages 221-227, 233-242
- INV201-101-25: Chapter 6 of Introduction to Stochastic Finance with Market Examples by Privault

Question:

Let $(\Omega, \mathcal{F}, \mathbb{Q})$ be a probability space and let $W(t)$ be a standard Brownian motion with respect to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Suppose that the risk-free rate is 0% and that $M(t)$ is a risky asset such that:

$$M(t) = M(0)e^{\alpha W(t) - \frac{\alpha^2 t}{2}},\ 0 < M(0) < \infty, \text{ and } \alpha \text{ is a constant.}$$

(a) Show that $M(t)$ is a $\mathbb{Q}$-martingale using each of the following approaches:

a. Deriving the stochastic dynamics of $M(t)$.

b. Applying the definition of a martingale.

Consider another risky asset $A(t) = A(0)e^{\vartheta W(t) - \frac{\vartheta^2 t}{2}}, \quad 0 < A(0) < \infty$ and ϑ is a constant and $\vartheta \neq \alpha$.

Define a new measure $\mathbb{Q}^A$ so that A is the numeraire asset under $\mathbb{Q}^A$.

(b) Write down the Radon-Nikodym derivative of $\mathbb{Q}^A$ with respect to $\mathbb{Q}$.

(c) Determine, using Ito's lemma and Girsanov Theorem, whether the normalized process $\frac{M(t)}{A(t)}$ is a $\mathbb{Q}$-martingale or a $\mathbb{Q}^A$-martingale.

(d) Derive an expression for today's price of an exchange option with payoff $P(T) = max[0, M(T) - A(T)]$.

QFI QF Spring 2023 Question 3

Learning Outcomes:

d) Understand Stochastic Calculus theory and technique used in pricing derivatives

e) Understand and apply the concepts of risk-neutral measure, forward measure, normalization, and the market price of risk

Source References:

- *Options, Futures, and Other Derivatives*, Hull, John C., 11th Edition, 2021, Chapters 28, pages 675
- *Problems and Solutions in Mathematical Finance: Stochastic Calculus*, Chin, Eric, Nel, Dian and Olafsson, Sverrir, 2014, pages 52, 221-227

Question:

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\{W_t\}_{t\geq 0}$ and $\{V_t\}_{t\geq 0}$ be standard Brownian motions with respect to the filtration $\mathcal{F}_t$. Let S_t and E_t denote a Canadian asset price quoted in USD and the USD-to-CAD exchange rate, respectively, with the following dynamics:

$$dS_t = \mu S_t dt + \sigma_S S_t dW_t$$
$$dE_t = \sigma_E E_t dV_t$$
$$dW_t dV_t = \rho dt$$

where $\mu, \sigma_S, \sigma_E, \rho$ are constant and $\rho > -1$.

(a) Show that

$$Z_t = \frac{\sigma_S W_t + \sigma_E V_t}{\sqrt{\sigma_S{}^2 + \sigma_E{}^2 + 2\rho\sigma_S\sigma_E}}$$

is a $\mathbb{P}$-standard Brownian motion.

Let r be the Canadian risk-free rate.

(b) Determine whether $\ln(S_t E_t)$ follows an arithmetic Brownian motion under the measure $\mathbb{P}$ or not.

(c) Show that $e^{-rt} S_t E_t$ is a martingale under the risk-neutral measure $\mathbb{Q}$ using Girsanov Theorem, with the numeraire being CAD risk-free assets.

QFI QF Spring 2023 Question 5

Learning Outcomes:

i) Calibrate a model to observed prices of traded securities including fitting to a given yield curve

Source References:

- Calibrating Interest Rate Models (Section 1.1-4.3 excl 4.1.2)

Question:

Using the current yield curve and cap prices you have calibrated the Hull-White model below

$$dr_t = (\theta_t - \gamma r_t)dt + \sigma dX_t.$$

The data set contains 20 treasury bond prices and 5 cap prices. You fitted the following curve for $r(0, t)$ (continuously compounded yield between 0 and t)

$$r(0,t) = 0.01091858598 + 0.01251008594\, t - 0.000140114635\, t^2 + 0.005654825\, t^3$$

and estimated values of γ, σ to be 0.19, 0.0196 respectively.

(a) Explain whether the fitted model is a true arbitrage-free model.

(b) Derive an expression for the instantaneous forward rate at time 0 $f(0,t)$.

(c) Derive an expression for θ_t .

(d) Compute $E[r_{1.25}|r_1 = 0.03\%]$, given $f(0{,}1.25) = 0.036068$.

QFI QF Spring 2023 Question 7

Learning Outcomes:

d) Understand Stochastic Calculus theory and technique used in pricing derivatives

Source References:

- *Problems and Solutions in Mathematical Finance: Stochastic Calculus*, Chin, Eric, Nel, Dian and Olafsson, Sverrir, 2014, pages 132-137

Question:

You have been asked to consider the following interest rate model for valuing a zero-coupon bond.

The Cox, Ingersoll and Ross (CIR) model for the short-term interest rate r_t:

$dr_t = \gamma^*(\overline{r^*} - r_t)dt + \sqrt{\alpha r_t}dX_t$ where γ^*, $\overline{r^*}$ and α are positive constant

with $\gamma^*\overline{r^*} > \frac{1}{2}\alpha$, and X_t is a Standard Brownian motion.

Let $Y_t = \ln(r_t)$.

(a)

(i) Show by using Ito's lemma that

$$dY_t = \left[\left(\gamma^* \overline{r^*} - \frac{1}{2}\alpha\right) e^{-Y_t} - \gamma^*\right] dt + \sqrt{\alpha} e^{-\frac{Y_t}{2}} dX_t.$$

(ii) Explain why the drift term of dY_t is positive if Y_t gets too far below from 0.

QFI QF Fall 2023 Question 2

Learning Outcomes:

d) Understand Stochastic Calculus theory and technique used in pricing derivatives

Source References:

- *Problems and Solutions in Mathematical Finance: Stochastic Calculus*, Chin, Eric, Nel, Dian and Olafsson, Sverrir, 2014, pages 52, 72-73

Question:

Let $(\Omega, \mathcal{F}, \mathbb{Q})$ be a probability space and let $W(t)$ be a standard Brownian motion with respect to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$.

(a) Evaluate the following expressions for $0 < s < t < u$:

a. $E^{\mathbb{Q}}(W(s)\, W(t) W(u))$

b. $E^{\mathbb{Q}}(W(t) W(u) \mid \mathcal{F}_s)$

Consider the following processes:

- $V(t)$, another $\mathbb{Q}$-Brownian motion that is independent of $W(t)$ and
- $X(t)$, defined as $(V(t))^2 W(t) - \int_0^t W(s) ds$.

(b) Determine whether $X(t)$ is a martingale under $\mathbb{Q}$ using Ito's lemma.

(c) Determine whether $X(t)$ is a martingale under $\mathbb{Q}$ using the definition of a martingale.

QFI QF Fall 2023 Question 5

Learning Outcomes:

h) Understand and apply numerical discretization methods to price options including Euler-Maruyama discretization and transition density methods

Source References:

- Calibrating Interest Rate Models (Section 1.1-4.3 excl 4.1.2)

Question:

You are simulating real-world interest rate paths on every trading day from the following Vasicek model with parameters:

$$dr_t = \gamma(\bar{r} - r_t)dt + \sigma dX_t$$

where $\bar{r} = 5\%, \gamma = 0.3, \sigma = 0.06$, and X_t is a standard Brownian motion.

Assume that there are 252 trading days in a year.

You just simulated r_t as 1% for trading day t.

(a) Calculate the probability of simulating a negative interest rate for the next trading day.

You generated a random number -1.96 from the standard normal distribution.

(b) Calculate the simulated rate for the next trading day using

(i) the Euler-Maruyama discretization method.

(ii) the transition density method.

(c) Compare and contrast the Euler-Maruyama discretization method and the transition density method for simulating interest rate paths in general and in this particular case for Vasicek model.

QFI QF Spring 2024 Question 1

Learning Outcomes:

a) Understand the principles of no-arbitrage and replication in asset pricing.

b) Understand Arrow-Debreau security and the distinction between complete and incomplete markets

f) Understand option pricing techniques

Source References:

- *Options, Futures, and Other Derivatives*, Hull, John C., 11th Edition, 2021, Chapter 13, pages 292-293
- INV201-100-25: Chapter 5 of Financial Mathematics – A Comprehensive Treatment, Campolieti

Question:

Let $S(0) = 100$ be the price of a security at time 0. After 6 months, the price of the security can be either 120 or 60. Let r be the nominal semi-annual risk-free rate of return.

(a) Determine the range of r such that there are no arbitrage possibilities.

For the remainder of the question, set $r = 0.06$.

(b) Calculate and interpret the state prices.

(c) Calculate the no-arbitrage price of a European call option with strike price of 100 that expires in 6 months.

(d) Describe two general situations in which arbitrage opportunities can arise.

Consider the following derivative of the security: the derivative pays 22 in 6 months when the price of the security is 120; the derivative pays 10 in 6 months when the price of the security is 60.

(e) Construct a replicating portfolio and use it to price the derivative.

QFI QF Spring 2024 Question 5

Learning Outcomes:

d) Understand Stochastic Calculus theory and technique used in pricing derivatives

e) Understand and apply the concepts of risk-neutral measure, forward measure, normalization, and the market price of risk

Source References:

- *Options, Futures, and Other Derivatives*, Hull, John C., 11th Edition, 2021, Chapters 28, pages 671-672
- *Problems and Solutions in Mathematical Finance: Stochastic Calculus*, Chin, Eric, Nel, Dian and Olafsson, Sverrir, 2014, pages 128-130, 221-227
- Understanding the Connection Between Real-World and Risk-Neutral Generators, SOA Research, Aug 2022, Sections 1-5, and Appendices A & D

Question:

The Vasicek model of interest rates is

$$dr_t = \gamma(\bar{r} - r_t)dt + \sigma dX_t$$

where γ and σ are positive constants, and X is a standard Brownian motion.

Let F be a filtration generated by r.

(a) Show that

(i) $E(r_t|F_s) = r_s e^{-\gamma(t-s)} + \bar{r}(1 - e^{-\gamma(t-s)})$

(ii) $Var(r_t|F_s) = \frac{\sigma^2}{2\gamma}\left[1 - e^{-2\gamma(t-s)}\right]$

Suppose that under the Vasicek model:

The short rate r is 4% and its real-world process is

$$dr_t = 0.1(0.05 - r_t)dt + 0.01dX_t$$

The risk-neutral process is

$$dr_t = 0.1(0.09 - r_t)dt + 0.01dX_t$$

(b) Determine the market price of interest risk.

The price of a zero-coupon bond with $1 principal at time t with maturity date T is given by

$$Z(r,t;T) = e^{A(t;T)-B(t;T)r}, \text{where } B(t;T) = \frac{1 - e^{-\gamma(T-t)}}{\gamma}.$$

Let Z be the price process with $T = 10$.

(c) Compute the drift and the diffusion of $\frac{dZ}{Z}$ for the risk-neutral process.

(d) Compute the drift and the diffusion of $\frac{dZ}{Z}$ for the real-world process.

QFI QF Spring 2024 Question 6

Learning Outcomes:

h) Understand and apply numerical discretization methods to price options including Euler-Maruyama discretization and transition density methods

i) Calibrate a model to observed prices of traded securities including fitting to a given yield curve

Source References:

- Calibrating Interest Rate Models (Section 1.1-4.3 excl 4.1.2)

Question:

You have one-month daily treasury bill yields (annualized) over 500 consecutive trading days in the daily_data table and 20 treasury bond yields with various maturities in the bond_data table. There are 252 trading days per year. You would like to fit the Vasicek model for the data sets.

Partially completed R codes and outputs are attached to the end of this question.

(a) Describe the assumptions made in the chosen real-world parameter estimation method.

You are using the real-world Vasicek model, $dr = \gamma(\bar{r} - r)dt + \sigma\, dX$ where X is standard Brownian motion.

(b) Estimate the parameters of your model.

(c) Describe the procedure employed in risk-neutral model calibration.

You are using the risk-neutral Vasicek model, $dr = \gamma(\bar{r} - r)dt + \sigma\, dY$ where Y is standard Brownian motion.

(d) Estimate the parameters of your new model.

(e) Determine whether the fitted models are adequate.

QFI QF Fall 2024 Question 1

Learning Outcomes:

d) Understand Stochastic Calculus theory and technique used in pricing derivatives

Source References:

- *Problems and Solutions in Mathematical Finance: Stochastic Calculus*, Chin, Eric, Nel, Dian and Olafsson, Sverrir, 2014, pages 52, 132-137

Question:

Let $\{B_t: 0 \le t \le T\}$ be a Wiener process with respect to the filtration $\{I_t: 0 \le t \le T\}$, where $T > 0$ is some future date.

Let $\{X_t: 0 \le t \le T\}$ and $\{Y_t: 0 \le t \le T\}$ be stochastic processes defined as follows:

$$X_t = \int_0^t 1_{\{B_u > 0\}} dB_u$$
$$Y_t = \int_0^t 1_{\{B_u < 0\}} dB_u$$

where 1_A is the indicator function for event A.

(a) Calculate $E[X_t^2]$ for $t < T$.

(b) Calculate $E[X_t Y_t]$ for $t < T$.

(c)

(i) List the three properties of a martingale.

(ii) Determine whether $\{X_t Y_t : 0 \leq t \leq T\}$ is a martingale with respect to the filtration $\{I_t : 0 \leq t \leq T\}$ by verifying whether all the three properties listed in part (c)(i) hold.

QFI QF Fall 2024 Question 2

Learning Outcomes:

a) Understand the principles of no-arbitrage and replication in asset pricing.

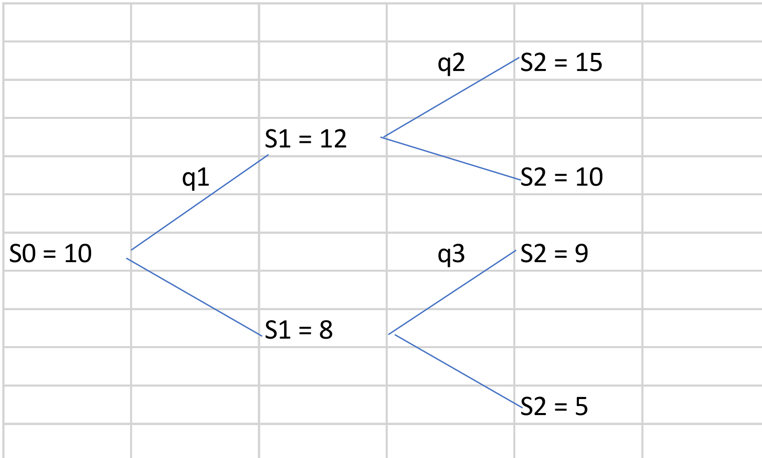
b) Understand Arrow-Debreau security and the distinction between complete and incomplete markets

d) Understand Stochastic Calculus theory and technique used in pricing derivatives

Source References:

- *Options, Futures, and Other Derivatives*, Hull, John C., 11th Edition, 2021, Chapter 13, pages 294-298
- INV201-100-25: Chapter 5 of Financial Mathematics – A Comprehensive Treatment, Campolieti

Question:



Consider a two-period discrete time model where the interest rate for a period is r and the price of the risky asset evolves according to the above diagram.

(a)

(i) Determine the range of r for which this model is arbitrage-free.

(ii) Assess whether this model is complete for the range of r in part (a)(i).

Consider a look-back call option which has the following payoff:

$$C = (\max(S_1, S_2) - K)^+$$

The strike price of the option is $K = 11$.

(b) Calculate the fair price of this option when $r = 1/9$ using the risk-neutral measure.

QFI QF Fall 2024 Question 7

Learning Outcomes:

h) Understand and apply numerical discretization methods to price options including Euler-Maruyama discretization and transition density methods

i) Calibrate a model to observed prices of traded securities including fitting to a given yield curve

Source References:

- Calibrating Interest Rate Models (Section 1.1-4.3 excl 4.1.2)

Question:

You are given one-month daily treasury bill yields (annualized) over 500 consecutive trading days in the daily_data table. There are 252 trading days per year. You would like to fit the CIR model,

$$dr = \gamma(\bar{r} - r)dt + \sqrt{\alpha r}\, dX$$

for the data set.

For this model you are considering the method based on Euler discretization and the method based on the transition density function.

(a) Compare and contrast these two methods.

You are given the attached R code snippets with outputs.

(b) Calculate the estimates of $\gamma, \bar{r}$, and α based on Euler discretization.

(c) Write estimates of $\gamma, \bar{r}$, and α based on the transition density method.

(d) Recommend an estimate method between Euler discretization method and the transition density method.

Partially completed R codes and outputs.

```
guess = c(4,0.1,0.1)
model1 = optim(c(gamma=4,rbar=0.1,alpha=0.1),
               CIR.log.lik,NULL,
               daily_data$Yield,method="BFGS")
model1$par

##     gamma      rbar     alpha
## 7.8697698 0.0533068 0.2674647

options()$nwarnings

## [1] 50

model1$count

## function gradient
##       83       18
```

Partially completed R codes and outputs.

```
# use the data frame
# display header and first three rows
head(daily_data,3)

##        Yield
## 1 0.0200000
## 2 0.0186188
## 3 0.0181607

# display header and last two rows
tail(daily_data,2)

##          Yield
## 499 0.1018005
## 500 0.1038554

N= nrow(daily_data)

euler.est= matrix(NA,3,1)
x1 = daily_data$Yield[1:(N-1)]^(-0.5)
x2 = daily_data$Yield[1:(N-1)]^(0.5)
y = daily_data$Yield[2:N]*x1
model=lm(y~0+x1+x2)
summary(model)

##
## Call:
## lm(formula = y ~ 0 + x1 + x2)
##
## Residuals:
##       Min        1Q    Median        3Q       Max
## -0.037402 -0.010118 -0.000279  0.009438  0.040784
##
## Coefficients:
##     Estimate Std. Error t value Pr(>|t|)
## x1 0.0003346  0.0002136   1.567    0.118
## x2 0.9968652  0.0049121 202.941   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.01455 on 497 degrees of freedom
## Multiple R-squared:  0.9961,Adjusted R-squared:  0.9961
## F-statistic: 6.304e+04 on 2 and 497 DF,  p-value: < 2.2e-16
```

Learning Objective 3: The candidate will understand various applications and risks of derivatives

QFI QF Fall 2020 Question 6

Learning Outcomes:

a) Understand the Greeks of derivatives

b) Understand static and dynamic hedging

c) Understand delta hedging, and the interplay between hedging assumptions and hedging outcomes

e) Understand how hedge strategies may fail

Source References:

- *Options, Futures, and Other Derivatives*, Hull, John C., Pearson, 2021, Chapters 19, 26, pages 421-422, 632-634
- *The Volatility Smile*, Derman, Emanuel and Miller, Michael, 2016, Chapters 3, 5
- INV201-104-25: Which Free Lunch Would You Like Today, Sir?

Question:

Consider an exotic option A with the following payoff:

- $K+S$, if $S>2K$
- $2S-K$, if $K<S\leq 2K$
- $2(K-S)+S$, if $S\leq K$

where S is the underlying asset paying no dividend and K is a constant.

Let B denote the portfolio consisting of one long position in option A and one short position in the underlying asset S, i.e. $B=A-S$.

(a)

(i) Sketch the payoff graph for the portfolio B.

(ii) Construct a static hedging strategy for option A, with plain vanilla options and the underlying asset *S*.

(b) Construct a dynamic delta-hedging strategy for this exotic option A.

(c) List pros and cons of static hedging strategies and dynamic hedging strategies.

Assume that S follows

$$dS = \mu_s S dt + \sigma_s S dZ$$

where μ_s and σ_s are deterministic, and Z is a standard Wiener process.

Given a self-financing portfolio $\Sigma = a\left[V - \frac{\partial V}{\partial S}S\right]$, where

- V: general option on the underlying asset S
- a: rebalancing factor, satisfying $\left[V - \frac{\partial V}{\partial S}S\right]da = aSd\left[\frac{\partial V}{\partial S}\right]$

(d) Show that $rV = \frac{\partial V}{\partial t} + \frac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S}$ using the law of one price and Ito's Lemma, where r denotes the constant risk-free rate.

Assume that Black-Scholes-Merton (BSM) holds, and the option V is delta-hedged.

Use the following notations:

- $\sigma_{S,I}$: implied volatility
- $\sigma_{S,R}$: realized volatility
- Δ_I: delta based on implied volatility
- Δ_R: delta based on realized volatility
- Γ_I: gamma based on implied volatility

(e)

(i) Show that the profit and loss function P&L of the hedged portfolio satisfies the following when the hedge is constructed with realized volatility $\sigma_{S,R}$:

$$d(P\,\&\,L) = \frac{1}{2}S^2\Gamma_I\left[\sigma_{s,R}{}^2 - \sigma_{s,I}{}^2\right]dt + (\Delta_I - \Delta_R)\left[(\mu_s - r)Sdt + \sigma_{s,R}SdZ\right]$$

(ii) Determine $d(P\&L)$, if V is hedged with implied volatility $\sigma_{s,I}$ instead.

Describe key P&L characteristics, when hedging with realized volatility vs. implied volatility, using the results in parts (i) and (ii) to support your answer.

QFI QF Fall 2020 Question 10

Learning Outcomes:

f) Identify and evaluate embedded options in liabilities (e.g., indexed annuity, structured product based variable annuity, and variable annuity guarantee riders including GMxB, etc.)

g) Demonstrate an understanding of hedging for embedded option in liabilities

Source References:

- INV201-108-25: Mitigating Interest Rate Risk in Variable Annuities: An Analysis of Hedging Effectiveness under Model Risk

Question (Excel can be used to do the calculation):

You were asked to construct a self-financing delta-rho dynamic hedging strategy for a portfolio of variable annuities (VA) contracts. Assume that:

- L_t is the net value of the VA contracts at quarter t
- Π_t is the value of the hedging portfolio at quarter t
- At time 0, $L_0 = 0$
- The bank account's interest rate for one quarter (or 3 months) is 0.5%

You borrowed $500M to establish the strategy with the following asset choices. At quarter 1, the value of the hedging portfolio has increased to $4.5M.

Time (t) in quarter	Stock	Zero-coupon Bond
0	$200	$100
1	$203	$101

For part (a), identify the formula used (with reference to the formula # in the Formula Sheet as provided.)

(a) Calculate the position in each of the three assets at time 0.

The response for this part is required on the paper provided to you.

(b) Define the objective of the hedging strategy in terms of the insurer's hedged loss at maturity.

ANSWER:

Suppose that the hedge position is determined based on the Black-Scholes-Vasicek (BSV) model for parts (c) and (d).

(c) State one problem with using the forward-looking approach to calibrate the stock volatility.

ANSWER:

Next, 100,000 daily market scenarios were projected under each of the three different market models below (where the Brownian motions driving the interest rate and equity processes are independent of each other). These models are data-generating models used to simulate the impact of model risk on the insurer's hedged losses (where model risk refers to deviation between the data-generating model and insurer's hedging model).

Model	Equity Model	Interest Rate Model
A	Black-Scholes	One-factor CIR
B	Black-Scholes	Three-factor CIR
C	Heston	Three-factor CIR

The table below presents the Conditional Tail Expectation (CTE) 95% measure of the insurer's hedged loss at maturity under each market model. However, the names of the corresponding market models are not disclosed in the results below.

Table	Model X	Model Y	Model Z
CTE 95%	1.8	0.5	4.0

(d)

(i) Identify the sources of model risk in your hedging strategy under each of Models A, B, and C.

ANSWER:

(ii) Identify the corresponding market model by matching Model X, Y, and Z to Model A, B, or C. Justify your answer.

ANSWER:

You asked your actuarial student to graph the insurer's hedged loss against the stock market volatility for each market model (i.e., Model A, B, C). In the report, you noticed the insurer's hedged loss is mostly centered at zero (without any trends) in all three models.

(e) (*1 point*) Explain whether you agree with the student's result.

ANSWER:

You expect the interest rate will steadily rise throughout the term of the VA contracts.

(f) (*1 point*)

(i) Explain how a delta-only hedging strategy would affect the insurer's hedged loss if your expectation becomes a reality.

ANSWER:

(ii) Explain how a wrong expectation would affect the insurer's hedged loss after modifying the hedge strategy.

ANSWER:

QFI QF Spring 2021 Question 12

Learning Outcomes:

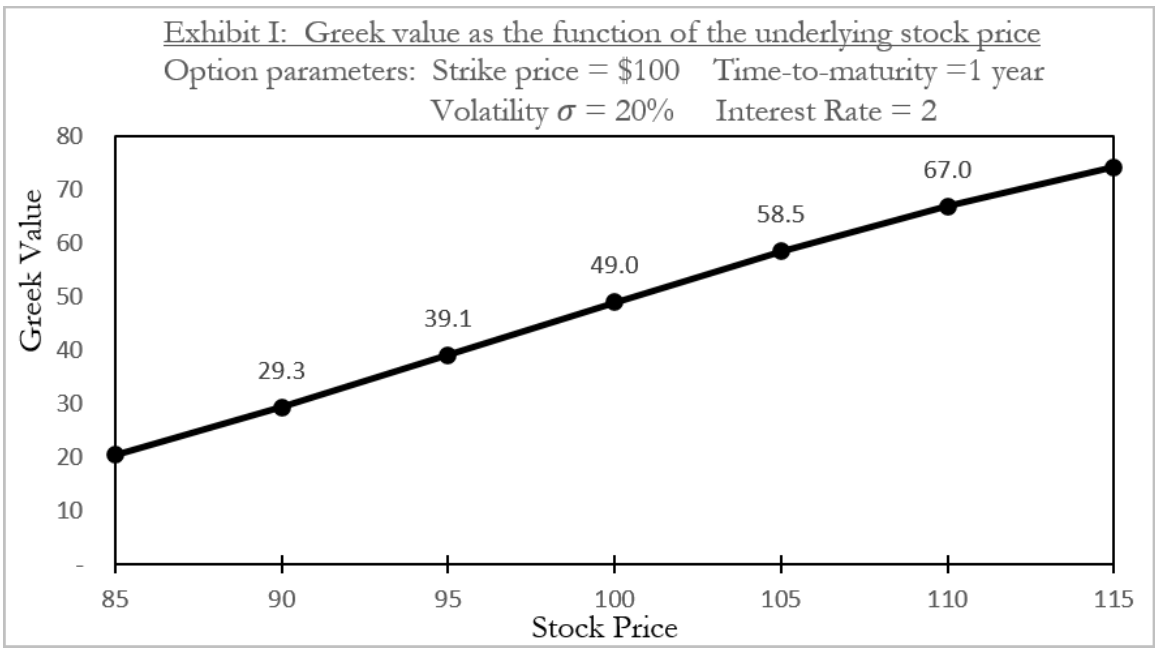
a) Understand the Greeks of derivatives

Source References:

- *Options, Futures, and Other Derivatives*, Hull, John C., 11th Edition, 2021, Chapter 19

Question:

Exhibit I below shows the Greek of a European call option on a non-dividend-paying stock based on the Black-Scholes-Merton (BSM) model.



(a) (*0.5 points*) Determine which Greek (Delta, Gamma, Vega, Rho, or Theta) Exhibit I represents. Justify your answer. (Here Theta is defined as the derivative of the option value with respect to the passage of time.)

(b) (*2 points*) Draw "Line A" in Exhibit I to show the same Greek of a European put option that has the same parameters as the one in Exhibit I. Indicate the Greek value in "Line A" at stock price = 100. You need not show other values in "Line A" but comment on the slope of this line.

(c) (*2 points*) Draw "Line B" in Exhibit I to show the same greek of a European put option that has the same parameters as in Exhibit I, except that the time-to-maturity is 1 month. Indicate the Greek value in "Line B" at stock price = 85. You need not show other values in "Line B" but comment on the slope of this line.

Exhibit II below shows Vega and Gamma for a European option on a non-dividend-paying stock. These Greek values are derived from the BSM model with the same strike price, volatility, interest rate, and time-to-maturity as in Exhibit I.

Exhibit II: Vega and Gamma with respect to the underlying stock price

Stock price	60	X
Vega (shown as the change in the option value to 1 percentage point change of the volatility, e.g., from 25% to 26%)	0.2401	0.2548
Gamma	0.0267	0.0159

(d) (*1.5 points*) Determine the stock price X in Exhibit II.

QFI QF Spring 2021 Question 13

Learning Outcomes:

c) Understand delta hedging, and the interplay between hedging assumptions and hedging outcomes

d) Understand the concepts of realized versus implied volatility

e) Understand derivatives mishaps

Source References:

- *The Volatility Smile*, Derman, Emanuel and Miller, Michael, 2016, Ch. 3, 5, 6

Question:

You are an actuary working on an investment product, where the interest credited is based on the growth rate of an underlying stock index S over a 1-year period, and crediting parameters including cap rate c and participation rate p. The interest credited will be floored at a non-negative guaranteed rate g.

The interest crediting strategy for this product per \$1 at time t is specified as:

$$Interest\ Credited_t = \max\left\{\min\left[\left(\frac{S_t}{S_{t-1}} - 1\right)\cdot p, c\right], g\right\}$$

Given the following data:

- Cap rate c is 5%
- Participation rate p is 90%
- Guaranteed rate g is 1%
- Continuous risk-free rate r is 5%
- Implied volatility σ is 20%

(a) Derive the replicating portfolio using options for the interest credited above the guaranteed rate, i.e. $Interest\ Credited_t - g$. Specify each option, including position, option type, term, and strike ratio K / S_{t-1}.

(b) Sketch the payoff of the replicating portfolio against the index growth rate $\left(\frac{S_t}{S_{t-1}} - 1\right)$.

Given the interest crediting period is from Dec 31, 2018 to Dec 31, 2019, index value $S_{t-1} = 1000$ on Dec 31, 2018, and $S_t = 1080$ on Dec 31, 2019, respectively. Assume 1000 of initial investment:

(c)

(i) Calculate the interest credited on Dec 31, 2019.

(ii) Calculate the cost of the replicating portfolio for the interest credited above the guaranteed rate on Dec 31, 2018.

The response for this part is to be provided in the Excel spreadsheet.

Assume the implied volatility σ is in absence of transaction costs. Given the transaction cost per share of stock k is 0.52% and the replicating portfolio is rebalanced every week.

(d)

(i) Calculate the effective volatility $\tilde{\sigma}$ that covers the transaction costs for long and short option positions, respectively. Assume 52 weeks per year and $\pi = 3.14$.

(ii) Justify the calculation of effective volatility regarding to each option position.

Regarding to rebalancing:

(e)

(i) Describe the relationship between hedging frequency and the profit.

Describe strategies that can be used for rebalancing.

QFI QF Fall 2021 Question 11

Learning Outcomes:

c) Understand delta hedging, and the interplay between hedging assumptions and hedging outcomes

Source References:

- *The Volatility Smile*, Derman, Emanuel and Miller, Michael, 2016, Chapter 7

Question:

Table 1 below shows the market data of a European call option on a non-dividend paying stock XYZ.

Table 1: Market data on XYZ stock price and its call option

Option maturity as of Day 1 = 3 years Option strike price = 80 Option size = 1 (i.e. One option contract has 1 underlying share) No changes in interest rate from Day 1 to Day 30						
Day	1	2	3	4	…	30
Stock price	80	70	75	82		80
Option price	12.25	12.25	12.22	12.30		12.07
Option delta	0.610	0.535	0.562	0.638		0.608

On Day 1, you sold 1000 of this option and immediately started delta-hedging. On Day 4, you liquidated all your options and shares of the XYZ stock. All your trades are conducted at the prices shown in Table 1. Transaction cost (including the borrowing cost, if any, that you incurred as part of your hedging strategy) should be ignored for this question.

(a) Calculate your cumulative total profit or loss on Day 4 under the following circumstances, respectively:

(i) You rebalanced your hedge position daily.

(ii) You never rebalanced your hedge position.

The response for this part is to be provided in the Excel spreadsheet.

Your analyst compiled Table 2 below that shows his explanations for the observations on the option and stock prices in Table 1. However, he was unable to explain observation 4.

Table 2:

Observation of the option and stock price movement	Analyst's explanation
1. The option price did not change from Day 1 to Day 2, while the stock price decreased from Day 1 to Day 2	The sticky strike rule
2. The option price decreased from Day 2 to Day 3, while the stock price increased from Day 2 to Day 3	The sticky delta rule
3. Both the option price and the stock price increased from Day 3 to Day 4	The local volatility model
4. The option price on Day 30 is lower than on Day 1, while the stock price on Day 30 is same as on Day 1	

(b) Determine whether each of the three explanations provided is valid or not. Explain why.

The response for this part is required on the paper provided to you.

(c) Provide your explanation for observation 4.

The response for this part is required on the paper provided to you.

QFI QF Fall 2021 Question 12

Learning Outcomes:

a) Understand the Greeks of derivatives

b) Understand static and dynamic hedging

c) Understand delta hedging, and the interplay between hedging assumptions and hedging outcomes

Source References:

- *Options, Futures, and Other Derivatives*, Hull, John C., Pearson, 2021, Chapter 19, 26
- *The Volatility Smile*, Derman, Emanuel and Miller, Michael, 2016, Ch. 3, 5, 6

Question:

For an exotic option *E* with the following payoff:

- m (S – 150)+ 50, if S > 150
- 2 (S – 100)+ 50, if S < 100
- 50, otherwise,

where S is an underlying equity with no dividend; *m* is constant.

(a)

(i) Sketch the payoff graph for option *E* using $m = 2$.

(ii) Build a static hedging strategy with vanilla options to hedge the equity risk.

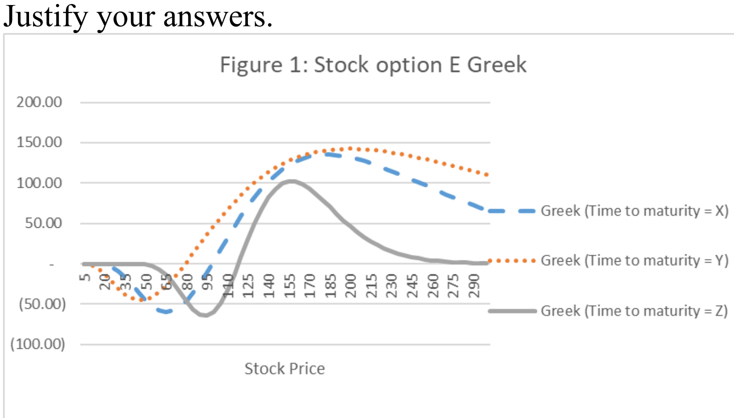
The response for this part is required on the paper provided to you.

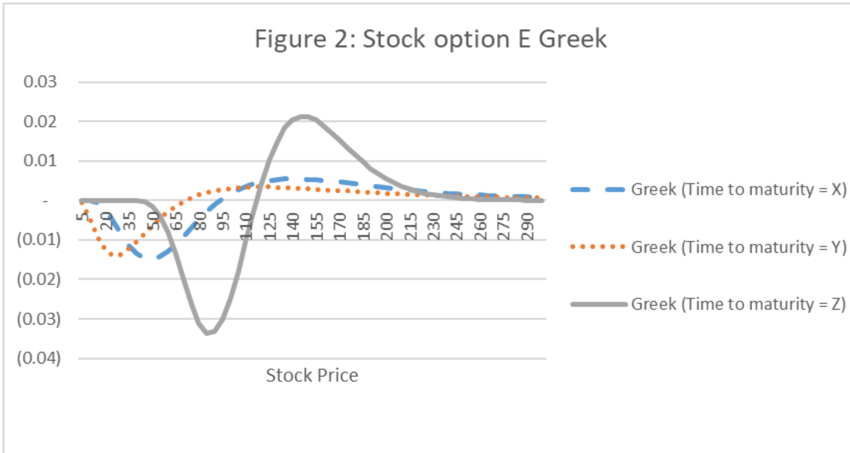
(b)

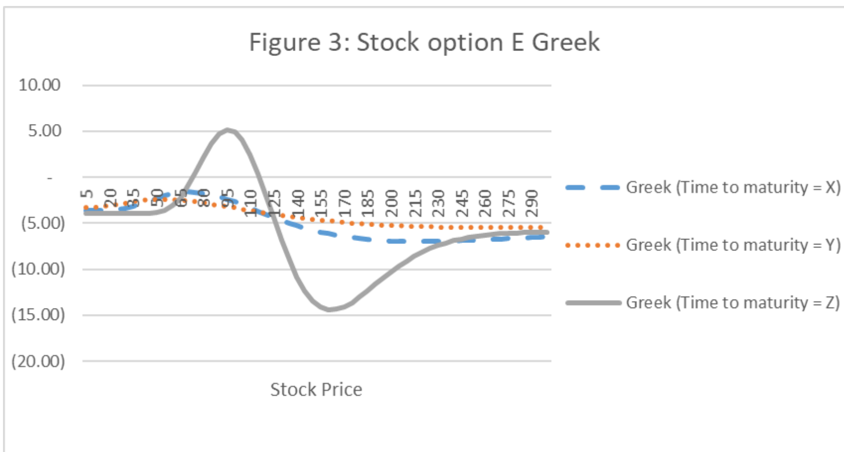
(i) Define the following Greeks: Delta, Gamma, Vega, and Theta.

(ii) Sketch Delta graph for option *E* using $m = 2$ and justify your answers.
(Hint: Build from vanilla options.)

(iii) Determine which figure corresponds to Gamma, Vega, and Theta, respectively. Justify your answers.







The response for this part is required on the paper provided to you.

Your coworker proposes to trade on the convexity of volatility skew with option *E*.

(c)

(i) Explain what the volatility skew is.

(ii) List three reasons why the volatility skew exists.

(iii) Explain why option *E* is not the suitable vehicle to trade on convexity of volatility skew.

The response for this part is required on the paper provided to you.

You are given the following for vanilla options on non-dividend paying stock.

Underlying S	100	100	100	100
Strike K	50	100	150	200

Volatility σ	27.7%	23.3%	21.2%	21.4%
T-t	5	5	5	5
r	2%	2%	2%	2%
d_1	1.59	0.45	-0.41	-1.00
d_2	0.97	-0.07	-0.88	-1.48
$N(d_1)$	0.94430	0.67450	0.34188	0.15909
$N(d_2)$	0.83487	0.47287	0.18905	0.06979
$N'(d_1)$	0.11236	0.36014	0.36718	0.24241
$N'(d_2)$	0.24836	0.39802	0.27053	0.13395

Your coworker is asking for your help to build a Vega-neutral option E^* to trade on the convexity of volatility skew, expecting

- Increase in the volatility convexity;
- Higher future volatility than the current implied volatility;
- Symmetric payoff against stock price, centered at the current level.

Option E^* = Option E – 2 Call (strike = 100) + m Put (strike = K^*)

(d)

(i) Determine K^* for option E^*.

The response for this part is to be provided in the Excel spreadsheet.

(ii) Solve for m so that option E^* is Vega-neutral.

The response for this part is to be provided in the Excel spreadsheet.

The stock price decreases from 100 to 80 the next day, impacting the implied volatilities as below. (Time decay of 1 day is ignored.)

Underlying S	80	80	80	80
Strike K	50	100	150	200
Volatility σ	27.7%	30.0%	38.9%	63.1%
T-t*	5	5	5	5
r	2%	2%	2%	2%
d_1	1.23	0.15	-0.17	0.13
d_2	0.61	-0.52	-1.04	-1.28
$N(d_1)$	0.86590	0.56034	0.43147	0.55071
$N(d_2)$	0.61503	0.30189	0.14857	0.09956
$N'(d_1)$	0.18700	0.39437	0.39304	0.39572
$N'(d_2)$	0.33067	0.34868	0.23167	0.17493

(e)

(i) Calculate the gain or loss of option E^*.

The response for this part is to be provided in the Excel spreadsheet.

(ii) Demonstrate how option E^* is an effective vehicle to take position on volatility convexity, given the result in part (e)(i).

The response for this part is required on the paper provided to you.

QFI QF Spring 2022 Question 2

Learning Outcomes:

a) Understand the Greeks of derivatives

Source References:

- *Options, Futures, and Other Derivatives*, Hull, John C., Pearson, 2021, Chapter 19

Question:

Assume the non-dividend paying stock price S_t follows a Geometric Brownian Motion with constant volatility $\sigma = 10\%$. Let the continuously compounded risk-free rate of the market be $r = 2\%$. All expectations are under the risk-neutral measure $\mathbb{Q}$.

Consider a special European-style option on S_t that expires at time $t = 5$ years. Let V_t represent the option price at time t, for $t \leq 5$. At expiry, the payoff for this option is $V_5 = min\{S_3, S_5\}$.

(a) Show that:

(i) $V_5 = S_5 \mathbb{I}_{\{S_3 \geq S_5\}} + S_3 \mathbb{I}_{\{S_3 < S_5\}}$ where $\mathbb{I}_{\{A\}} = \begin{cases} 1 \text{ if } A \text{ is true} \\ 0 \text{ if } A \text{ is false} \end{cases}$.

(ii) $P[S_3 < S_5] = 0.583$ under $\mathbb{Q}$ measure.

Assume from this point on that $t < 3$.

(b) Show that:

(i) $E_t\left[S_3 \mathbb{I}_{\{S_3 < S_5\}}\right] = 0.619\, e^{-0.02t} S_t$.

(ii) $E_t\left[S_5 \mathbb{I}_{\{S_3 \geq S_5\}}\right] = 1.03\, E_t[S_3] E\left[e^{\sqrt{0.02}Z} \mathbb{I}_{\{Z \leq -0.21\}}\right]$ with Z a standard normal random variable.

After further work, you have determined that:

$$E_t\left[S_5 \mathbb{I}_{\{S_3 \geq S_5\}}\right] = 0.401\, e^{-0.02t} S_t.$$

(c) Calculate V_t and its Delta.

Your coworker claims that the special European-style option considered above can be Delta- and Gamma-hedged till its expiration by using a suitable short position in the underlying asset only.

(d) Critique your coworker's claim.

QFI QF Spring 2022 Question 11

Learning Outcomes:

c) Understand delta hedging, and the interplay between hedging assumptions and hedging outcomes

e) Understand derivatives mishaps

Source References:

- *The Volatility Smile*, Derman, Emanuel and Miller, Michael, 2016, Chapters 6 & 7

Question:

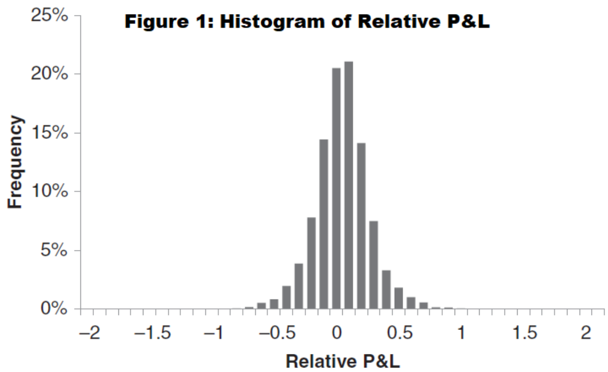
Consider a European option on a non-dividend paying stock. Assume that

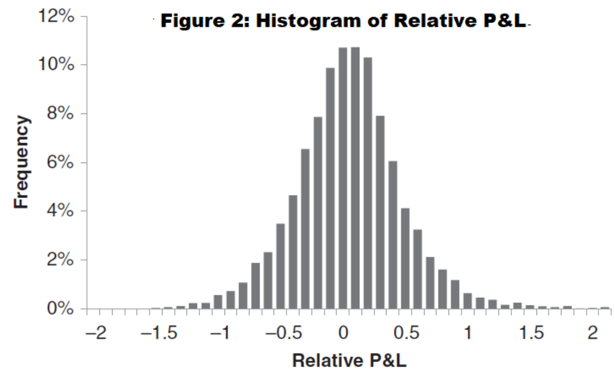
- The stock price S follows Geometric Brownian Motion:
  $dS = \mu S dt + \sigma S dW, \ where\ \mu > 0,\ \sigma = 20\%.$
- There are no transaction costs.

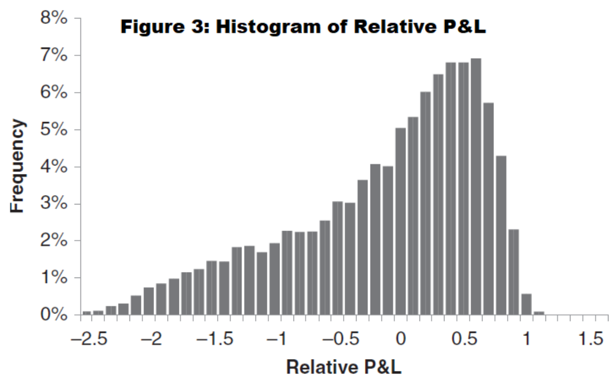
You are given the following four delta hedging strategies on the option:

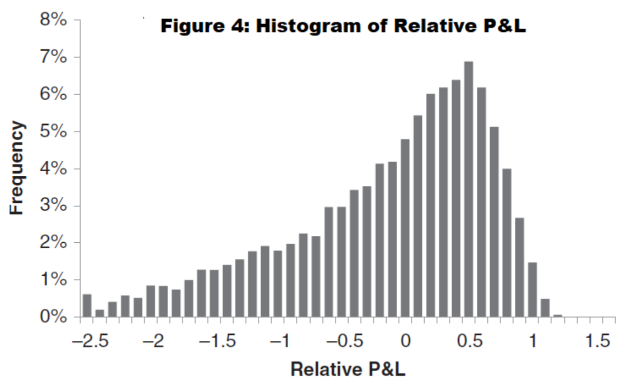
Strategy 1: Use 20% volatility to determine Delta and rebalance hedge position daily.
Strategy 2: Use 40% volatility to determine Delta and rebalance hedge position daily.
Strategy 3: Use 20% volatility to determine Delta and rebalance hedge position weekly.
Strategy 4: Use 40% volatility to determine Delta and rebalance hedge position weekly.

According to the stock price process, your analyst ran Monte Carlo simulations and produced the following histograms of Relative P&L of each strategy, where Relative P&L refers to the Profit/Loss that is measured relative to what would be the Black-Scholes-Merton (BSM) fair value of the option if you replicated continuously at the realized volatility. That is, Relative P&L = Present Value of Payoff – BSM Fair Value.









(a) Explain which Strategy is associated with Figure 1 and Figure 2, respectively.

(b) Explain why Figure 3 looks similar to Figure 4.

Assume that transaction cost is proportional to the value of the stocks traded.

(c)

(i) Sketch the histograms of relative P&L for Strategy 1 and Strategy 3, respectively. Note: You need not mark any values on your *x*-axis and *y*-axis. The key is to show the shape or contour of the histogram.

(ii) Explain the key drivers for the differences in the histogram.

Let m_1 and m_3 be the mean value of relative P&L of Strategy 1 and Strategy 3, respectively.

Let s_1 and s_3 be the standard deviations of relative P&L of Strategy 1 and Strategy 3, respectively.

(d) Compare m_1 vs. m_3 vs. 0. Justify your ranking.

(e) Compare s_1 vs. s_3. Justify your ranking.

QFI QF Spring 2022 Question 12

Learning Outcomes:

a) Understand the Greeks of derivatives

Source References:

- *Options, Futures, and Other Derivatives*, Hull, John C., Pearson, 2021, Chapter 19

Question:

Assume the Black-Scholes (B-S) framework.

The current B-S price of a *T*-year European call option with strike price K on a non-dividend paying stock is:

$$C = SN(d_1) - Ke^{-rT}N(d_2)$$

where:

$N(\cdot) = \text{cumulative normal distribution}$
$S = \text{the current price of the stock}$
$r = \text{the continuously compounded risk-free interest rate}$
$\sigma = \text{the volatility of the stock's continuously compounded returns}$

$$d_1 = \frac{\ln\frac{S}{K} + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$
$$d_2 = \frac{\ln\frac{S}{K} + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

You are also given that:

$$SN'(d_1) = Ke^{-rT}N'(d_2)$$

(a) Show that the Gamma of the European call is:

$$\text{Gamma} = N'(d_1)\frac{1}{S\sigma\sqrt{T}}$$

(b) Prove that the Gamma of a European call is equal to the Gamma of an otherwise equivalent European put.

(c) Identify whether each of the following statements is true or false. Briefly justify your answer.

(i) Gamma approaches 0 for deep-in-the money calls.

(ii) Gamma approaches 1 for deep-out-of-the-money puts.

(iii) For an out-of-the-money option with an underlying asset price that is exhibiting low volatility, Gamma is expected to be relatively low.

(iv) For an option that happens to be right at-the-money very near to the expiry date, a stable Gamma is likely to be observed.

QFI QF Spring 2022 Question 14

Learning Outcomes:

f) Identify and evaluate embedded options in liabilities (e.g., indexed annuity, structured product based variable annuity, and variable annuity guarantee riders including GMxB, etc.)

g) Demonstrate and understand target volatility funds and hedging for embedded options

Source References:

- INV201-106-25: Variable Annuity Volatility Management: An Era of Risk-Control

Question:

Volco VA company (VVA) is looking to manage the volatility associated with its Variable Annuity (VA) and Equity-Indexed Annuity (EIA) businesses. One of the fund managers suggested that the company consider managed-volatility funds as part of its new product line. He has asked the actuarial area to analyze the feasibility of these fund types, with a particular interest in target and capped volatility funds.

The fund manager is interested in the performance of a target volatility fund under a market scenario. He has asked an analyst to generate the following equity prices and forward volatilities for 4 years (Table 1), so that you can calculate the corresponding fund returns.

Table 1	Years (t)			
	0	1	2	3
S_t	100	88	105	110
σ_t	20%	40%	10%	30%

Table 2	Years (t)			
	0	1	2	3
Equity %	75%	α_1	α_2	α_3
Bond %	25%	1 -α_1	1 -α_2	1 -α_3
Equity Price	100	88	105	110
Bond Index Price	100	103.05	106.18	109.42
Target Vol Fund Price	100	X	100.16	Y

(a) Calculate the resulting target volatility fund prices X and Y in Table 2, assuming a continuously compounded risk-free rate of 3%, a target volatility of 15% and a maximum equity % of 200%.

The response for this part is to be provided in the Excel spreadsheet.

The fund manager is concerned about how the fund volatility will perform under both high and low volatility scenarios.

Table 3	
Stock Returns	Realized Volatility
Negative	$>\sigma_C$
Positive	$= \sigma_T$
Negative	$<\sigma_T$

(b) Compare the relative performance of the target volatility fund, capped volatility fund, and underlying asset under the scenarios in Table 3, where the target volatility = σ_T and cap volatility = σ_C and $\sigma_T < \sigma_C$.

The response for this part is to be provided in the Excel spreadsheet.

As part of the analysis, the Hedging Actuary has asked you to perform an investigation of the properties of the target volatility fund.

(c) Explain whether the following statements are True or False:

(i) Call options on a target volatility fund should be cheaper than or equal to the equivalent call options on the underlying risky-asset.

(ii) Call options on a capped volatility fund should be cheaper than or equal to the equivalent call options on the underlying risky-asset.

The response for this part is to be provided in the Excel spreadsheet.

QFI QF Fall 2022 Question 13

Learning Outcomes:

a) Understand the Greeks of derivatives

c) Understand delta hedging, and the interplay between hedging assumptions and hedging outcomes

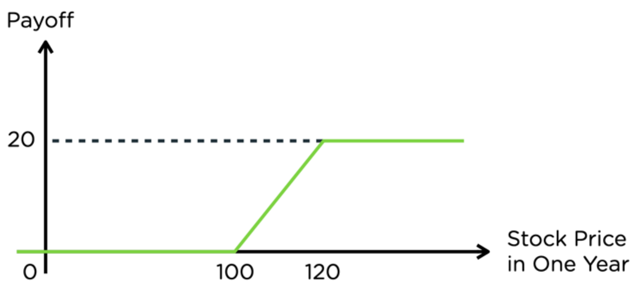
d) Understand the concepts of realized versus implied volatility.

Source References:

- *Options, Futures, and Other Derivatives*, Hull, John C., Pearson, 2021, Chapter 19
- *The Volatility Smile*, Derman, Emanuel and Miller, Michael, 2016, Ch. 5, 6

Question:

Consider a one-year contingent claim on Stock XYZ, a non-dividend-paying stock. The payoff of the contingent claim in one year is as follows:



(a) Construct a strategy to replicate the payoff of the contingent claim with only European options on Stock XYZ.

Assume that:

- The Black-Scholes framework applies.
- The continuously compounded risk-free interest rate is 0%.
- The price of Stock XYZ is 110.
- The implied volatility used for both valuation and hedging is 30%.

(b) Compare and contrast realized volatility and implied volatility.

(c)

(i) Calculate the Delta of this contingent claim.

(ii) Explain why the Delta is positive.

Regarding a long position in the contingent claims, your colleague made the following comments:

- Comment 1: As the price of the underlying stock moves away from the price range within the two strike prices, we expect the Delta of the contingent claim to converge to zero.
- Comment 2: The net Gamma exposure of the contingent claim is always positive.

(d) Assess each of your colleague's comments above.

Your firm enters into a long position in 100 of contingent claims described above and immediately delta hedges the position using the underlying shares. You are given:

- Initially, the price of the stock is 110, the price of the 100-strike European call is 18.14, and the price of the 120-strike European call is 9.28.
- One day later, the price of the stock is 130, the price of the 100-strike European call is 33.56, and the price of the 120-strike European call is 20.40.

Assume your firm has not rebalanced the hedge and only whole shares can be bought or sold.

(e)

(i) Calculate the profit or loss at the end of the next day from Delta hedging.

(ii) Explain why the profit or loss is not zero from Delta hedging.

QFI QF Spring 2023 Question 10

Learning Outcomes:

f) Identify and evaluate embedded options in liabilities (e.g., indexed annuity, structured product based variable annuity, and variable annuity guarantee riders including GMxB, etc.)

g) Demonstrate and understand target volatility funds and hedging for embedded options

Source References:

- INV201-105-25: An Introduction to Computational Risk Management of Equity-Linked Insurance, Feng, 2018 (sections 1.2-1.3, 4.7 & 6.2-6.3)
- INV201-106-25: Variable Annuity Volatility Management: An Era of Risk-Control

Question:

For its proposed GMMB rider offering, QFI Life is assessing 3 volatility management strategies with investments referencing an equity (S).

GMMB	Guaranteed	100% deposit
	Term	2 Years
	Current Price (S_0)	100

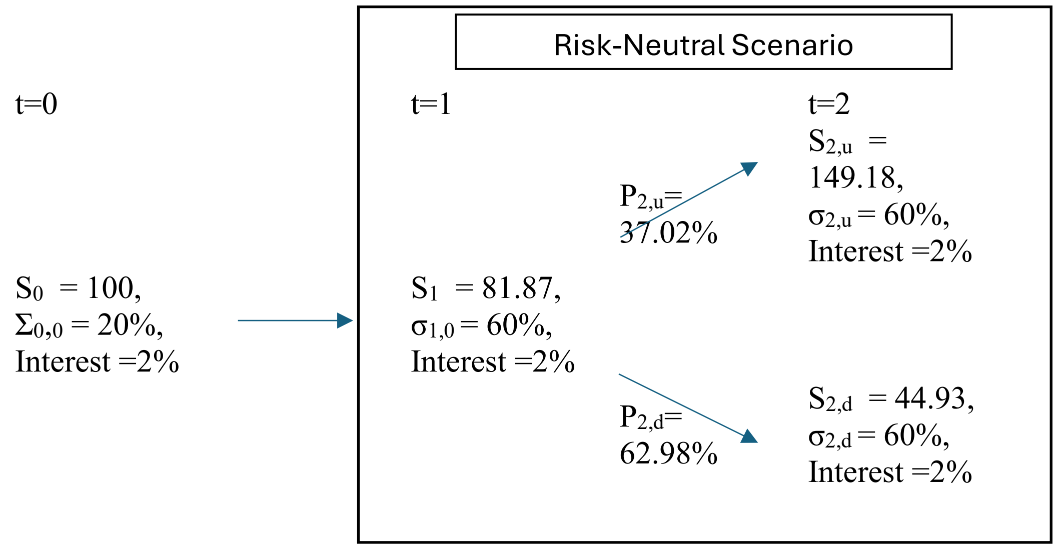
Reference Equity (S)	Current Volatility ($\sigma_{0,0}$)	20%
	Dividend (d)	0%
Other Assumptions	Death Rate (q)	0%
	Lapse Rate (l)	0%

Volatility Management Strategies

1) Asset Transfer Program:
   - Guarantee Ratio (G%) = $1 - \frac{investment\ value}{guaranteed\ value}$
   - Asset Allocation to cash = Max [G%, 0%]
     (Note: allocation is to cash as opposed to fixed income)
   - Rider Fee = 0%
   - Rebalancing on an annual basis
2) Capped volatility fund:
   - Capped volatility at 60%
   - Equity ratio = min(100%, $\frac{60\%}{realized\ equity\ volatility}$)
   - Rider Fee = 0%
   - Rebalancing on an annual basis
     (Note: Allocation is to cash as opposed to fixed income)
3) VIX-Indexed Fee:
   - Rider Fee = 0bps, if t=0; max[0bps, 200bps * ($\sigma_{t,j}$ -20%)], if t= 1, 2
   - $\sigma_{t,j}$ is the time-t volatility at stock level $S_{t,j}$ (where j =0, u, d)
   - Fee charged to investment value at the beginning of the year
   - Equity allocation is 100%
   - Fund Value = equity price (S) less rider fee

(a) Describe the principal objectives for an insurer in designing an equity-based guarantee.

Your co-worker, Robin asks your help to build a prototype for risk-neutral scenario simulation. The risk-neutral scenario (from $t = 1$ to $t = 2$) is given below:



The interest rate is assumed to be continuously compounded and cash earns 0% interest rate. It is also assumed that the realized volatility turns out to be equal to the implied volatility, which is defined above.

(b) Calculate the guarantee cost at the end of year 1 (t=1) for the GMMB rider under each of the 3 volatility management strategies. (Initial deposit = $100)

Robin simulates 10,000 scenarios and analyzes the 2 key metrics for each of the volatility management strategies but forgets to label the results. (Initial deposit = $100)

Strategies	Guarantee Cost (t=0)	Hedge P&L (Cumulative)
100% Static allocation in Equity (S)	12.30	Loss of 4.6%
B	10.01	Loss of 1.2%
C	12.45	Loss of 5.0%
D	12.14	Loss of 1.5%
E	12.18	Loss of 4.1%

(c) Identify the 4 volatility management strategies from the table above including no volatility management strategy.

Robin re-simulates the guaranteed cost under the two scenarios below:

Strategies	Guarantee Cost (t=0)	
	Volatility = 10%	Volatility = 40%
Asset Transfer Program	4.38	10.35
Capped volatility fund	7.55	20.91
VIX-Indexed Fee	7.35	21.50

(d)

(i) Calculate the Vega under each of the 3 volatility management strategies (Hint: use finite difference approximation).

(ii) Explain how low Vega can benefit the hedge program.

(iii) Propose a volatility management strategy from the insurer's perspective based on the results in part (c) and (d) (i).

QFI's target market are high-net-worth clients, who are interested in, and willing to pay for upside investment potential. Your manager, Joe considers implementing the asset transfer program strategy.

(e) Critique whether Joe's proposal meets the needs of the clients in the target market.

QFI QF Spring 2023 Question 11

Learning Outcomes:

f) Identify and evaluate embedded options in liabilities (e.g., indexed annuity, structured product based variable annuity, and variable annuity guarantee riders including GMxB, etc.)

g) Demonstrate and understand target volatility funds and hedging for embedded options

Source References:

- INV201-106-25: Variable Annuity Volatility Management: An Era of Risk-Control
- INV201-108-25: Mitigating Interest Rate Risk in Variable Annuities: An Analysis of Hedging Effectiveness under Model Risk

Question:

You are the actuary at XYZ in charge of constructing a hedging strategy for your company's VA product with a guaranteed minimum maturity benefit (GMMB) rider.

Your company uses a Heston model to simulate the underlying equity market dynamics, under the risk-neutral measure $\mathbb{Q}$:

$$dS_t = rS_t dt + \sqrt{v_t} S_t dW_t^S$$
$$dv_t = \kappa(\theta - v_t)dt + \sigma_v \sqrt{v_t} dW_t^v$$

where W_t^S and W_t^v are Wiener processes with correlation $\rho_v < 0$, and $r, \kappa, \theta, \sigma_v$ are constant with $2\kappa\theta > \sigma_v^2$.

You are calibrating the instantaneous variance process for hedging VA liabilities. You are considering the following approaches:

(i) Implied volatility surface computed from exchange traded vanilla options.

(ii) 1-year VIX index.

(iii) Historical realized volatility.

(a) Explain the considerations when using each of the approaches above.

The rider fee (α) is currently assessed as a percentage of the policyholder's account value, such that the insurer's expected present value of prospective rider fees for the GMMB at time *t* is:

$$Y_t = E^{\mathbb{Q}}\left[\int_t^T e^{-r(s-t)} \alpha A_s ds\right] {}_{T-t}p_{x+t}$$
$$= A_t\left(1 - e^{-\alpha(T-t)}\right) {}_{T-t}p_{x+t}$$

where ${}_{T-t}p_{x+t}$ = probability of a policyholder, who is *(x+t)* years old, surviving in the next *T-t* years.

You are given that the delta and vega of the GMMB liability net of fees are:

$$delta_t = \frac{\partial \Psi(t, T, A, G, v)}{\partial A} {}_{T-t}p_{x+t} - \left(1 - e^{-\alpha(T-t)}\right) {}_{T-t}p_{x+t}$$

$$vega_t = \frac{\partial \Psi(t, T, A, G, v)}{\partial (\varsigma)} {}_{T-t}p_{x+t}$$

$where\ \varsigma = v^{\frac{1}{2}}$ and

$\Psi(t, T, A, G, v)$
$=$ time
$- t$ price of a put option with maturity T and strike price G , written on the account value A

Your manager has asked you to explore changing the rider fee as a percentage of the guaranteed payment.

(b) Show that the insurer's expected present value of prospective rider fees becomes:

$$Y_t = \alpha G \left(\frac{1 - e^{-r(T-t)}}{r} \right) {}_{T-t}p_{x+t}$$

(c) Explain whether the following has increased, decreased, or remained the same after this change, from the insurer's perspective.

(i) Delta of the liability net of rider fees.

(ii) Vega of the liability net of rider fees.

You are asked to implement a new product feature, where the rider fee is an annual rate of the guaranteed payment paid continuously and indexed to the instantaneous variance:

$$\alpha_t = m + \lambda v_t$$

where m is a base fee, and $\lambda > 0$ is a scaling factor.

You are given for $t < s$:

$$E^{\mathbb{Q}}[v_s] = v_t e^{-\kappa(s-t)} + \theta\left(1 - e^{-\kappa(s-t)}\right)$$

(d) Show that the fair value of prospective fees at time t, as defined as the risk-neutral expected present value of fees that will be collected by the insurer before the contract's maturity at time T, is:

$$L_t = G\left[(m + \lambda\theta)\left(\frac{1 - e^{-r(T-t)}}{r}\right) + \lambda(v_t - \theta)\left(\frac{1 - e^{-(r+\kappa)(T-t)}}{r + \kappa}\right)\right] {}_{T-t}p_{x+t}$$

(e) Explain whether you agree or disagree with the following statements made by your analyst.

(i) “The new rider fee is not a function of A_t, therefore it is not sensitive to changes in the account value.”

“The new rider fee has a positive Vega.”

QFI QF Fall 2023 Question 7

Learning Outcomes:

e) Understand how hedge strategies may fail

Source References:

- INV201-104-25: Which Free Lunch Would You Like Today, Sir?

Question:

As an option trader, you look at how to make profit from volatility arbitrage. To manage your risk, you use the Black-Scholes based Delta hedge and rebalance your hedges daily.

For the volatility used for hedging, you consider one of the following two choices:

- Implied volatility
- Actual volatility

(a) List the pros and the cons of hedging with implied volatility and actual volatility.

(b) Choose the most appropriate volatility for hedging under each of the following two constraints.

(i) Mark to model

(ii) Mark to market

Assume the following:

- The risk-free interest rate is 0.
- The spot price of a non-dividend-paying stock XYZ is 100.

- While the implied volatility σ(implied) of the at-the-money 1-year call option on XYZ is 20%, you predict that the actual volatility σ(actual) of XYZ for the next year will be 30%.

(c) Design a volatility arbitrage to make money assuming that your prediction is correct and that you hedge with actual volatility.

(d) Calculate the final profit from the arbitrage executed in part (c).

The response for this part is to be provided in the Excel spreadsheet.

QFI QF Fall 2023 Question 9

Learning Outcomes:

c) Understand delta hedging, and the interplay between hedging assumptions and hedging outcomes

e) Understand how hedge strategies may fail

Source References:

- *The Volatility Smile*, Derman, Emanuel and Miller, Michael, 2016, Ch. 3

Question:

The Vanna Hull White Company (VHW) is a financial services company that specializes in boutique financial options. An agent from the VHW has approached your company to purchase $1,000,000 of a 1-year guarantee structured product featuring participation in the geometric average of a major stock index as described below.

$$P = \max\left(I, I * \left(1 + p_{dpa}\left(e^{\frac{1}{T}\int_0^T \ln\left(\frac{S_t}{S_0}\right)dt} - 1\right)\right)\right)$$

where I = initial investment amount, p_{dpa}= participation rate

Assume the current risk-free interest rate r = 3%, the current price of the underlying index (S_0) = 100, the current implied volatility (σ) = 20%, and the dividend rate q = 0%.

You are given that the price of a geometric mean Asian call option is equivalent to a vanilla European call option with a volatility of $\sigma_a = \frac{\sigma}{\sqrt{3}}$ and a dividend rate of $\frac{1}{2}\left(r + \frac{\sigma_a^2}{2}\right)$.

The time-t price of a European call option (for Black-Scholes model) is

$$N(d_1)\, S_t e^{-q(T-t)} - N(d_2)\, K^* e^{-r(T-t)}$$

where $d_1 = \dfrac{\ln\left(\frac{S_t}{K}\right)+\left(r - q + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$, $d_2 = d_1 - \sigma\sqrt{T-t}$,

S_t = index price at time t, K = strike price, T = maturity of the option.

The company is considering purchasing at-the-money (ATM) options and zero-coupon bonds (ZCBs) to hedge the 1-year guarantee structured product.

(a)

(i) Identify the type of options which should be purchased.

(ii) Calculate the values in the table below, (assuming a Black-Scholes framework):

	Value
Risk Budget	
Number of ZCBs to purchase (1,000 Face value amount)	
Number of ATM options to purchase	
Fair participation rate	

A fellow actuary has expressed concern over how the options may perform if market volatilities increase significantly and has asked you to look into options to reduce the volatility risk.

(b)

(i) (*2 points*) Determine the Vega of the Asian call options above.

(ii) (*1 point*) Explain the value of the above Vega in relation to the Vega of a European call option and why this relation intuitively makes sense.

The company's CRO has approached you to determine a Delta-Vega hedge strategy for the Asian options.

(c) Determine an initial Delta-Vega hedge position using an ATM 1-year European call option and the underlying stocks.

QFI QF Fall 2023 Question 10

Learning Outcomes:

f) Identify and evaluate embedded options in liabilities (e.g., indexed annuity, structured product based variable annuity, and variable annuity guarantee riders including GMxB, etc.)

g) Demonstrate an understanding of hedging for embedded option in liabilities

Source References:

- INV201-105-25: An Introduction to Computational Risk Management of Equity-Linked Insurance, Feng, 2018 (sections 1.2-1.3, 4.7 & 6.2-6.3)
- INV201-108-25: Mitigating Interest Rate Risk in Variable Annuities: An Analysis of Hedging Effectiveness under Model Risk

Question:

For a variable annuity (VA) contract with a GMDB rider, the following are defined:

- The underlying asset for the fund is S_t, whose risk neutral SDE is $dS_t = rS_t dt + \sigma S_t dW_t$
  where r is the constant risk-free rate and σ is the constant volatility of S_t.
- F_t is value of the fund at time t. A fee of m is charged continuously on the fund value while the contract is in-force and thus $F_t = S_t e^{-mt}$.
- Before maturity of the contract at time T, the maximum of F_t and a guaranteed amount $G = S_0$ is paid upon death of the insured.
- Probability of the insured (contract issued at age x) surviving until time t is ${}_tp_x$, and the constant force of mortality is $\mu_{x+t} = \frac{d\ {}_tp_x/dt}{{}_tp_x}$.
- Mortality probability is independent of the fund return distribution.
- A zero-coupon bond with notional of 1 and maturity T has price $P_t = e^{-r(T-t)}$ at time t.
- The net liability L_t is value of the VA contract in excess of the account balance less the value of prospective fee income. Specifically, it can be expressed formulaically as

$$L_t = {}_tp_x(\Omega_t - \Upsilon_t) - {}_tp_x E^{\mathbb{Q}}\left[\int_t^T mF_s e^{-r(s-t)}{}_{s-t}p_{x+t} ds\right]$$

$$\Omega_t = E^{\mathbb{Q}}\left[\int_t^T e^{-r(T-t)} \max\,(G, F_s)_{s-t}p_{x+t}\mu_{x+s}ds\right]$$
$$\Upsilon_t = E^{\mathbb{Q}}\left[\int_t^T F_s e^{-r(T-t)}{}_{s-t}p_{x+t}\mu_{x+s}ds\right]$$

(a) Derive the no-arbitrage value of the net liability L_t at time t.

Given that the Delta and Rho of L_t are as follow:

$$Delta = \frac{\partial L}{\partial S} = \int_t^T Ge^{-m(s-t)}[N(d_1) - 1]\ {}_s p_x \mu_{x+s} ds - m\int_t^T e^{-ms}\ {}_s p_x ds$$
$$Rho = {\partial L}/{\partial r} = -\int_t^T G(s-t)e^{-r(s-t)}N(-d_2)\ {}_s p_x \mu_{x+s} ds$$

where $d_1 = \frac{ln\frac{S_t}{G} + (s-t)(r-m+\frac{\sigma^2}{2})}{\sigma\sqrt{s-t}}$ and $d_2 = d_1 - \sigma\sqrt{s-t}$

(b) Derive the positions of stock, zero-coupon bond and money market account for a portfolio Π_t that hedges the Delta and Rho of the net liability in part (a).

Assume that the hedging portfolio Π_t was developed based on the SDE $dS_t = r_t S_t dt + \sigma S_t dW_t$, where r_t was deterministic but time-dependent and calibrated to observed term structure.

The following steps are done to test the effectiveness of the hedging portfolio Π_t:

- Real world scenarios of S_t are simulated based on the SDE $dS_t = \mu S_t dt + \sigma S_t dW_t$.
- The interest rates are simulated stochastically using the two interest models below. Moreover, as a control, the interest rate is not simulated and assumed to follow the deterministic path r_t.
- Assume continuous re-balancing of the hedging portfolio.
- Effectiveness is assessed as the hedging gain/loss at time T (i.e., $\Pi_T - L_T$).

Control	Interest rate model 1	Interest rate model 2
Deterministic interest rate r_t	One-factor CIR model	Three-factor CIR model

(c) Describe the hedging effectiveness you expect to observe under each of the 3 models of simulating interest rates (specified in the table above). Explain your reasoning.

Assume that the hedging portfolio Π_t was developed based on a short rate r_t that follows the one-factor Vasicek model calibrated to the observed term structure.

(d) Describe changes in hedging effectiveness in comparison to part (c) for the Interest rate model 1 and the Interest rate model 2.

QFI QF Fall 2024 Question 8

Learning Outcomes:

c) Understand delta hedging, and the interplay between hedging assumptions and hedging outcomes

d) Understand the concepts of realized versus implied volatility.

e) Understand derivatives mishaps

Source References:

- *The Volatility Smile*, Derman, Emanuel and Miller, Michael, 2016, Chapters 3, 5-7

Question:

Assume that the dynamics of a stock S_t is described the following SDE:

$$dS_t = rS_t dt + \sigma_R S_t dW_t$$

where r is the constant risk-free rate, σ_R is the realized volatility of S_t, and W_t is a Brownian motion.

You bought a call option with strike K and maturity at time T on S_t at the value C_t, delta-hedged the position based on the implied volatility Σ. Assume the cost of borrowing is the risk-free rate r.

(a) Calculate the gain or loss of the hedged portfolio dV_t over an infinitesimal period dt.

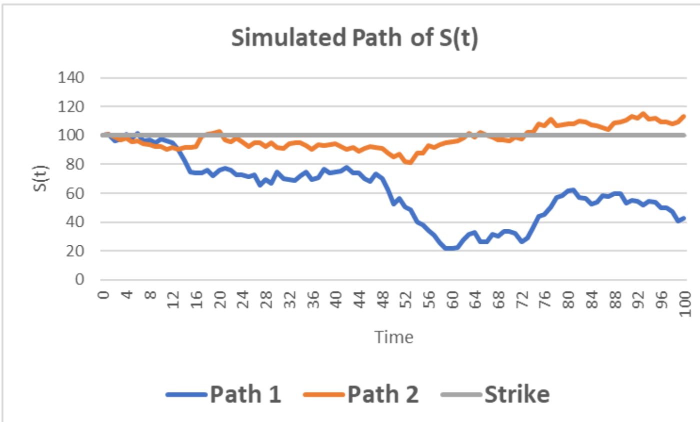
Assume that the hypothetical value of the call option is C_t^R if it was valued using the realized volatility, and you have delta-hedged the long position in C_t based on the realized volatility σ_R.

(b)

(i) Prove that the gain or loss of the hedged portfolio dV_t over an infinitesimal period dt is $dV_t = e^{rt}d[e^{-rt}(C_t - C_t^R)]$

(ii) Derive the present value of the total gain or loss to maturity $\int_t^T e^{r(s-t)}dV_s$.

Given two simulated paths of S_t from the given SDE:



(c) Compare $\int_0^{100} e^{r(s-t)}dV_s$ between the two paths if they materialize respectively, assuming

(i) The portfolio is delta-hedged based on σ_R.

(ii) The portfolio is delta-hedged based on Σ.

QFI QF Fall 2024 Question 12

Learning Outcomes:

f) Identify and evaluate embedded options in liabilities (e.g., indexed annuity, structured product based variable annuity, and variable annuity guarantee riders including GMxB, etc.)

g) Demonstrate an understanding of hedging for embedded option in liabilities

Source References:

- INV201-105-25: An Introduction to Computational Risk Management of Equity-Linked Insurance, Feng, 2018 (sections 1.2-1.3, 4.7 & 6.2-6.3)

Question:

Your company is reviewing different product features for a Single Premium EIA

- Choice of Indexes: S&P 500 Price Return Index (*S(T)* is level of index at time T)
- Payoff is $S(T)^{\alpha}$ where α ≤ 1 is a constant
- $S(T)$ follows a geometric Brownian motion under risk-neutral measure, $\mathbb{Q}$ with volatility, σ, and initial value $S(0) = 1$.

You would like to set the guarantee rate *(g)* such that the continuously compounded interest credited is 25 bps less than that of the expected value under the participation rate (α ≤ 1), to cover some potential hedging costs associated with new product. Formulaically,

$$\mathbb{E}[S(T)^{\alpha}]e^{-0.0025T} = e^{gT}$$

Let r = 4%, α = 50%, σ = 20%.

(a) Show that the guarantee rate is 1.25%.

(b) Derive the $\mathbb{Q}$-probability that the EIA credits the guaranteed rate in a single year.

You are considering implementing a cap such that the probability of $S(T)^{\alpha}$ exceeding the cap rate is no more than 10% in a single year. Final cap rate is rounded to the nearest whole %.

(c) Derive the appropriate cap rate.

After further discussion, your team has decided to offer the product with a cliquet design, including the cap rate. The guarantee rate is 1.25%. You are given that

$$\mathbb{E}\left[S(1)^{0.5}\mathbb{I}_{\{.025<\ln S(1)\leq .28\}}\right] = 0.42109$$

(d) Calculate the risk-neutral price for a 5-year cliquet EIA.

(e) Critique the following statement made by your analyst:

"By setting the cap rate such that the probability that $S(T)^{\alpha}$ exceeds cap rate is no more than 10% in a single year, we should expect to pay the cap rate approximately once every ten years."