CURATED PAST EXAM ITEMS
- Solutions -

INV 201 – Quantitative Finance

Important Information:

- These curated past exam items are intended to allow candidates to focus on past SOA fellowship assessments. These items are organized by topic and learning objective with relevant learning outcomes, source materials, and candidate commentary identified. We have included items that are relevant in the new course structure, and where feasible we have made updates to questions to make them relevant.
- Where an item applies to multiple learning objectives, it has been placed under each applicable learning objective.
- Candidate solutions other than those presented in this material, if appropriate for the context, could receive full marks. For interpretation items, solutions presented in these documents are not necessarily the only valid solutions.
- Learning Outcome Statements and supporting syllabus materials may have changed since each exam was administered. New assessment items are developed from the current Learning Outcome Statements and syllabus materials. The inclusion in these curated past exam questions of material that is no longer current does not bring such material into scope for current assessments.
- Thus, while we have made our best effort and conducted multiple reviews, alignment with the current system or choice of classification may not be perfect. Candidates with questions or ideas for improvement may reach out to education@soa.org. We expect to make updates annually.

Course INV 201

Curated Past Exam Solutions

All Learning Objectives

1. **Key Types of Derivatives**
2. **Valuation of Derivatives**
3. **Applications and Risks of Derivatives**

Table of Contents

Learning Objective 1: The candidate will understand key types of derivatives

QFI QF Fall 2021 Question 5

Learning Outcomes:

a) Understand the payoffs of basic derivative instruments

Source References:

- *Options, Futures, and Other Derivatives*, Hull, John C., 11th Edition, 2021, Chapters 1, 7, pages 28-30, 39, 183-185

Commentary on Question:

In general, the candidates performed well, particularly in identifying the input data, the formula and the calculation of the 2-year swap rate and its value, and the forward swap rate of the 2-year forward swap contract with expiry of 1 year in parts c), d), and e).

However, the results were below expectation in part b) in the determination of an arbitrage opportunity as to the price of the forward 1-year zero-coupon bond, and in a less proportion, in part a) for the definitions of a forward rate agreement, and interest rate swaps.

Other comments will be made under the appropriate section.

Solution:

(a) Describe forward rate agreements, forward contracts, and interest rate swaps.
Commentary on Question:

For the forwards rate agreement, many did refer to an exchange of a fixed rate versus a floating rate. This was not the completed answer, and the candidates should have been more precise and refer more specially to an exchange of the forward rate versus the future spot rate.

Also, for the interest rate swap, we expected the candidates to go further than describing an exchange of fixed cash flows for floating rate cash flows. To obtain more grading points, the candidates should have completed this answer with *mentioning the usual reference of the LIBOR floating rate, the swap rate itself, and the value of the contract at initiation.*

Forward Rate Agreement: A FRA is a contract between two counterparties to exchange one cash flow in the future, namely, the forward rate in exchange of the future spot rate.

Forward contract: This is a contract between two counterparties in which they agree that at some predetermined date, they will exchange a security, such as a Treasury note, for a cash price that is also predetermined at initiation of the contract.

Interest Rate Swap: A swap is a contract between two counterparties to exchange cash flows in the future. In a fixed-for-floating swap a counterparty pays a fixed coupon while the other pays a rate linked to a floating rate, typically the LIBOR rate. The fixed rate is called the swap rate, and it is set at initiation of the contract so that the value of the swap is zero.

(b) Determine arbitrage strategy based on the above data.

Commentary on Question:

The calculations and the determination of an arbitrage opportunity as to the price of the forward 1-year zero-coupon quoted by the bank bond ($96,08) versus the implied 1-year forward bond price (($95.60) were well done, but there were fewer good results in the determination of the strategy to take advantage of the situation.

In particular, some candidates but not such many, did complete the answer by describing the cash flows at duration 0, 1, and 2. This was giving additional grading points.

The 2-year bond price is exp(-3.5%*2)*100 = 93.239
The 1 year bond price is exp(-2.5%*1)*100 = 97.531
The implied 1 year forward bond price = 93.239/97.531*100 = 95.600
Yes, there is an arbitrage opportunity.

The company should sell a 1 year bond for $93.71(=96.08*.97531) with notional of $96.08 and agree with the bank to sell the forward bond at $96.08. Buy 2 year zero coupon bond at $95.60

At year 0 – receive $93.71 for the 1 year bond;
pay $93.24 for the 2 year bond
net cash proceeds = $93.71 -$93.24 = $0.47 from the portfolio

At year 1 – pay $96.08 from the 1 year bond purchase
receive $96.08 for selling $100 notioanal 1 year zero coupond bond
net cash proceed is 0

At year 2 - pay $100 from the 1 year zero coupon bond
receive $100 for the 2 year zero coupon bond
net cash proceed is 0

(c) Calculate the 2-year swap rate and the value of the swap at time 0.

Commentary on Question:

Usually well answered.

Swap value at initiation is 0.

$Z(t,T) = e^{-r(t,T)(T-t)}$

Z(0,.5) = exp(-2%*0.5) = 0.99005
Z(0,1) = exp(-2.5%*1) = 0.97531
Z(0,1.5) = exp(-3%*1.5) = 0.95600
Z (0,2) = exp (-3.5%*2) = 0.93239

$$c = n * \left(\frac{1 - Z(0,T_M)}{\sum_{j=1}^{M} Z(0,T_j)}\right) = 2 * \frac{1 - 0.93239}{0.99005 + 0.97531 + 0.95600 + 0.93239} = 3.51\%$$

(d) Calculate the value of the 2-year swap in part (c) at time 0.5 (after cash payment).
Commentary on Question:

Usually well answered to the exception that some have not considered the value of for the P_{FR} (T, T).

At $T_{0.5}$

Z(0.5,1) = exp(-1%*.5) = 0.99501
Z(0.5,1.5) = exp(-1.5%*1) = 0.98511
Z (0.5,2) = exp (-2%*1.5) = 0.97045

Value of swap = Value of floating rate bond – Value of fixed rate bond

$$V_{swap}(t; c, T) = P_{FR}(t,T) - P_c(t,T)$$

$P_{FR}(T_i,T) = 100$

$$= 100 - \left(\frac{c}{2} * 100 * \sum_{j=t+1}^{M} Z(T_i,T_j) + 100 * Z(T_i,T_M)\right)$$

$= 100 - (\frac{3.51}{2} * 100 *$ (0.99501+0.98511+0.97045) +100*0.97045) = -2.22

(e) Calculate the forward swap rate of the 2-year forward swap contract with expiry of 1 year.

Commentary on Question:

Some have used the values of the Z's instead of the F's.

Z(0,.5) = exp(-2%*0.5) = 0.99005
Z(0,1) = exp(-2.5%*1) = 0.97531
Z(0,1.5) = exp(-3%*1.5) = 0.95600
Z(0,2) = exp(-3.5%*2) = 0.93239
Z(0,2.5) = exp(-3.5%*2.5)=0.91622
Z(0,3) = exp(-4%*3)=0.88692
F(0,1,1.5) = Z(0.1.5)/Z(0,1) = 0.980199
F(0,1,2) = Z(0.2)/Z(0,1) = 0.955997
F(0,1,2.5) = Z(0.2.5)/Z(0,1) = 0.939413
F(0,1,3) = Z(0.3)/Z(0,1) = 0.909373

$$f_2^s(0,T,T^*) \quad = \quad 2 \times \frac{1 - F(0,T,T^*)}{\sum_{j=1}^{m} F(0,T,T_j)}$$

Where

$$F(t,T_1,T_2) = \frac{Z(t,T_2)}{Z(t,T_1)}$$

= 2*(1-0.909373) / (0.980199+0.955997+0.939413+0.909373)

= 4.79%

Learning Objective 2: The candidate will understand the principles and techniques for the valuation of derivatives

QFI QF Fall 2020 Question 1

Learning outcomes:

d) Understand Stochastic Calculus theory and technique used in pricing derivatives

Source references:

- *Options, Futures, and Other Derivatives*, Hull, John, 11th Edition, 2021, Chapter 14, pages 318-321, 327
- *Problems and Solutions in Mathematical Finance: Stochastic Calculus*, Chin, Eric, Nel, Dian and Olafsson, Sverrir, 2014, pages 52, 57-58, 128-130, 132-137

Commentary on Question:

Overall, parts (a), (b), and (d) were answered well by most candidates. Part (c) presented some challenge, as did correctly justifying the normal distribution in part (e).

Solution:

(a) Show that X_t satisfies the stochastic differential equation

$$dX_t = \theta f(t) X_t dW_t$$

Commentary on Question:

Most candidates answered this part well. The statement of the question included an extra minus sign in front of the M_t term in the definition of X_t, which was a typo.

From Ito's Formula,

$$dX_t = \frac{\partial X_t}{\partial t}dt + \frac{\partial X_t}{\partial W_t}dW_t + \frac{1}{2}\frac{\partial^2 X_t}{\partial W_t^2} + \cdots = \frac{\partial X_t}{\partial t}dt + \left(\frac{\partial X_t}{\partial M_t}\frac{\partial M_t}{\partial W_t}\right)dW_t + \frac{1}{2}\frac{\partial}{\partial W_t}\left(\frac{\partial X_t}{\partial M_t}\frac{\partial M_t}{\partial W_t}\right)dt$$

$$= \left[\frac{\partial X_t}{\partial t} + \frac{1}{2}\frac{\partial}{\partial W_t}\left(\frac{\partial X_t}{\partial M_t}\frac{\partial M_t}{\partial W_t}\right)\right]dt + \left(\frac{\partial X_t}{\partial M_t}\frac{\partial M_t}{\partial W_t}\right)dW_t.$$

Now

$$\frac{\partial X_t}{\partial M_t} = \theta \exp\{\theta M_t - \frac{1}{2}\int_0^t f(s)^2 ds\} = \theta X_t$$

and

$$\frac{\partial M_t}{\partial W_t} = f(t)$$

and

$$\frac{\partial}{\partial W_t}\left(\frac{\partial X_t}{\partial M_t}\frac{\partial M_t}{\partial W_t}\right) = \theta^2 f(t)^2 X_t.$$

Finally,

$$\frac{\partial X_t}{\partial t} = -\frac{1}{2}\theta^2 f(t)^2 X_t$$

resulting in

$$dX_t = \theta f(t) X_t dW_t.$$

(b) Show that M_t ~ Normal(0, $\int_0^t f(s)^2 ds$) for any $t > 0$.

Commentary on Question:

Most candidates attempted the alternative solution but did not receive full credit. Most did not remark that the integrand is deterministic thus implying normality.

Write the preceding in integral form and take expectations:

$$\int_0^t dX_s = \int_0^t \theta f(s) X_s dW_s$$

so

$X_t - X_0 = \int_0^t \theta f(s) X_s dW_s$.

Hence

$$E(X_t) - E(X_0) = E(\int_0^t \theta f(s) X_s dW_s) = 0$$

implying $E(X_t) = E(X_0) = 1$. Thus

$$E\left(e^{\theta M_t}\right) = \exp\left(\frac{1}{2}\theta^2 \int_0^t f(s)^2 ds\right)$$

Hence $M_t \sim Normal(0, \int_0^t f(s)^2 ds)$.

Alternative Solution: As M_t is an Ito integral with deterministic integrand, M_t is normally distributed. Now E[M_t]=0 and Var(M_t)=$\int_0^t f(s)^2 ds$ by Ito isometry.

(c) Show that $Z_t = yt + (1-t)(z + \int_0^t \frac{1}{1-s} dW_s)$ for $0 \le t < 1$.

Commentary on Question:

Candidates struggled on this part of the question. Most used the alternative solution to arrive at the result.

Since $\frac{\partial Y_t}{\partial t} = \frac{y - Z_t}{(1-t)^2}, \frac{\partial Y_t}{\partial Z} = \frac{-1}{1-t}, \frac{\partial^2 Y_t}{\partial Z^2} = 0$, applying Ito's lemma we have

$$dY_t = \frac{\partial Y_t}{\partial t} dt + \frac{\partial Y}{\partial Z} dZ_t + \frac{1}{2}\frac{\partial^2 Y_t}{\partial Z^2}(dt)^2$$
$$= \frac{y - Z_t}{(1-t)^2} dt - \frac{dZ_t}{1-t} = -\left(\frac{1}{1-t}\right) dW_t$$

Take integrals to obtain

$$\int_0^t dY_s = -\int_0^t \frac{1}{1-s} dW_s$$

so

$Y_t - Y_0 = -\int_0^t \frac{1}{1-s} dW_s$.

Hence

$$\frac{y-Z_t}{1-t} = y - z - \int_0^t \frac{1}{1-s} dW_s$$

so

$$Z_t = yt + (1-t)\left(z + \int_0^t \frac{1}{1-s} dW_s\right).$$

Alternatively:

$$\frac{dZ_t}{1-t} = \frac{y-Z_t}{(1-t)^2} + \frac{dW_t}{1-t}$$

$$\frac{dZ_t}{1-t} + \frac{Z_t dt}{(1-t)^2} = \frac{y\,dt}{(1-t)^2} + \frac{dW_t}{1-t}$$

$$d\left(\frac{Z_t}{1-t}\right) = \frac{y\,dt}{(1-t)^2} + \frac{dW_t}{1-t}$$

$$\int_0^t d\left(\frac{Z_t}{1-t}\right) = \int_0^t \frac{y\,dt}{(1-t)^2} + \int_0^t \frac{dW_s}{1-s}$$

$$\frac{Z_t}{1-t} - z = \frac{y}{1-t} - y + \int_0^t \frac{dW_s}{1-s}$$

so

$$Z_t = yt + (1-t)\left(z + \int_0^t \frac{1}{1-s} dW_s\right).$$

(d) Find the mean and the variance of Z_t for $0 \le t < 1$.

Commentary on Question

Most candidates were able to derive the mean and variance correctly.

Using the properties of stochastic integrals,

$$E\left(\int_0^t \frac{1}{1-s} dW_s\right) = 0$$

and

$$E\left[\left(\int_0^t \frac{1-t}{1-s} dW_s\right)^2\right] = E\left((1-t)^2 \int_0^t \frac{1}{(1-s)^2} ds\right) = t(1-t)$$

Thus the mean of Z_t is $E(Z_t|Z_0 = z) = yt + z(1-t)$ and the variance is $Var((Z_t|Z_0 = z) = t(1-t)$.

(e) Show that Z_t follows a normal distribution for $0 < t < 1$.

Commentary on Question:

Most candidates did not adequately justify normality as it does not follow from being an Ito integral only. Candidates had to comment that the integrand was deterministic.

Now since $\int_0^t \left(\frac{1}{1-s}\right)^2 ds = \frac{t}{1-t} < \infty$ for $0 < t < 1$, $\int_0^t \frac{1}{1-s} dW_s$ is in the form required for part (b), so $\int_0^t \frac{1}{1-s} dW_s$ follows a normal distribution.

Alternatively, noting that $\int_0^t \frac{1}{1-s} dW_s$ is square integrable and the integrand is deterministic allows one to conclude that the Ito integral is normal.

QFI QF Fall 2020 Question 2

Learning outcomes:

d) Understand Stochastic Calculus theory and technique used in pricing derivatives

e) Understand and apply the concepts of risk-neutral measure, forward measure, normalization, and the market price of risk

Source references:

- *Options, Futures, and Other Derivatives, H*ull, John, 11th Edition, 2021, Chapter 28, pages 671-672
- *Problems and Solutions in Mathematical Finance: Stochastic Calculus*, Chin, Eric, Nel, Dian and Olafsson, Sverrir, 2014, pages 72-73, 221-227

Solutions:

(a) Establish a condition on μ_1, μ_2, σ_1 and σ_2 such that both $X_t e^{-rt}$ and $Y_t e^{-rt}$ are martingales under the risk-neutral measure $\mathbb{Q}$.

Commentary on Question:

This part proved to be the most challenging. Candidates who weren't able to derive the desired condition received partial credit for correct steps.

By product rule, we have

$$d(X_t e^{-rt}) = X_t e^{-rt}((\mu_1 - r)dt + \sigma_1 dW_t) = X_t e^{-rt}\sigma_1 \left(\frac{\mu_1 - r}{\sigma_1} dt + dW_t\right)$$

$$d(Y_t e^{-rt}) = Y_t e^{-rt}((\mu_2 - r)dt + \sigma_2 dW_t) = Y_t e^{-rt}\sigma_2 \left(\frac{\mu_2 - r}{\sigma_2} dt + dW_t\right)$$

To make both $X_t e^{-rt}$ and $Y_t e^{-rt}$ are martingales under the risk-neutral measure Q, we need to define the following Wiener process under Q

$$dW_t^* = \frac{\mu_1 - r}{\sigma_1} dt + dW_t = \frac{\mu_2 - r}{\sigma_2} dt + dW_t$$

Hence we need

$$\frac{\mu_1 - r}{\sigma_1} = \frac{\mu_2 - r}{\sigma_2}$$

(b) Derive the Radon-Nikodym derivative $\frac{d\mathbb{Q}}{d\mathbb{P}}$ by assuming that the condition in part (c) holds.

Commentary on Question:

Candidates did well on this part.

Suppose the condition given in part (d) holds. Let

$$\alpha = \frac{\mu_1 - r}{\sigma_1}$$

and

$$dW_t^* = \alpha dt + dW_t$$

Then by the Girsanov theorem, we have

$$\frac{dQ}{dP} = \xi_T = e^{-\frac{\mu_1 - r}{\sigma_1} W_T - \frac{(\mu_1 - r)^2}{2\sigma_1^2} T}$$

QFI QF Fall 2020 Question 3

Learning outcomes:

d) Understand Stochastic Calculus theory and technique used in pricing derivatives

Source references:

- *Options, Futures, and Other Derivatives, H*ull, John, 11th Edition, 2021, Chapter 14, page 327, 329

Solution:

(a) Show that for all $i, j = 0, 1, \dots, n-1$

(i) $E\left[\left(\Delta W_{t_i}\right)^4\right] = 3h^2$ using Ito's lemma.

(ii) $E\left[\left(\Delta W_{t_i}\right)^2\left(\Delta W_{t_j}\right)^2\right] = h^2$ if $i < j$.

Commentary on Question:

Candidates performed as expected on part (a). To receive full credit for part (a)(i), Ito's Lemma must be used. To receive full credit for part (a)(ii), independence must be clearly specified or implied.

(i)

From Ito's lemma:

$$d\left(\left(W_t - W_{t_i}\right)^4\right) = 4\left(W_t - W_{t_i}\right)^3 dW_t + 6\left(W_t - W_{t_i}\right)^2 dt.$$

Integrating over (t_i, t_{i+1}),we have:

$$\left(W_{t_{i+1}} - W_{t_i}\right)^4 = 4\int_{t_i}^{t_{i+1}}\left(W_t - W_{t_i}\right)^3 dW_t + 6\int_{t_i}^{t_{i+1}}\left(W_t - W_{t_i}\right)^2 dt$$

It follows that:

$$E\left[\left(\Delta W_{t_i}\right)^4\right] = 0 + 6\int_{t_i}^{t_{i+1}} E\left[\left(W_t - W_{t_i}\right)^2\right]\, dt = 6\int_{t_i}^{t_{i+1}}(t - t_i)dt$$
$$= 3(t_{i+1} - t_i)^2 = 3h^2$$

(ii)

Since ΔW_{t_i} and ΔW_{t_j} are independent when $i < j$

$$E\left[\left(\Delta W_{t_i}\right)^2\left(\Delta W_{t_j}\right)^2\right] = E\left[\left(\Delta W_{t_i}\right)^2\right]E\left[\left(\Delta W_{t_j}\right)^2\right] = h^2$$

QFI QF Fall 2020 Question 7

Learning Outcomes:

f) Understand option pricing techniques

j) Define and explain the concept of volatility smiles and describe several approaches for modeling smiles, including stochastic volatility, local- volatility, jump-diffusions

Source References:

- *The Volatility Smile*, Derman, Emanuel and Miller, Michael, 2016, Chapter 8, pages 144-145, 163-164

- *Options, Futures, and Other Derivatives*, Hull, John, 11th Edition, 2021, Chapters 15, 20, pages 352-353, 451

Commentary on Question:

This question was intended to measure candidates' understanding of equity derivative and volatility smile. Most candidates well understood the volatility smile but showed a lack of knowledge of the practical equity derivative application.

Solution:

(a)

(i) Explain volatility smiles.

Commentary on Question:

Most candidates performed well in part (a). The purpose of part (a) was to measure the understanding of volatility smile.

(i) When plot the market implied volatility vs the strike price or in-the-moneyness of the options, we often observe implied volatility is non-constant as function of strikes, with lower implied volatility near the at-the-money (ATM) strike, and higher implied volatility for both lower and higher strikes.

(b) Describe the most salient characteristics of the equity volatility smile.

Commentary on Question:

Candidates demonstrated modest understanding in part (b).

- Its most notable character is the negative slope as a function of the strike.
- The negative slope is generally steeper for short expiration.
- Implied volatility and index returns are negatively correlated.
- Equity smile is often a smirk than a smile – increase and decrease in implied volatility are often asymmetric. skew is partially due to an asymmetry in the way equity index movement: large negative returns are much more frequent than large positive returns.
- There is also a demand component that contributes to smile, people are willing to pay additional premium for hedge of large movement.

(c) Identify the trades of the replicating portfolio.

Commentary on Question:

Most candidates performed poorly in part (c). Some candidates showed a lack of understanding of replicating portfolio construction.

By offering return of premium and a cap on T&T growth, the index annuity longs an at-the-money call and shorts an out-of-the-money call at 5 delta.

Assume no lapse or redemption before renewal, to hedge this liability, company should buy an at-the-money call and sell an out-the-money call at 105%, at inception.

(d) Calculate the price for the replicating portfolio and determine whether the budget is sufficient for the hedging, using the fitted implied volatility function IV(K) provided.

Commentary on Question:

The candidates who had a right approach in part (c) also performed well in part (d). However, many candidates made mistakes in calculation.

$$C(S, K, t, \sigma, r) = SN(d_1) - Ke^{-rt}N(d_2)$$

$$d_{1,2} = \frac{\ln\left(\frac{S}{K}\right) + \left(r \pm \frac{\sigma^2}{2}\right)t}{\sigma\sqrt{\tau}}$$

$$N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}y^2}\, dy$$

For at-the-money call: S=100, K=100, r =3%, d=0%, t=1, σ=15% + (100-100) *1.4%=15%

Plug into formula above, C(100) = 7.49

For Out-of-money call: S=100, K=105, r =3%, d=0%, t=1; σ=15% + (105-100) *1.4%=22%

Plug into formula above, C(105) = 7.93

The hedge portfolio price = 7.49 – 7.93 = -0.44 < 0.5

Yes, the budget is sufficient

(e) Explain the reasonableness of the implied volatility function IV(K) in the context of smile arbitrage.

Commentary on Question:

Most candidates performed poorly in part (e). Most candidates didn't approach the question from the arbitrage-free perspective.

When a portfolio of options with non-negative (non-positive) payoff actually has a negative (positive) market price while using the volatility smile, the volatility smile is considered not arbitrage-free.

Since we are long an at-the-money call (paying $5.80) and short an out-the-money call (receiving $6.50), we are receiving $0.7 by constructing this portfolio, while the portfolio will have a non-negative payoff (call spread).

Therefore, arbitrage exists, due to unreasonable volatility smile.

(f)

(i) Identify types of market conditions that would negatively affect the ability to manage the product with the added guarantee.

(ii) Suggest a modeling approach to better measure the risk.

(i) The embedded option in this index annuity is basically a call spread (long ATM call + short OTM call), with the new product design feature, the price of the portfolio is 0.5 = ATM call – OTM call, where we could back out the OTM call strike, however which is floored at 3%.

The possible challenge of such product design is budget is not enough to offer a cap of at least 3%, then the product has to be offered at a loss or below the expected profit level.

(ii) It is important for the pricing actuaries to understand the market condition where such risk exists.

This could be achieved by model the market condition: interest rate, equity level, equity volatility stochastically, especially equity volatility smile, as it drives the difference between ATM call and OTM call.

With, stochastically volatility models, volatility can change through time, are a function of time, index level, strike level.

Pros: automatically create a volatility smile – is appropriate for pricing exotic option, it could also match the term structure of the volatility.

Cons: cannot replicate European options, only can approximate. The calibration can be unstable, resulting in jumps in mark-to-market Profit/Loss;

can be calibrated using vanilla option or exotic option, but not both at the same time.

QFI QF Fall 2020 Question 8

Learning Outcomes:

d) Understand Stochastic Calculus theory and technique used in pricing derivatives

e) Understand and apply the concepts of risk-neutral measure, forward measure, normalization, and the market price of risk

Source References:

- *Options, Futures, and Other Derivatives*, Hull, John, 11th Edition, 2021, Chapter 31, pages 721-724
- *Problems and Solutions in Mathematical Finance: Stochastic Calculus*, Chin, Eric, Nel, Dian and Olafsson, Sverrir, 2014, pages 132-137

Solution:

(a) Explain why interest rates are always positive in this model.

Commentary on Question:

Candidates generally answered this question okay. The most common error or omitted portion were not commenting on the technical condition was true to ensure the drift term cannot force it negative.

When the interest rate r_t is moving toward zero, the diffusion part $\sqrt{\alpha r_t}$ declines, and it becomes in fact zero when r_t hits zero.

When r_t = 0, the only term left is $dr_t = \gamma(\bar{r})$>0. Thus, the next step will be for sure that r_t increases (because the change $dr_t > 0$).

One important caveat is that to ensure the interest rate process is always positive (and well defined), we must have the following technical condition satisfied: $\gamma\bar{r} > \frac{1}{2}\alpha$

That is, the term that "pulls up" the interest rate when r_t hits zero, " $\gamma\bar{r}$," must be large enough.

(b) Show that $r_t = e^{-\gamma t} r_0 + \bar{r}(1 - e^{-\gamma t}) + \sqrt{a} e^{-\gamma t} \int_0^t e^{\gamma s} \sqrt{r_s}\, dX_s$.

Commentary on Question:

Candidates performed well on this question. Clearer answers were those that stated they were using Ito's lemma and mentioned integrating both sides.

Using the Ito`s lemma on $Z_t = e^{\gamma t} r$, we have

$$d(e^{\gamma t} r_t) = \frac{\partial}{\partial t}(e^{\gamma t} r_t)dt + \frac{\partial}{\partial r}(e^{\gamma t} r)|_{r=r_t} dr_t + \frac{1}{2}\frac{\partial^2}{\partial r^2}(e^{\gamma t} r)|_{r=r_t} \ (dr_t)^2$$

$= \gamma e^{\gamma t} r_t dt + e^{\gamma t}[(\gamma(\bar{r} - r_t)dt + \sqrt{\alpha r_t} dX_t]$

$=\gamma(\bar{r})e^{\gamma t}\ dt + e^{\gamma t}\sqrt{\alpha r_t} dX_t$

Integrating the above expressions on both sides from 0 to t

$$\int_0^t d(e^{\gamma s} r_s) = \int_0^t \gamma(\bar{r})e^{\gamma s}\, ds + \int_0^t e^{\gamma s}\sqrt{\alpha r_s} dX_s$$

$$r_t = e^{-\gamma t} r_0 + \bar{r}(1 - e^{-\gamma t}) + \sqrt{\alpha}\, e^{-\gamma t} \int_0^t e^{\gamma s} \sqrt{r_s}\, dX_s$$

(c) Determine $E[r_t]$ and $Var[r_t]$.

Commentary on Question:

Candidates performed okay on this part of the question. Most had no issue with the Expectation and the beginning of the Variance including the relation with Ito's Isometry. Most struggled with substituting back in the E[t] to push to the final equation.

$E[r_t] = e^{-\gamma t} r_0 + \bar{r}(1 - e^{-\gamma t}) + E[\sqrt{\alpha}\, e^{-\gamma t} \int_0^t e^{\gamma s} \sqrt{r_s}\, dX_s]$

$E[r_t] = e^{-\gamma t} r_0 + \bar{r}(1 - e^{-\gamma t})$ *as* $E[\sqrt{\alpha}\, e^{-\gamma t} \int_0^t e^{\gamma s} \sqrt{r_s}\, dX_s] = 0$

$Var[r_t] = E[r_t - E(r_t)]^2 = \alpha\, e^{-2\gamma t} E[\left(\int_0^t e^{\gamma s} \sqrt{r_s}\, dX_s\right)^2]$

using Ito's Isometry

$= \alpha\, e^{-2\gamma t} \int_0^t e^{2\gamma s}\, E[r_s] ds$

Substituting $E[r_t]$ *gives:*

$\alpha\, e^{-2\gamma t} \int_0^t e^{2\gamma s}\, [e^{-\gamma s} r_0 + \bar{r}(1 - e^{-\gamma s})]\, ds = (\alpha\, \bar{r}) \frac{1}{2\gamma}(1 - e^{-2\gamma t} - 2e^{-\gamma t} + 2e^{-2\gamma t}) + \frac{\alpha}{\gamma} r_0 e^{-2\gamma t}(e^{\gamma t} - 1)$ *or*

$$\frac{\alpha\bar{r}}{2\gamma}(1 - e^{-\gamma t})^2 + \frac{\alpha}{\gamma}\, r_0 e^{-\gamma t}(1 - e^{-\gamma t})$$

(d) Express $\frac{\partial Z}{\partial t}$, $\frac{\partial Z}{\partial r}$ and $\frac{\partial^2 Z}{\partial r^2}$ in terms of $Z(r_t, t, T)$, $A(t,T)$ and $B(t,T)$.

Commentary on Question:

Candidates performed well on this section. Most common mistake was with the $\frac{\partial Z}{\partial t}$ term either including an incorrect extra term or omitting the r term on $\frac{\partial B}{\partial t}$ component.

$$Z(r,t,T) = e^{A(t,T) - B(t,T) r}$$

$$\frac{\partial Z}{\partial t} = \left[\frac{\partial A}{\partial t} - \frac{\partial B}{\partial t} r\right] Z(r,t,T)$$

$$\frac{\partial Z}{\partial r} = -B(t,T)\, Z(r,t,T)$$

$$\frac{\partial^2 Z}{\partial r^2} = B^2(t,T)\, Z(r,t,T)$$

(f) Show that

(i) $\frac{\partial A}{\partial t} = \gamma \bar{r} B(t,T)$

(ii) $\frac{\partial B}{\partial t} = \gamma B(t,T) + \frac{1}{2} a B(t,T)^2 - 1$

Commentary on Question:

Candidates performed poorly on this question. A good portion of the candidate did not attempt this part of the question. An alternate solution was also accepted using given formulas and is present below.

Primary Solution:

Plugging the results of part (e) to the fundamental pricing equation

$$\frac{\partial Z}{\partial t} + \frac{\partial Z}{\partial r}\gamma(\bar{r} - r) + \frac{1}{2}\frac{\partial^2 Z}{\partial r^2} r\alpha = rZ$$

we have

$$\left[\frac{\partial A}{\partial t}-\frac{\partial B}{\partial t}r\right]Z - BZ\,\gamma(\bar{r}-r)+\frac{1}{2}B^2 Z\,r\alpha = \, rZ$$

It follows that

$$\left[\frac{\partial A}{\partial t}-\frac{\partial B}{\partial t}r\right] - B(t,T)\,\gamma(\bar{r}-r)+\frac{1}{2}B^2(t,T)r\alpha = r$$

Rearranging the terms

$$\left[\frac{\partial A}{\partial t}-B(t,T)\,\gamma(\bar{r})\right]-\left(\frac{\partial B}{\partial t}-B(t,T)\,\gamma-\frac{1}{2}B^2(t,T)\alpha+1\right)r=0$$

In order to have the above expression =0 for all t and r

$\frac{\partial A}{\partial t}-B(t,T)\,\gamma(\bar{r})=0$ *This implies* $\frac{\partial A}{\partial t}=B(t,T)\,\gamma(\bar{r})$

$\left(\frac{\partial B}{\partial t}-B(t,T)\,\gamma-\frac{1}{2}B^2(t,T)\alpha+1\right)=0$

This implies

$$\frac{\partial B}{\partial t}=\gamma B(t,T)\ +\frac{1}{2}\alpha B^2(t,T)-1$$

QFI QF Fall 2020 Question 12

Learning Outcomes:

d) Understand Stochastic Calculus theory and technique used in pricing derivatives

e) Understand and apply the concepts of risk-neutral measure, forward measure, normalization, and the market price of risk

Source References:

- *Options, Futures, and Other Derivatives*, Hull, John, 11th Edition, 2021, Chapters 14, 28, pages 327, 675
- *Problems and Solutions in Mathematical Finance: Stochastic Calculus*, Chin, Eric, Nel, Dian and Olafsson, Sverrir, 2014, pages 97-98, 221-227

Commentary on Question:

This question is to test candidates on how to apply Ito's Lemma and the concept of Martingale.

Solution:

(a) Show, using Ito's Lemma, that

$$\frac{d\widetilde{V}}{\widetilde{V}} = (\sigma_Z^2 - \sigma_V \sigma_Z)dt + (\sigma_V - \sigma_Z)dX_t$$

Commentary on Question:

Overall, candidates did well on this part. Alternative approaches such as using product rule, quotient rule, etc. were also awarded full marks provided derivation was done correctly.

From Ito's lemma it's straightfoward to have

$$d(\ln V) = \left(r - \frac{\sigma_V^2}{2}\right)dt + \sigma_V dX_t$$

$$d(\ln Z) = \left(r - \frac{\sigma_Z^2}{2}\right)dt + \sigma_Z dX_t$$

It follows that

$$d\left(\ln\tilde{V}\right) = d(\ln V) - d(\ln Z) = \left(\frac{\sigma_Z^2}{2} - \frac{\sigma_V^2}{2}\right)dt + (\sigma_V - \sigma_Z)dX_t$$

In turn, from Ito's lemma it's straightfoward to have

$$\frac{d\tilde{V}}{\tilde{V}} = \left(\left(\frac{\sigma_Z^2}{2} - \frac{\sigma_V^2}{2}\right) + \frac{(\sigma_V - \sigma_Z)^2}{2}\right)dt + (\sigma_V - \sigma_Z)dX_t$$

$$= (\sigma_Z^2 - \sigma_V \sigma_Z)dt + (\sigma_V - \sigma_Z)dX_t$$

(b) Show that $\widetilde{V}$ is a martingale under $\mathbb{Q}^Z$ using:

(i) the result in part (a);

(ii) the Feynman-Kac theorem.

Commentary on Question:

Candidates did well on part (b)(i). However, very few candidates were able to apply Feynman-Kac theorem correctly. Partial marks were awarded for stating the theorem and identifying R(r)=0.

(i) By differntiating $\tilde{X}_t = X_t - \int_0^{\mathrm{t}} \sigma_Z(r,u)du$

$$\widetilde{dX}_t = dX_t - \sigma_Z(r,t)dt,$$

$$dX_t = \widetilde{dX}_t + \sigma_Z(r,t)dt$$

plugging it in the dX_t of result of (a) yields the driftless martingale.

$$\frac{d\tilde{V}}{\tilde{V}} = (\sigma_Z^2 - \sigma_V\sigma_Z)dt + (\sigma_V - \sigma_Z)dX_t$$

$$= (\sigma_V - \sigma_Z)\widetilde{dX}_t$$

(ii) From the Feynman-Kac theorem, it implies $R = 0$ in the following equation

$$R(r)\tilde{V} = 0 \; = \; \frac{\partial \tilde{V}}{\partial t} \; + \; \frac{\partial \tilde{V}}{\partial r} \; (m^*(r,t) \; + \; \sigma_Z \, (r,t) \, s(r,t)) \; + \; \frac{1}{2} \frac{\partial^2 \tilde{V}}{\partial r^2} \; s(r,t)^{\,2}$$

Hence

$$\tilde{V}(r,t;\,T) = E_f^*\left(e^{-\int_t^T R(u)du} \;\; \tilde{V}(r,T;T)|r_t\right) = E_f{}^*\left(\tilde{V}(r,T;T)|r_t\right),$$

which is a martingale.

(c) Derive expressions for σ_z and σ_v in terms of $s(r,t)$, V, and Z.

Commentary on Question:

Candidates did very poorly on this part.

By the Ito's lemma and the Fundamental Pricing equation,

$$dV = \left(\frac{\partial V}{\partial t} + \frac{\partial V}{\partial r} m^*(r,t) \; + \; \frac{1}{2} \frac{\partial^2 \tilde{V}}{\partial r^2} \; s(r,t)^2\right) dt + \frac{\partial V}{\partial r} s(r,t) dX_t$$

$$= rVdt + \frac{\partial V}{\partial r} s(r,t) dX_t$$

$$= rVdt + \sigma_V V dX_t$$

with

$$\sigma_V = \frac{1}{V}\left(\frac{\partial V}{\partial r}\right) s(r,t).$$

For other security $Z(r,t)$,

$$dZ = \left(\frac{\partial Z}{\partial t} + \frac{\partial Z}{\partial r} m^*(r,t) \; + \; \frac{1}{2} \frac{\partial^2 Z}{\partial r^2} \; s(r,t)^2\right) dt + \frac{\partial Z}{\partial r} s(r,t) dX_t$$

$$= rZdt + \sigma_Z Z dX_t$$

with

$$\sigma_Z = \frac{1}{Z}\left(\frac{\partial Z}{\partial r}\right)s(r,t).$$

QFI QF Spring 2021 Question 1

Learning Outcomes:

d) Understand Stochastic Calculus theory and technique used in pricing derivatives

Source References:

- *Options, Futures, and Other Derivatives*, Hull, John C., 11th Edition, 2021, Chapters 14, 28, pages 329, 671-675
- *Problems and Solutions in Mathematical Finance: Stochastic Calculus*, Chin, Eric, Nel, Dian and Olafsson, Sverrir, 2014, pages 72-73, 221-227

Commentary on Question:

The focus of this question is understanding the differences and implications of real-world versus risk-neutral probability measures by applying Ito's lemma, Girsanov's theorem, and the Radon-Nikodym ("R-N") derivative. Candidates struggled to show this understanding, especially for parts (c) and (d). Detailed commentaries are listed underneath each part.

Solution:

(a) Determine the market price of risk for all $t \le 1$.

Commentary on Question:

There is a typo in the question, where the "S" is missing in the process of $\mathrm{dS_t}$ *when* $0 \le \mathrm{t} \le 0.5$. *The correct process should be* $\mathrm{dS_t} = 0.05\mathrm{S_t dt} + 0.2\mathrm{S_t dW_t}$. *But most candidates identified the typo. Overall, most candidates were able to calculate the correct market price of risk. Credits were also given to the answers using the wrong process as stated in the question.*

The market price of risk is defined as

$$\lambda_t = \frac{\mu_t - r_t}{\sigma_t}$$

where μ_t is the stock price drift rate, r_t is the risk-free rate, and σ_t is the stock price volatility. By plugging in the numbers, we get

$$\lambda_t = \begin{cases} \dfrac{0.05 - 0.01}{0.2} = 0.2 & if\ 0 \leq t \leq 0.5 \\ \dfrac{-0.05 - 0.01}{0.3} = -0.2 & if\ 0.5 < t \leq 1 \end{cases}$$

(b) Calculate $E^{\mathbb{P}}\left[S_1 | S_{0.5}\right]$.

Commentary on Question:

Candidates were able to apply Ito's Lemma to get $\mathrm{dlnS_t}$ *and to express* $\mathrm{S_t}$ *in terms of* $\mathrm{S_{0.5}}$. *But they had difficulties calculating the expected value. Some candidates also demonstrated that they did not understand* $S_{0.5}$ *is a known quantity and can be treated as constant in the expectation.*

For $0.5 < t \leq 1$, we apply Ito's lemma and get

$$dlnS_t = \left(\mu_t - \frac{\sigma_t^2}{2}\right)dt + \sigma_t dW_t = -0.095\, dt + 0.3dW_t$$

which gives

$$lnS_t - lnS_{0.5} = -0.095(t - 0.5) + 0.3(W_t - W_{0.5})$$

or

$$S_t = S_{0.5}e^{-0.095(t-0.5)+0.3(W_t - W_{0.5})}$$

Hence

$$E^P[S_t|S_{0.5}] = S_{0.5}E^P[e^{-0.095(t-0.5)+0.3(W_t-W_{0.5})}] = S_{0.5}e^{-0.095(t-0.5)+0.045(t-0.5)}$$

which leads to $E^P[S_1|S_{0.5}] = S_{0.5}e^{-0.025}$.

(c) Derive the Radon-Nikodym derivative of the risk-neutral measure $\mathbb{Q}$ with respect to the real-world measure $\mathbb{P}$.

Commentary on Question:

Candidates performed poorly on this question. One source for this question (Chin et al) has many typos related to the Radon-Nikodym derivative. (page 222-223, 225, 239, 240). Whereas the other source - page 194-196, 203, 205, 218, 219 and 234 of (Chin el al) have the correct Radon-Nikodym derivatives. Credit was given to answers using the wrong R-N derivative as stated in the incorrect source.

For candidates that provided a general form of the R-N derivative, a common mistake made was to use λ_t with the incorrect sign on the first integral component.

Most candidates were not able to derive the correct derivative when $0.5 < t \leq 1$.

Solution 1 – Based on the correct R-N derivative form from the source page 218.

The Radon-Nikodym derivative is calculated as

$$Z_s = e^{-\int_0^s \lambda_t dW_t - \frac{1}{2}\int_0^s \lambda_t^2 dt}$$

where λ_t is the market price of risk as $\lambda_t = \frac{\mu_t - r_t}{\sigma_t}$, we get

$$\lambda_t = \begin{cases} 0.2, & 0 \leq t \leq 0.5 \\ -0.2, & 0.5 < t \leq 1 \end{cases}$$

Plugging in the numbers, we get

$$\int_0^s \lambda_t dW_t = \begin{cases} 0.2W_s & if\ 0 \leq s \leq 0.5 \\ -0.2(W_s - 2W_{0.5}) & if\ 0.5 < s \leq 1 \end{cases}$$

and

$$\int_0^s \lambda_t^2 dt = 0.04s$$

Hence

$$Z_s = \begin{cases} e^{-0.2W_s - 0.02s} & if\ 0 \leq s \leq 0.5 \\ e^{0.2(W_s - 2W_{0.5}) - 0.02s} & if\ 0.5 < s \leq 1 \end{cases}.$$

Or

The Radon-Nikodym derivative is calculated as

$$Z_s = e^{\int_0^s X_t dW_t - \frac{1}{2}\int_0^s X_t^2 dt}$$

where $X_t = -\lambda_t = \frac{r_t - \mu_t}{\sigma_t}$, we get

$$X_t = \begin{cases} -0.2, & 0 \leq t \leq 0.5 \\ 0.2, & 0.5 < t \leq 1 \end{cases}$$

Plugging in the numbers, we get

$$\int_0^s X_t dW_t = \begin{cases} -0.2W_s & if\ 0 \leq s \leq 0.5 \\ 0.2(W_s - 2W_{0.5}) & if\ 0.5 < s \leq 1 \end{cases}$$

and

$$\int_0^s X_t^2 dW_t = 0.04s$$

Hence

$$Z_s = \begin{cases} e^{-0.2W_s - 0.02s} & if\ 0 \leq s \leq 0.5 \\ e^{0.2(W_s - 2W_{0.5}) - 0.02s} & if\ 0.5 < s \leq 1 \end{cases}$$

(d) Show that $\left\{S_t e^{-0.01t} : 0 \leq t \leq 1\right\}$ is a $\mathbb{Q}$-martingale.

Commentary on Question:

Candidates had the most difficulty with this part. Many were able to prove the no drift condition, but failed to mention Girsanov's theorem or the R-N derivative as justification for the substitution of a different standard Wiener process under an equivalent measure. A complete response should demonstrate and justify the relationship between the two Wiener processes.

For ease of presentation, we use r_t, μ_t, and σ_t to denote the risk-free rate, the drift rate of the stock price, and the volatility of the stock price. Let $Y_t = S_t e^{-\int_0^t r_u du}$.

Applying Ito's lemma on Y_t and using the fact that

$$dS_t = \mu_t S_t dt + \sigma_t S_t dW_t$$

we obtain

$$\begin{aligned} dY_t = e^{-\int_0^t r_u du} dS_t - r_t Y_t dt &= e^{-\int_0^t r_u du}(\mu_t S_t dt + \sigma_t S_t dW_t) - r_t S_t e^{-\int_0^t r_u du} dt \\ &= e^{-\int_0^t r_u du}(\mu_t S_t dt - r_t S_t dt + \sigma_t S_t dW_t) \\ &= \sigma_t S_t e^{-\int_0^t r_u du}\left(dW_t + \frac{\mu_t - r_t}{\sigma_t} dt\right) \end{aligned}$$

Now let

$$\widetilde{W}_t = W_t + \int_0^t \frac{\mu_u - r_u}{\sigma_u} du$$

By Girsanov's theorem, there exists an equivalent measure defined by the R-N derivative, so that $\widetilde{W}_t$ is a standard Wiener process on the same filtration.

We then have

$$dY_t = \sigma_t S_t e^{-\int_0^t r_u du} d\widetilde{W}_u$$

Since dY_t does not have the dt term, Y_t is a martingale under the risk-neutral measure $\mathbb{Q}$.

QFI QF Spring 2021 Question 2

Learning Outcomes:

d) Understand Stochastic Calculus theory and technique used in pricing derivatives

Source References:

- *Options, Futures, and Other Derivatives*, Hull, John C., 11th Edition, 2021, Chapters 14, 28, pages 327, 675
- *Problems and Solutions in Mathematical Finance: Stochastic Calculus*, Chin, Eric, Nel, Dian and Olafsson, Sverrir, 2014, pages 128-130

Solution:

(a) Evaluate $Var^{\mathbb{P}}\left[|W_t|\right]$.

Commentary on Question:

Most candidates failed to work out the integrals.

$E^{\mathbb{P}}[|W_t|] = \int_{-\infty}^{\infty} |w| \frac{1}{\sqrt{2\pi t}} e^{-\frac{w^2}{2t}} dw$

$E^{\mathbb{P}}[|W_t|]= 2\int_0^{\infty} |w| \frac{1}{\sqrt{2\pi t}} e^{-\frac{w^2}{2t}} dw$

$$E^{\mathbb{P}}[|W_t|] = -2\int_0^{\infty} \frac{d}{dw}\left(\frac{\sqrt{t}}{\sqrt{2\pi}} e^{-\frac{w^2}{2t}}\right) dw = 2\frac{\sqrt{t}}{\sqrt{2\pi}} = \sqrt{\frac{2t}{\pi}}$$

$Var^{\mathbb{P}}[|W_t|] = E^{\mathbb{P}}[|W_t|^2] - E^{\mathbb{P}}[|W_t|]^2$ (definition of variance)

$Var^{\mathbb{P}}[|W_t|] = E^{\mathbb{P}}[W_t^2] - E^{\mathbb{P}}[|W_t|]^2$ $(as\ E^{\mathbb{P}}[|W_t|^2] = E^{\mathbb{P}}[W_t^2])$

$Var^{\mathbb{P}}[|W_t|] = t - 2t/\pi$

(b) Determine integer *k* that makes W_t^k a martingale.

Commentary on Question:

Most of the candidates got this part right. Some did not provide the k = 0 solution.

By Ito's Lemma, $dW_t^k = kW_t^{k-1}dW_t + \frac{1}{2}k(k-1)W_t^{k-2}(dW_t)^2$

$$= kW_t^{k-1}dW_t + \frac{1}{2}k(k-1)W_t^{k-2}dt$$

We need drift term to be zero to make the process a martingale.

When k=0 or 1, the drift term=0.

So if k=0, 1, then the process is a martingale.

QFI QF Spring 2021 Question 3

Learning Outcomes:

d) Understand Stochastic Calculus theory and technique used in pricing derivatives

Source References:

- *Options, Futures, and Other Derivatives*, Hull, John C., 11th Edition, 2021, Chapters 14, 28, pages 327, 675
- *Problems and Solutions in Mathematical Finance: Stochastic Calculus*, Chin, Eric, Nel, Dian and Olafsson, Sverrir, 2014, pages 128-130

Commentary on Question:

This question tests candidates' knowledge of Ito's lemma, Ito's isometry, martingales, and basic properties of Brownian Motion. Most candidates did well on this question. Some candidates did not state what rules and formulas they were applying to from step to step.

Solution:

(a) Derive $E\left[W_s^3 W_t\right]$ for $t > s$.

Commentary on Question:

Most candidates did well on this part. The few candidates who did badly tried to decompose the wrong term and failed to state the independence of W_s^3 and $W_t - W_s$.

By the properties of Brownian motion, we have

$E[W_s^3 W_t]$

$= E[W_s^3(W_s + W_t - W_s)]$

$= E[W_s^4] + E[W_s^3(W_t - W_s)]$

$= E[W_s^4] + E[W_s^3]E[(W_t - W_s)]$

$= E[W_s^4]$

Let $Z = \frac{W_s}{\sqrt{s}}$, which is a standard normal distribution. Then $E[Z^4] = 3$.

This gives

$E[W_s^4] = s^2 E[Z^4] = 3s^2$

(b) Determine the value of c such that $W_t^3 - ctW_t$ is a martingale.

Commentary on Question:

Most candidates did well on this part. Some candidates pursued the alternate solution of using Ito's lemma and setting the drift term to 0.

Let $M_t = W_t^3 - ctW_t$. Then we have

$E[M_t|F_s] = E[(W_s + W_t - W_s)^3|F_s] - ctE[W_t|F_s]$

$= E[W_s^3|F_s] + 3E[W_s^2(W_t - W_s)|F_s] + 3E[W_s(W_t - W_s)^2|F_s] + E[(W_t - W_s)^3|F_s] - ctE[W_s|F_s] - ctE[W_t - W_s|F_s]$

$= W_s^3 + 0 + 3W_s(t - s) + 0 - ctW_s + 0$

$= M_s$ if c = 3

(c) Show that $X_t = \int_0^t W_u du$ is not a martingale.

Commentary on Question:

Most candidates did well on this part. However, some candidates mistook the integral $\int_0^t W_u du$ for $\int_0^t W_u \, dW_u$ and said it is a martingale. Some candidates failed to apply stochastic integrals clearly and effectively to show that X_t is not a martingale.

By Product Rule, we have

$$X_t = tW_t - \int_0^t u\, dW_u$$

Let $s \le t$. Since the Ito integral is a martingale, we have

$$E[X_t|F_s] = E[tW_t|F_s] - E\left[\int_0^t ud\,W_u|F_s\right] = tW_s - \int_0^s ud\,W_u = X_s + (t-s)W_s \ne X_s$$

for t>s. Hence X_t is not a martingale.

(d) Calculate

(i) $E[V^2]$

(ii) $E[VY]$

Commentary on Question:

Most candidates did well on this part. Some candidates lost points for not mentioning Ito's Isometry in part (i) or not stating the independence of V and G in part (ii).

By Ito's isometry

(i)

$$E[V^2] = \int_0^1 e^{-2s} ds = \frac{1}{2}(1 - e^{-2})$$

(ii)

Let $G = \int_1^2 e^{-s} dW_s$

Y = V +G

$E[VY] = E[V(V+G)]$

Since V and G are independent,

$= E[V^2] + E[VG]$

$= E[V^2] + E[V]E[G]$

$= E[V^2]$

$= \frac{1}{2}(1 - e^{-2})$

QFI QF Spring 2021 Question 4

Learning Outcomes:

d) Understand Stochastic Calculus theory and technique used in pricing derivatives

f) Understand option pricing techniques

Source References:

- *Options, Futures, and Other Derivatives*, Hull, John C., 11th Edition, 2021, Chapter 14, page 327
- *Problems and Solutions in Mathematical Finance: Stochastic Calculus*, Chin, Eric, Nel, Dian and Olafsson, Sverrir, 2014, pages 221-227
- INV201-101-25: Chapter 6 of Introduction to Stochastic Finance with Market Examples by Privault

Commentary on Question:

The objective in this question was to test Ito's Lemma as applied to the valuation of derivatives on a security that is driven by a Weiner Process. Most candidates performed above average and partial credit was given for answers with calculation errors or missing steps.

Solution:

(a) Show, using Ito's lemma, that $\sigma = 0.3$.

Commentary on Question:

Candidates performed well on this question. An alternative solution was also accepted.

Let $V = S^c$. Find the partial derivatives:

- $\frac{\partial V}{\partial S} = (S)^{c-1}c = VS^{-1}c$
- $\frac{\partial^2 V}{\partial S^2} = (S)^{c-2}c(c-1) = VS^{-2}c(c-1)$
- $\frac{\partial V}{\partial t} = 0$

Apply Ito's Lemma:

$$dV = \frac{\partial V}{\partial S}(dS) + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(dS)^2 + \frac{\partial V}{\partial t}(dt)$$
$$= (VS^{-1}c)(0.045\, S\, dt + \sigma S dW_t) + \frac{1}{2}\big(VS^{-2}c(c-1)\big)(\sigma S\, dW_t)^2 + 0$$

$$= (Vc)(0.045\,dt + \sigma\,dW_t) + \frac{1}{2}\big(Vc(c-1)\big)(\sigma^2\,dt)$$
$$= V\left[0.045c + \frac{1}{2}c(c-1)\sigma^2\right]dt + Vc\sigma dW_t$$
$$\frac{dV}{V} = \left[0.045\,c + \frac{1}{2}c(c-1)\sigma^2\right]dt + c\sigma\,dW_t$$
$$\frac{dS^c}{S^c} = \left[0.045\,c + \frac{1}{2}c(c-1)\sigma^2\right]dt + c\sigma\,dW_t$$

Compare the coefficient of dt and dW_t:

- $0.045\,c + \frac{1}{2}c(c-1)\sigma^2 = 0.18$
- $c\sigma = 0.6 \Rightarrow c = \frac{0.6}{\sigma}$

Substitute the second equation into the first:

$$0.045\left(\frac{0.6}{\sigma}\right) + \frac{1}{2}\left(\frac{0.6}{\sigma}\right)\left(\frac{0.6}{\sigma} - 1\right)\sigma^2 = 0.18$$

This can be written as

$$\left(\frac{0.027}{\sigma}\right) + 0.3\left(\frac{0.6}{\sigma} - 1\right)\sigma = 0.18$$

or $\sigma^2 = 0.09$ which implies $\sigma = 0.3$ since it is positive.

Alternative Solution:

Using the solution formula to the Geometric Brownian Motion:

$$(S_t)^c = (S_0)^c e^{c\left(r - \frac{1}{2}\sigma^2\right)t + c\sigma W_t}$$
$$= (S_0)^c e^{c\left(.045 - \frac{1}{2}\sigma^2\right)t + c\sigma W_t}$$

$$(S_t)^c = (S_0)^c e^{\left(0.18 - \frac{1}{2}0.6^2\right)t + 0.6W_t}$$
$$= (S_0)^c e^{0.6W_t}$$

Compare the coefficient of dt and dW_t:

- $0.045\,c - \frac{1}{2}c\sigma^2 = 0$
- $c\sigma = 0.6$

Solve the system of equations:

$$\sigma^2 = 2 * 0.045$$

$\sigma^2 = 0.09$ which implies $\sigma = 0.3$ since it is positive

$$c = \frac{0.6}{\sigma} = \frac{0.6}{0.3} = 2$$

(b) Calculate the time-0 no-arbitrage price of this derivative security.

Commentary on Question:

Candidates performed ok on this part of the question. Common mistakes were to forget the discount term when computing the time-0 no-arbitrage price and failing to convert to a standard normal random variable before applying the formula given in the question.

Use the following equivalency:

$$\frac{dS_t}{S_t} = r\,dt + \sigma\,dW_t \Leftrightarrow S_t = S_0 e^{\left(r-\frac{1}{2}\sigma^2\right)t+\sigma W_t}$$
$$\frac{dS_t}{S_t} = 0.045\,dt + 0.3\,dW_t \Leftrightarrow S_t = 1e^{\left(0.045-\frac{1}{2}(0.3)^2\right)t+0.3W_t} = e^{0.3W_t}$$

Thus, we have $S_3 = e^{0.3W_3}$, where $W_3 \sim N(0,3)$.

The expected value of the derivative security under the risk-neutral probability measure is:

$$\begin{aligned} E[S_3(\ln S_3)^2] &= E[e^{0.3W_3}(\ln e^{0.3W_3})^2] \\ &= E[e^{0.3W_3}(0.3W_3)^2] \\ &= 0.09E[e^{0.3W_3}W_3^2] \end{aligned}$$

Since $Z \sim N(0,1)$, it follows that $W_3 = Z\sqrt{3}$, and thus:

$$\begin{aligned} E[S_3(\ln S_3)^2] &= 0.09E\left[e^{0.3Z\sqrt{3}}\left(Z\sqrt{3}\right)^2\right] \\ &= 0.09(3)E\left[e^{0.3\sqrt{3}\cdot Z}Z^2\right] \\ &= 0.09(3)\cdot\left(1+\left(0.3\sqrt{3}\right)^2\right)e^{0.5\left(0.3\sqrt{3}\right)^2} \\ &= 0.39246 \end{aligned}$$

The time-0 no-arbitrage price is:

$$E[S_3(\ln S_3)^2]\cdot e^{-3r} = 0.39246 \cdot e^{-3(0.045)} = 0.3429$$

QFI QF Spring 2021 Question 7

Learning Outcomes:

d) Understand Stochastic Calculus theory and technique used in pricing derivatives

i) Calibrate a model to observed prices of traded securities including fitting to a given yield curve

Source References:

- *Problems and Solutions in Mathematical Finance: Stochastic Calculus*, Chin, Eric, Nel, Dian and Olafsson, Sverrir, 2014, pages 128-130, 132-137
- Calibrating Interest Rate Models (Section 1.1-4.3 excl 4.1.2)

Commentary on Question:

The purpose of this question is to test candidates' understanding of the Vasicek model and calibration used in practice. Most of candidates understood the Vasicek model well but almost all candidates did not perform well in the calibration problem.

Solution:

(a)

(i) Solve the stochastic differential equation.

(ii) Identify the distribution of r_t by providing its mean and variance.

Consider $F(t, r_t) = e^{at} r_t$.

Since $\frac{\partial F}{\partial t} = ae^{at} r_t, \frac{\partial F}{\partial r} = e^{at}, \frac{\partial^2 F}{\partial r^2} = 0$, Ito's lemma gives us

$$dF = ae_t^{at} r_t dt + e^{at} dr_t = [ae^{at} r_t + e^{at}(\nu - ar_t)]dt + e^{at}\sigma dX_t = e^{at}\nu dt + e^{at}\sigma dX_t$$

$$F(t, r_t) = F(0, r_0) + \nu \int_0^t e^{as} ds + \int_0^t e^{as} \sigma dX_s$$

$$\text{where } F(0, r_0) = r_0$$

$$\therefore r_t = e^{-at} r_0 + \nu \int_0^t e^{a(s-t)} ds + \sigma e^{-at} \int_0^t e^{as} dX_s = \mu_t + \sigma e^{-at} \int_0^t e^{as} dX_s$$

$$\text{where } \mu_t = e^{-at} r_0 + \nu \int_0^t e^{a(s-t)} ds$$

The mean of r_t is μ_t.

The variance is:

$$\sigma^2 e^{-2at}\int_0^t e^{2as}dt \text{ (by Ito Isometry)} = \frac{\sigma^2 e^{-2at}}{2a}(e^{2at}-1) = \frac{\sigma^2(1-e^{-2at})}{2a}$$

Also, it shows Gaussian distribution.

(b) Show that the limiting distribution of r_t as t approaches infinity is $N\left(\frac{\nu}{a}, \frac{\sigma^2}{2a}\right)$

$$\lim_{t\to\infty} E[r_t] = \lim_{t\to\infty}\left[e^{-at}r_0 + \nu\int_0^t e^{a(s-t)}ds\right] = \lim_{t\to\infty}\left[e^{-at}r_0 + \frac{(1-e^{-at})\nu}{a}\right] = \frac{\nu}{a}$$

$$\lim_{t\to\infty} Var[r_t] = \lim_{t\to\infty}\frac{\sigma^2(1-e^{-2at})}{2a} = \frac{\sigma^2}{2a}$$

(c) Demonstrate that the interest rate, r_{t+m}, follows the same distribution. Hint: Use time frame $(m, t+m)$ from solution of part (a).

Commentary on Question:

Quite a few candidates expressed r_{t+m} with an initial value of r_0 instead of r_m.. Then, they took a limit value as in part (b) to get the desired answer.

Assume r_m is stochastic, independent of the Brownian motion X_t. If we have that $r_m \sim N\left(\frac{\nu}{a}, \frac{\sigma^2}{2a}\right)$, independent of X_t, then we have:

$$F(t+m, r_{t+m}) = F(m, r_m) + \nu\int_m^{t+m} e^{as}ds + \int_m^{t+m} e^{as}\sigma dX_s$$

$$\text{where } F(m, r_m) = r_m e^{am}$$

$$\therefore r_{t+m} = e^{-a(t)}r_m + \nu e^{-a(t+m)}\int_m^{t+m} e^{as}ds + \sigma e^{-a(t+m)}\int_m^{t+m} e^{as}dX_s$$

$$E[r_{t+m}] = e^{-a(t)}E[r_m] + \frac{\left(1-e^{-a(t)}\right)\nu}{a} = \frac{\nu}{a}$$

$$Var[r_{t+m}] = e^{-2a(t)}Var[r_m] + \frac{\sigma^2\left(1-e^{-2a(t)}\right)}{2a} = \frac{\sigma^2}{2a}$$

(d)

(i) Estimate the parameters for interest rate process above.

(ii) Describe for the estimation of arbitrage free parameters using the table below observed in the market.

From

$$dr_t = [\nu - ar_t]dt + \sigma dX_t$$

It can be written in discrete manner

$$r_{t+\delta} - r_t = -ar_t\delta + \nu\delta + \sigma\varepsilon_t\sqrt{\delta}, \varepsilon_t \sim N(0,1)$$

$$r_{t+\delta} = (1 - a\delta)r_t + \nu\delta + \sigma\varepsilon_t\sqrt{\delta}, \varepsilon_t \sim N(0,1)$$

According to coefficient of regression from the hint,

$$\beta = 1 - a\delta,$$

$$\alpha = \nu\delta,$$

$$Var(r_{t+\delta}) = \sigma^2\delta$$

with $\delta = 0.25$ from the table

$$\beta = \frac{20 \cdot \sum_{i=1}^{20} r_{i-1} \cdot r_i - \sum_{i=1}^{20} r_i \cdot \sum_{i=1}^{20} r_{i-1}}{20 \cdot \sum_{i=1}^{20} r_{i-1}^2 - \left(\sum_{i=1}^{20} r_{i-1}\right)^2} = 0.089788$$

$$\alpha = \frac{\left(\sum_{i=1}^{20} r_i - \beta \sum_{i=1}^{20} r_{i-1}\right)}{20} = 0.037312$$

Therefore,

$$a = \frac{1-\beta}{\delta} = \frac{1 - 0.089788}{0.25} = 3.6408$$

$$\nu = \frac{\alpha}{\delta} = \frac{0.03737}{0.25} = 0.14927,$$

$$Var(r_{t+\delta}) = \frac{1}{20} \cdot \sum_{i=1}^{20} r_i^2 - \left(\frac{1}{20}\sum_{i=1}^{20} r_i\right)^2 = 0.000266$$

$$\sigma = \sqrt{\frac{Var(r_{t+\delta})}{\delta}} = \sqrt{\frac{0.000266}{0.25}} = 0.032628$$

For the arbitrgae free parameter estimation, it can be found by minimizing the errors between the arbitrage zero coupon bond prices in parametric formula and observed zero coupon bond prices.

For instance, a^*, υ^* can be searched by minimizing

$$J(a^*, v^*) = \sum_{i=1}^{n} \left(Z^{Vasicek}(0, T_i, a^*, v^*) - Z^{Data}(0, T_i) \right)$$

Each term in the parenthesis is the model's pricing error for each maturity T_i, that is, the distance between the model price and the data. If the model works well, each pricing error should be small, and thus also the sum of the pricing errors squared for nonlinear least square search.

QFI QF Spring 2021 Question 10

Learning Outcomes:

d) Understand Stochastic Calculus theory and technique used in pricing derivatives

e) Understand and apply the concepts of risk-neutral measure, forward measure, normalization, and the market price of risk

Source References:

- *Options, Futures, and Other Derivatives*, Hull, John C., 11th Edition, 2021, Chapters 29, 31, pages 688-68, 722-723
- *Problems and Solutions in Mathematical Finance: Stochastic Calculus*, Chin, Eric, Nel, Dian and Olafsson, Sverrir, 2014, pages 132-137

Commentary on Question:

Overall, candidates performed as expected on this question. There was a mistake in the given equation for A(t;T). However, credit was given when it was due; candidates were not penalized for using the correct or incorrect version equation. The model solution showed the work assuming candidates used the equation for A(t;T) given in the question.

Solution:

(a) Compare $m(r,t)$ with an arbitrage-free parameter $m^*(r,t)$ and explain the meaning of the parameters when $m^*(r,t) = \gamma^*(\bar{r}^* - r)$.

Commentary on Question:

Partial credit was awarded for candidates who demonstrated some understandings with regard to risk-neutral vs. real world as well as mean reversion.

The drift $m^*(r,t)$ provides arbitrage-free bond return, while $m(r,t)$ does not.

Vasicek model assumes that $m^*(r,t)$ has the same form as the drift rate of the original interest rate process:

$$m^*(r,t) = \gamma^*(\bar{r}^* - r)$$

where $\gamma^*, \bar{r}^*$ are two constants, which γ^* controls the sensitivity of the long-term bond prices to variation in the short-term rates.

(b) Show that $E\left[\frac{dZ/dt}{Z}\right] = E(r_t) + \frac{\sigma^2 B}{2\gamma^*}(1-\gamma^*)$ using Ito's lemma.

Commentary on Question:

Partial credit was awarded for candidates who showed the appropriate partial derivatives and applying them using Ito's Lemma.

$$\frac{\partial Z}{\partial t} = (A' - B'r)Z, \qquad \frac{\partial Z}{\partial r} = -BZ, \qquad \frac{\partial^2 Z}{\partial r^2} = B^2 Z$$

where

$$\frac{\partial B}{\partial t} = B' = -e^{-r^*(T-t)}$$

$$\frac{\partial A}{\partial t} = A' = (1 + B')\left(\bar{r}^* - \frac{\sigma^2}{2\gamma^*}\right) - \frac{\sigma^2 B(t;T)' B(t;T)}{2\gamma^*}$$

By Ito' lemma:

$$dZ = \left(\frac{\partial Z}{\partial t} + \frac{\partial Z}{\partial r}\gamma^*(\bar{r}^* - r_t) + \frac{\sigma^2}{2}\frac{\partial^2 Z}{\partial r^2}\right)dt + \frac{\partial Z}{\partial r}\sigma dX_t$$

Note that $\gamma^* B = 1 + B'$, thus:

$$A' = \gamma^* B\bar{r}^* - \frac{\sigma^2}{2}B - \frac{\sigma^2(\gamma^* B - 1)B(t;T)}{2\gamma^*} = \gamma^* B\bar{r}^* - \frac{\sigma^2}{2\gamma}B - \frac{\sigma^2}{2}B^2 + \frac{\sigma^2 B(t;T)}{2\gamma^*}$$

So:

$$\frac{\partial Z}{\partial t} = \left[B\gamma^*\bar{r}^* - \frac{\sigma^2}{2}B^2 - B'r\right]Z$$

Therefore:

$$\frac{dZ}{Z} = \left[\gamma^* B\bar{r}^* - \frac{\sigma^2}{2}B - \frac{\sigma^2}{2}B^2 + \frac{\sigma^2 B}{2\gamma^*} - B'r_t - B\gamma^*(\bar{r}^* - r_t) + \frac{\sigma^2}{2}B^2\right]dt + \frac{1}{Z}\frac{\partial Z}{\partial r}\sigma dX_t$$

$$E\left[\frac{\frac{dZ}{dt}}{Z}\right] = \left[-B'E(r_t) + B\gamma^* E(r_t) - \frac{\sigma^2}{2}B + \frac{\sigma^2 B}{2\gamma^*}\right] = (-B' + B\gamma^*)E(r_t) + \frac{\sigma^2 B}{2\gamma^*}(1-\gamma^*)$$

$$= E(r_t) + \frac{\sigma^2 B}{2\gamma^*}(1-\gamma^*)$$

(c) Compute $E\left[\frac{dZ/dt}{Z}\right]$ on zero-coupon bond with 10 years to maturity.

Commentary on Question:

The zero-coupon bond prices given were not correct for the given risk-neutral parameters. However, credit was given where it was due.

$B(0,10) = \frac{1}{0.4653}(1 - e^{-0.4653*10}) = 2.129$

$$E[r_0] = r_0 = 2\%$$

Using the result from part (b), we have:

$$E\left[\frac{dZ/dt}{Z}\right] = 2\% + \frac{2.21\%^2 \times 2.129}{2 \times 0.4653}(1 - 0.4653) = 2.0597\ \%$$

(d) Calculate the value of a call option with 1 year to maturity $(T_0 = 1)$, strike price $K = 0.9$, written on a zero-coupon bond with 5 years to maturity.

Under the Vasicek model, a European call open with strike price K and maturity T_0 on a zero coupon maturiing on $T_B > T_0$is given by:

$$V(r_0, 0) = Z(r_0, 0; T_B)N(d_1) - KZ(r_0, 0; T_0)N(d_2)$$

$$d_1 = \frac{1}{S}log\left(\frac{Z(r_0, 0; T_B)}{KZ(r_0, 0; T_0)}\right) + \frac{S}{2}$$

$$d_2 = d_1 - S$$

$$S = B(T_0; T_B) * \sqrt{\frac{\sigma^2}{2\gamma^*}(1 - e^{-2\gamma^* T_0})}$$

Thus, we have:

$$B(1;5) = \frac{1}{\gamma^*}\left(1 - e^{-\gamma^*(5-1)}\right) = 1.815$$

$$S(1{,}5) = 1.815 * \sqrt{\frac{2.21\%^2}{2 * 0.4653}(1 - e^{-2*0.4653})} = 0.03236$$

$$d_1 = \frac{1}{0.03236} log\left(\frac{0.898}{0.9 * 0.975}\right) + \frac{0.03236}{2} = 0.7298$$

$$d_2 = 0.7298 - 0.03236 = 0.6975$$

The value of the call option is:

$$V = 0.898 * N(d_1) - 0.9 * 0.975 * N(0.6975) = 0.02425$$

QFI QF Fall 2021 Question 1

Learning Outcomes:

d) Understand Stochastic Calculus theory and technique used in pricing derivatives

Source References:

- *Options, Futures, and Other Derivatives*, Hull, John C., 11th Edition, 2021, Chapter 14, page 327
- *Problems and Solutions in Mathematical Finance: Stochastic Calculus*, Chin, Eric, Nel, Dian and Olafsson, Sverrir, 2014, pages 128-130

Commentary on Question:

Candidates performed fairly on this question.

Solution:

(a) Explain why X_t is a normally distributed random variable.

Commentary on Question:

Candidates performed fairly well on this part. To receive full credit, candidates needed to note that we have a Riemann sum and the linear combinations of normal variables are normal and normality is preserved in the limit.

By Ito's lemma or Ito's product rule we have

$$d(uW_u) = W_u du + u dW_u,$$

which gives

$$X_t = \int_0^t u\, dW_u - tW_t$$

Both terms on the right-hand side are normally distributed, hence the result.

(b) Compute $E[X_t]$ and $Var[X_t]$.

Commentary on Question:

Candidates performed poorly on this part. Most candidates did not work through all the details of the whole derivation.

By Ito's lemma or Ito's product rule we have

$$d(uW_u) = W_u du + u dW_u,$$

which gives

$$tW_t = X_t + \int_0^t u\, dW_u$$

$$X_t = tW_t - \int_0^t u d\, W_u.$$

Hence

$$E[X_t] = E[tW_t] - E\left[\int_0^t u d\, W_u\right] = 0.$$

We also have

$$Var(X_t) = Var(tW_t) + Var\left(\int_0^t u d\, W_u\right) - 2Cov\left(tW_t, \int_0^t u d\, W_u\right)$$

$$= t^2 Var(W_t) + Var\left(\int_0^t u d\, W_u\right) - 2tCov\left(W_t, \int_0^t u d\, W_u\right)$$

Note that

$$Var(W_t) = t$$

From Ito's isometry, we get

$$Var\left(\int_0^t u d\, W_u\right) = \int_0^t u^2 d\, u = \frac{1}{3}t^3$$

In addition from Ito's isometry, we have

$$2Cov\left(W_t, \int_0^t u d\, W_u\right) = Var\left(W_t + \int_0^t u d\, W_u\right) - Var(W_t) - Var\left(\int_0^t u d\, W_u\right)$$

$$= Var\left(\int_0^t d\, W_u + \int_0^t u d\, W_u\right) - t - \frac{1}{3}t^3 = Var\left(\int_0^t (u+1) d\, W_u\right) - t - \frac{1}{3}t^3$$

$$= \int_0^t (u+1)^2 du - t - \frac{1}{3}t^3 = t^2$$

Hence we have

$$Var(X_t) = \frac{1}{3}t^3$$

Alternative method for calculating $2Cov\left(W_t, \int_0^t ud\,W_u\right)$:
Since $W_t = \int_0^t d\,W_u$, from Ito's isometry

$$2Cov\left(W_t, \int_0^t ud\,W_u\right) = 2Cov\left(\int_0^t d\,W_u, \int_0^t ud\,W_u\right)$$
$$= 2\int_0^t udu = t^2$$

Alternative Solution:

Work with X_t directly, which clearly has mean zero, and then evaluate the expectation of its square as a double integral using the known covariance function of a Wiener process.

$E\left[\int_0^t W_u du\right] = 0$
$Var\left(\int_0^t W_u du\right) = E\left[\left(\int_0^t W_u du\right)^2\right] - \left(E\left[\int_0^t W_u du\right]\right)^2$
$Var\left(\int_0^t W_u du\right) = E[(\left(\int_0^t W_s ds\right)(\left(\int_0^t W_u du\right)] - \left(E\left[\int_0^t W_u du\right]\right)^2$

$= E[(\left(\int_{s=0}^{s=t}\int_{u=o}^{u=t} E[W_s W_u] du\, ds\right)$

$E[W_s W_u] = mi\,n(s,u)$
$Var\left(\int_0^t W_u du\right) = \int_{s=0}^{s=t}\int_{u=o}^{u=t} \min(s,u)\,du\,ds$
$= \int_{s=0}^{s=t}\int_{u=o}^{u=s} udu + \int_{s=0}^{s=t}\int_{u=s}^{u=t} sdu\,ds$
$= \int_{s=0}^{s=t}\frac{1}{2}s^2 ds + \int_0^t s(t-s)ds$
$= \frac{1}{3}t^3$

Let Y_t be defined as

$$Y_t = \int_0^t \sqrt{|W_u|}\,dW_u.$$

(c) Compute $Var[Y_t]$.

Commentary on Question:

Candidates performed fairly well on this part. Most candidates were able to obtain partial credit by identifying the need to use Ito isometry.

We know that $E(Y_t) = 0$ because Y_t is an Ito integral for all $0 < t < T$.

Therefore, $Var(Y_t) = E[Y_t^2] = \int_0^t E(|W_u|)\,du$ by Ito isometry.

Now: $E(|W_u|) = \int_{-\infty}^{\infty} |w| \frac{1}{\sqrt{2\pi u}} e^{-\frac{w^2}{2u}} \, dw = 2\int_0^{\infty} w \frac{1}{\sqrt{2\pi u}} e^{-\frac{w^2}{2u}} dw = \sqrt{\frac{2u}{\pi}}$.

Finally: $Var(Y_t) = \int_0^t \sqrt{\frac{2u}{\pi}} du = \sqrt{\frac{2}{\pi}} \frac{2}{3} t^{3/2} = \sqrt{\frac{8}{9\pi}} t^{3/2}$.

QFI QF Fall 2021 Question 2

Learning Outcomes:

b) Understand Arrow-Debreau security and the distinction between complete and incomplete markets

f) Understand option pricing techniques

Source References:

- INV201-100-25: Chapter 5 of Financial Mathematics – A Comprehensive Treatment, 2nd Edition, Campolieti
- INV201-101-25: Chapter 6 of Introduction to Stochastic Finance with Market Examples by Privault

Solution:

(a) Show that $\mathbb{P}$ and $\mathbb{Q}$ are equivalent probability measures on the probability space implied by the price process A_t.

Commentary on Question:

This question tests the understanding of the definition of equivalent probability measures. Candidates performed as expected on this part.

From the two trees, we see that the sample space and the event space are the same for the probability measures $\mathbb{P}$ and $\mathbb{Q}$. Additionally, $\mathbb{P}(A_t) = 0 \Leftrightarrow \mathbb{Q}(A_t) = 0$ for every event A_t in the event space.

Note: The following statements are equivalent:

- $\mathbb{P}(A_t) = 0 \Leftrightarrow \mathbb{Q}(A_t) = 0$ *for every event* A_t *in the event space.*
- *If an event cannot occur under the* $\mathbb{P}$ *measure, then it also cannot occur under the* $\mathbb{Q}$ *measure, and vice versa.*
- $\mathbb{P}$ *and* $\mathbb{Q}$ *are absolutely continuous with respect to each other.*
- $\mathbb{P} \ll \mathbb{Q}$ *and* $\mathbb{Q} \ll \mathbb{P}$.

(b) Determine if the price process A_t is a:

(i) $\mathbb{Q}$-martingale.

(ii) $\mathbb{P}$-martingale.

Commentary on Question:

Candidates performed well on this part. Most candidates reached the correct conclusions, although many candidates did not check all the conditional expectations under the $\mathbb{Q}$ measure.

Part (i)

A discrete process $X = \{X_n: n = 0,1,2, \dots\}$ is a martingale relative to $(\Omega, \mathcal{F}, \mathbb{P})$ if for all n:

(a) $E(X_{n+1}|\mathcal{F}_n) = X_n$;

(b) $E(|X_n|) < \infty$;

(c) X_n is $\mathcal{F}_n$-adapted.

The last two conditions can be seen trivially from the tree. To check the first condition, with $r = 0$, we can calculate:

$$E^{\mathbb{Q}}(A_2|A_1 = 120) = \frac{3}{5} \times 144 + \frac{2}{5} \times 84 = 120 = A_1$$

$$E^{\mathbb{Q}}(A_2|A_1 = 60) = \frac{4}{7} \times 84 + \frac{3}{7} \times 28 = 60 = A_1$$

$$E^{\mathbb{Q}}(A_1|A_0 = 100) = \frac{2}{3} \times 120 + \frac{1}{3} \times 60 = 100 = A_0$$

Hence, the process A_t is a $\mathbb{Q}$-martingale.

Part (ii)

$E^{\mathbb{P}}(A_2|A_1 = 120) = \frac{1}{2} \times 144 + \frac{1}{2} \times 84 = 114 \neq 120$. Under $\mathbb{P}$, the process A_t violates the martingale property $E(X_{n+1}|\mathcal{F}_n) = X_n$, so it is not a $\mathbb{P}$-martingale.

(c) Calculate the values of the Radon-Nikodym derivative $\frac{d\mathbb{Q}}{d\mathbb{P}}$ for all paths through the tree, (i.e. up-up, up-down, down-up, down-down nodes).

Commentary on Question:

Candidates performed below expectation on this part. Many candidates did not write down the derivatives for $t = 0$ and $t = 1$. Some candidates used a wrong formula for $t = 2$.

The Radon-Nikodym derivative on the finite sample space $\omega \in \Omega$ is a random variable defined as $\frac{d\mathbb{Q}}{d\mathbb{P}}(\omega) = \frac{\mathbb{Q}(\omega)}{\mathbb{P}(\omega)}$. The sample paths of the tree are ω. Therefore, at each node of the tree, we can calculate the following:

$$\frac{d\mathbb{Q}}{d\mathbb{P}}(stationary;\ t = 0) = 1$$

$$\frac{d\mathbb{Q}}{d\mathbb{P}}(up;\ t = 1) = \frac{2/3}{1/2} = \frac{4}{3}$$

$$\frac{d\mathbb{Q}}{d\mathbb{P}}(down;\ t = 1) = \frac{1/3}{1/2} = \frac{2}{3}$$

$$\frac{d\mathbb{Q}}{d\mathbb{P}}(up, up;\ t = 2) = \frac{2/3 \times 3/5}{1/2 \times 1/2} = \frac{8}{5}$$

$$\frac{d\mathbb{Q}}{d\mathbb{P}}(up, down;\ t = 2) = \frac{2/3 \times 2/5}{1/2 \times 1/2} = \frac{16}{15}$$

$$\frac{d\mathbb{Q}}{d\mathbb{P}}(down, up;\ t = 2) = \frac{1/3 \times 4/7}{1/2 \times 1/2} = \frac{16}{21}$$

$$\frac{d\mathbb{Q}}{d\mathbb{P}}(down, down;\ t = 2) = \frac{1/3 \times 3/7}{1/2 \times 1/2} = \frac{4}{7}$$

(d) Evaluate the process $\xi_t = E^{\mathbb{P}}\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\middle|\mathcal{F}_t\right)$ at time $t = 1$ for both up and down nodes where $\mathcal{F}_t$ is the filtration history up to time *t*.

Commentary on Question:

Candidates performed as expected on this part. Partial credits were awarded if candidates wrote down the correct formula but used incorrect results from the last part.

$$E^{\mathbb{P}}\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\middle|A_1 = 120\right) = \frac{1}{2}\times\frac{8}{5}+\frac{1}{2}\times\frac{16}{15} = \frac{4}{3}$$

$$E^{\mathbb{P}}\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\middle|A_1 = 60\right) = \frac{1}{2}\times\frac{16}{21}+\frac{1}{2}\times\frac{4}{7} = \frac{2}{3}$$

(e) Show numerically that $E^{\mathbb{Q}}[X] = E^{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}X\right]$ at time 0 by using the results in part (d).

Commentary on Question:

Candidates performed as expected on this part. Most candidates were able to calculate $\mathrm{E}^{\mathbb{Q}}[\mathrm{X}]$. *Some candidates did not have the incorrect formulae for* $E^{\mathbb{P}}\left(\frac{d\mathbb{Q}}{d\mathbb{P}}X\right)$.

$$E^{\mathbb{Q}}(X) = \frac{2}{3}*\left(\frac{3}{5}*20+\frac{2}{5}*20\right)+\frac{1}{3}*\left(\frac{4}{7}*20+\frac{3}{7}*0\right) = 20*\frac{6}{7} = 17.1429$$

$$E^{\mathbb{P}}\left(\frac{d\mathbb{Q}}{d\mathbb{P}}X\right) = \frac{1}{2}\times\frac{1}{2}\times\frac{d\mathbb{Q}}{d\mathbb{P}}(up,up;\ t=2)\times 20 +$$

$$\frac{1}{2}\times\frac{1}{2}\times\frac{d\mathbb{Q}}{d\mathbb{P}}(up,down;\ t=2)\times 20 +$$

$$\frac{1}{2}\times\frac{1}{2}\times\frac{d\mathbb{Q}}{d\mathbb{P}}(down,up;\ t=2)\times 20 +$$

$$\frac{1}{2}\times\frac{1}{2}\times\frac{d\mathbb{Q}}{d\mathbb{P}}(down,down;\ t=2)\times 0$$

$$= \frac{1}{2}\times\frac{1}{2}\times 20\times\left(\frac{d\mathbb{Q}}{d\mathbb{P}}(up,up;\ 2)+\frac{d\mathbb{Q}}{d\mathbb{P}}(up,down;\ 2)+\frac{d\mathbb{Q}}{d\mathbb{P}}(down,up;\ 2)\right)$$

$$= \frac{1}{2}\times\frac{1}{2}\times 20\times\left(\frac{8}{5}+\frac{16}{15}+\frac{16}{21}\right)$$

$$= 17.1429 = E^{\mathbb{Q}}(X)$$

QFI QF Fall 2021 Question 3

Learning Outcomes:

c) Understand put-call parity and price bounds

d) Understand Stochastic Calculus theory and technique used in pricing derivatives

e) Understand and apply the concepts of risk-neutral measure, forward measure, normalization, and the market price of risk

f) Understand option pricing techniques

Source References:

- *Options, Futures, and Other Derivatives*, Hull, John C., 11th Edition, 2021, Chapters 11, 14, 28, pages 255-256, 327, 675
- *Problems and Solutions in Mathematical Finance: Stochastic Calculus*, Chin, Eric, Nel, Dian and Olafsson, Sverrir, 2014, pages 186-188, 221-227

Commentary on Question:

This question tests candidates' understanding of Ito Lemma and martingales. Candidates did well overall in this question. Several candidates were able to come up with alternative solutions in some parts.

Solution:

(a)

(i) Determine the stochastic differential equation satisfied by the discounted price process $S_t^d = B_t^{-1} S_t$.

(ii) Explain why $\pi_T^d(X) = \pi_t^d(X) + \int_t^T \alpha_u \sigma S_u^d dW_u^{\mathbb{Q}}$.

(iii) Show that the discounted derivative prices $\pi_t^d(X), t < T$ form a $\mathbb{Q}$-martingale using part (a) (ii).

Commentary on Question:

Candidates did well in part (i) and (iii). A common oversight was not using Ito Lemma in part (ii).

Part (i)

$S_t^d = B_t^{-1} S_t = e^{-rt} S_t.$

By Ito's lemma:

$dS_t^d = -re^{-rt} S_t dt + re^{-rt} S_t dt + \sigma e^{-rt} S_t dW_t^Q = \sigma S_t^d dW_t^Q.$

Part (ii)

Self-financing means:

$\pi_u(X) = \alpha_u S_u + \beta_u B_u$ implies $d\pi_u(X) = \alpha_u dS_u + \beta_u dB_u$.

Using Ito's Lemma,

$d\pi_u^d(X) = d\left(e^{-ru}\pi_u(X)\right) = -re^{-ru}\pi_u(X)du + e^{-ru}d\pi_u(X) = -re^{-ru}(\alpha_u S_u + \beta_u B_u)du + e^{-ru}\alpha_u dS_u + e^{-ru}\beta_u dB_u = \alpha_u \sigma S_u^d dW_u^Q = \alpha_u dS_u^d$

Integrate the above equation from t to T

$\pi_T^d(X) - \pi_t^d(X) = \int_t^T \alpha_u dS_u^d$

$X^d - \pi_t^d(X) = \int_t^T \alpha_u dS_u^d$ Replicating portfolio

Part (iii)

If s < t then
Using part a(ii)
$\pi_t^d(X) = \pi_s^d(X) + \int_s^t \alpha_u dS_u^d$
Since conditional expectation of the stochastic integral = 0
⇨ $E[\pi_t^d(X)] = \pi_s^d(X)$
So this is a martingale.

(b) Prove that $C_t(K,T) - P_t(K,T) = S_t - Ke^{-r(T-t)}, t < T$.

Commentary on Question:

Candidates did well in this part. Alternate solutions were accepted for full credit as long as they didn't assume the Put-Call parity formula as given.

$C_T - P_T = max(S_T - K, 0) - max(K - S_T, 0)$
$= S_T - K$
$\pi_t(X) = e^{-t(T-t)}E_Q[X|F_t]$ ------ Eq. 1
Apply Eq. 1
$C_t - P_t = e^{-r(T-t)}E[S_T|F_t] - K\, e^{-r(T-t)}$
$C_t - P_t = S_t - K\, e^{-r(T-t)}$
Since $e^{-r(T-t)}E[S_T|F_t] = S_t$ using part (a) (i)
$C_t = C_t(K,T)\ and\ P_t = P_t(K,T)$

(c) Show that $\pi_t^d(V) = P_t(K,T) + C_t\left(Ke^{-r(T-T_c)}, T_c\right), t < T_c$.

Commentary on Question:

Candidates were able to start this part successfully, but most were not able to connect risk-neutral expectation to earn full credit.

Use the put-call parity from part (c),
$max(P_{T_c}, C_{T_c}) = max(P_{T_c}, P_{T_c} + S_{T_c} - Ke^{-r(T-T_c)})$
$max(P_{T_c}, C_{T_c}) = P_{T_c} + max(S_{T_c} - Ke^{-r(T-T_c)}, 0)$

Therefore:
$e^{-r(T_c-t)} E[max(P_{T_c}, C_{T_c})|F_t] = e^{-r(T_c-t)} E[P_{T_c} + max(S_{T_c} - Ke^{-r(T-T_c)}, 0)|F_t]$

$= e^{-r(T_c-t)} E[P_{T_c}|F_t] + e^{-r(T_c-t)} E[max(S_{T_c} - Ke^{-r(T-T_c)}, 0)|F_t]$
And $e^{-r(T_c-t)} E[P_{T_c}|F_t] = P_t(K, T)$ using part (b)

$e^{-r(T_c-t)} E[max(S_{T_c} - Ke^{-r(T-T_c)}, 0)|F_t] = C_t(Ke^{-r(T-T_c)}, T_c)$

$\pi_t^d(V)$
$= P_t(K, T) + C_t(Ke^{-r(T-T_c)}, T_c)$.

QFI QF Fall 2021 Question 8

Learning Outcomes:

d) Understand Stochastic Calculus theory and technique used in pricing derivatives

e) Understand and apply the concepts of risk-neutral measure, forward measure, normalization, and the market price of risk

i) Calibrate a model to observed prices of traded securities including fitting to a given yield curve

Source References:

- *Options, Futures, and Other Derivatives*, Hull, John C., 11th Edition, 2021, Chapters 14, 31, pages 327, 721-723
- *Problems and Solutions in Mathematical Finance: Stochastic Calculus*, Chin, Eric, Nel, Dian and Olafsson, Sverrir, 2014, pages 132-137, 221-227
- Calibrating Interest Rate Models (Section 1.1-4.3 excl 4.1.2)

Commentary on Question:

This question generally tests candidates' understanding of the following concepts:

- *Ito's Lemma*
- *Bond pricing*
- *Yield curve*

- *Implications of a zero floor for interest rate under different scenarios*

Most candidates attempted the question and performed as expected.

Solution:

(a) Show that

$$\frac{r_t B_1 - \frac{\partial B_1}{\partial t} - \frac{1}{2}\frac{\partial^2 B_1}{\partial r^2}\sigma^2}{\frac{\partial B_1}{\partial r}} = \frac{r_t B_2 - \frac{\partial B_2}{\partial t} - \frac{1}{2}\frac{\partial^2 B_2}{\partial r^2}\sigma^2}{\frac{\partial B_2}{\partial r}}$$

Commentary on Question:

Most candidates were able to list out the key points in this question:

- *Condition to eliminate the interest rate risk*
- *Risk free portfolio should earn risk-free rate*

Full marks were awarded to candidates who showed all the derivation.

Using Ito's Lemma:

$$dB_i = \frac{\partial B_i}{\partial t}dt + \frac{\partial B_i}{\partial r}dr_t + \frac{1}{2}\frac{\partial^2 B_i}{\partial r^2}(dr_t)^2$$

$$= \frac{\partial B_i}{\partial r}(a(b - r_t)dt + \sigma dW_t) + \frac{\partial B_i}{\partial t}dt + \frac{1}{2}\frac{\partial^2 B_i}{\partial r^2}(a(b - r_t)dt + \sigma dW_t)^2$$

$$= \left(\frac{\partial B_i}{\partial r}a(b - r_t) + \frac{\partial B_i}{\partial t} + \frac{1}{2}\frac{\partial^2 B_i}{\partial r^2}\sigma^2\right)dt + \frac{\partial B_i}{\partial r}\sigma dW_t$$

Plugging to $dP = \eta dB_1 - \theta dB_2$:

$$dP = \eta\left[\left(\frac{\partial B_1}{\partial r}a(b - r_t) + \frac{\partial B_1}{\partial t} + \frac{1}{2}\frac{\partial^2 B_1}{\partial r^2}\sigma^2\right)dt + \frac{\partial B_1}{\partial r}\sigma dW_t\right]$$

$$-\theta\left[\left(\frac{\partial B_2}{\partial r}a(b - r_t) + \frac{\partial B_2}{\partial t} + \frac{1}{2}\frac{\partial^2 B_2}{\partial r^2}\sigma^2\right)dt + \frac{\partial B_2}{\partial r}\sigma dW_t\right]$$

For interest rate risk to be elimated,

$$\frac{\partial P}{\partial r} = \left(\eta\frac{\partial B_1}{\partial r} - \theta\frac{\partial B_2}{\partial r}\right)a(b - r_t) = 0$$

Choose η and θ such that $\eta \frac{\partial B_1}{\partial r} = \theta \frac{\partial B_2}{\partial r}, i.e., \theta = \eta \frac{\frac{\partial B_1}{\partial r}}{\frac{\partial B_2}{\partial r}}$

$$dP = \left[\eta\left(\frac{\partial B_1}{\partial t} + \frac{1}{2}\frac{\partial^2 B_1}{\partial r^2}\sigma^2\right) - \theta\left(\frac{\partial B_2}{\partial t} + \frac{1}{2}\frac{\partial^2 B_2}{\partial r^2}\sigma^2\right)\right] dt + \sigma(\eta\frac{\partial B_1}{\partial r} - \theta\frac{\partial B_2}{\partial r})dW_t$$

Then dP is riskless, its deterministic return should equal to risk-free rate:$dP = rPdt$.

$$\left[\eta\left(\frac{\partial B_1}{\partial t} + \frac{1}{2}\frac{\partial^2 B_1}{\partial r^2}\sigma^2\right) - \theta\left(\frac{\partial B_2}{\partial t} + \frac{1}{2}\frac{\partial^2 B_2}{\partial r^2}\sigma^2\right)\right] dt = r(\eta B_1 - \theta B_2)dt$$

Since $\theta = \eta \frac{\frac{\partial B_1}{\partial r}}{\frac{\partial B_2}{\partial r}}$, we have

$$\eta r\left(B_1 - \frac{\partial B_1/\partial r}{\partial B_2/\partial r}B_2\right) = \eta\left[\left(\frac{\partial B_1}{\partial t} + \frac{1}{2}\frac{\partial^2 B_1}{\partial r^2}\sigma^2\right) - \frac{\partial B_1/\partial r}{\partial B_2/\partial r}\left(\frac{\partial B_2}{\partial t} + \frac{1}{2}\frac{\partial^2 B_2}{\partial r^2}\sigma^2\right)\right]$$

$$r\left(B_1 - \frac{\partial B_1/\partial r}{\partial B_2/\partial r}B_2\right) = \left(\frac{\partial B_1}{\partial t} + \frac{1}{2}\frac{\partial^2 B_1}{\partial r^2}\sigma^2\right) - \frac{\partial B_1/\partial r}{\partial B_2/\partial r}\left(\frac{\partial B_2}{\partial t} + \frac{1}{2}\frac{\partial^2 B_2}{\partial r^2}\sigma^2\right)$$

$$rB_1 - \frac{\partial B_1}{\partial t} - \frac{1}{2}\frac{\partial^2 B_1}{\partial r^2}\sigma^2 = \frac{\partial B_1/\partial r}{\partial B_2/\partial r}\left(rB_2 - \frac{\partial B_2}{\partial t} - \frac{1}{2}\frac{\partial^2 B_2}{\partial r^2}\sigma^2\right)$$

$$\frac{rB_1 - \frac{\partial B_1}{\partial t} - \frac{1}{2}\frac{\partial^2 B_1}{\partial r^2}\sigma^2}{\partial B_1/\partial r} = \frac{rB_2 - \frac{\partial B_2}{\partial t} - \frac{1}{2}\frac{\partial^2 B_2}{\partial r^2}\sigma^2}{\partial B_2/\partial r}$$

(b) Show that the price of a default-free discount bond satisfies the following partial differential equation

$$\frac{\partial B}{\partial r}\left[a\left(b - r_t\right) - \sigma\lambda\right] + \frac{\partial B}{\partial t} + \frac{1}{2}\frac{\partial^2 B}{\partial r^2}\sigma^2 - r_t B = 0$$

Commentary on Question:

Candidates did well in this question.

From

$$\lambda(r_t, t) = \frac{a(b - r_t) - m(t)}{\sigma}$$

We have $m(t) = a(b - r_t) - \sigma\lambda$.

$$\frac{r_t B - \frac{\partial B}{\partial t} - \frac{1}{2}\frac{\partial^2 B}{\partial r^2}\sigma^2}{\frac{\partial B}{\partial r}} = a(b - r_t) - \sigma\lambda$$

Therefore, we have PDE:

$$\frac{\partial B}{\partial r}[a(b - r_t) - \sigma\lambda] + \frac{\partial B}{\partial t} + \frac{1}{2}\frac{\partial^2 B}{\partial r^2}\sigma^2 - r_t B = 0$$

(c) Describe the key features of this interest rate model.

Commentary on Question:

Full marks were given to candidates who identified 4 key features. Most candidates were able to list at least two key features.

- Interest rate is mean reverting, which means that if they diverge too much from a central value, they tend to revert back to it.
- The model has constant volatility
- The model implies that the statistical distribution of interest rates in the future is normal.
- It gives positive probability to negative nominal interest rate.
- The solution to the fundamental pricing equation under the Vasicek model is known in closed form

(d) Explain how to estimate the interest rate model parameters, using the given data. Identify the estimated parameters that can be used in pricing interest rate derivative.

Commentary on Question:

Most candidates stated the estimation of risk-neutral parameters, not the real-world parameters.

Most candidates did not identify the estimated parameters that can be used in pricing interest rate derivatives.

- The volatility σ can be estimated directly from the time series of interest rate r_t.
- Compute long-run mean of spot rate over the sample period.

- Obtain speed of mean reversion by regressing the changes in interest rate.
- The volatility σ can be used in the bond pricing formula. The estimated long run mean and speed of mean reversion are not relevant for pricing interest rate securities.

(e) Calculate the default-free discount bond price with 30-year maturity with *r* = 0.1%, 5%, and 10%, respectively.

Commentary on Question:

Most candidates did not perform well in this question. Candidates used the formula from Pietro textbook, and partial marks were awarded.

Using the formula:

$$B = e^{\frac{1}{a}(1-e^{-aT})(R-r)-TR-\frac{\sigma^2}{4a^3}(1-e^{-aT})^2}$$

$$R = b - \frac{\sigma\lambda}{a} - \frac{\sigma^2}{a^2}$$

a = 0.25
T = 30
b = 0.05
σ = 0.015
λ = -0.1

Spot rate	30-year zero coupon bond
0.1%	0.254079
5%	0.208879
10%	0.171034

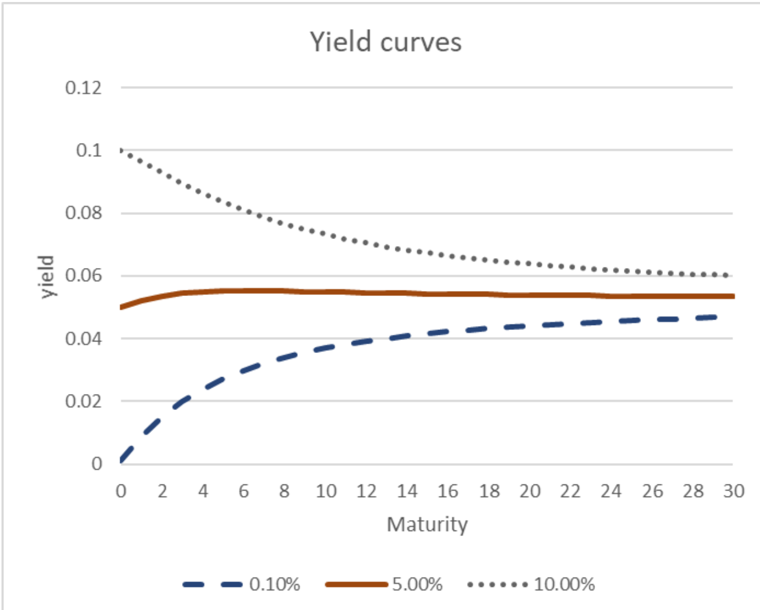
(f) Generate the yield curves for the same set of spot rates in part (e) with different maturities,1 through 30 years.

Commentary on Question:

Candidates performed below expectation for this question. About half of the candidates did not attempt this question. Full marks were awarded for candidates who used the yield formula to solve the question.

$$yield = -\frac{\ln(B)}{T}$$

Spot rate	1	5	10	30 years
0.1%	0.71%	2.343%	3.383%	4.567%
5%	5.045%	5.14%	5.182%	5.220%
10%	9.469%	7.994%	7.018%	5.886%



QFI QF Fall 2022 Question 1

Learning Outcomes:

d) Understand Stochastic Calculus theory and technique used in pricing derivatives

Source References:

- *Options, Futures, and Other Derivatives*, Hull, John C., 11th Edition, 2021, Chapters 14, 28, pages 327, 675
- *Problems and Solutions in Mathematical Finance: Stochastic Calculus*, Chin, Eric, Nel, Dian and Olafsson, Sverrir, 2014, pages 52, 72-73

Solution:

(a) List the criteria for a stochastic process to be a martingale with respect to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$.

Commentary on Question:

Most candidates were able to list the three criteria. However, some candidates forgot the absolute value when stating the second criterion.

The three criteria for $0 \leq s \leq t \leq T$ are:

$E^{\mathbb{P}}(V_t|\mathcal{F}_s) = V_s$;

The equality holds almost surely.

$E^{\mathbb{P}}[|V_t|] < \infty$;

And the 3rd criterion is that V_t is $\mathcal{F}_t$-measurable

(b) Derive a stochastic differential equation for X_t using Ito's Lemma.

Commentary on Question:

Most candidates did this part correctly. Some candidates did not use the correct notation that is specific to this problem.

By Ito's Lemma, one can derive

$$dX_t = W_t{}^2 dt + \frac{\partial \alpha_t}{\partial t} dt + \frac{\partial \alpha_t}{\partial W_t} dW_t + \frac{1}{2} \frac{\partial^2 \alpha_t}{\partial W_t^2} dt$$

(c) Identify an appropriate β_t, if it exists, that makes X_t a martingale.

Commentary on Question:

Many candidates were able to derive the correct function. Most candidates were able to state that a martingale requires the drift to be zero.

From part (b), it follows that $W_t{}^2 + \frac{\partial \alpha_t}{\partial t} + \frac{1}{2} \frac{\partial^2 \alpha_t}{\partial W_t^2} = 0 \quad (*)$
implies that X_t is a martingale.

Since $\alpha_t = -tW_t{}^2 + \beta_t$ where β_t is deterministic, we have

$dX_t = (W_t{}^2 - W_t{}^2)dt + \frac{\partial \beta_t}{\partial t} dt - 2\mathrm{t}W_t dW_t - tdt.$

Therefore, by letting $\beta_t = \frac{1}{2}\mathrm{t}^2$ and $\alpha_t = -tW_t{}^2 + \frac{1}{2}\mathrm{t}^2$, X_t becomes a martingale.

(d) Calculate $E(W_t{}^4)$ using Ito's Lemma.

Commentary on Question:

Most candidates did this part correctly.

We use Ito's Lemma on $f(W_t) = W_t^{\ 4}$.

$$dW_t^{\ 4} = 4W_t^{\ 3}dW_t + 6W_t^{\ 2}dt$$

therefore

$$W_t^{\ 4} = 4\int_0^t W_s^{\ 3}dW_s + 6\int_0^t W_s^{\ 2}ds$$

We then obtain

$$E(W_t^{\ 4}) = 6E\left(\int_0^t W_s^{\ 2}ds\right) \qquad (**).$$

One can evaluate (**) as follows (By Fubini's theorem):

$$E\left(\int_0^t W_s^{\ 2}ds\right) = \int_0^t E(W_s^{\ 2})ds = \int_0^t sds = \frac{1}{2}t^2.$$

Alternatively, a candidate can use part (c) in the following manner:

Since X_t is a martingale,

$$E\left(\int_0^t W_s^{\ 2}ds\right) + E(\alpha_t) = 0 \ \Rightarrow E\left(\int_0^t W_s^{\ 2}ds\right) = E\left(tW_t^{\ 2} - \frac{1}{2}\mathrm{t}^2\right) = tE(W_t^{\ 2}) - \frac{1}{2}\mathrm{t}^2 = \frac{1}{2}\mathrm{t}^2.$$

In either case, the result is:

$$E(W_t^{\ 4}) = 3t^2.$$

QFI QF Fall 2022 Question 2

Learning Outcomes:

a) Understand the principles of no-arbitrage and replication in asset pricing

e) Understand and apply the concepts of risk-neutral measure, forward measure, normalization, and the market price of risk

f) Understand option pricing techniques

Source References:

- *Options, Futures, and Other Derivatives*, Hull, John C., 11th Edition, 2021, Chapter 13, pages 296-298
- *Problems and Solutions in Mathematical Finance: Stochastic Calculus*, Chin, Eric, Nel, Dian and Olafsson, Sverrir, 2014, pages 221-227
- Understanding the Connection Between Real-World and Risk-Neutral Generators, SOA Research

Solution:

(a) Determine the range of α so that there is no arbitrage opportunity.

Commentary on Question:

Candidates did poorly on this part. Many failed to recognize the conditions needed to avoid arbitrage.

If one invests S_{t-1} in the risk-free asset at the beginning of year t, then the expected return is $1.05S_{t-1}$.

Therefore, the following inequality should be satisfied to avoid arbitrage:$\alpha S_{t-1} < 1.05S_{t-1} < 1.3S_{t-1}$ at $t = 1, 2, \ldots$

Thus, $\alpha < 1.05$.

(b) Derive the price S_4 as a function of d and give the possible range of S_4.

Commentary on Question:

Candidates did well on this part.

$S_4 = 100\alpha^{d}1.3^{4-d}$.

Thus, $100\alpha^{4} < S_4 < 285.61$

(c) Calculate the real-world probability that the double barrier option will be exercised.

Commentary on Question:

Candidates did poorly on this part. Many incorrectly states that 5 down movements were needed to exercise the barrier option.

Since the upper barrier is 290, the barrier can be hit only if all 5 stock price movements are up.

In the same manner, the lower barrier 70 can be hit only if the first 4 movements are all down.

Therefore, the real-world probability of the double barrier option being exercized is:

$$(0.65)(0.6)(0.55)(0.5)^2 + (0.35)(0.4)(0.45)(0.5) = 8.5125\%.$$

(d) Calculate the risk-neutral probability of an up-movement in the price of the asset S.

Commentary on Question:

Candidates did well on this part.

Let us define the risk-neutral probability of up-movement as q.

Then, we can solve the following equation for q:

$1.05 = q \times 1.3 + (1 - q) \times 0.9.$

Thus, $q = \frac{15}{40} = 37.5\%$.

(e) Calculate the price Z_0 of the double barrier option at $t = 0$ under the risk-neutral measure.

Commentary on Question:

Candidates did well on this part. Candidates that incorrectly performed previous parts were not penalized for the same mistakes again.

The risk-neutral expected payoff of the option at time 5 is

$$E^{Q}[Z] = 100(q^5 + (1 - q)^4) = 16.$$

The time-0 price of the option is:

$$Z_0 = \frac{E^{Q}[Z]}{(1.05)^5} = 12.54.$$

QFI QF Fall 2022 Question 4

Learning Outcomes:

d) Understand Stochastic Calculus theory and technique used in pricing derivatives

Source References:

- *Options, Futures, and Other Derivatives*, Hull, John C., 11th Edition, 2021, Chapter 14, page 327
- *Problems and Solutions in Mathematical Finance: Stochastic Calculus*, Chin, Eric, Nel, Dian and Olafsson, Sverrir, 2014, pages 52, 128-130

Solution:

(a) Show that $Z_t = \int_0^t W_u du$ is a normally distributed random variable.

Commentary on Question:

Most candidates were able to show that $\mathrm{Z_t}$ *was normal; a full credit response required the use of the mean square limit as justification.*

We can use integration by parts to show $d(tW_t) = tdW_t + W_t dt$. Therefore,

$$Z_t = \int_0^t W_u \, du = tW_t - \int_0^t u \, dW_u = \int_0^t (t-u) \, dW_u$$

The last integral is the mean square limit of $\sum_{k=1}^{n}(t-u_{k-1})\,(W_k - W_{k-1})$for a subdivision $0 = u_0 < u_1 < \cdots < u_n = t$ and $W_k \coloneqq W_{u_k}$. This sum represents a linear combination of independent normally distributed random variables, which is also normal in the limit.

<u>Alternative solution:</u>

$Z_t = \int_0^t W_u \, du$ is the mean square limit of $\sum_{k=1}^{n} W_k(u_k - u_{k-1})$ for a subdivision $0 = u_0 < u_1 < \cdots < u_n = t$ and $W_k \coloneqq W_{u_k}$. This sum represents a linear combination of independent normally distributed random variables, which is also normal in the limit.

(b) Determine whether Y_t is a Wiener process with respect to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$.

Commentary on Question:

Most candidates were able to derive the expectation and variance of Y_t *correctly but were not able to identify the incremental variance as the reason it fails to be a Wiener process. Some candidates incorrectly deduced* Y_t *was a Wiener process based on the expectation and variance.*

In order to be a Wiener process with respect to filtration $\{\mathcal{F}_t\}_{t\geq 0}$, Y_t should satisfy:

i. $Y_0 = 0$ and has continuous sample paths
ii. For $t > 0, s > 0, Y_{t+s} - Y_t \sim N(0, s)$ [stationary increment]
iii. For $t > 0, s > 0, Y_{t+s} - Y_t \perp Y_t$ [independent increment]

We start with the expectation and variance of Y_t:

$E[Y_t] = E\left[\frac{\sqrt{3}}{t} Z_t\right] = E\left[\frac{\sqrt{3}}{t}\int_0^t (t-u)\, dW_u\right] = \frac{\sqrt{3}}{t} E\left[\int_0^t (t-u)\, dW_u\right] = 0$, as it is an Ito integral.

$$Var[Y_t] = Var\left[\frac{\sqrt{3}}{t} Z_t\right] = E\left[\left(\frac{\sqrt{3}}{t} Z_t\right)^2\right] = \frac{3}{t^2} E\left[{Z_t}^2\right] = \frac{3}{t^2}\int_0^t (t-u)^2\, du = \frac{1}{t^2} t^3 = t$$

by Ito's isometry.

We can use the above to determine the incremental variance:

$$\begin{aligned}
Var[Y_{t+s} - Y_t] &= Var\left[\frac{\sqrt{3}}{t+s} Z_{t+s} - \frac{\sqrt{3}}{t} Z_t\right] \\
&= Var\left[\frac{\sqrt{3}}{t+s} Z_{t+s}\right] + Var\left[\frac{\sqrt{3}}{t} Z_t\right] - 2Cov\left[\frac{\sqrt{3}}{t+s} Z_{t+s}, \frac{\sqrt{3}}{t} Z_t\right] \\
&= (t+s) + t - \frac{6}{t(t+s)} E\left[\int_0^{t+s} (t+s-u)\, dW_u \int_0^t (t-u)\, dW_u\right] \\
&= 2t + s \\
&- \frac{6}{t(t+s)}\left[E\left[\int_t^{t+s} (t+s-u)\, dW_u \int_0^t (t-u)\, dW_u\right]\right. \\
&\left. + E\left[\int_0^t (t+s-u)\, dW_u \int_0^t (t-u)\, dW_u\right]\right] \\
&= 2t + s - \frac{6}{t(t+s)}\int_0^t (t+s-u)\,(t-u) du \\
&= 2t + s - \frac{6}{t(t+s)}\left(\frac{st^2}{2} + \frac{t^3}{3}\right) = \frac{s^2}{t+s} \neq s
\end{aligned}$$

So, Y_t fails to have the stationary increment property and is therefore not a Wiener process.

<u>Alternative solution:</u>

For a Wiener process, W_t, we expect $Cov[W_{t+s}, W_t\ ] = t$. However,

$$ov\left[\frac{\sqrt{3}}{t+s}Z_{t+s},\frac{\sqrt{3}}{t}Z_t\right]=\frac{3}{t(t+s)}E\left[\int_0^{t+s}(t+s-u)\,dW_u\int_0^t(t-u)\,dW_u\right]$$
$$=\frac{3}{t(t+s)}\left[E\left[\int_{\mathrm{t}}^{t+s}(t+s-u)\,dW_u\int_0^t(t-u)\,dW_u\right]\right.$$
$$\left.+E\left[\int_0^t(t+s-u)\,dW_u\int_0^t(t-u)\,dW_u\right]\right]$$
$$=\frac{3}{t(t+s)}\int_0^t(t+s-u)\,(t-u)du=\frac{3}{t(t+s)}\left(\frac{st^2}{2}+\frac{t^3}{3}\right)=\frac{3st+2t^2}{2(t+s)}$$
$$=t+\frac{st}{2(t+s)}\neq t$$

Therefore, Y_t is not a Wiener process.

(c) Derive an expression for dG_t in terms of dY_t.

Commentary on Question:

Few candidates performed well on this section. Most failed to establish any relationship between $\mathrm{G_t}$ *and* $\mathrm{Y_t}$.

We can easily derive an expression for $\ln S_u$, given the process satisfies the GBM model:

$$S_u=S_0e^{\left(\mu-\frac{1}{2}\sigma^2\right)u+\sigma W_u}\Leftrightarrow\ln S_u=\ln S_0+\left(\mu-\frac{1}{2}\sigma^2\right)u+\sigma W_u$$

We can substitute the above into the expression for $\ln G_t$:

$$\ln G_t=\frac{1}{t}\int_0^t\left(\ln S_0+\left(\mu-\frac{1}{2}\sigma^2\right)u+\sigma W_u\right)du$$

$$=\ln S_0+\left(\mu-\frac{1}{2}\sigma^2\right)\frac{t}{2}+\frac{\sigma}{t}\int_0^t W_u\,du$$
$$=\ln S_0+\left(\mu-\frac{1}{2}\sigma^2\right)\frac{t}{2}+\frac{\sigma}{\sqrt{3}}Y_t$$

This allows us to re-write the process, G_t as:

$$G_t=S_0e^{\left(\mu-\frac{1}{2}\sigma^2\right)\frac{t}{2}+\frac{\sigma}{\sqrt{3}}Y_t}$$

By applying Ito's Lemma to G_t, we find:

$$dG_t = G_t\left[\frac{1}{2}\left(\mu - \frac{1}{2}\sigma^2\right)dt + \frac{\sigma}{\sqrt{3}}dY_t + \frac{1}{2}\left(\frac{\sigma^2}{3}\right)dt\right],$$

or equivalently:

$$\frac{dG_t}{G_t} = \frac{1}{2}\left(\mu - \frac{1}{6}\sigma^2\right)dt + \frac{\sigma}{\sqrt{3}}dY_t$$

QFI QF Spring 2023 Question 2

Learning Outcomes:

d) Understand Stochastic Calculus theory and technique used in pricing derivatives

e) Understand and apply the concepts of risk-neutral measure, forward measure, normalization, and the market price of risk

f) Understand option pricing techniques

Source References:

- *Options, Futures, and Other Derivatives*, Hull, John C., 11th Edition, 2021, Chapters 14, 28, pages 327, 675
- *Problems and Solutions in Mathematical Finance: Stochastic Calculus*, Chin, Eric, Nel, Dian and Olafsson, Sverrir, 2014, pages 221-227, 233-242
- INV201-101-25: Chapter 6 of Introduction to Stochastic Finance with Market Examples by Privault

Commentary on Question:

In general, this part of the question was the one found most accessible for candidates. Most candidates derived part i) correctly. A handful found part ii) difficult. Many used their results in i) to justify ii), despite being asked explicitly to show "using the definition" which means the martingale property must be shown explicitly. Others did not adequately demonstrate the martingale property and mixed up when you could remove arguments outside of the expectation. Many failed to show that E(abs(M(t)) could be simplified to M(0)

Solution:

(a) Show that $M(t)$ is a $\mathbb{Q}$-martingale using each of the following approaches:

(i) Deriving the stochastic dynamics of $M(t)$.

(ii) Applying the definition of a martingale.

(i)

$$dM(t) = \alpha M(t)dW(t) - \frac{\alpha^2}{2}M(t)dt + \frac{\alpha^2}{2}M(t)dt$$

$= \alpha M(t)dW(t)$

which is driftless and hence $M(t)$ is a martingale under $\mathbb{Q}$.

(ii)

By definition,

$$E^{\mathbb{Q}}(|M(t)|) = E^{\mathbb{Q}}\left(\left|M(0)e^{\alpha W(t)-\frac{\alpha^2 t}{2}}\right|\right)$$

$$= M(0)e^{-\frac{\alpha^2 t}{2}}E^{\mathbb{Q}}\left(e^{\alpha W(t)}\right) = M(0) < \infty$$

Clearly $M(t)$ is $\mathcal{F}_t$-measureable

For $0 < s < t$

$$E_s^{\mathbb{Q}}\big(M(t)\big) = E_s^{\mathbb{Q}}\left(M(0)e^{\alpha W(t)-\frac{\alpha^2 t}{2}}\right)$$

$=M(0)e^{-\frac{\alpha^2 t}{2}}E_s^{\mathbb{Q}}\left(e^{\alpha(W(t)-W(s)+W(s))}\right)$

$=M(0)e^{-\frac{\alpha^2 t}{2}}e^{\alpha W(s)}E_s^{\mathbb{Q}}\left(e^{\alpha(W(t)-W(s))}\right)$, since $W(s)$ is $\mathcal{F}_s$-measureable

$=M(0)e^{-\frac{\alpha^2 t}{2}}e^{\alpha W(s)}e^{\frac{\alpha^2}{2}(t-s)}$, since $W(t)-W(s)$ is normal distribution mean 0 and variance t-s.

$= M(0)e^{\alpha W(s)-\frac{\alpha^2 s}{2}} = M(s)$

Hence by definition, $M(t)$ is a $\mathbb{Q}$-martingale

(b) Write down the Radon-Nikodym derivative of $\mathbb{Q}^A$ with respect to $\mathbb{Q}$.

Commentary on Question:

Of all parts, candidates did the poorest on this section. Most left it blank and/or incorrect. A handful knew the standard definition of the RN derivative but very few successfully provided the correct expression. Another small handful knew the definition but incorrectly derived an expression involving the other asset/numeraire pair and not Q and the risk-free asset.

Candidates could apply a number of approaches to derive the RN-derivative, but since the question states "write-down", they will receive full marks for writing down the correct expression and simplifying to the final answer.

Approach 1 From question a), we have shown $dM(t)$ is driftless under $\mathbb{Q}$ and so is the bank account numeraire, trivially, since r = *0*. $A(t)$ has the equivalent form of $A(t)$ and hence is also driftless under $\mathbb{Q}$

Hence, we can write the RN-derivative as a ratio of asset numeraires:

$$\frac{d\mathbb{Q}^A}{d\mathbb{Q}} = \frac{A(t)/A(0)}{e^{0*t}/e^{0*0}} = e^{\vartheta W(t) - \frac{\vartheta^2 t}{2}}$$

Approach 2 The result follows directly from Girsanov's Theorem since $W(t)$ is a $\mathbb{Q}$-standard Brownian Motion and $-\vartheta$ is constant and therefore $\mathcal{F}_t$-adapted

$$Z_t = e^{-(-\vartheta)W(t) - \frac{(-\vartheta)^2 t}{2}} = e^{\vartheta W(t) - \frac{\vartheta^2 t}{2}} = \frac{d\mathbb{Q}^A}{d\mathbb{Q}}$$

(c) Determine, using Ito's lemma and Girsanov Theorem, whether the normalized process $\frac{M(t)}{A(t)}$ is a $\mathbb{Q}$-martingale or a $\mathbb{Q}^A$-martingale.

Commentary on Question:

Overall, candidates performed fairly well on this question with most recognizing it was not a Q-martingale but was a $\mathbb{Q}^A$-martingale. Some lost marks by not including clear statements that it was not a Q martingale (and instead relied upon the examiner to infer the candidate knew because it had a drift term in the stochastic dynamics it was not a martingale). Some made errors in the Q dynamics but correctly applied Girsanov's theorem to find the dynamics under $\mathbb{Q}^A$ and were not penalized again for the mistake under Q. Some candidates missed full marks as they did not adequately *explain that it was Girsanov's theorem that let them incorporate the $\mathbb{Q}^A$ Brownian motion, instead just applying the transformation without any justification.*

Let $\widetilde{M}(t) = M(t)/A(t)$. *First simplify the expression and then apply Ito's Lemma*

$$\widetilde{M}(t) = \frac{M(0)e^{\alpha W(t) - \frac{\alpha^2 t}{2}}}{A(0)e^{\vartheta W(t) - \vartheta^2 t/2}} = \widetilde{M}(0)e^{(\alpha-\vartheta)W(t) - \frac{(\alpha^2-\vartheta^2)t}{2}}$$

$$d\widetilde{M}(t) = (\alpha - \vartheta)\widetilde{M}(t)dW(t) - \frac{(\alpha^2 - \vartheta^2)}{2}\widetilde{M}(t)dt + \frac{(\alpha - \vartheta)^2}{2}\widetilde{M}(t)dt$$
$$= (\vartheta^2 - \alpha\vartheta)\widetilde{M}(t)dt + (\alpha - \vartheta)\widetilde{M}(t)dW(t)$$

This is not a martingale under $\mathbb{Q}$ *as the dynamics are not driftless*

We have already established in parts a) and b) that

$Z_t = \frac{d\mathbb{Q}^A}{d\mathbb{Q}} = e^{\vartheta W(t) - \frac{\vartheta^2 t}{2}}$ *is a* $\mathbb{Q}$ *martingale*

Using Girsanov's theorem, we know there is a $\mathbb{Q}^A$ *standard Brownian Motion, say* $\widetilde{W}(t)$ *such that* $\widetilde{W}(t) = W(t) + (-\vartheta)t$

So, $d\widetilde{M}(t) = (\vartheta^2 - \alpha\vartheta)\widetilde{M}(t)dt + (\alpha - \vartheta)\widetilde{M}(t)\left(d\widetilde{W}(t) + \vartheta dt\right)$

$= \vartheta^2\widetilde{M}(t)dt - \alpha\vartheta\widetilde{M}(t)dt + \alpha\vartheta\widetilde{M}(t)dt - \vartheta^2\widetilde{M}(t)dt + (\alpha - \vartheta)\widetilde{M}(t)d\widetilde{W}(t) = (\alpha - \vartheta)\widetilde{M}(t)d\widetilde{W}(t)$

Hence $\widetilde{M}(t)$ *is driftless and a* $\mathbb{Q}^A$*-martingale*

(d) Derive an expression for today's price of an exchange option with payoff $P(T) = max[0, M(T) - A(T)]$.

Commentary on Question:

More than half of candidates attempt this question; however, few attempts were successful, and overall marks were low on this part. A very small minority of candidates were able to achieve near full marks. There was a very similar (and harder) question on the Spring 2022 paper that would have prepared candidates for this one.

The purpose of this question is to test the application of change of numeraire, either explicitly, or based on the equivalency of asset/numeraire pairs:

$P(0) = E^{\mathbb{Q}}(M(T) - A(T))^+$

This expression cannot easily be evaluated under $\mathbb{Q}$

Rewriting it, $P(0) = E^{\mathbb{Q}}\left(A(T)\frac{(M(T)-A(T))}{A(T)}\right)^{+}$

$$= E^{\mathbb{Q}}\left(A(T)\left(\widetilde{M}(T)-1\right)\right)^{+}$$

$$= E^{\mathbb{Q}}\left(A(0)\frac{A(T)}{A(0)}\left(\widetilde{M}(T)-1\right)\right)^{+}$$

$$= A(0)E^{\mathbb{Q}}\left(\frac{d\mathbb{Q}^{A}}{d\mathbb{Q}}\left(\widetilde{M}(T)-1\right)\right)^{+}$$

$$= A(0)E^{\mathbb{Q}^{A}}\left(\left(\widetilde{M}(T)-1\right)^{+}\right)$$

But $\widetilde{M}(T)$ *is a* $\mathbb{Q}^{A}$ *-martingale and hence the above expression has the familiar Black-76 formula for a call option struck on* $\widetilde{M}(T)$ *at K=1*

$$P(0) = A(0)\left(\widetilde{M}(0)N(d_1) - N(d_2)\right)$$

$$d_1 = \frac{log\left(\widetilde{M}(0)\right) + \frac{(\alpha-\vartheta)^2 T}{2}}{(\alpha-\vartheta)\sqrt{T}}$$

$$d_2 = d_1 - (\alpha-\vartheta)\sqrt{T}$$

QFI QF Spring 2023 Question 3

Learning Outcomes:

d) Understand Stochastic Calculus theory and technique used in pricing derivatives

e) Understand and apply the concepts of risk-neutral measure, forward measure, normalization, and the market price of risk

Source References:

- *Options, Futures, and Other Derivatives*, Hull, John C., 11th Edition, 2021, Chapter 28, page 675

- *Problems and Solutions in Mathematical Finance: Stochastic Calculus*, Chin, Eric, Nel, Dian and Olafsson, Sverrir, 2014, pages 52, 221-227

Solution:

(a) Show that

$$Z_t = \frac{\sigma_S W_t + \sigma_E V_t}{\sqrt{\sigma_S{}^2 + \sigma_E{}^2 + 2\rho\sigma_S\sigma_E}}$$

is a $\mathbb{P}$-standard Brownian motion.

Commentary on Question:

Candidates generally performed well on this part. Candidates who stated $\mathrm{Z_t}$ *satisfied the properties solely due to* $\mathrm{W_t}$ *and* $\mathrm{V_t}$ *satisfying them only received partial credit as the question directed candidates to "show" the properties are satisfied*

For Z_t to be a $\mathbb{P}$-standard Brownian motion, it must satisfy:

1. $Z_0 = 0$ and Z_t has continuous sample paths,
2. $Z_t \sim N(0, t)$, and
3. $Z_{t+s} - Z_t \perp Z_t$ for $s > 0$

Given that W_t and V_t are standard Brownian motions, we find that:

$$Z_0 = \frac{\sigma_S W_0 + \sigma_E V_0}{\sqrt{\sigma_S{}^2 + \sigma_E{}^2 + 2\rho\sigma_S\sigma_E}} = \frac{\sigma_S(0) + \sigma_E(0)}{\sqrt{\sigma_S{}^2 + \sigma_E{}^2 + 2\rho\sigma_S\sigma_E}} = 0$$

Continuous sample paths follow from being a linear combination of two standard Brownian motions.

For second property, Z_t is normal given that it is a linear combination of two normal distributions. However, we must verify the expectation and variance.

Similar to the first property,

$$E[Z_t] = E\left[\frac{\sigma_S W_t + \sigma_E V_t}{\sqrt{\sigma_S{}^2 + \sigma_E{}^2 + 2\rho\sigma_S\sigma_E}}\right] = \frac{1}{\sqrt{\sigma_S{}^2 + \sigma_E{}^2 + 2\rho\sigma_S\sigma_E}}\{\sigma_s E[W_t] + \sigma_E E[V_t]\} = 0$$

$$Var(Z_t) = E[Z_t^2] = E\left[\left(\frac{\sigma_S W_t + \sigma_E V_t}{\sqrt{{\sigma_S}^2 + {\sigma_E}^2 + 2\rho\sigma_S\sigma_E}}\right)^2\right]$$
$$= \frac{1}{{\sigma_S}^2 + {\sigma_E}^2 + 2\rho\sigma_S\sigma_E} E[{\sigma_S}^2 W_t^2 + {\sigma_E}^2 V_t^2 + 2\sigma_S W_t \sigma_E V_t]$$
$$= \frac{1}{{\sigma_S}^2 + {\sigma_E}^2 + 2\rho\sigma_S\sigma_E} \{{\sigma_S}^2 E[W_t^2] + {\sigma_E}^2 E[V_t^2] + 2\sigma_S\sigma_E E[W_t V_t]\}$$
$$= \frac{1}{{\sigma_S}^2 + {\sigma_E}^2 + 2\rho\sigma_S\sigma_E} \{{\sigma_S}^2 t + {\sigma_E}^2 t + 2\sigma_S\sigma_E\rho t\} = t$$

For the final property, given that $Z_{t+s} - Z_t$ and Z_t are jointly normally distributed, it suffices to show $Cov(Z_{t+s} - Z_t\ , Z_t)$ = $E[(Z_{t+l} - Z_t)Z_t] = 0$.

With Z_t being $\mathcal{F}_t$-measurable, $E[(Z_{t+l} - Z_t)Z_t] = E[Z_t E[Z_{t+l} - Z_t|\mathcal{F}_t]] = 0$.

(b) Determine whether $\ln(S_t E_t)$ follows an arithmetic Brownian motion under the measure $\mathbb{P}$ or not.

Commentary on Question:

Candidate who attempted to perform Ito's Lemma directly on $\ln(\mathrm{S_t E_t})$ *often left out a cross-term. Those who first determined the form of* $\mathrm{d(S_t E_t)}$ *or followed the alternate approach were most successful. Some candidates conflated the notion of arithmetic Brownian motion and martingale. A clear conclusion was required for full credit.*

By the product rule, $d(\mathrm{S_t E_t}) = \mathrm{E_t dS_t + S_t dE_t + dS_t dE_t}$.

Substituting yields, $d(\mathrm{S_t E_t}) = \mathrm{S_t E_t}\left((\mu + \rho\sigma_S\sigma_E)\mathrm{dt} + \sqrt{{\sigma_S}^2 + {\sigma_E}^2 + 2\rho\sigma_S\sigma_E}\,dZ_t\right)$.

We can apply Ito's Lemma to the above to get the desired SDE:

$$d\big(ln(\mathrm{S_t E_t})\big) = \left(\mu - \frac{\sigma_S^2}{2} - \frac{\sigma_E^2}{2}\right)\mathrm{dt} + \sqrt{{\sigma_S}^2 + {\sigma_E}^2 + 2\rho\sigma_S\sigma_E}\,dZ_t.$$

The final result does follow an arithmetic Brownian motion under measure $\mathbb{P}$.

Alternate approach:

$d\big(ln(\mathrm{S_t E_t})\big) = d(lnS_t) + d(lnE_t)$, where under Ito's Lemma

$d(lnS_t) = \left(\mu - \frac{\sigma_S^2}{2}\right)dt + \sigma_S dW_t$ and $d(lnE_t) = -\frac{\sigma_E^2}{2}dt + \sigma_E dV_t$

Thus, $d\left(ln(S_t E_t)\right) = \left(\mu - \frac{\sigma_S^2}{2} - \frac{\sigma_E^2}{2}\right)dt + \sigma_S dW_t + \sigma_E dV_t$, which is an equivalent result.

(c) Show that $e^{-rt}S_tE_t$ is a martingale under the risk-neutral measure $\mathbb{Q}$ using Girsanov Theorem, with the numeraire being CAD risk-free assets.

Commentary on Question:

For full credit, candidates needed to show an understanding of Girsanov's Theorem as well as the form of a martingale under risk-neutral measure $\mathbb{Q}$. Many candidates only received partial credit for calculating the SDE of $e^{-rt}S_tE_t$.

S_tE_t by definition represents the Canadian asset price in CAD, making r the associated risk-free rate.

$$d(e^{-rt}S_tE_t) = -re^{-rt}S_tE_tdt + e^{-rt}[E_tdS_t + dE_tS_t + dE_tdS_t,]$$
$$= e^{-rt}S_tE_t\left((\mu - r + \rho\sigma_S\sigma_E)dt + \sqrt{\sigma_S{}^2 + \sigma_E{}^2 + 2\rho\sigma_S\sigma_E}dZ_t\right)$$

By applying Girsanov's Theorem to change measure $\mathbb{P}$ to an equivalent risk-neutral measure $\mathbb{Q}$, we can utilize a process, $\tilde{\tilde{Z}}_t$, which is a standard Brownian motion under that measure. By letting,

$$d\tilde{\tilde{Z}}_t = dZ_t + \frac{(\mu - r + \rho\sigma_S\sigma_E)}{\sqrt{\sigma_S{}^2 + \sigma_E{}^2 + 2\rho\sigma_S\sigma_E}}$$

we find under $\mathbb{Q}$, $d(e^{-rt}S_tE_t) = e^{-rt}S_tE_t\left(\sqrt{\sigma_S{}^2 + \sigma_E{}^2 + 2\rho\sigma_S\sigma_E}d\tilde{\tilde{Z}}_t\right)$.

As there is no drift, $e^{-rt}S_tE_t$ is a $\mathbb{Q}$-martingale.

QFI QF Spring 2023 Question 5

Learning Outcomes:

i) Calibrate a model to observed prices of traded securities including fitting to a given yield curve

Source References:

- Calibrating Interest Rate Models (Section 1.1-4.3 excl 4.1.2)

Commentary on Question:

The question tested candidates on quantitative tools and techniques for modeling the term structure of interest rates with the Hull-White model. Candidates performed well on part (e) which tested their ability to calculate the price of an option. Candidates performed poorly on parts (a)-(d).

Solution:

(a) Explain whether the fitted model is a true arbitrage-free model.
Commentary on Question:

Most candidates answered this question incorrectly and stated that the Hull-White model is an arbitrage-free model.

The fitted yield curve is obtained by fitting a third degree polynomial to 20 points.

It may have done through least square or "lm" using R.

It is very unlikely the fit will be perfect as least square fit just minimizes the errors, not making them zero.

Also gamma and sigma are obtained by fitting five cap prices so those estimates with the 3 degree polynomial would be unlikely to produce a perfect fit to yield curve and cap prices, so the calibrated model is not truly arbitrage free.

(b) Derive an expression for the instantaneous forward rate at time 0 $f(0,t)$.
Commentary on Question:

Candidates performed below expectation on this question. A common mistake was to take the derivative of r(0,t).

$$f(0,t) = \frac{\partial}{\partial t}\left(t\, r(0,t)\right)$$

$$f(0,t) = 0.01091858598 + 0.01251008594 * 2 * t - 0.000140114635 * 3 * t^2 + 0.005654825 * 4 * t^3$$

$$f(0,t) = 0.010918586 + 0.025020172t - 0.0004203439t^2 + 0.0226193t^3$$

(c) Derive an expression for θ_t .
Commentary on Question:

Many candidates were able to identify the correct initial formula to use. Most candidates struggled to convert the formula to decimal numbers.

$$\theta_T = \frac{\partial f(0,T)}{\partial T} + \gamma f(0,T) + \frac{\sigma^2}{2\gamma}(1 - exp(-2\gamma T))$$

As f(0,T) has been calculated as percentage points and σ^2 is in the equation, we need to convert f(0,T) and σ to deimal numbers before using the formula.

$$\theta_t = \sum_{i=1}^{n} a_i i(i+1)t^{i-1} + \sum_{i=0}^{n} \gamma a_i (i+1)t^i + \frac{\sigma^2}{2\gamma}(1 - exp(-2\gamma t))$$

$$\theta_t = 0.02709470321 + 0.0039131448472t + 0.06777803465805t^2 + 0.004297667t^3 + 0.001010947(1 - exp\,(-0.38t)$$

(d) Compute $E[r_{1.25}|r_1 = 0.03\%]$, given $f(0{,}1.25) = 0.036068$.

Commentary on Question:

Most candidates performed poorly on this question. To receive full credit, candidates needed to use the appropriate formula and calculate the expectation correctly. An alternative answer was accepted if candidates assumed r_1=3%.

$$E[r_{t+s}|r_t] = r_t exp(-\gamma s) + f(0, s+t) - f(0,t)exp(-\gamma s) + \frac{\sigma^2}{2\gamma^2}\left[1 - exp(-\gamma s) + exp\big(-2\gamma(t+s)\big) - exp\big(-\gamma(2t+s)\big)\right]$$

$E[r_{1.25}|r_1 = o.o3\%] =$ 0.000286 + 0.036068 – 0.055441 +0.005321*0.016137

$$E[r_{1.25}|r_1 = o.o3\%] = -1.9\%$$

QFI QF Spring 2023 Question 7

Learning Outcomes:

d) Understand Stochastic Calculus theory and technique used in pricing derivatives

Source References:

- *Problems and Solutions in Mathematical Finance: Stochastic Calculus*, Chin, Eric, Nel, Dian and Olafsson, Sverrir, 2014, pages 132-137

Solution:

(a)

(i) Show by using Ito's Lemma that

$$dY_t = [\left(\gamma^* \overline{r^*} - \frac{1}{2}\alpha\right) e^{-Y_t} - \gamma^*]dt + \sqrt{\alpha} e^{-\frac{Y_t}{2}} dX_t$$

(ii) Explain why the drift term of dY_t is positive if Y_t gets too far below from 0.

Commentary on Question:

Candidates performed well on this question part. Most candidates were able to correctly recall and apply Ito's Lemma to derive the process for dY_t. Candidates needed to specify that the inequality $\gamma\ \bar{r} > \frac{1}{2}\alpha$ contributes to positive drift to receive full credit.

(i) By Ito's Lemma, $dY_t = \frac{1}{r} dr - \frac{1}{2r^2}(dr)^2$

$$= e^{-Y_t}\left(\gamma(\bar{r} - r_t)dt + \sqrt{\alpha r_t} dX_t\right) -$$

$$\frac{1}{2}\frac{1}{r_t^2}\ \alpha r_t\ dt$$

Note: $dt^2 = dt\ dXt = 0, dX_t^2 = dt,\ r_t = e^{Y_t}, \sqrt{r_t} = e^{\frac{Y_t}{2}}$

$$= e^{-Y_t}\left(\gamma(\bar{r} - r_t)dt + \sqrt{\alpha r_t} dX_t\right) -$$

$$\frac{1}{2}\frac{1}{r_t}\ \alpha\ dt$$

$$= e^{-Y_t}\left(\gamma(\bar{r} - e^{Y_t})dt + \sqrt{\alpha} e^{\frac{Y_t}{2}} dX_t\right) -$$

$$\frac{1}{2} e^{-Y_t}\ \alpha\ dt$$

$$= \left(\gamma\bar{r} - \frac{1}{2}\alpha\right) e^{-Y_t} - \gamma]dt + \sqrt{\alpha} e^{-\frac{Y_t}{2}} dX_t$$

(ii) As $\gamma\ \bar{r} > \frac{1}{2}\alpha$, if Y_t gets too far below from 0, the drift term of dY_t will become strongly positive as e^{-Y_t} will be very large.

QFI QF Fall 2023 Question 2

Learning Outcomes:

d) Understand Stochastic Calculus theory and technique used in pricing derivatives

Source References:

- *Problems and Solutions in Mathematical Finance: Stochastic Calculus*, Chin, Eric, Nel, Dian and Olafsson, Sverrir, 2014, pages 52, 72-73

Commentary on Question:

This question tests candidates' understanding of the properties of Brownian motions, Ito's lemma, and martingales.

Solution:

(a) Evaluate the following expressions for $0 < s < t < u$:

(i) $E^{\mathbb{Q}}(W(s)\, W(t)W(u))$

(ii) $E^{\mathbb{Q}}(W(t)W(u) \mid \mathcal{F}_s)$

Commentary on Question:

Many candidates did well on this part by applying the independence properties of increments of Brownian motions.

This question is straight bookwork from Neftci and Chin. For both of these parts, use independent increments to simplify the expressions.

Part (i)

$E^{\mathbb{Q}}(W(t) \times W(u) \times W(s))$

$= E^{\mathbb{Q}}\big((W(u) + W(t) - W(t)) \times W(t) \times W(s)\big)$

$= E^{\mathbb{Q}}\big((W(u) - W(t)) \times W(t) \times W(s)\big) + E^{\mathbb{Q}}(W^2(t) \times W(s)) = \boldsymbol{A} + \boldsymbol{B}$

Since $\big(W(u) - W(t)\big)$ is independent of $W(t)$ and $W(s)$ we may go with

$$\boldsymbol{A} = E^{\mathbb{Q}}\Big(\big(W(u) - W(t)\big) \times W(t) \times W(s)\Big)$$

$$= E^{\mathbb{Q}}\big(W(u) - W(t)\big)E^{\mathbb{Q}}\big(W(t) \times W(s)\big) = 0$$

$\boldsymbol{B} = E^{\mathbb{Q}}(W^2(t) \times W(s))$

$= E^{\mathbb{Q}}((W(t) - W(s) + W(s))^2 \times W(s))$

$= E^{\mathbb{Q}}((W(t) - W(s))^2 \times W(s)) + 2E^{\mathbb{Q}}\big((W(t) - W(s)) \times W^2(s)\big) + E^{\mathbb{Q}}(W^3(s))$

$= E^{\mathbb{Q}}((W(t) - W(s))^2) \times E^{\mathbb{Q}}(W(s)) + 2E^{\mathbb{Q}}(W(t) - W(s)) \times E^{\mathbb{Q}}(W^2(s)) + E^{\mathbb{Q}}(W^3(s))$

$= (t - s) \times 0 + 0 \times s + 0$

Hence $E^{\mathbb{Q}}(W(t) \times W(u) \times W(s)) = 0$

Part (ii)

Use the expressions obtained in part (ii) before the split of independent increments. For notation, denote $E^{\mathbb{Q}}(X|\mathcal{F}_s) = E_s^{\mathbb{Q}}(X)$

We may simply go with:

$$E_s^{\mathbb{Q}}\big(W(t) \times W(u)\big) = E_s^{\mathbb{Q}}\big(W(t) \times W(u)\big)$$

$$= E_s^{\mathbb{Q}}\Big(W(t) \times \big(W(u) - W(t) + W(t)\big)\Big)$$

$$= E_s^{\mathbb{Q}}\Big(W(t) \times \big(W(u) - W(t)\big)\Big) + E_s^{\mathbb{Q}}\big(W^2(t)\big)$$

$= 0 + E_s^{\mathbb{Q}}\Big(\big(W(t) - W(s)\big)^2\Big) + \big(W(s)\big)^2$ where in the last equality we use the fact that $E_s^{\mathbb{Q}}\big(W(t)\big) = W(s)$ and $E(X^2) = E[(X - \mu)^2] + \mu^2$

ence, $E_s^{\mathbb{Q}}\big(W(t) \times W(u)\big) = (t - s) + W^2(s)$

(b) Determine whether $X(t)$ is a martingale under $\mathbb{Q}$ using Ito's lemma.

Commentary on Question:

Many candidates were able to derive the correct formula by applying Ito's lemma and got the right answer.

This question is an application of the multivariate Ito's Lemma:

$$dX(t) = d\left((V(t))^2 \times W(t) - \int_0^t W(p)dp \right)$$

Let $\boldsymbol{A} *= \ d((V(t))^2 \times W(t))$

$= 2V(t)W(t)dV(t) + (V(t))^2 dW(t) + .5 * 2W(t)dt$

Let $\boldsymbol{B} *= d\left(-\int_0^t W(s)ds\right)$

$= -(W(t)dt)$

$\boldsymbol{A} * + \boldsymbol{B} *= 2V(t)W(t)dV(t) + ((V(t))^2)dW(t)$

As V and W are independent and the SDE is driftless, X(t) is a martingale.

(c) Determine whether $X(t)$ is a martingale under $\mathbb{Q}$ using the definition of a martingale.

Commentary on Question:

Most candidates did poorly in this part. Many candidates were able to list the three conditions of martingales. However, few were able to prove the second property.

To obtain full marks, candidates need to show $X(t)$ satisfies the full definition of a martingale and all 3 parts.

Criteria 1 – adaptability

Clearly $X(t)$ is adapted to $\mathcal{F}_t$

Criteria 2 – $E^{\mathbb{Q}}(|X(t)|) < \infty$. Note, there is more than one way to demonstrate this.

$$E^{\mathbb{Q}}(|X(t)|) = E^{\mathbb{Q}}\left(\left| (V(t))^2 \times W(t) - \int_0^t W(p)dp \right| \right)$$

By the triangle inequality, we have:

$$\leq E^{\mathbb{Q}}(|(V(t))^2 \times W(t)|) + E^{\mathbb{Q}}\left(\left| \int_0^t W(p)dp \right| \right)$$

By independence of V and W, and abs(integral) ≤ integral(abs)

$$\leq E^{\mathbb{Q}}(|(V(t))^2|) \times E^{\mathbb{Q}}(|W(t)|) + \left| \int_0^t E^{\mathbb{Q}}|W(p)|dp \right|$$

Evaluate $E^{\mathbb{Q}}(|W(t)|) = 2\sqrt{t}\int_0^{\infty} W(1)\frac{1}{\sqrt{2\pi}}e^{-\frac{W^2(1)}{2}}dW(1)$

$$= -\sqrt{\frac{2t}{\pi}}\int_0^{\infty}\frac{\partial e^{-\frac{W^2(1)}{2}}}{\partial W(1)}dW(1) = \sqrt{\frac{2t}{\pi}}$$

So, $E^{\mathbb{Q}}(|X(t)|) \leq t\sqrt{\frac{2t}{\pi}} + \int_0^t \sqrt{\frac{2p}{\pi}}dp = t\sqrt{\frac{2t}{\pi}} + \frac{2}{3}\sqrt{\frac{2}{\pi}}\, t^{3/2} = \frac{5}{3}\sqrt{\frac{2}{\pi}}t^{3/2} < \infty$

This is unnecessarily complicated.

From $E(X^2) = \left(E(X)\right)^2 + E\left(\left(X - E(X)\right)^2\right)$ we know $E^{\mathbb{Q}}(|W(t)|) \leq \sqrt{E^{\mathbb{Q}}\left(W^2(t)\right)} = \sqrt{t}$.

Thus

$$E^{\mathbb{Q}}\left(\left|\left(V(t)\right)^2\right|\right) \times E^{\mathbb{Q}}(|W(t)|) + \left|\int_0^t E^{\mathbb{Q}}|W(p)|dp\right|$$

$$\leq t \times \sqrt{t} + \left|\int_0^t \sqrt{p}dp\right| = \left(1 + \frac{2}{3}\right)t^{\frac{3}{2}} < \infty$$

Criteria 3 – martingale property

$$E_s^{\mathbb{Q}}(X(t)) = E_s^{\mathbb{Q}}\left((V(t))^2 \times W(t) - \int_0^t W(p)dp\right)$$

By the independence of V and W, and by splitting the integral we have

$$= E_s^{\mathbb{Q}}((V(t))^2) \times E_s^{\mathbb{Q}}(W(t)) - E_s^{\mathbb{Q}}\left(\int_0^s W(p)dp + \int_s^t W(p)dp\right)$$

Consider the first part of the equation. $W(t)$ is a Brownian motion and a martingale. i.e., $E_s^{\mathbb{Q}}\left(W(t) - W(s) + W(s)\right) = W(s)$. $E_s^{\mathbb{Q}}\left(\left(V(t)\right)^2\right) = E_s^{\mathbb{Q}}\left(\left(V(t) - V(s) + V(s)\right)^2\right) = E_s^{\mathbb{Q}}((V(t) - V(s))^2) + 2E_s^{\mathbb{Q}}\left(V(s)(V(t) - V(s))\right) + E_s^{\mathbb{Q}}\left(\left(V(s)\right)^2\right) = t - s + \left(V(s)\right)^2$

So, the first part of the equation is $W(s)\left(t - s + \left(V(s)\right)^2\right)$

Next, consider the second part of the equation and use the measurability of the first integral and indepence of the integrand on (s,t) of $\mathcal{F}_s$

$$E_s^{\mathbb{Q}}\left(\int_0^s W(p)dp + \int_s^t W(p)dp\right)$$

$$= \int_0^s W(p)dp + \int_s^t E_s^{\mathbb{Q}}\big(W(p) - W(s) + W(s)\big)dp$$

$$= \int_0^s W(p)dp + W(s)\int_s^t dp = \int_0^s W(p)dp + (t-s)W(s)$$

Putting the two parts together, we have

$$E_s^{\mathbb{Q}}(X(t)) = W(s)\left(t - s + \big(V(s)\big)^2\right) - \left(\int_0^s W(p)dp + (t-s)W(s)\right)$$

$$= W(s)\big(V(s)\big)^2 - \int_0^s W(p)dp = X(s)$$

Hence, $X(t)$ is a martingale

QFI QF Fall 2023 Question 5

Learning Outcomes:

h) Understand and apply numerical discretization methods to price options including Euler-Maruyama discretization and transition density methods

Source References:

- Calibrating Interest Rate Models (Section 1.1-4.3 excl 4.1.2)

Commentary on Question:

This question tests candidates' understanding of interest rate calibrations. Most of the candidates earned full or partial credits from part (a) and (b), but only a few candidates earned partial credits from part (c).

Solution:

(a) Calculate the probability of simulating a negative interest rate for the next trading day.

Commentary on Question:

Most of the candidates were able to use the correct formula for this question. Candidates earned partial credits if they used the correct formula but failed to calculate the final numbers. Full credit will be given to candidates who calculated the correct value.

$P[r_{t+s} < 0|r_t] = \Phi(-\frac{\bar{r}+(r_t-\bar{r})e^{-\gamma s}}{\sigma\sqrt{\frac{(1-e^{-2\gamma s})}{2\gamma}}})$

Substituting values for paramteres

$$z = \frac{\bar{r} + (r_t - \bar{r})e^{-\gamma s}}{\sigma\sqrt{\frac{(1 - e^{-2\gamma s})}{2\gamma}}} = 2.66113$$

Then the probability is $\Phi(-2.66113) = 0.0039$

(b) Calculate the simulated rate for the next trading day using

(i) the Euler-Maruyama discretization method.

(ii) the transition density method.

Commentary on Question:

Some of candidates earned partial credits for using the correct formulas and parameters but only a few candidates calculated the correct values.

(i) Under Euler-Maruyama discretization

$$r(i) = \alpha + \beta r(i-1) + \epsilon_i, i = 1,2, \ldots$$

Where $\alpha = \gamma\bar{r}\Delta, \beta = 1 - \gamma\Delta$ and $\epsilon_i \sim N(0, {\sigma^*}^2)$ with $\sigma^* = \sigma\sqrt{\Delta}$ With given parameters

$$\alpha = 0.3 * 0.05 * \frac{1}{252} = 5.952 * 10^{-5}$$

$$\beta = 1 - 0.05 * \frac{1}{252} = 0.9980$$

$$\sigma^* = 0.06 * \sqrt{\frac{1}{252}} = 0.0037796$$

$$r(i-1) = 0.01$$

$$r(i) = 5.952 * 10^{-5} + 0.9980 * 0.01 - 1.96 * 0.0037796$$

$$r(i) = 0.002631$$

(ii) With the transition density method next random number from the Vasicek is given by

$$r_{t+s} = \bar{r} + (r_t - \bar{r})e^{-\gamma s} + \left(\frac{\sigma^2}{2\gamma}(1 - e^{-2\gamma s})\right)^{\frac{1}{2}} Z$$

With the given parameters

$$(r_t - \bar{r})e^{-\gamma s} = (0.01 - 0.05)e^{-\frac{0.3}{252}} = -0.03995241$$

$$\left(\frac{\sigma^2}{2\gamma}(1 - e^{-2\gamma s})\right)^{\frac{1}{2}} = 0.03777$$

$$r_{t+s} = 0.05 - 0.03995241 + 0.03777 * (-1.96)$$

$$r_{t+s} = 0.002644$$

(c) Compare and contrast the Euler-Maruyama discretization method and the transition density method for simulating interest rate paths in general and in this particular case for Vasicek model.

Commentary on Question:

*Not many candidates attempted this question and some of them successfully identified Euler-Maruyama method is an approximating method and Transition density method is an exact method. However, few candidates pointed out that the differences between those two methods are minimal when s is small in the Vasicek model.*The Euler-Maruyama method is based on the first order discretization of a stochastic differential equation (or simiple discretization)

i.e.

$$dr_t = a(r_t)dt + b(r_t)dX_t$$

is approximated using

$$r_{t+\Delta} - r_t \approx a(r_t)\Delta + b(r_t)\sqrt{\Delta}Z$$

Where Δ is a small time step and Z is a standard normal random variable with mean 0 and variance 1.

Essentially in simulation we are assuming $r_{t+\Delta}|r_t$ is normally distributed with mean $r_t + a(r_t)\Delta$ and variance $b(r_t)^2\Delta$. So even if the original process doesn't take negative values, the approximation may give negative values.

Transition density method relies on the exact distribution of $r_{t+\Delta}|r_t$. So it is an exact method not an approximation. The disadvantage of this method is the exact distribution may not be available for many cases.

In the Vasicek method as we saw in part (a) and (b) the difference is minimal. That is because the exact distribution of $r_{t+s}|r_t$ is normal and since for small values of s

$$e^{-\gamma s} \approx 1 - \gamma s$$

With that

$$\bar{r} + (r_t - \bar{r})e^{-\gamma s} \approx \bar{r} + (r_t - \bar{r})(1 - \gamma s)$$

$$\bar{r} + (r_t - \bar{r})e^{-\gamma s} \approx \bar{r}\gamma s + r_t(1 - \gamma s)$$

and $\left(\frac{\sigma^2}{2\gamma}(1 - e^{-2\gamma s})\right) \approx \sigma^2 s$. These are the mean and variance in the Euler-Maruyama discretization method.

QFI QF Spring 2024 Question 1

Learning Outcomes:

a) Understand the principles of no-arbitrage and replication in asset pricing.

b) Understand Arrow-Debreau security and the distinction between complete and incomplete markets

Source References:

- *Options, Futures, and Other Derivatives*, Hull, John C., 11th Edition, 2021, Chapters 13, pages 292-293
- INV201-100-25: Chapter 5 of Financial Mathematics – A Comprehensive Treatment, Campolieti

Commentary on Question:

This question tests candidates' understanding of arbitrage with a simple one-period model.

Solution:

(a) Determine the range of r such that there are no arbitrage possibilities.

Commentary on Question:

Most candidates were able to derive an accurate range of the risk-free rate. A few candidates treated the rate as an annual rate and obtained incorrect ranges.

Let Q_u and Q_d be the risk-neutral probabilities that the security will go up and go down after 6 months, respectively. Then by the arbitrage theorem, we have

$$Q_u + Q_d = 1$$
$$Q_u \frac{120}{1+r} + Q_d \frac{60}{1+r} = 100$$

From the above equations, we can get

$$60Q_u + 60 = 100(1+r)$$

Since $0 \leq Q_u \leq 1$, we have

$$60 \leq 100(1+r) \leq 120$$

which gives the following no-arbtrage range of r

$$-0.4 \leq r \leq 0.2$$

Alternatively:

$$0 \leq Q_u = \frac{1+r-d}{u-d} = \frac{0.4+r}{0.6} \leq 1 \Rightarrow -0.4 \leq r \leq 0.2.$$

(b) Calculate and interpret the state prices.

Commentary on Question:

Most candidates were able to obtain and interpret the state prices correctly.

Let ψ_u and ψ_d denote the state prices corresponding to the up state and the down state, respectively. Then by the arbitrage theorem, we have

$$1 = (1+0.06)\psi_u + (1+0.06)\psi_d$$
$$100 = 120\psi_u + 60\psi_d$$

Solving the above equations gives the state prices:

$$\psi_u = 0.7233$$
$$\psi_d = 0.2201$$

The state prices can be interpreted as follows:

1. ψ_u is the price investors are willing to pay for an insurance policy that pays 1 in the up state and nothing in the down state.
2. ψ_d is the price investors are willing to pay for an insurance policy that pays 1 in the down state and nothing in the up state.

(c) Calculate the no-arbitrage price of a European call option with strike price of 100 that expires in 6 months.

Commentary on Question:

Most candidates obtained the correct price. A few candidates used a wrong risk-free rate.

Let Q_u and Q_d be the risk-neutral probabilities that the security will go up and go down after 6 months, respectively. Let C be the no-arbitrage price of the option. Since $r = 0.06$, the arbitrage theorem gives

$$Q_u + Q_d = 1$$
$$Q_u \frac{120}{1.06} + Q_d \frac{60}{1.06} = 100$$
$$Q_u \frac{(120-100)^+}{1.06} + Q_d \frac{(60-100)^+}{1.06} = C$$

Solving the first two equations gives

$$Q_u = 0.7667$$
$$Q_d = 0.2333$$

Plugging the risk-neutral probabilities into the third equation, we get

$$C = 0.7667 \times \frac{20}{1.06} = 14.46$$

Alternatively:

$$Q_u = \frac{1+r-d}{u-d} = \frac{0.46}{0.6} = 0.7667$$

(d) Describe two general situations in which arbitrage opportunities can arise.

Commentary on Question:

Most candidates gave correct cases when arbitrage opportunities occur.

Arbitrage opportunities can arise in two different fashions:
One can make a series of investments with no current net commitment, yet expect to make a positive profit.

A portfolio can ensure a negative net commitment today, while yielding nonnegative profits in the future.

(e) Construct a replicating portfolio and use it to price the derivative.

Commentary on Question:

Many candidates did not obtain the replicating portfolio correctly. Some candidates used options to replicate the derivative.

Let x be the number of shares of the security and let y be the money deposited/borrowed. The portfolio should replicate the payoffs of the derivative in both the up and the down states.

$$120x + 1.06y = 22$$
$$60x + 1.06y = 10$$

Solving the equations gives

$$x = 0.2$$
$$y = -1.8868$$

That is, the replicating portfolio consists of 0.2 shares of the security and 1.8868 is borrowed. The total value of the replicating portfolio at time 0 is

$$100 \times 0.2 - 1.8868 = 18.1132$$

Hence the value of the derivative at time 0 should be 18.1132.

QFI QF Spring 2024 Question 5

Learning Outcomes:

d) Understand Stochastic Calculus theory and technique used in pricing derivatives

e) Understand and apply the concepts of risk-neutral measure, forward measure, normalization, and the market price of risk

Source References:

- *Options, Futures, and Other Derivatives*, Hull, John C., 11th Edition, 2021, Chapter 28, pages 671-672
- *Problems and Solutions in Mathematical Finance: Stochastic Calculus*, Chin, Eric, Nel, Dian and Olafsson, Sverrir, 2014, pages 128-130, 221-227
- Understanding the Connection Between Real-World and Risk-Neutral Generators, SOA Research, Aug 2022, Sections 1-5, and Appendices A & D

Commentary on Question:

This question tests candidates' understanding of the fundamentals of stochastic differential and model calibration under both risk-neutral and real-world measures. Most candidates performed very well in part (a) and (b), but not many candidates earned points in part (c) and (d).

Solution:

(a) Show that

(i) $E(r_t|F_s) = r_s e^{-\gamma(t-s)} + \bar{r}(1 - e^{-\gamma(t-s)})$

(ii) $Var(r_t|F_s) = \frac{\sigma^2}{2\gamma}\left[1 - e^{-2\gamma(t-s)}\right]$

Commentary on Question:

Most Candidates successfully derived the formulas of the expectation and variance.

$$dr_u = \gamma(\bar{r} - r_u)du + \sigma dX_u$$

$$e^{-\gamma(t-u)}dr_u = e^{-\gamma(t-u)}\gamma(\bar{r} - r_u)du + e^{-\gamma(t-u)}\sigma dX_u$$

$$e^{-\gamma(t-u)}dr_u + e^{-\gamma(t-u)}\gamma r_u du = e^{-\gamma(t-u)}\gamma(\bar{r})du + e^{-\gamma(t-u)}\sigma dX_u$$

$$d\left(e^{-\gamma(t-u)}r_u\right) = e^{-\gamma(t-u)}\gamma(\bar{r})du + e^{-\gamma(t-u)}\sigma dX_u$$

Integration on both sides of the equation from s to t

$r_t = r_s e^{-\gamma(t-s)} + \bar{r}\left(1 - e^{-\gamma(t-s)}\right) + \sigma \int_s^t e^{-\gamma(t-u)}\, dX_u$

$E(r_t|F_s) = r_s e^{-\gamma(t-s)} + \bar{r}(1 - e^{-\gamma(t-s)})$

$as\ E\left\{\sigma \int_s^t e^{-\gamma(t-u)}\, dX_\mathrm{u}\right\} = 0$

$Var(r_t|F_s) = E\{[(r_t - E(r_t)^2|F_s\}$
$=\sigma^2\left[\int_s^t e^{-\gamma(t-u)}\, dX_\mathrm{u}\right]^2$

By Ito Isometry,

$=\sigma^2 \int_s^t e^{-2\gamma(t-u)}\, du$

$$= \frac{\sigma^2}{2\gamma}\left[1 - e^{-2\gamma(t-s)}\right]$$

(b) Determine the market price of interest risk.

Commentary on Question:

Most candidates successfully identified the formula of market price.

$\lambda(r,t) = \frac{1}{\sigma}(\gamma(\bar{r} - r) - \gamma^*(\overline{r^*} - r))$ = -0.004/0.01=−0.4

(c) Compute the drift and the diffusion of $\frac{dZ}{Z}$ for the risk-neutral process.

Commentary on Question:

Less than half of the candidates successfully identified the diffusion term in the risk-neutral process and calculated correctly. Candidates earned partial credits if they can identify the correct formula or claim the correct drift term.

The drift of $\frac{dZ}{Z}$ or the instantaneous return of the bond, in the risk-neutral world is 4%.

$$\frac{dZ(t,T)}{Z(t,T)} = r_t dt + \sigma_Z(t,T)dX_t$$

$$\sigma_Z(t,T) = -B(t;T)\sigma$$

The diffusion of $\frac{dZ}{Z}$ in the risk-neutral world
$= -B(0;10)\sigma$
$= -\frac{1-e^{-0.1 \times 10}}{0.1}$ x 0.1
=-0.063212

(d) Compute the drift and the diffusion of $\frac{dZ}{Z}$ for the real-world process.

Commentary on Question:

Few candidates preformed perfectly in this part by using the correct equation to move from risk-neutral process to real-world process. Candidates earned partial credits if they can state the drift term didn't change from part (c).

From part (d)

$\frac{dZ}{Z} = 0.04dt - 0.063212Z\ dX$

When we move to the real world, the return increases by the product of the market price of dZ risk and -0.063212.

The bond price process becomes:

$$\frac{dZ}{Z} = [0.04 + (-0.4\,x - 0.063212)]dt - 0.063212\,dX$$

$$\frac{dZ}{Z} = 0.065285\,dt - 0.063212\,dX$$

The drift increases from 4% to 6.5285% as we move from the risk-neutral world to the real world

The diffusion in the real world

$= -B(0;10)\sigma$

$= -\frac{1-e^{-0.1\,x\,10}}{0.1}$ x 0.1

=-0.063212

QFI QF Spring 2024 Question 6

Learning Outcomes:

h) Understand and apply numerical discretization methods to price options including Euler-Maruyama discretization and transition density methods

i) Calibrate a model to observed prices of traded securities including fitting to a given yield curve

Source References:

- Calibrating Interest Rate Models (Section 1.1-4.3 excl 4.1.2)

Commentary on Question:

Overall, candidates either left the question blank or attempted only part of it. For those that attempted this question, candidates performed better where there were formulas involved. However, when it came to the qualitative aspects of the question, there were fewer candidates that were able to provide appropriate explanations.

Solution:

(a) Describe the assumptions made in the chosen real-world parameter estimation method.

Commentary on Question:

Full marks were given if each of the following assumptions were provided. Many candidates recognized the need for using the maximum likelihood function and regression. Many candidates failed to identify that a major assumption is that the rates follow a normal distribution.

Assumptions:

- Daily yields of 3 months anualized rates follow normal distribution.
  - $r_{t+s}|r_t$ is normally distributed with mean $\bar{r} + (r_t - \bar{r})exp\,(-\gamma s)$ and variance $\frac{\sigma^2}{2\gamma}(1 - exp(-2\gamma s))$
  - The conditional pdf of $r_{i\Delta}|r_{(i-)\Delta}, i = 1,2, \ldots$ is normal.
  - We can write the likelihood function of the sample.
  - Minimizing likelihood function is equivalent to regressing $$y = (r_\Delta, r_{2\Delta}, \ldots, r_{n\Delta})^T$$ on $x = \left(r_0, r_\Delta, \ldots, r_{(n-1)\Delta}\right)^T$.
  - Above is true if the contribution of r_0 to the likelihood function is small or in another word sample is very large.

(b) Estimate the parameters of your model.

Commentary on Question:

In order to receive full marks, the candidate needed to identify the formulas below and accurately use them to obtain the correct results. Full credit was given for candidates that used Euler's method to approximate gamma.

Estimated regression parameters and Vasicek model parameters are related as follows

$$\gamma = -\frac{ln(\hat{\beta}^*)}{\Delta}$$
$$\bar{r} = \frac{\hat{\alpha}^*}{1 - \hat{\beta}^*}$$
$$\sigma = \sqrt{\frac{2\gamma\hat{\sigma}^{*2}}{1 - \hat{\beta}^{*2}}}$$

From the given R output

$$\hat{\alpha}^* = 0.0001815, \hat{\beta}^* = 0.9964045, \hat{\sigma}^* = 0.001300116$$

Plugging these values in the formulas we obtain

$$\gamma = 0.90769249, \bar{r} = 0.05049257, \sigma = 0.02067589$$

(c) Describe the procedure employed in risk-neutral model calibration.

Commentary on Question:

Many candidates identified at least one aspect of the procedure below. Partial credit was given in most cases if the candidate could identify the need for non-linear regression, least-squares regression, and minimizing the difference between modeled rates vs. market rates. Few candidates identified the initial formula below or the sensitivity of the initial guess.

In the method Vasicek yield rates are calculated using the formula

$$r(t) = \frac{\left(r(0)B(0;t) - A(0;t)\right)}{t}$$

In the A(0;t) and B(0;t) are as given in the formula.
Then use the non-linear least square regression method to minimize the distance between obsereved values of r(t) with the expected values of r(t) with respect to parameters.

In studies this method performs better than fitting observed bond prices to its theoretical prices under Vasicek model.

Also, the non-linear leastsquare estimation method is quite sensitive to initial guesses, many different initial guesses

(d) Estimate the parameters of your new model.

Commentary on Question:

Full marks were given if the formula below was identified and used appropriately. Many candidates used the formula directly without commentary for why the formula was appropriate.

The output does not contain estimated model parameters but it contains standard error and t-value; multiplying these items together we obtain estimates.

$$\gamma^* = 0.031458 * 15.49 = 0.487311$$
$$\bar{r}^* = 0.001385 * 50.90 = 0.070482$$

(e) Determine whether the fitted models are adequate.

Commentary on Question:

Full credit was given if the candidate could appropriately identify that both the real-world and risk-neutral world models are appropriate. Many candidates commented on the p-values, however not very many candidates commented on R-squared.

From the diagnostic statistics for realworld estimate we see that p-value for the test H_0: $\beta^* = 0$ is almost zero so that test is rejected with certainty.

Also R-squared is close to 1 so model describes the data almost perfectly.

For the risk neutral parameter estimation both p-values are close to zero so model is perfect.

QFI QF Fall 2024 Question 1

Learning Outcomes:

d) Understand Stochastic Calculus theory and technique used in pricing derivatives

Source References:

- *Problems and Solutions in Mathematical Finance: Stochastic Calculus*, Chin, Eric, Nel, Dian and Olafsson, Sverrir, 2014, pages 52, 132-137

Commentary on Question:

This question tests candidates' knowledge of Ito's isometry, martingales, and Jensen's inequality. Most candidates were able to answer part of the question. However, few candidates scored high.

Solution:

(a) Calculate $E[X_t^2]$ for $t < T$.

Commentary on Question:

Most candidates did well on this part.

By Ito's isometry, we have

$$E[X_t^2] = E\left[\left(\int_0^t 1_{\{B_u>0\}} dB_u\right)^2\right] = E\left[\int_0^t 1_{\{B_u>0\}}^2 \, du\right] = E\left[\int_0^t 1_{\{B_u>0\}} \, du\right] = \int_0^t E\left[1_{\{B_u>0\}}\right] du$$

Note that

$$E\left[1_{\{B_u>0\}}\right] = P(B_u > 0) = \frac{1}{2}$$

Combining the above results, we get

$$E[X_t^2] = \int_0^t \frac{1}{2} du = \frac{1}{2} t$$

(b) Calculate $E[X_t Y_t]$ for $t < T$.

Commentary on Question:

Most candidates were able to apply the correlation property of Ito integral and get the correct answer.

Note that $1_{\{B_u>0\}}1_{\{B_u<0\}} = 0$ for $u \in (0,T)$.

By the correlation property of Ito integral, we have

$$E[X_tY_t] = E\left[\int_0^t 1_{\{B_u>0\}}dB_u \int_0^t 1_{\{B_u<0\}}dB_u\right] = E\left[\int_0^t 1_{\{B_u>0\}}1_{\{B_u<0\}}du\right] = E\left[\int_0^t 0du\right] = 0$$

(c)

(i) List the three properties of a martingale.

(ii) Determine whether $\{X_tY_t: 0 \le t \le T\}$ is a martingale with respect to the filtration $\{I_t: 0 \le t \le T\}$ by verifying whether all the three properties listed in part (c)(i) hold.

Commentary on Question:

Most candidates were able to list the conditions of martingales. However, few candidates were able to prove the second and the third properties of martingales.

(i)

The three properties of a martingale $\{S_t: 0 \le t \le T\}$ are:

1. S_t is adapted to a filtration $\{I_t: 0 \le t \le T\}$
2. Unconditional forecast is finite, i.e., $E[|S_t|] < \infty$
3. $E[S_u|I_t] = S_t$ for $t < u$

(ii)

We show that $\{X_tY_t: 0 \le t \le T\}$ is a martingale.

By the definition of X_t and Y_t, we know that the process X_tY_t is adapted to the filtration $\{I_t: 0 \le t \le T\}$.

Second, we show that $E[|X_tY_t|]$ is finite. This can be done as follows.

$$E[|X_tY_t|] \le E\left[\frac{X_t^2+Y_t^2}{2}\right] = \frac{1}{2}t < \infty$$

Finally, we show that for $0 \le s < t \le T$, $E[X_tY_t|I_s] = X_sY_s$.

Note that

$$X_tY_t = X_sY_s + X_s(Y_t - Y_s) + (X_t - X_s)Y_s + (X_t - X_s)(Y_t - Y_s)$$

Hence

$$\begin{aligned} E[X_tY_t|I_s] &= E[X_sY_s + X_s(Y_t - Y_s) + (X_t - X_s)Y_s + (X_t - X_s)(Y_t - Y_s)|I_s] \\ &= E[X_sY_s|I_s] + E[X_s(Y_t - Y_s)|I_s] + E[(X_t - X_s)Y_s|I_s] + E[(X_t - X_s)(Y_t - Y_s)|I_s] \end{aligned}$$

Since X_sY_s, X_s, and Y_s are known at time s, we have $E[X_sY_s|I_s] = X_sY_s$. In addition, X_t and Y_t are martingales as they are Ito integrals. We have

$$E[X_s(Y_t - Y_s)|I_s] = X_sE[Y_t - Y_s|I_s] = X_sE[Y_t - Y_s] = 0$$

$$E[(X_t - X_s)Y_s|I_s] = Y_sE[X_t - X_s|I_s] = 0$$

By the correlation property of Ito integrals, we have

$$\begin{aligned} E[(X_t - X_s)(Y_t - Y_s)|I_s] = E[(X_t - X_s)(Y_t - Y_s)] &= E\left[\int_s^t 1_{\{B_u>0\}}dB_u \int_s^t 1_{\{B_u<0\}}dB_u\right] \\ &= E\left[\int_s^t 1_{\{B_u>0\}}1_{\{B_u<0\}}du\right] = E\left[\int_s^t 0du\right] = 0 \end{aligned}$$

Combining the above results, we just proved that $E[X_tY_t|I_s] = X_sY_s$.

QFI QF Fall 2024 Question 2

Learning Outcomes:

b) Understand Arrow-Debreau security and the distinction between complete and incomplete markets

Source References:

- *Options, Futures, and Other Derivatives*, Hull, John C., 11th Edition, 2021, Chapter 13, pages 294-298

- INV201-100-25: Chapter 5 of Financial Mathematics – A Comprehensive Treatment, 2nd Edition, Campolieti

Commentary on Question:

Candidates did relatively well on this problem. Part (a)(ii) was the one which was missed by the majority of candidates.

Solution:

(a)

(i) Determine the range of r for which this model is arbitrage-free.

(ii) Assess whether this model is complete for the range of r in part (a)(i).

(i)

The model is arbitrage free if the following equations are satisfied simultaneously:

At S_0=10: 10 x (1 + r) = 12 x q_1 + 8 x (1 - q_1)

At S_1 = 12: 12 x (1+ r) = 15 x q_2 + 10 x (1 – q_2)

At S_1 =8: 8 x (1 + r) = 9 x q_3 +5 x (1 – q_3)

Solving them for q_1, q_2 and q_3 we get:

$$q_1 = \frac{1 + r - \frac{8}{10}}{\frac{12}{10} - \frac{8}{10}} = \frac{2 + 10r}{4}$$

$$q_2 = \frac{1 + r - 10/12}{15/12 - 10/12} = \frac{2 + 12r}{5}$$

$$q_3 = \frac{1 + r - 5/8}{9/8 - 5/8} = \frac{3 + 8r}{4}$$

Since each q_i must be in the (0,1) interval, replacing q_i with 0 and 1 in the above yields:

$-1/5 < r < 1/5$

$-1/6 < r < 1/4$

$-3/8 < r < 1/8$

The intersection of the 3 intervals (-1/6, 1/8) gives the values of r for which the model is arbitrage free.

Some candidates did not substitute 0 and 1 for the q_s, others did not intersect the 3 intervals for r, these candidates receive partial credits

(ii) The model is complete when r $\in \left(-\frac{1}{6}, \frac{1}{8}\right)$, since each r in this interval produces equivalent risk-neutral measure

This part was missed by the majority of candidates.

(b) Calculate the fair price of this option when $r = 1/9$ using the risk-neutral measure.

Fair Price of this option

$=\frac{1}{(1+r)^2} E^Q[(\max(S_1, S_2) - K)^+]$

$=\left(\frac{9}{10}\right)^2 ((\max(12, 15) - 11)^+ q_1 q_2 + (\max(12, 10) - 11)^+ q_1(1 - q_2))$

$+\left(\frac{9}{10}\right)^2 ((\max(8, 9) - 11)^+ (1 - q_1) q_3 + (\max(8, 5) - 11)^+ (1 - q_1)(1 - q_3))$

$= \left(\frac{9}{10}\right)^2 \left(4 q_1 q_2 + 1 q_1 (1 - q_2)\right)$

In the case that r = 1/9, we have from part (b) that

$$q_1 = \frac{2}{4} + \frac{10}{4}\left(\frac{1}{9}\right) = \frac{7}{9} \qquad q_2 = \frac{2}{5} + \frac{12}{5}\left(\frac{1}{9}\right) = \frac{2}{3}$$

Therefore, fair price of this option =

$$\left(\frac{9}{10}\right)^2 \left(4\left(\frac{7}{9}\right)\left(\frac{2}{3}\right) + 1\left(\frac{7}{9}\right)\left(\frac{1}{3}\right)\right) = \frac{189}{100}$$

Almost all candidates worked on this part. Common mistake here was using continuous compounding rather than discrete one. Some candidates did not calculate the option payoff correctly. In both cases, partial credit was given if the rest of the calculations were correct.

QFI QF Fall 2024 Question 7

Learning Outcomes:

h) Understand and apply numerical discretization methods to price options including Euler-Maruyama discretization and transition density methods

i) Calibrate a model to observed prices of traded securities including fitting to a given yield curve

Source References:

- Calibrating Interest Rate Models (Section 1.1-4.3 excl 4.1.2)

Commentary on Question:

This question tests candidates' knowledge on the model calibration techniques.

Solution:

(a) You have one-month daily treasury bill yields (annualized) over 500 consecutive trading days in the daily_data table. There are 252 trading days per year. You would like to fit the CIR model,

$$dr = \gamma(\bar{r} - r)dt + \sqrt{\alpha r}\, dX$$

for the data set.

For this model you are considering the method based on Euler discretization and the method based on the transition density function.

Compare and contrast these two methods.

Commentary on Question:

Candidates performed below expectation on this part. Partial credits were awarded to candidates who have identified each component in the model solution.

Euler discretiztion involves discretizing the CIR SDE

$$dr_t = \gamma(\bar{r} - r_t)\, dt + \sqrt{\alpha\, r_t}\, dX_t$$

as

$$r_{t+\Delta} - r_t = \gamma(\bar{r} - r_t)\Delta + \sqrt{r\_t}\epsilon_{t+\Delta}$$

$$\epsilon_{t+\Delta} \sim N(0, \sqrt{\alpha\Delta})$$

By writing $r(i) = r_{i\Delta}, i = 0,1, \dots, n$ and

$$\alpha_1 = \gamma\bar{r}\Delta$$

$$\beta_1 = 1 - \gamma\Delta$$

$$\sigma = \sqrt{\alpha\Delta}$$

We can write

$$r(i) = \alpha_1 + \beta_1 r(i-1) + \sqrt{r(i-1)}\epsilon_i$$

$$\frac{r(i)}{\sqrt{r(i-1)}} = \frac{\alpha_1}{\sqrt{r(i-1)}} + \beta_1\sqrt{r(i-1)} + \epsilon_i, \qquad i = 1,2, \dots n$$

Therefore by writing

$$y_i = \frac{r(i)}{\sqrt{r(i-1)}}, \qquad x_{1i} = \frac{1}{\sqrt{r(i-1)}}, \qquad x_{2i} = \sqrt{r(i-1)}$$

The model becomes a multiple linear regression model with no intercept

$$y_i = \alpha_1 x_{1i} + \beta_1 x_{2i} + \epsilon_i$$

Maximum Likelihood Method based on Transition density

This method relies on the fact that probability density function of $r_{t+s}|r_t$ is a constant multiplier of non-central chisquared distribution. Normally we ignore the contribution from the pdf of r_0 to the likelihood function as the sample is large.

MLE is exact and should be more accurate than Euler method.

However, maximizing log likelihood function requires numerical optimization method. These are very sensitive to initial guess. The calculation of non-central chi-square in R appears to be not very stable.

(b) Calculate the estimates of $\gamma, \bar{r}$ and α based on Euler discretization.

Commentary on Question:

Candidates performed as expected on this part.

From R output

$$\alpha_1 = 0.0003346$$

$$\beta_1 = 0.9968652$$

$$\sigma = 0.01455$$

Using $\gamma = \frac{1-\beta_1}{\Delta}$

$$\gamma = 0.7899795$$

Using $\bar{r} = \frac{\alpha_1}{1-\beta_1}$

$$\bar{r} = 0.1067507$$

Using $\alpha = \frac{\sigma^2}{\Delta}$

$$\alpha = 0.0533412$$

(c) Write estimates of $\gamma, \bar{r}$ and α based on the transition density method.

Commentary on Question:

Candidates performed as expected on this part.

From the output

$$\gamma = 7.86976,$$

$$\bar{r} = 0.053306,$$

$$\alpha = 0.26746$$

(d) Recommend an estimate method between Euler discretization method and the transition density method.

Commentary on Question:

Candidates performed below expectation on this part. Partial credit was given for each component answered correctly.

From the second output even though MLE method converges, there are some warnings; warnings could be problamtics.

Two estimates are vastly different.

The diagnostics statistics for the Euler method indicates it's a good fit. However no diagnostics statistics are provided for the MLE method other than the warnings.

Based on all these considerations, the Euler estimate is recommended.

Learning Objective 3: The candidate will understand various applications and risks of derivatives

QFI QF Fall 2020 Question 6

Learning Outcomes:

a) Understand the Greeks of derivatives

b) Understand static and dynamic hedging

c) Understand delta hedging, and the interplay between hedging assumptions and hedging outcomes

e) Understand how hedge strategies may fail

Source References:

- *Options, Futures, and Other Derivatives*, Hull, John C., Pearson, 2021, Chapters 19, 26, pages 421-422, 632-634
- *The Volatility Smile*, Derman, Emanuel and Miller, Michael, 2016, Chapters 3, 5
- INV201-104-25: Which Free Lunch Would You Like Today, Sir?

Solution:

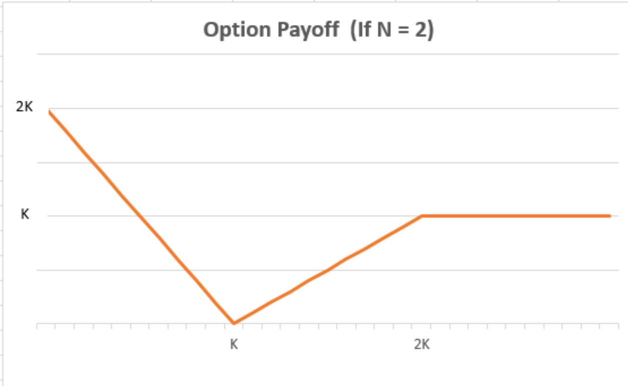
(a)

(i) Sketch the payoff graph for the portfolio *B*.

(ii) Construct a static hedging strategy for option *A*, with plain vanilla options and the underlying asset *S*.

Commentary on Question:

Overall, candidates did well on this part of the question. Most candidates were able to graph the option payoffs successfully for portfolio B and construct the hedging strategy for A.



(i)
(ii)

B can be replicated by Long 2 Puts at K, Long 1 Call at K, and Short One Call at 2K. Since B is A – S, we also need to Long 1 Share of S.

(b) Construct a dynamic delta-hedging strategy for this exotic option *A*.

Commentary on Question:

Most candidates knew how to take the derivative of the result in part (a) but were unable to proceed from there to correct the hedging strategy.

From (i), A= C(K) – C(2K) + 2P(K) + S

$$\frac{\partial A}{\partial S} = \frac{\partial C(K)}{\partial S} - \frac{\partial C(2K)}{\partial S} + \frac{\partial P(K)}{\partial S} + \frac{\partial S}{\partial S}$$

$$\frac{\partial C(K)}{\partial S} - \frac{\partial C(2K)}{\partial S} + \frac{2 * \partial[C(K) - S]}{\partial S} + \frac{\partial S}{\partial S}, (Put - Call\ parity)$$

$$(2+1)\frac{\partial C(K)}{\partial S} - \frac{\partial C(2K)}{\partial S} - 2 + 1$$

$$(3)\frac{\partial C(K)}{\partial S} - \frac{\partial C(2K)}{\partial S} - (1)$$

Delta hedging this option by

- $shorting\,(3)\frac{\partial C(K)}{\partial S} - \frac{\partial C(2K)}{\partial S} - (1)\ unit\ of\ underlying\ asset\ S$
- $cash\ balance\left[(3)\frac{\partial C(K)}{\partial S} - \frac{\partial C(2K)}{\partial S} - (1)\right]S - B$

(c) List pros and cons of static hedging strategies and dynamic hedging strategies.

Commentary on Question:

Most candidates did well on this question. Some candidates did not comment on the availability of assets or list both pros and cons.

Static Hedging Strategy:

Pro:

1. No need to be rebalanced.

2. Do no rely on theoretical models: No assumption for the future behavior of underlying assets, is required.

Con:

1. The options required for hedging strategy might not be available in market.

Dynamic Hedging Strategy:

Pro:

1. More practical in reality, as the strategy can be built with securities available in market.

Con:

1. Require constant rebalancing.

2. Hedging error if the assumptions made for the future behavior of underlying assets deviate from reality.

(d) Show that $rV = \frac{\partial V}{\partial t} + \frac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S}$ using the law of one price and Ito's Lemma, where *r* denotes the constant risk-free rate.

Commentary on Question:

Many candidates did not recognize that the rebalancing factor was also a function of stock price and treated it as a constant in solving this question. Candidates who received full points recognized this fact.

Solve dΣ through Ito lemma

$$d\Sigma = d\left[\alpha\left[V - \frac{\partial v}{\partial s}S\right]\right] = Vd\alpha + \alpha dV - \left[\alpha Sd\left(\frac{\partial v}{\partial s}\right) + \frac{\partial v}{\partial s}Sd\alpha + \frac{\partial v}{\partial s}\alpha dS\right]$$

$$\alpha\left[dV - \frac{\partial V}{\partial S}dS\right] + Vd\alpha - \frac{\partial V}{\partial S}Sd\alpha - \alpha Sd\left(\frac{\partial V}{\partial S}\right)$$

$$\alpha\left[dV - \frac{\partial v}{\partial s}dS\right]\left[V - \frac{\partial v}{\partial s}S\right]d\alpha = \alpha Sd\left[\frac{\partial v}{\partial s}\right])$$

$$\alpha\left[\frac{\partial V}{\partial S}dS+\frac{1}{2}\sigma_s^2\frac{\partial^2 V}{\partial S^2}dt+\frac{\partial V}{\partial t}dt-\frac{\partial V}{\partial S}dS\right]$$

$$\alpha\left[\frac{1}{2}\sigma_s^2\frac{\partial^2 V}{\partial S^2}dt+\frac{\partial V}{\partial t}dt\right]$$

The portfolio Σ has no risk, as the random facor has been fully hedged. Based on the rule of one price, the return of the portfolio Σ should be equal to risk free rate of r.

$d\Sigma=\alpha\left[\frac{1}{2}\sigma_s^2\frac{\partial^2 V}{\partial S^2}+\frac{\partial V}{\partial t}\right]dt=r\Sigma dt=r\alpha\left[V-\frac{\partial V}{\partial S}S\right]dt$

$$\rightarrow\left[\frac{1}{2}\sigma_s^2\frac{\partial^2 V}{\partial S^2}+\frac{\partial V}{\partial t}\right]=r\left[V-\frac{\partial V}{\partial S}S\right]$$

$\rightarrow rV=\frac{\partial V}{\partial t}+\frac{1}{2}{\sigma_s}^2S^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}$

(e)

(i) Show that the profit and loss function P&L of the hedged portfolio satisfies the following when the hedge is constructed with realized volatility $\sigma_{s,R}$:

$$d(P\&L)=\frac{1}{2}S^2\Gamma_I\left[{\sigma_{s,R}}^2-{\sigma_{s,I}}^2\right]dt+(\Delta_I-\Delta_R)\left[(\mu_s-r)Sdt+\sigma_{s,R}SdZ\right]$$

(ii) Determine $d(P\&L)$, if V is hedged with implied volatility $\sigma_{s,I}$ instead.

(iii) Describe key P&L characteristics, when hedging with realized volatility vs. implied volatility, using the results in parts (i) and (ii) to support your answer.

Commentary on Question:

Many candidates were unsure how to complete the first proof or use the result in part (i) to determine the value in part (ii). Most candidates successfully listed several drivers of the P&L using realized and implied volatility.

(i) From (d) $rV=\frac{\partial V}{\partial t}+\frac{1}{2}{\sigma_s}^2S^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}$

$\rightarrow rV=\frac{\partial V}{\partial t}+\frac{1}{2}{\sigma_{s,I}}^2S^2\Gamma_I+rS\Delta_I$

$\rightarrow \frac{\partial V}{\partial t}=rV-\frac{1}{2}{\sigma_{s,I}}^2S^2\Gamma_I-rS\Delta_I$ --- *(1)*

Hedged portfolio: V *(long option)* $-\Delta_R S$ *(short* Δ_R *unit of* S*)* $+[\Delta_R S-V]$ *(cash balance)*

$dP \wedge L = d[V - \Delta_R S] - [V - \Delta_R S]rdt$ ---- *(2)*

$$dV = \Delta_I\left(dS\middle|\sigma_{s,R}\right) + \frac{\partial V}{\partial t}dt + \frac{1}{2}{\sigma_{s,R}}^2 S^2 \Gamma_I dt$$

$\Delta_I\left[\mu_s S dt + \sigma_{s,R} S dz\right] + \frac{\partial V}{\partial t}dt + \frac{1}{2}{\sigma_{s,R}}^2 S^2 \Gamma_I dt$ ---- *(3)*

$d\Delta_R S = \Delta_R\left(dS\middle|\sigma_{s,R}\right) = \Delta_R\left[\left[\mu_s S dt + \sigma_{s,R} S dz\right]\right]$ ---- *(4)*

(1), (3), (4) → (2)

$dP \wedge L = \Delta_I\left[\mu_s S dt + \sigma_{s,R} S dz\right] + \left[rV - \frac{1}{2}{\sigma_{s,I}}^2 S^2 \Gamma_I - rS\Delta_I\right]dt$

$\frac{+1}{2}{\sigma_{s,R}}^2 S^2 \Gamma_I dt - \Delta_R\left[\left[\mu_s S dt + \sigma_{s,R} S dz\right]\right]$

$-[V - \Delta_R S]rdt$

This can be simplified and rearranged to

$$d(P\,\&\,L) = \frac{1}{2}S^2\Gamma_I\left[{\sigma_{s,R}}^2 - {\sigma_{s,I}}^2\right]dt + (\Delta_I - \Delta_R)\left[(\mu_s - r)Sdt + \sigma_{s,R}SdZ\right]$$

(ii) From i), if hedged with implied volatility, $\Delta_I = \Delta_R$ *so the last term cancels leaving*

$$\frac{1}{2}\left({\sigma_{s,R}}^2 - {\sigma_{s,I}}^2\right)S^2\Gamma_I dt$$

(iii)

Realized volatility:

- *Know exactly what profit to get at expiration.*

Hedged portfolio

= V *(long option)* $-\Delta_R S$ *(short* Δ_R *unit of S) +* $[\Delta_R S - V]$ *(cash balance)*

= V_I *(priced with* $\sigma_{s,I}$*) -* V_R *(replicated with* $\sigma_{s,R}$*) + Cash Balance*

- *The P&L could fluctuate during the life of option.*

$dP\&L$ *contains a stochastic term* $(\Delta_I - \Delta_R)\ \sigma_{s,R} S dz$

Implied volatility:

- *No fluctuation in P&L during the life of option.*
  $dP\&L = \frac{1}{2}\left({\sigma_{s,R}}^2 - {\sigma_{s,I}}^2\right)S^2\Gamma_I dt$*, which has no stochastic term.*

- *Make profit as long as on the right side of trade. (i.e. long option if* ${\sigma_{s,R}}^2 > {\sigma_{s,I}}^2$*)*

$dP\&L = \frac{1}{2}\left({\sigma_{s,R}}^2 - {\sigma_{s,I}}^2\right)S^2\Gamma_I dt > 0$*, given the gamma of option (*$\Gamma_I > 0$*)*

- *Unable to predict how much money you will make.*

QFI QF Fall 2020 Question 10

Learning Outcomes:

f) Identify and evaluate embedded options in liabilities (e.g., indexed annuity, structured product based variable annuity, and variable annuity guarantee riders including GMxB, etc.)

g) Demonstrate an understanding of hedging for embedded option in liabilities

Source References:

- INV201-108-25: Mitigating Interest Rate Risk in Variable Annuities: An Analysis of Hedging Effectiveness under Model Risk

Commentary on Question:

This question is to test candidates' understanding on the features of guaranteed riders of a VA contract and hedging strategies against equity, interest rate and volatility risk.

Solution:

(a) Calculate the position in each of the three assets at time 0.

Commentary on Question:

A few candidates answered this question and very few of them successfully identify the position. Most candidates left this part as blank.

With $L_0 = 0$, $\prod_0$ (for a perfect hedge) should be 0:

0 = Δ_t * 200 + n_t * 100 - 500M(1)

The self-financing hedge portfolio should satisfy the following:

$$\Pi_{t+h} = \left(\Pi_t - \Delta_t S_t - n_t P_{t,t+T^B}\right) B_{t+h}/B_t + \Delta_t S_{t+h} + n_t P_{t+h,t+T^B}.$$

4.5M = (0 - Δ_t * 200 – n_t * 100) * 1.005 + Δ_t * 203 + n_t * 101

And

507M = Δ_t * 203 + n_t * 101 (2)

By solving the two equations, we can get Δ_t = 2M and n_t= 1M

So,

- A position of 2M in stock
- A position in 1M in zero-coupon bond
- A position in bank account with total borrowing of $500M such that the hedging portfolio is self-financing

(b) Define the objective of the hedging strategy in terms of the insurer's hedged loss at maturity.

Commentary on Question:

More than half of candidates performed well in this question. The remaining candidates failed to correctly describe that the objective of a hedging strategy is to offset the insurer's unhedged loss at maturity with the terminal value of the hedging portfolio.

The insurer's hedged loss at maturity is: $HL_t = L_t - \Pi_t$

The objective of the hedging portfolio is to offset L_t, and therefore to result in a hedged loss of approximately zero at maturity

(c) State one problem with using the forward-looking approach to calibrate the stock volatility.

Commentary on Question:

Only a few candidates successfully identified relevant problems that describe the difficulties of using the forward-looking approach to calibrate the stock volatility. Candidates should compare the differences in features between VA contracts and traded derivatives in market (outlined in the solution below).

The problems of the approach include:

- VAs have long-term maturities, while forward-looking measures are extracted from shorter-term traded options (which may involve unsound extrapolation)
- Two models that are well calibrated to the implied volatility vanilla option surface may lead to very different prices and hedge ratios for exotic option
- Therefore, there is no guarantee that implied volatilities from traded vanilla options will consist in appropriate volatility inputs when hedging VAs with non-vanilla features, such as GMWBs

(d)

(i) Identify the sources of model risk in your hedging strategy under each of Models A, B, and C.

(ii) Identify the corresponding market model by matching Model X, Y, and Z to Model A, B, or C. Justify your answer.

Commentary on Question:

For candidates who answered this question, candidates performed well in part (i). They successfully identified the features of those three models. They also performed fairly in part (ii) but some of them didn't provide any justification on the matches, and some of them didn't correctly understand the relationship between the impact of model risk on the effectiveness of hedging strategies and the level of resulting CTE.

(i)

Model A: difference in interest rate model (CIR vs Vasicek)

Model B: changes in the slope and curvature of term structure are not accounted for in the hedging strategy, but are reflected in the model

Model C: stochastic volatility and change in the slope and curvature of term structure are not accounted for in the hedging strategy, but are reflected in the model

(ii)

Model X: CTE 95% of 1.8 is for Model B.

Model Y: CTE 95% of 0.5 is for Model A.

Model Z: CTE 95% of 4 is for Model C.

Justification:

Since the insurer always uses the BSV model to establish its hedging, the three data-generating models give rise to varying degrees of model risk. Market model with higher level of deviation from the BSV model will get the less effective hedging results.

(e) Explain whether you agree with the student's result.

Commentary on Question:

Only a few candidates successfully identified the student is wrong. For some of those who disagreed with the students, they failed to provide appropriate justifications. Successful candidates noticed the hedge strategy didn't hedge market volatility and considered its impact on hedged loss.

Do not agree with the result.

Because this hedge does not protect against Vega risk, the hedged loss should not be mostly centered around zero for varying degrees of stock market volatility. The hedged loss should be centered at zero around the unconditional volatility with a trend line for varying degrees of stock market volatility.

(f)

(i) Explain how a delta-only hedging strategy would affect the insurer's hedged loss if your expectation becomes a reality.

(ii) Explain how a wrong expectation would affect the insurer's hedged loss after modifying the hedge strategy.

Commentary on Question:

About a half of candidates performed well in this question. The remaining candidates failed to understand the impact of interest rate on the VA guaranteed riders

(i) If the interest rates rise steadily throughout the term of the VA contracts, the value of the guarantees offered by the insurer will decrease, resulting in a net gain for the insurer if rho risk is not hedged (i.e., delta-only strategy).

(ii) If interest rates turn out to be low and stable, a delta-rho hedge strategy can reduce the insurer's exposure to large hedging losses, as compared to delta-only hedge.

QFI QF Spring 2021 Question 12

Learning Outcomes:

a) Understand the Greeks of derivatives

Source References:

- *Options, Futures, and Other Derivatives*, Hull, John C., 11th Edition, 2021, Chapter 19

Commentary on Question:

This question tests candidates' understanding of option Greeks.

Solution:

(a) Determine which Greek (Delta, Gamma, Vega, Rho, or Theta) Exhibit I represents. Justify your answer. (Here Theta is defined as the derivative of the option value with respect to the passage of time.)

Commentary on Question:

Candidates performed as expected.

Exhibit I shows Rho because:

i. Delta is bounded by 1;

ii. Gamma and Vega exhibit bell-shape around at-the-money stock price of $100;

iii. Theta is negative;

Since none of the above pattern fits Exhibit I, it is Rho.

(b) Draw "Line A" in Exhibit I to show the same Greek of a European put option that has the same parameters as the one in Exhibit I. Indicate the Greek value in "Line A" at stock price = 100. You need not show other values in "Line A" but comment on the slope of this line.

Commentary on Question:

Candidates performed below expectations on this part. Partial credit was given when a candidate's answer to part (b) is consistent with the answer to part (a), even though the answer to part (a) is incorrect.

Line A (blue line)

$Call\ Rho = K(T-t)e^{-r(T-t)}N(d_2)$

$$Put\ Rho = K(T-t)e^{-r(T-t)}N(-d_2) = K(T-t)e^{-r(T-t)} - K(Tle\ -t)e^{-r(T-t)}N(d_2)$$

$Put\ Rho = K(T-t)e^{-r(T-t)} - Call\ Rho$

Since $K(T-t)e^{-r(T-t)}$ is a constant, the shape of the Put Rho is same as the Call Rho, but with an oppisite (negative) slope.

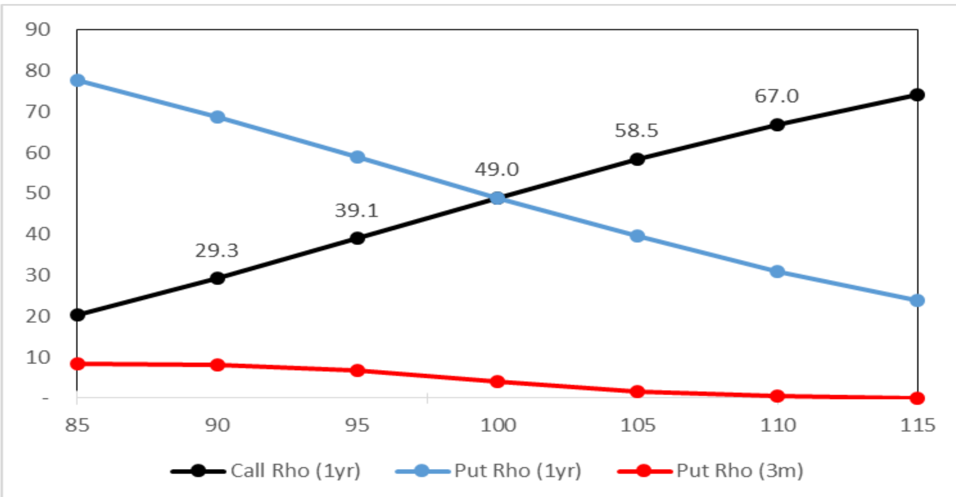
At stock price = 100 = strike price

$$d_2 = \frac{ln\frac{S}{K} + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} = \frac{(2\% - \frac{20\%^2}{2})}{20\%} = 0$$

$N(d_2) = N(-d_2) = 0.5$

$Put\ Rho = Call\ Rho = 49.0$

Line A intersects with "Call Rho" at stock price = 100



(c) Draw "Line B" in Exhibit I to show the same Greek of a European put option that has the same parameters as in Exhibit I, except that the time-to-maturity is 1 month. Indicate the Greek value in "Line B" at stock price = 85. You need not show other values in "Line B" but comment on the slope of this line.

Commentary on Question:

Candidates performed below expectations on this part. Partial credit was given when a candidate's answer to part (b) is consistent with the answer to part (a), even though the answer to part (a) is incorrect.

Line B (red line)

For 1 month maturity, at stock price =85:

$$d_2 = \frac{ln\frac{85}{100} + \left(2\% - \frac{20\%^2}{2}\right)(1/12)}{20\%\sqrt{1/12}} = -0.8126$$

$$N(-d_2) = 0.7918$$

1 month Put Rho = 100 $* \left(\frac{1}{12}\right) * e^{-2\%\frac{1}{12}} * 0.7918 = 8.30$

"Line B" starts at below the "Call Rho" line with a negative slope. For ease of reference, "Line B" is shown in part (b).

(d) Exhibit II below shows Vega and Gamma for a European option on a non-dividend-paying stock. These Greek values are derived from the BSM model with the same strike price, volatility, interest rate, and time-to-maturity as in Exhibit I.

Exhibit II: Vega and Gamma with respect to the underlying stock price

Stock price	60	X
Vega (shown as the change in the option value to 1 percentage point change of the volatility, e.g., from 25% to 26%)	0.2401	0.2548
Gamma	0.0267	0.0159

Determine the stock price X in Exhibit II.

Commentary on Question:

Candidates performed below expectations on this part.

Note: The Vega and Gamma values in Exhibit II are derived in the same manner as the Rho in Exhibit I, but they are not based on the option parameters in Exhibit I. Nevertheless, credit was given if X was solved correctly by using the option parameters in Exhibit I.

$$Vega = 0.01 * S\sqrt{T-t}N'(d_1)$$

$$Gamma = \frac{N'(d_1)}{S\sigma\sqrt{T-t}}$$

$$\frac{Vega}{Gamma} = 0.01 * S^2\sigma(T-t) \qquad (1)$$

$$\frac{0.2401}{0.0267} = 0.01 * 60^2\sigma(T-t) \qquad (2)$$

$$\frac{0.2548}{0.0159} = 0.01 * X^2\sigma(T-t) \qquad (3)$$

$$Use\ equations\ (2)\ and\ (3)\ to\ get\ X = 60 * \sqrt{\frac{0.2548}{0.0159} * \frac{0.0267}{0.2401}} = 80$$

QFI QF Spring 2021 Question 13

Learning Outcomes:

c) Understand delta hedging, and the interplay between hedging assumptions and hedging outcomes

d) Understand the concepts of realized versus implied volatility

e) Understand derivatives mishaps

Source References:

- *The Volatility Smile*, Derman, Emanuel and Miller, Michael, 2016, Chapters 3, 5, 6

Solution:

(a) Derive the replicating portfolio using options for the interest credited above the guaranteed rate, i.e. $Interest\ Credited_t - g$. Specify each option, including position, option type, term, and strike ratio K / S_{t-1}.

Commentary on Question:

Points are awarded for both deriving the formula and correct description of the replication portfolio. Detailed description of the portfolio is required, including, strike ratio, option term, option type.

$$Interest\ Credited_t - Guar$$

$$= max\left\{min\left[\left(\frac{S_t}{S_{t-1}} - 1\right) * Par, Cap\right], Guar\right\} - Guar,$$

$$= par * max\left\{min\left[\left(\left(\frac{S_t}{S_{t-1}} - 1\right) - \frac{Guar}{par}\right), \frac{Cap}{par} - \frac{Guar}{par}\right], 0\right\},$$

$$= par * max\left\{min\left[\begin{array}{c}\left(\left(\frac{S_t}{S_{t-1}} - 1\right) - \frac{Guar}{par}\right), \\ \left(\left(\frac{S_t}{S_{t-1}} - 1\right) - \frac{Guar}{par}\right) - \left(\left(\frac{S_t}{S_{t-1}} - 1\right) - \frac{Cap}{par}\right)\end{array}\right], 0\right\}$$

Denote $G = \left(\left(\frac{S_t}{S_{t-1}} - 1\right) - \frac{Guar}{par}\right)$, $C = \left(\left(\frac{S_t}{S_{t-1}} - 1\right) - \frac{Cap}{par}\right)$,

as $Cap > Guar, G > C$.

Thus, $Interest\ Credited_t - Guar$

$= par * max[min(G, G - C), 0]$

$= par * \{max[0, -min(G, G - C)] + min(G, G - C)\}$

$= par * \{max[0, max(-G, C - G)] - max(-G, C - G)\}$

$= par * \{max[G, max(0, C)] - max(0, C)\}$

$= par * \{max[0, max(G, C)] - max(0, C)\}$

$= par * \{max(0, G) - max(0, C)\}$, as $G > C$

$= par * \left\{max\left[\frac{S_t}{S_{t-1}} - \left(1 + \frac{Guar}{par}\right), 0\right] - max\left[\frac{S_t}{S_{t-1}} - \left(1 + \frac{Cap}{par}\right), 0\right]\right\}$.

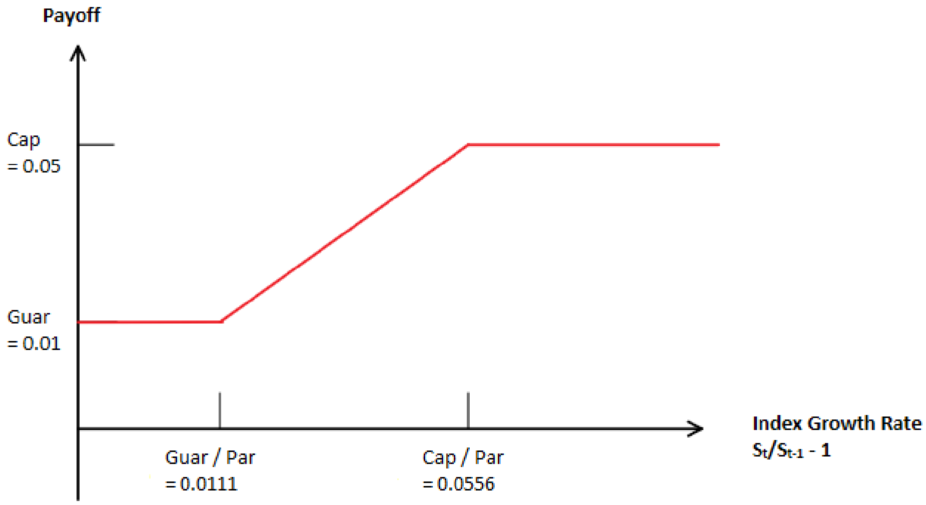
Therefore, the interest credited above the guaranteed rate can be replicated by p units of call spread, with the following options:

- Long position of a 1-year term European call option, with strike ratio $\frac{K_L}{S_{t-1}} = 1 + \frac{Guar}{par} = 1 + \frac{0.01}{0.9} = 1.0111$
- Short position of a 1-year term European call option, with strike ratio $\frac{K_S}{S_{t-1}} = 1 + \frac{Cap}{par} = 1 + \frac{0.05}{0.9} = 1.0556$

(b) Sketch the payoff of the replicating portfolio against the index growth rate $\left(\frac{S_t}{S_{t-1}} - 1\right)$.

Commentary on Question:

Correct shape of the curve as well as identification of both the turning points are required for full credit.



Turning point:

$$\left(\frac{Guar}{Par}, Guar\right) = (0.0111, 0.01)$$

and

$$\left(\frac{Cap}{Par}, Cap\right) = (0.0556, 0.05).$$

(c)

(i) Calculate the interest credited on Dec 31, 2019.

(ii) Calculate the cost of the replicating portfolio for the interest credited above the guaranteed rate on Dec 31, 2018.

Commentary on Question:

For part (i), interest percentage as well as dollar amount need to be specified.

For part (ii), each step needs to be shown clearly and demonstrate how each parameter is calculated.

(i)

$Interest\ Credited_t$

$$= max\left\{min\left[\left(\frac{S_t}{S_{t-1}} - 1\right) * Par, Cap\right], Guar\right\},$$

$= max\left\{min\left[\left(\frac{1080}{1000} - 1\right) * 90\%, 5\%\right], 1\%\right\},$

$= max\{min[(7.2\%, 5\%)], 1\%\}$

$= 5\%.$

Therefore, interest credited per \$1000 of investment $= 5\% * 1000 = \$50.$

(ii)

From (a), the interest crediting strategy can be replicated by the following call spread:

$$par * \left\{max\left[\frac{S_t}{S_{t-1}} - \left(1 + \frac{Guar}{par}\right), 0\right] - max\left[\frac{S_t}{S_{t-1}} - \left(1 + \frac{Cap}{par}\right), 0\right]\right\}.$$

The cost of the replicating portfolio is the option value of this call spread at $t - 1$.

Using Black-Scholes model to calculate the option value,

$$C(S_t, t) = S_t N(d_1) - Ke^{-r(T-t)} N(d_2),$$

where

$$d_1 = \frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)\right],$$

$$d_2 = d_1 - \sigma\sqrt{T-t}.$$

Option value of the long position of a 1-year term European call option with strike ratio $K_L/S_{t-1} = 1.0111$ is:

$$C_L(S_{t-1}, t-1) = S_{t-1}N(d_1) - K_L e^{-r} N(d_2),$$

where

$$K_L = S_{t-1}\left(1 + \frac{Guar}{par}\right) = 1000 \ * 1.0111 = 1011.11$$

$$d_1 = \frac{1}{\sigma}\left[\ln\left(\frac{1}{1+\frac{Guar}{par}}\right) + \left(r + \frac{\sigma^2}{2}\right)\right] = \frac{1}{0.2}\left[\ln\left(\frac{1}{1+\frac{0.01}{0.9}}\right) + \left(0.05 + \frac{0.2^2}{2}\right)\right] = 0.2948$$

$N(d_1) = 0.6141.$

$d_2 = d_1 - \sigma = 0.2948 - 0.2 = 0.0948, N(d_2) = 0.5359.$

$C_L(S_{t-1}, t-1) = 1000 * 0.6141 - 1011.11 * e^{-0.05} * 0.5359 = \$98.7.$

Option value of the short position of a 1-year term European call option with strike ratio $K_S/S_{t-1} = 1.0556$ is:

$C_S(S_{t-1}, t-1) = S_{t-1}N(d_1) - K_S e^{-r}N(d_2)$,

where

$K_S = S_{t-1}\left(1 + \frac{Cap}{par}\right) = 1000\ * 1.0556 = 1055.56.$

$d_1 = \frac{1}{\sigma}\left[\ln\left(\frac{1}{1+\frac{Cap}{par}}\right) + \left(r + \frac{\sigma^2}{2}\right)\right] = \frac{1}{0.2}\left[\ln\left(\frac{1}{1+\frac{0.05}{0.9}}\right) + \left(0.05 + \frac{0.2^2}{2}\right)\right] = 0.0797,$

$N(d_1) = 0.5319.$

$d_2 = d_1 - \sigma = 0.0797 - 0.2 = -0.1203,$

$N(d_2) = 1 - N(-d_2) = 1 - 0.5478 = 0.4522.$

$C_S(S_{t-1}, t-1) = 1000 * 0.5319 - 1055.56 * e^{-0.05} * 0.4522 = \$77.8.$

Therefore, the total cost = $par * [C_L(S_{t-1}, t-1) - C_S(S_{t-1}, t-1)] = 0.9 * (\$98.7 - \$77.8) = \$18.8.$

(d)

(i) Calculate the effective volatility $\tilde{\sigma}$ that covers the transaction costs for long and short option positions, respectively. Assume 52 weeks per year and $\pi = 3.14$.

(ii) Justify the calculation of effective volatility regarding to each option position.

Commentary on Question:

For part (i), solutions using the variance formula $\sigma^2 \pm 2\sigma k\sqrt{\frac{2}{\pi dt}}$ *are awarded full credit as well.*

(i) The effective volatility $\tilde{\sigma}$ for long call position = $\sigma - k\sqrt{\frac{2}{\pi dt}} = 20\% - 0.52\% \times \sqrt{\frac{2}{3.14} \times \frac{52}{1}} = 17.00\%.$

(ii) The effective volatility $\tilde{\sigma}$ for short call position = $\sigma + k\sqrt{\frac{2}{\pi dt}} = 20\% + 0.52\% \times \sqrt{\frac{2}{3.14} \times \frac{52}{1}} = 23.00\%.$

(ii)

When you long an option, you should pay less than the fair BSM value, since the hedging cost will diminish your P&L. Fora long position, the effective volatility is reduced.

When you short an option, you must ask for more money to cover your hedging costs, and therefore you should have sold it for a greater price than the BSM value. For a short position, the effective volatility should be enhanced.

(e)

(i) Describe the relationship between hedging frequency and the profit.

(ii) Describe strategies that can be used for rebalancing.

Commentary on Question:

For part (i), the candidate needs to mention smaller hedging error leads to more certainty regarding the profit. "Frequent rebalancing reduces hedging error" does not answer the question.

For part (ii), reasonable description of benchmarks that trigger rebalancing are accepted.

(i)

The more you rebalance:

- the smaller the hedging error, the more certain about the profit,
- but the greater the cost and the smaller the expected profit as the more of profit is given away in transaction costs.

(ii)

Rebalancing strategies:

- Rebalancing at regular intervals: set a time interval and rebalance at the end of every time step, no matter how little or how much additional options must be traded.
- Rebalancing Triggered by changes in the hedge ratio: set a trigger rate and rebalance only after a substantial change in the hedge ratio has occurred, where the trigger rate is hit.

QFI QF Fall 2021 Question 11

Learning Outcomes:

b) Understand static and dynamic hedging

Source References:

- *The Volatility Smile*, Derman, Emanuel and Miller, Michael, 2016, Chapter 7

Commentary on Question:

This question tests candidates' understanding of delta hedging and volatility smile.

Solution:

(a) Calculate your cumulative total profit or loss on Day 4 under the following circumstances, respectively:

(i) You rebalanced your hedge position daily.

(ii) You never rebalanced your hedge position.

Commentary on Question:

Candidates performed below expectations in this part. Some were able to achieve full credit when they set up the Excel sheet correctly. Partial credit was awarded if the candidate's answer was correct for hedging of 1 option (rather than 1,000 options as asked by the question).

The hedge is to buy stocks when the call option is sold. Total hedge position consists of short calls and long stocks

I. With daily rebalancing: total gain = 459, as shown below:

Day = t	1	2	3	4	Total
Stock price = S_t	80	70	75	82	
Option price = O_t	12.25	12.25	12.22	12.30	
Option delta = D_t	0.610	0.535	0.562	0.638	
# of short options = NO_t	-1000	-1000	-1000	0	
# of long stocks = $NS_t = -NO_t * D_t$	610	535	562	0	
Gain from stocks = $GS_t = NS_{t-1} * (S_t - S_{t-1})$		-6100	2675	3934	509
Gain from options = $GO_t = NO_{t-1} * (O_t - O_{t-1})$		0	30	-80	-50
Total gain = $GS_t + GO_t$					**459**

II. Without daily rebalancing:

Total gain = 600 * (82 – 80) – 1000 * (12.30 – 12.25) = 1170

(b) Determine whether each of the three explanations provided is valid or not. Explain why.

Commentary on Question:

Candidates performed below expectations on this part. Only a small portion of candidates were able to justify why each of the analyst's three explanations is valid or not.

Overall:

Because there is no change of interest rate and the effect of the time decay over 1 day is small (in light of the option maturity of 3 years), the answer below ignores the effect of interest rate and time decay.

Observation 1

For a given strike, the "sticky strike rule" says that the implied vol does not change with the stock price. If this were true, the option price on Day 2 should have decreased due to decrease of the stock price. Since this is not the case, the sticky strike rule cannot explain Observation 1.

Observation 2

For a given strike, the "sticky delta rule" says that the implied vol increases when the stock price rises. If this were true, the option price on Day 3 should have increased due to increase of the stock price and the implied vol from Day 2 to Day 3. Since this is not the case, the sticky delta rule cannot explain Observation 2.

Observation 3

For a given strike, the local volatility model says that the implied vol falls when the stock price rises. This has two effects on the call option price: (i) when stock price rises, it increases the call option value; (ii) when the implied vol falls, it decreases the call option value. When these two effects happen simultaneously as in the local volatility model, it's possible for the option price to increase due to (i) outweighing (ii). So "local volatility model" can explain Observation 3.

(c) Provide your explanation for observation 4.

Commentary on Question:

Candidates performed as expected on this part.

Because there is no change in the stock price, interest rate and delta between Day 1 and Day 30, the decrease of the option price can be explained by option theta, or time decay.

QFI QF Fall 2021 Question 12

Learning Outcomes:

a) Understand the Greeks of derivatives

b) Understand static and dynamic hedging

c) Understand delta hedging, and the interplay between hedging assumptions and hedging outcomes

Source References:

- *Options, Futures, and Other Derivatives*, Hull, John C., Pearson, 2021, Chapters 19, 26
- *The Volatility Smile*, Derman, Emanuel and Miller, Michael, 2016, Chapters 3, 5, 6

Solution:

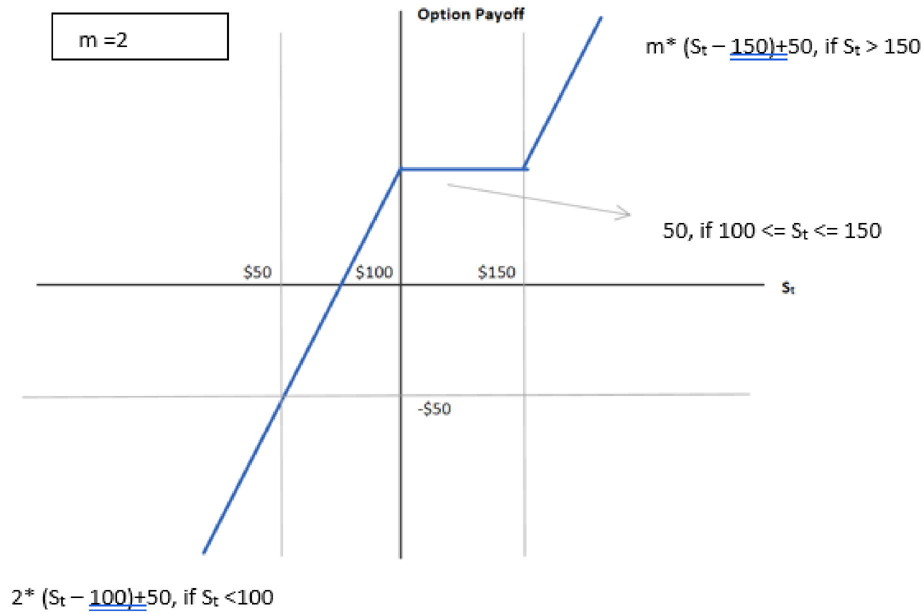
(a)

(i) Sketch the payoff graph for option *E* using $m = 2$.

(ii) Build a static hedging strategy with vanilla options to hedge the equity risk.

Commentary on Question:

Candidates did okay for this question. Most candidates were able to sketch the payoff and identify the embedded vanilla options. However, some candidates failed to recognize that the long position in call was subjected to equity risk. Some candidates were also confused about static hedging strategy with option vs. dynamic hedging with underlying assets, or failed to use the opposite position to hedge the portfolio.



ii)
m *(S – 150)+50, if 150 < S ;
0 , otherwise
= m* Max [S – 150, 0] + [50 | 150 < S]
→ Long m unit of call option with strike at $150 + [50 | 150 < S] - (1)

2 *(S – 100)+ 50, if S < 100;
0 , otherwise
= - 2* Max [100 - S, 0] + [50 | S < 100]
→ Short 2 unit of put option with strike at $100 + [50 | S < 100] - (2)

(1) + (2) + [50 | otherwise]
→ m * C(150) – 2 * P(100) + 50

Static Hedging Strategy: hold opposite postion to offset the equity exposure.

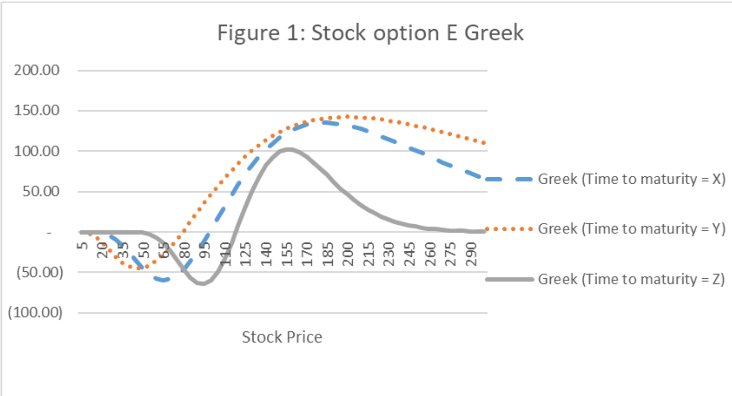
m=2
Short 2 unit of C(150) and long 2 unit of P(100)
Cash has no equity exposure

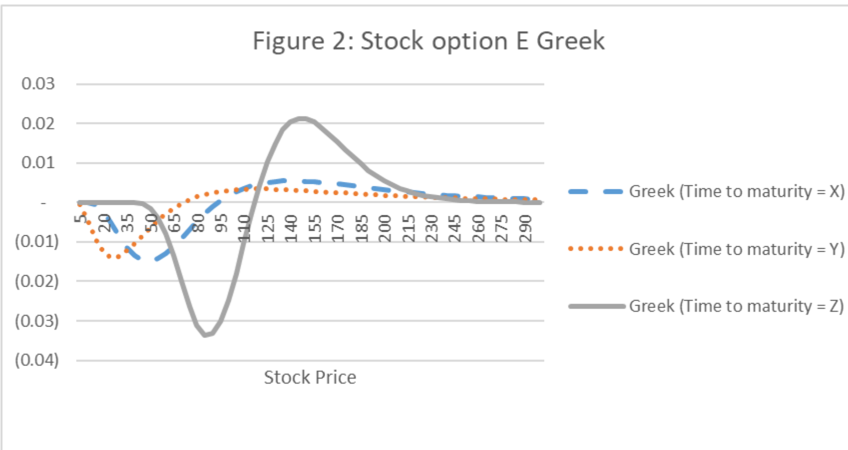
(b)

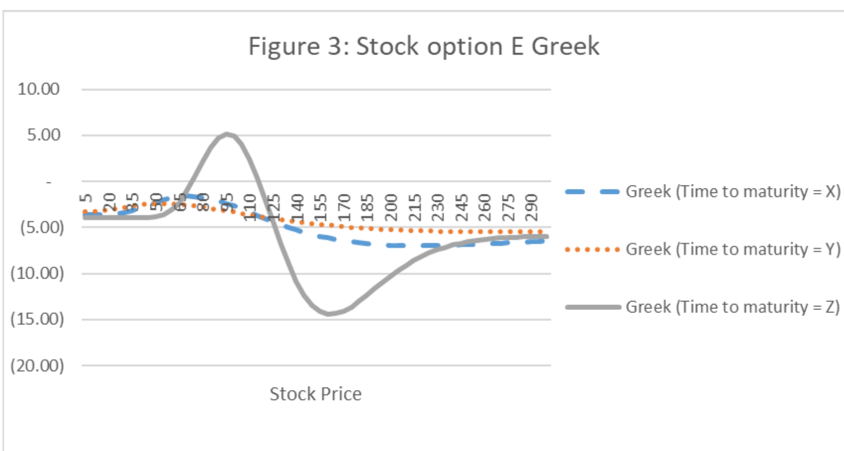
(i) Define the following Greeks: Delta, Gamma, Vega, and Theta.

(ii) Sketch Delta graph for option *E* using $m = 2$ and justify your answers.
(Hint: Build from vanilla options.)

(iii) Determine which figure corresponds to Gamma, Vega, and Theta, respectively. Justify your answers.







Commentary on Question:

Candidates did well for the part b(i).

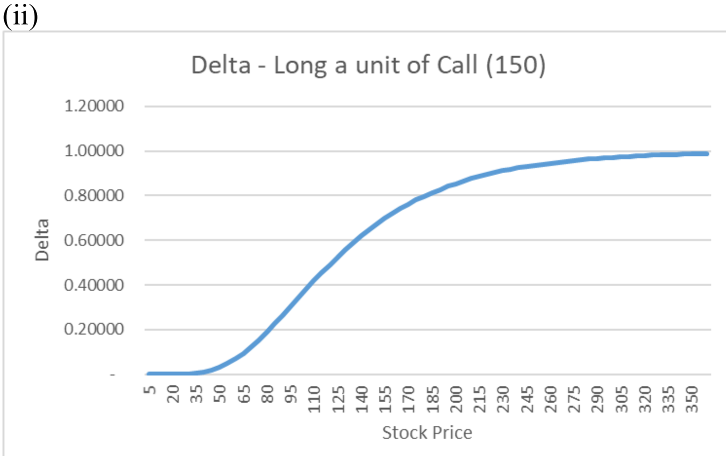
For b(ii), most candidates knew that delta has to be positive (/ negative) for long position in call (/put). However, some didn't demonstrate their knowledge on how delta would behave when approaching the strike price, or got confused with the sign. In addition, some candidates didn't provide justification to support their answer, in that case, only partial credit is given.

For b(iii), most candidates were able to identify the chart for theta and knew the sign for theta to be opposite from gamma and Vega. However, most candidates failed to recognize that gamma spikes up at ATM and Vega diminishes as time approaches the maturity date, and use it to differentiate the chart or use these as support for the graphs they identified as either gamma or Vega.

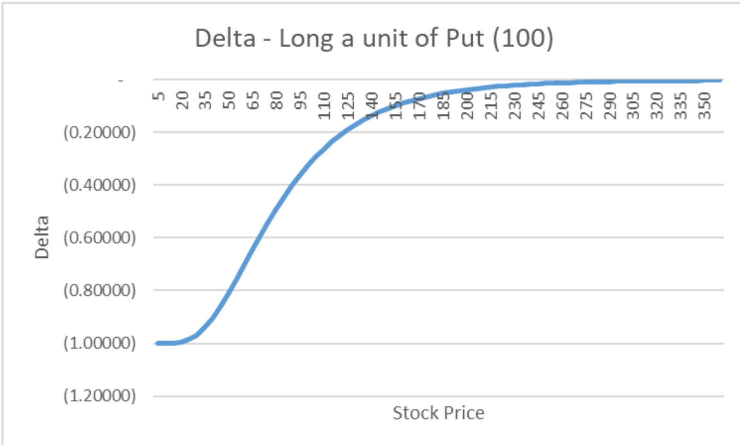
(i)

<table>
<tr><td>Delta</td><td rowspan="3">change in option price with relatively small changes in</td><td>price of underlying asset</td></tr>
<tr><td>Vega</td><td>volatility</td></tr>
<tr><td>Theta</td><td>passage of time.</td></tr>
<tr><td>Gamma</td><td>change in Delta with relatively small changes in</td><td>price of underlying asset</td></tr>
</table>

(ii)

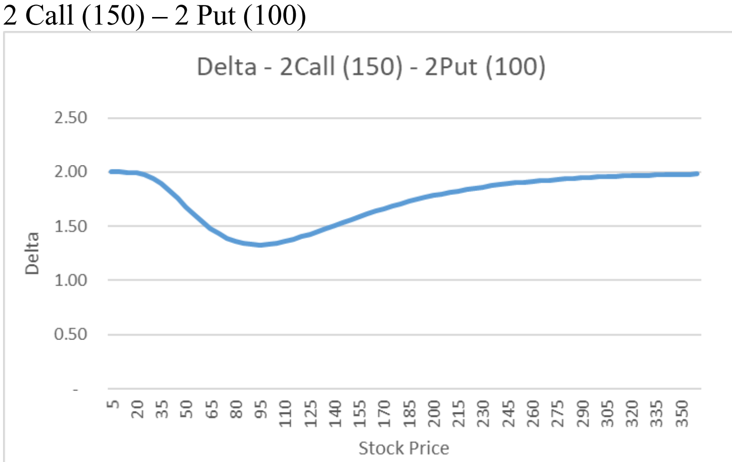


- Delta >0
- Delta grades from 0 (out of the money) to +1 (in the money)
- Delta is getting closer to 1 around $150.



- Delta <0
- Delta grades from 0 (out of the money) to -1 (in the money)
- Delta is getting closer to -1 around $100.

Cash has no delta, so the delta for option E is the delta for
2 Call (150) – 2 Put (100)



- Delta >0
- Delta <1, when stock price is around the range of (100, 150)
- Delta grades to 1 beyond 150, and beneth 100

(iii)
Figure 1 Vega
Figure 2 Gamma
Figure 3 Theta

Theta is negaive for a long positon of vallina option. Option E consists of a long call with strike price at 150 and a short put with strike price at 100.
→ negtive around $150, and positive around $100.
→ Only Figure 3 fits the profile.

Both Vega and Gamma are positive for a long position of vallina option.
→ positive around $150, and negative around $100.

→ Figure 1 and 2 fit the profile.

To differetiate Figure 1 vs. Figure 2:
Vega diminishes progressively with the reduction of time-to maturity regardless of the stock price level, which is not the case for Gamma.

- When the stock price is close to the strike price near the option expiry, a small change in stock price could quickly result in call option delta flipping between 0 and 1, that is, Gamma is highly unstable in this sitution.
- When the stock price is away from the strike price, Gamma dimishes with the reduction of the time-to maturity, similar to Vega.

→ Figure 1 is Vega, and Figure 2 is Gamma.
(c)

(i) Explain what the volatility skew is.

(ii) List three reasons why the volatility skew exists.

(iii) Explain why option *E* is not the suitable vehicle to trade on convexity of volatility skew.

Commentary on Question:

Most candidates could explain what volatility skew is and identify the high demand for OTM put against downside risks is one of the key drivers for volatility skew. However, most candidates failed to identify the other drivers.

For part ii), only some of candidates were able to identify that option E was equivalent to risk reversal with cash position, and was unsuitable to trade on convexity of volatility skew.

(i)
Volatility skew is a form of volatility smile, describing the relationship between implied volatilities (BSM) and strikes, where that downside strikes have greater implied volatility than upside strikes. The graph of implied volatility vs. strike price is thus showing a skew type of shape

Reason to have volatility skew:

- Demand component. Investors who own equities may want to hedge against large losses, and willing to pay extra (/risk premium) for the protections/ insurance against the risks of extrem events.
- Risk premium to comensate option sellers for the negative vega convexity that they take on from selling the out of the money options. ie. When the levels of volaitlity increases, the trader's negative vega position also increases. Seller than need to buy at higher volatility to rebalance, which cost more.
- Out of the money options are likely less liquid than at/in the money options, and thus harder to hedge.
- The risk of the price of the underlying asset having sudden/immediate jump is much larger for a deeply out of the money option than what a seller would take with an at the money options.

(ii)
Option E takes long vega position at strike price of $150, and short vega position at $100, which is equivalent to a risk reversal strategy. With all others being equal, traders can benefit from any increase in the implied volatility at the strike price of $150 relative to the strike price of $100.

To trade on volatiltiy convexity, traders need to long vega at OTM strikes on both sides, and short vega near the ATM strike (butterfly strategy), to benefit from the change in implied volatility becoming more (less) at OTM strike than ATM strike.

Option E is a vehicle to trade on the steepness/flateness of volatility skew, but not convexity.

(d)

(i) Determine K^* for option E^*.

(ii) Solve for m so that option E^* is Vega-neutral.

Commentary on Question:

Many candidates didn't attempt the question, but for those who attempted the question, most knew to set k = 50 to build a butterfly strategy, and tried to find m to achieve Vega-neutral.*

Most candidates assumed that m= 2 for part d), which is incorrect, but majority of the credit was still given if candidates performed the rest of the calculations correctly.

For candidates who didn't provide proper justification to support their answer, partial credit was given.

To trade on the convexity of volatility skew and benefit from increase in convexity, option E* needs to long Vega at the OTM strike and short/neutralize Vega near the ATM strike.

Option E* consists of
- 2 units of short call at strike = $100 (negative Vega)
- m units of short(/or long) put at strike = $k*
- Option E:

m units of long call at strike = $150 (positive Vega)
+ 2 units of short put at strike = $100 (negative Vega)

The strike price $k* needs to be OTM.

To have option E* with symmetric payoff centered at the current stock price, we need to have k* = $50 in long position, and have the same number of call at strike = $150 in long position.
- 2 unit of short call at $100 + 2 unit of short put at $100
  ➔ already have symmetric payoff centered at the current stock price

- m units of long call at $150 +m units of short(/or long) put at $50
  ➔ to be symmetric

To achieve Vega neutral,

$$\frac{\{Vega;\ 2(Short\ Put\ (100) + Short\ Call\ (100))\}}{\{Vega;\ m(Long\ Call\ (150) + Long\ Put\ (50))\}} = -1$$

Calculate Vega = $SN'(d_1)\sqrt{T-t}$

Vega on Put (100)= Vega on Call (100)
= 100* 0.36014 * sqrt (5) = 80.53

Vega on Call (150)
= 100 * 0.36718 *sqrt(5) = 82.1

Vega on Put (50)
= 100 *0.11236 * sqrt(5) = 25.12

$$\frac{\{Vega;\ 2(Short\ Put\ (100) + Short\ Call\ (100))\}}{\{Vega; N(Long\ Call\ (150) + Long\ Put\ (50))\}}$$
$$= \frac{-2 * (80.53 + 80.53)}{m * (82.1 + 25.12)} = \frac{-322.12}{107.22\ m}$$

$$Set\ \frac{-322.12}{107.22\ m} = -1 \ => m = 3$$

(e)

(i) Calculate the gain or loss of option E^*.

(ii) Demonstrate how option E^* is an effective vehicle to take position on volatility convexity, given the result in part (e)(i).

Commentary on Question:

Most candidates didn't attempt the question, but for those who did, most knew that the gain for option E should be calculated by reflecting the stock price change, and comment on the effectiveness of the option taking position on volatility convexity.*

(i)

Option E*= 3 * C(150) +3 * P (50) – 2 * P(100) – 2 * C(100)

The inital price of Option E*:

$C(150) = S * N(d_1)_{150} - 150 \exp(-2\% * 5) * N(d_2)_{150}$
$= 100 * 0.3419 - 150 \exp(-2\% * 5) * 0.1891$
$= 8.5242$

$C(100) = S * N(d_1)_{100} - 100 \exp(-2\% * 5) * N(d_2)_{100}$
$= 100 * 0.6745 - 100 \exp(-2\% * 5) * 0.4728$
$= 24.6693$

$P(100) = 100 \exp(-2\% * 5) * N(-d_2)_{100} - S * N(-d_1)_{100}$
$= 100 \exp(-2\% * 5) * 0.5272 - 100 * 0.3255$
$= 15.1530$

$P(50) = 50 \exp(-2\% * 5) * N(-d_2)_{100} - S * N(-d_1)_{50}$
$= 50 \exp(-2\% * 5) * N(-0.97) - 100 * N(-1.59)$
$= 50 \exp(-2\% * 5) * 0.1651 - 100 * 0.0557$
$= 1.8994$

→ $3 * 8.5242 + 3 * 1.8994 - 2 * 15.1530 - 2 * 24.6693 = -48.3735$

The price of Option E' after stock price decreases from 100 to 80:

$C(150)^* = S * N(d_1)^*_{150} - 150 \exp(-2\% * 5) * N(d_2)^*_{150}$
$= 80 * 0.4315 - 150 \exp(-2\% * 5) * 0.1486$
$= 14.3512$

$C(100)^* = S * N(d_1)^*_{100} - 100 \exp(-2\% * 5) * N(d_2)^*_{100}$
$= 80 * 0.5603 - 100 \exp(-2\% * 5) * 0.3019$
$= 17.5070$

$P(100)^* = 100 \exp(-2\% * 5) * N(-d_2)^*_{100} - S * N(-d_1)^*_{100}$
$= 100 \exp(-2\% * 5) * 0.6981 - 80 * 0.4397$
$= 27.9907$

$P(50)^* = 50 \exp(-2\% * 5) * N(-d_2)^*_{100} - S * N(-d_1)^*_{50}$
$= 50 \exp(-2\% * 5) * 0.3850 - 100 * 0.1341$
$= 6.6901$

→ $3 * 14.3512 + 3 * 6.6901 - 2 * 17.5070 - 2 * 27.9907 = -27.8715$

Gain on Option E*
= The price of option E* $_{\text{after S: 100 ->80}}$ – The price of option E* $_{\text{Initial}}$
= 20.502

(ii)
Following the decrease in stock price from 100 to 80, we have increase in implied volatility where the increase is more significant at out of the money strike than at the money strike.

Option E* long Vega at out of the money strike and short at at the money strike, and thus generates gains benefiting from the increase in volatility convexity

QFI QF Spring 2022 Question 2

Learning Outcomes:

a) Understand the Greeks of derivatives

Source References:

- *Options, Futures, and Other Derivatives*, Hull, John C., Pearson, 2021, Chapter 19

Commentary on Question:

This question attempts to test candidates' understanding of martingales and the valuation of non-standard options. Candidates' performance was uneven.

Solution:

(a) Show that:

(i) $V_5 = S_5 \mathbb{I}_{\{S_3 \geq S_5\}} + S_3 \mathbb{I}_{\{S_3 < S_5\}}$ where $\mathbb{I}_{\{A\}} = \begin{cases} 1 \text{ if } A \text{ is true} \\ 0 \text{ if } A \text{ is false} \end{cases}$.

(ii) $P[S_3 < S_5] = 0.583$ under $\mathbb{Q}$ measure.

Commentary on Question:

Most candidates did not receive full credit for part (i). Many simply re-stated the premise of the problem. Some mistakenly stated the indicator function was equivalent to a probability. To receive full credit, the indicator function needed to be explicitly incorporated within the proof.

Candidates who attempted part (ii) generally performed as expected. To receive full credit, candidates needed to demonstrate an understanding of the distribution of S_t. Credit was not given for correct final answers provided without justification.

(i) A straightforward calculation:

$$V_5 = min\{S_3, S_5\}$$

$$= \begin{cases} S_5 \text{ if } S_3 \geq S_5 \\ S_3 \text{ if } S_3 < S_5 \end{cases}$$

$$= \begin{cases} S_5 \text{ if } S_3 \geq S_5 \\ 0 \quad \text{if } S_3 < S_5 \end{cases} + \begin{cases} 0 \quad \text{if } S_3 \geq S_5 \\ S_3 \text{ if } S_3 < S_5 \end{cases}$$

$$= S_5 \mathbb{I}_{\{S_3 \geq S_5\}} + S_3 \mathbb{I}_{\{S_3 < S_5\}}.$$

(ii) Under the risk-neutral measure $\mathbb{Q}$, S_t follows a GBM with a drift equal to the risk-free rate. This is expressed in terms of the SDE $dS_t = rS_t dt + \sigma S_t dW_t$, which has the solution:

$$S_t = S_0 e^{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W_t}.$$

Therefore,

$$\begin{aligned} S_3 < S_5 &\Leftrightarrow {}^{S_5}\!/_{S_3} > 1 \\ &\Leftrightarrow e^{\left(0.02 - \frac{1}{2}(0.1)^2\right)(5-3) + 0.1(W_5 - W_3)} > 1 \\ &\Leftrightarrow (0.015)(2) + 0.1(W_5 - W_3) > 0 \\ &\Leftrightarrow W_5 - W_3 > -0.3 \end{aligned}$$

Given that $W_5 - W_3 \sim N(0,2)$, we obtain:

$$\mathbb{Q}[S_3 < S_5] = 1 - \Phi(-0.3/\sqrt{2}) = \Phi(0.21) = 0.583.$$

(b) Show that:

(i) $E_t\left[S_3 \mathbb{I}_{\{S_3 < S_5\}}\right] = 0.619\, e^{-0.02t} S_t.$

(ii) $E_t\left[S_5 \mathbb{I}_{\{S_3 \geq S_5\}}\right] = 1.03\, E_t[S_3] E\left[e^{\sqrt{0.02}Z} \mathbb{I}_{\{Z \leq -0.21\}}\right]$ with *Z* a standard normal random variable.

Commentary on Question:

Candidates who attempted part (i) did well. A key element of the solution is recognizing that the expectation of the indicator function is the probability of the indicated event.

Candidates performed poorly on part (ii). Most did not attempt a solution or wrote very minimal work that earned no credit. As implied by the statement candidates were asked to show, candidates needed to relate S_3 and S_5, similarly to the work expected in (b)(ii). In fact, much of the elements of a full credit response parallel that of the prior question.

(i)

$$\begin{aligned} E_t\left[S_3 \mathbb{I}_{\{S_3 < S_5\}}\right] &= E_t[S_3]\, E_t\left[\mathbb{I}_{\{S_3 < S_5\}}\right] \\ &= e^{0.02(3-t)} S_t E_t\left[\mathbb{I}_{\{S_3 < S_5\}}\right] \\ &= e^{0.06-0.02t}(0.583) \\ &= 0.619 e^{-0.02t} S_t \end{aligned}$$

since the expectation of an indicator function over a probability distribution is simply the probability of the indicated event, which was found in part (b)(ii).

(ii)

$$\begin{aligned} E_t\left[S_5 \mathbb{I}_{\{S_3 \geq S_5\}}\right] &= E_t\left[S_3 e^{\left(r-\frac{1}{2}\sigma^2\right)(5-3)+\sigma(W_5-W_3)} \mathbb{I}_{\{S_3 \geq S_5\}}\right] \\ &= e^{\left(0.02-\frac{1}{2}(0.1)^2\right)(2)} E_t\left[S_3 e^{0.1(W_5-W_3)} \mathbb{I}_{\{S_3 \geq S_5\}}\right] \end{aligned}$$

Using the fact that $S_3 < S_5 \Leftrightarrow Z > -0.21 \Rightarrow S_3 \geq S_5 \Leftrightarrow Z \leq -0.21$, the above is equivalent to

$$= 1.03 E_t[S_3] E[e^{\sqrt{0.02}Z} \mathbb{I}_{\{Z \leq -0.21\}}]$$

since $0.1(W_5 - W_3) \sim N\left(0, (0.1)^2(5-3)\right)$, i.e. $N(0, 0.02)$, and period from 3 to 5 years is independent from period t to 3 years.

(c) Calculate V_t and its Delta.

Commentary on Question:

Candidates performed reasonably well. A common mistake was not to include an appropriate discount factor in calculating V_t, thereby providing the expected payoff rather than the price. Candidates still received credit for their delta response if it was consistent with their answer for V_t.

V_t follows from part (c) and the statement, after discounting to time t. More specifically,

$$\begin{aligned} V_t &= e^{-0.02(5-t)}[0.619\, e^{-0.02t} S_t + 0.401\, e^{-0.02t} S_t] \\ &= e^{-0.1} S_t (0.619 + 0.401) \\ &= 0.92 S_t \end{aligned}$$

The delta of an option is the first partial derivative of the price with respect to the underlying stock price, i.e. $\frac{\partial V_t}{\partial S}$.

Thus, $\frac{\partial V_t}{\partial S} = \frac{\partial}{\partial S}(0.92S_t) = 0.92$,

which remains static over the period $t < 3$.

(d) Your coworker claims that the special European-style option considered above can be Delta- and Gamma-hedged till its expiration by using a suitable short position in the underlying asset only.

Critique your coworker's claim.

Commentary on Question:

Candidates performed poorly on this part. To receive full credit, responses needed to highlight that the nature of the Greeks of this option changes once S_3 is known and fixed. Candidates needed to understand that the responses in parts (c) and (d) assumed $t < 3$.

My coworker is wrong. During the period $\mathrm{t} < 3$, the delta of the option is constant and therefore, gamma is 0. After $\mathrm{t} = 3$, S_3 is fixed, the delta of the option will depend on S_t, and the gamma will be non-zero. Since the underlying has a gamma of 0, a position in the underlying asset only will not allow for gamma-hedging until expiration.

QFI QF Spring 2022 Question 11

Learning Outcomes:

c) Understand delta hedging, and the interplay between hedging assumptions and hedging outcomes

e) Understand derivatives mishaps

Source References:

- *The Volatility Smile*, Derman, Emanuel and Miller, Michael, 2016, Chapters 6, 7

Solution:

(a) Explain which Strategy is associated with Figure 1 and Figure 2, respectively.

Commentary on Question:

Candidates did well on this part of the question. Most candidates were able to identify the correct strategies.

Hedging discretely rather than continuously at the correct realized volatility introduces uncertainty in the hedging outcome but does not bias the final P&L — the expected value is zero. The hedging error decreases as we increase the number of times that we re-hedge the portfolio (i.e., as we measure the volatility more accurately), but only with the square root of n. In order to halve the hedging error, we need to quadruple the number of re-hedgings.

Since Strategy 1 and Strategy 3 are based on realized volatility, and Strategy 1 has higher rebalancing frequency than Strategy 3, we have:

Figure 1 = Strategy 1 (Relative P&L is narrowly around 0)

Figure 2 = Strategy 3 (Relative P&L is widely around 0)

(b) Explain why Figure 3 looks similar to Figure 4.

Commentary on Question:

Candidates did well on this part of the question. Most candidates were able to explain the common point between both figures.

Unless we rebalance an option at the realized volatility, increasing the frequency of replication will not significantly diminish the replication error in the P&L. The reason is evident from Chapter 5: If the option is not hedged at the realized volatility, the incremental P&L dP&L(I, R) in Equation 5.35 of Chapter 5 contains a term proportional to $(\Delta I - \Delta R)\, dS$. This dependence on dS introduces a random noise into the P&L whose standard deviation does not diminish with more frequent hedging.

We now know that Figure 3 and Figure 4 are associated with Strategy 2 and 4 because none of the two strategies is based on realized volatility. Therefore, the standard deviation of daily hedging histogram could be similar to that of weekly hedging histogram, thus Figure 3 and Figure 4 look similar to each other.

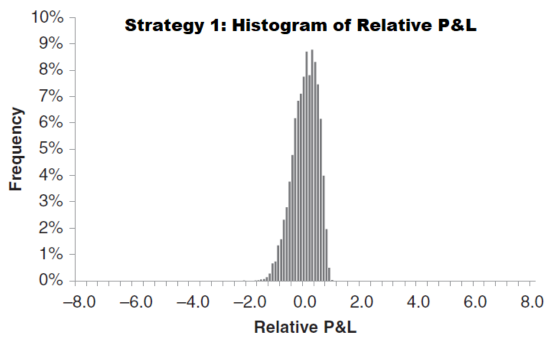
(c)

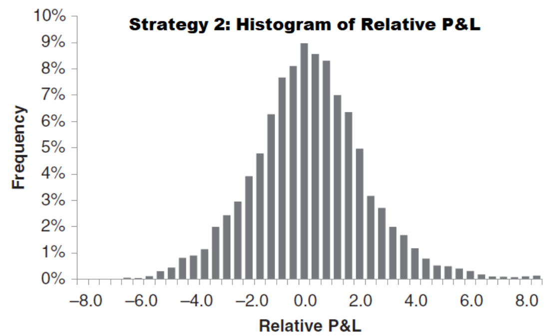
(i) Sketch the histograms of relative P&L for Strategy 1 and Strategy 3, respectively. Note: You need not mark any values on your *x*-axis and *y*-axis. The key is to show the shape or contour of the histogram.

(ii) Explain the key drivers for the differences in the histogram.

Commentary on Question:

Many candidates were able to identify key drivers for the differences between the strategies





When introducing transaction costs, increasing the hedging frequency can lower the variance of relative P&L, but will also increase the hedging cost at the same time. Transaction costs will shift the mean of both distributions below 0.

(d) Compare m_1 vs. m_3 vs. 0. Justify your ranking.

Commentary on Question:

Candidates performed fairly well on this question. Most candidates got the correct ranking.

The more you rebalance, the more of your profit you give away in transaction costs, so that the mean of the P&L distribution decreases. Hence:

m1 < m3 < 0 (Strategy 1 has higher expected loss than Strategy 3)

(e) Compare s_1 vs. s_3. Justify your ranking.

Commentary on Question:

Most candidates identified the less volatile strategy.

The more frequently you rebalance, the more accurately you replicate the option and the smaller the standard deviation (SD) of the profit and loss (P&L) histogram. The less you rebalance, the less profit you relinquish, but the less certain that profit is. Hence:

s1 < s3 (Strategy 1 has lower standard deviation than Strategy 3)

QFI QF Spring 2022 Question 12

Learning Outcomes:

a) Understand the Greeks of derivatives

Source References:

- *Options, Futures, and Other Derivatives*, Hull, John C., Pearson, 2021, Chapter 19

Commentary on Question:

Overall, candidates performed well on this question. For part a, to receive full credit, candidates must also show the derivation of the first derivative. For part e, to receive full credit, candidates must provide appropriate critique that is consistent with the source material.

Solution:

(a) Show that the Gamma of the European call is:

$$\text{Gamma} = N'(d_1)\frac{1}{S\sigma\sqrt{T}}$$

(b) Prove that the Gamma of a European call is equal to the Gamma of an otherwise equivalent European put.

Apply the put-call parity equation:

$$C - P = S - Ke^{-rT}$$

$$\frac{\partial C}{\partial S} - \frac{\partial P}{\partial S} = 1$$

Show that the second partial derivative for call is equal to that for put:

$$\frac{\partial^2 C}{\partial S^2} - \frac{\partial^2 P}{\partial S^2} = 0$$

$$\frac{\partial^2 C}{\partial S^2} = \frac{\partial^2 P}{\partial S^2} \Rightarrow \Gamma_{Call} = \Gamma_{Put}$$

(c) Identify whether each of the following statements is true or false. Briefly justify your answer.

(i) Gamma approaches 0 for deep-in-the money calls.

(ii) Gamma approaches 1 for deep-out-of-the-money puts.

(iii) For an out-of-the-money option with an underlying asset price that is exhibiting low volatility, Gamma is expected to be relatively low.

(iv) For an option that happens to be right at-the-money very near to the expiry date, a stable Gamma is likely to be observed.

(i) True. For deep-in-the-money calls, the delta has to stay close to +1. The delta will not change much irrespective of the change in the price of the underlying, and thus the rate of change (i.e., gamma) must be close to 0.

(ii) False. For deep out-of-the-money puts, the delta has to stay close to 0. Changes in delta will be strictly limited and so gamma must be close to 0.

(iii) True. Due to low volatility, the probability that the price of the underlying will cross the strike price before the expiry date is relatively low, so we should not expect a strong sensitivity of the delta of the option to changes in the price of the underlying asset.

(iv) False. For small increases or decreases in the price of the underlying, the option delta will quickly converge to 1 or 0 for call or to -1 to 0 for put, so gamma is very unstable.

QFI QF Spring 2022 Question 14

Learning Outcomes:

f) Identify and evaluate embedded options in liabilities (e.g., indexed annuity, structured product based variable annuity, and variable annuity guarantee riders including GMxB, etc.)

g) Demonstrate and understand target volatility funds and hedging for embedded options

Source References:

- INV201-106-25: Variable Annuity Volatility Management: An Era of Risk-Control

Solution:

(a) Calculate the resulting target volatility fund prices X and Y in Table 2, assuming a continuously compounded risk-free rate of 3%, a target volatility of 15% and a maximum equity % of 200%.

Commentary on Question:

Most candidates received at least partial credit. Some candidates did not use the correct formula to calculate X and Y and thus did not get the correct answers.

	t=0	t=1	t=2	t=3	T=4
Equity %	0.75	0.375	1.5	0.5	0.75
Bond %	0.25	0.625	-0.5	0.5	0.25
Equity Price	100	88	105	110	93
Bond Index Price	100	103.0455	106.1837	109.4174	112.7497
Target Vol Fund Price	100	91.7614	100.1554	105.7843	99.2209

(See formula in QFI 132-21, page 1520.)

X = 91.76

Y = 105.78

(b) Compare the relative performance of the target volatility fund, capped volatility fund, and underlying asset under the scenarios in Table 3, where the target volatility = σ_T and cap volatility = σ_C and $\sigma_T < \sigma_C$.

Commentary on Question:

Most candidates did well on this part. Candidates need to justify their answers to receive full credit. Some candidates only compare the performance of two out of the three returns and thus received partial credit.

Scenario 1:

Since the volatility level is above the cap (and thus above the target volatility), the equity allocation for both the target and capped volatility fund will be less than 1. Because the target volatility is below the cap, the target fund will have a lower equity allocation than the capped fund. Since the market returns are negative, this will result in:

Target fund return > capped fund return > underlying stock return

Scenario 2:

When the market volatility equals the target volatility, all 3 funds will have an equity allocation = 1, thus:

Target fund return = capped fund return = underlying stock return

(Note: There was a typo in the Excel spreadsheet provided at the exam. Candidates who answered correctly based on Scenario 2 in the Excel spreadsheet received full credit.)

Scenario 3:

Since the volatility level is below target volatility (and thus below the cap volatility), the capped fund will have an equity allocation of 100% while the target volatility fund will have an allocation >100%, so:

Underlying stock return = capped fund return > target fund return

(c) Explain whether the following statements are True or False:

(i) Call options on a target volatility fund should be cheaper than or equal to the equivalent call options on the underlying risky-asset.

(ii) Call options on a capped volatility fund should be cheaper than or equal to the equivalent call options on the underlying risky-asset.

Commentary on Question:

Candidates need to justify their answers to receive full credit.

(i) False. While call option prices increase with volatility, prices of call options on target volatility funds are only cheaper if the target vol is less than the market vol.

(ii) True. Call option prices increase with volatility, and the volatility on a capped vol fund is always equal to or less than the vol of the underlying asset.

QFI QF Fall 2022 Question 13

Learning Outcomes:

a) Understand the Greeks of derivatives

c) Understand delta hedging, and the interplay between hedging assumptions and hedging outcomes

d) Understand the concepts of realized versus implied volatility.

Source References:

- *Options, Futures, and Other Derivatives*, Hull, John C., Pearson, 2021, Chapter 19
- *The Volatility Smile*, Derman, Emanuel and Miller, Michael, 2016, Chapters 5, 6

Solution:

(a) Construct a strategy to replicate the payoff of the contingent claim with only European options on Stock XYZ.

Commentary on Question:

Most candidates were able to identify the correct European call positions to replicate the contingent claim. Partial credit is granted for a correct but incomplete specification of parameters (long/short, strike, maturity) strategy.

The strategy required to replicate the payoff of the contingent claim consists of the following positions:

- A long position in a 100-strike one-year European call
- A short position in a 120-strike one-year European call

(b) Compare and contrast realized volatility and implied volatility.

Commentary on Question:

Most candidates answered this question correctly and were able to provide the clear definition for the two types of volatilities.

Implied volatility is a parameter that matches the model option price to the market price using the Black-Scholes Model equation. Implied volatility is derived from the present and expected future data.

Realized volatility is a statistic that measures the standard deviation of returns for a past period. Realized volatility is derived from the past/historical data.

(c)

(i) Calculate the Delta of this contingent claim.

(ii) Explain why the Delta is positive.

Commentary on Question:

Most candidates understood that the Delta of the contingent claim was the sum of the Deltas of the long and short European call positions from part (a). Some candidates did not calculate the correct values of the delta for the long and short European call positions.

For the overall Delta of the contingent claim, it was important to understand the relationship of the Delta of the two European call positions and how that impacts the overall Delta of the contingent claim. Most candidates only noted one or the other.

The Delta of the contingent claim is calculated as the sum of the delta of the replicating strategy in which

$$\Delta_{\text{Call}(K=100)} = N(d_1) = N\left(\frac{\ln\frac{110}{100} + \left(0 + \frac{0.3^2}{2}\right)(1)}{0.3\sqrt{1}}\right) = N(0.46770) = 0.68$$

$$\Delta_{\text{Call}(K=120)} = N(d_1) = N\left(\frac{\ln\frac{110}{120} + \left(0 + \frac{0.3^2}{2}\right)(1)}{0.3\sqrt{1}}\right) = N(-0.14004) = 0.44432$$

Therefore, the Delta of the contingent claim is:

$$\begin{aligned}\Delta_{\text{Claim}} &= +\Delta_{\text{Call}(K=100)} - \Delta_{\text{Call}(K=120)} \\ &= +0.68 - 0.44432 \\ &= 0.23568 \approx \boxed{0.24}\end{aligned}$$

The overall Delta of the contingent claim is positive because:

- A call option with a lower strike will always have a Delta that is equal to or higher than a call with a higher strike.
- Since the call option that is long has a higher Delta than the call option that is short, the resulting net Delta of the contingent claim is positive.

(d) Regarding a long position in the contingent claims, your colleague made the following comments:

- Comment 1: As the price of the underlying stock moves away from the price range within the two strike prices, we expect the Delta of the contingent claim to converge to zero.
- Comment 2: The net Gamma exposure of the contingent claim is always positive.

Assess each of your colleague's comments above.

Commentary on Question:

For Comment 1, it could be successfully approached by either describing the Delta for the two European call options and the net impact of those two Deltas or describing the Delta of the contingent claim. Most candidates took the first approach.

For Comment 2, it is important to state when Gamma for the contingent claim goes from negative to positive and vice versa. Some candidates stated the Gamma could be negative without providing additional details.

Comment 1 is correct.

This is because any further movement of the price of the underlying asset above the higher strike of 120 or below the low strike of 100 will not have any meaningful effect on the contingent claim payoff.

Comment 2 is incorrect.

The net Gamma exposure of the contingent claim will switch from positive to negative when the underlying price moves from the lower strike of 100 to the higher strike of 120. Conversely, the net Gamma exposure of the contingent claim will switch from negative to positive when the underlying price moves from the higher strike of 120 to the lower strike of 100.

(e)

(i) Calculate the profit or loss at the end of the next day from Delta hedging.

(ii) Explain why the profit or loss is not zero from Delta hedging.

Commentary on Question:

The Delta for hedging the contingent claim is the same as from part (c). Most candidates calculated the payoff of the contingent claim correctly. Some candidates did not calculate the final loss due to either not scaling the number of shares needed for delta hedging or not using the correct Delta.

Regarding why the loss is not zero from Delta hedging. Most candidates identified that the other Greeks were not hedged. Not many candidates identified that the large movement in the underlying contributes to the non-zero loss.

To Delta hedge the 100 contingent claims, the firm needs to short 24 shares. The answer of 24 shares is derived from 100 contingent claims x 0.24 (delta of the contingent claim).

The profit or loss from Delta hedging is then calculated as:

$$\begin{aligned}\text{Profit} &= 100[(33.56 - 20.40) - (18.14 - 9.28)] - 24(130 - 110) \\ &= \boxed{-50}\end{aligned}$$

The loss is not zero from Delta hedging in that:

- There is a large movement in the price of the underlying.
- The firm has only Delta hedged and did not hedge the other Greeks.

QFI QF Spring 2023 Question 10

Learning Outcomes:

f) Identify and evaluate embedded options in liabilities (e.g., indexed annuity, structured product based variable annuity, and variable annuity guarantee riders including GMxB, etc.)

g) Demonstrate and understand target volatility funds and hedging for embedded options

Source References:

- INV201-105-25: An Introduction to Computational Risk Management of Equity-Linked Insurance, Feng, 2018 (sections 1.2-1.3, 4.7 & 6.2-6.3)
- INV201-106-25: Variable Annuity Volatility Management: An Era of Risk-Control

Commentary on Question:

The question is mainly trying to test the candidates understanding of the principles of volatility management strategies and ability to apply them when designing and managing a product with equity guarantee.

Solution:

(a) Describe the principal objectives for an insurer in designing an equity-based guarantee.

Commentary on Question:

Most candidates could list out the principal objectives for an insurer in designing an equity-based guarantee, but failed to demonstrate their understanding of these objectives with descriptions, especially for stabilizing ALM and hedging performance.

- Write profitable business:
  Do the volatility management strategies reduce the hedge cost (risk-neutral value) of the guaranteed?

- Stabilize ALM and hedging performance
  Do the volatility management strategies improve the key hedge ratio, in particular Vega?
  How well do volatility management strategies minimize hedge P&L losses during crisis?
  Can our risk management and hedge program effectively mirror the changing fund position? (i.e. less basis risk)

- Optimize capital requirement
  Do the volatility management strategies reduce Statutory reserve requirement (and volatility of reserve)?

(b) Calculate the guarantee cost at the end of year 1 (t=1) for the GMMB rider under each of the 3 volatility management strategies. (Initial deposit = $100)

Commentary on Question:

For Asset Transfer Program, some candidates were able to determine the percentage of portfolio that needed to be allocated in cash, given the volatility level. However, many failed to rebalance the portfolio based on the portfolio value at t=1.

For Capped volatility fund, many candidates knew that the portfolio remained 100% in equity as the level of volatility was still within the threshold at 60%.

For VIX-Indexed Fee, some candidates calculated the rider fee in bps correctly, but were unable to get to the correct dollar amount. These candidates failed to realize that the rider fee was charged at the beginning of the year (as stated in the question), and thus fee only incurred at t=1.

Very few candidates attempted to calculate the guaranteed cost by taking weighted averaged of the guaranteed payoff under the risk neutral probabilities and calculating the present value at t=1.

<u>Asset Transfer Program</u>

Rebalance at t = 1:

Guaranteed Ratio (G%) = 1- 81.87/100 = 18.13%

Allocation in equity (S) = 1 – G% = 81.87%

Allocation in cash = 18.13%

Equity: 0.8187 unit of Equity S ($67.03 =0.8187 * $81.87)

Cash: $14.84 (sold 0.1813 unit of Equity S = 0.1813 * $81.87)

Payoff at t = 2:

Node 2, u:

The investment value

= 0.8187 unit of equity S + cash = $149.18 * 81.87% + $14.87= $136.98

GMMB payoff = Max (100-136.98, 0) = 0

Node 2, d:

The investment value

= 0.8187 unit of equity S + cash = $44.93 * 81.87% + $14.87 = $51.62

GMMB payoff = Max (100-51.62, 0) = $48.37

Guaranteed cost at the end of year 1 (t=1)

= NPV (Guaranteed Payoff) - Rider Fee

= (0 * 37.02% + 48.37 * 62.98%) * exp(-2%) - 0 = $29.86

- Capped Volatility Fund ($\sigma_{capped} = 60\%$)
- Rebalance at t = 1:
- Allocation in equity (S) = $\sigma_{capped}/\ \sigma_{s1}$ = 60%/60% = 100%
- Allocation in cash = 0 %
- Equity: 1 unit of Equity S ($81.87 =1 * $81.87)

Payoff at t = 2:

Node 2, u:

The investment value

= 1 unit of equity S + 0 cash = $149.18

GMMB payoff = Max (100-149.18, 0) = 0

Node 2, d:

The investment value

= 1 unit of equity S + cash = $44.93

GMMB payoff = Max (100-44.93, 0) = $55.07

Guaranteed cost at the end of year 1 (t=1)

= NPV (Guaranteed Payoff) - Rider Fee

= (0 * 37.02% + 55.07 * 62.98%) * exp(-2%) - 0 = $33.99

<u>VIX- Index Fee</u>

Fee charged at t = 1:

Rider Fee = Max [0 bps, 200bps *(60% - 20%)] = 80 bps

Investment value *80bps = $81.87 * 80bps = 0.655

Rebalance at t = 1:

Sold 0.008 unit of equity S for the charged rider fee.

1- [(81.87 – 0.655)/ 81.87] = 1- 0.992 = 0.008

Payoff at t = 2:

Node 2, u:

The investment value

= 0.992 unit of equity S = $149.18 * 0.992 = 147.99

GMMB payoff = Max (100-147.99, 0) = 0

Node 2, d:

The investment value

= 0.992 unit of equity S = $44.93 * 0.992

GMMB payoff = Max (100-44.57, 0) = $55.43

Guaranteed cost at the end of year 1 (t=1)

= (0 * 37.02% + 55.43 * 62.98%) * exp(-2%) – 0.655

= 34.22 - 0.655 = $33.56

(c) Identify the 4 volatility management strategies from the table above including no volatility management strategy.

Commentary on Question:

Many candidates were able to identify strategy C and strategy B to be no volatility strategy and Asset Transfer Program respectively, but failed to provide justifications.

Lots of candidates failed to differentiate between D and E by recognizing that Capped Volatility could create protection against "tail spike" in volatility, and thus more effectively reducing the hedge P&L than VIX- Index Fee.

Strategy C: no volatility management strategy (or leverage on volatility).

- Higher guaranteed cost and hedge loss than the strategy of 100% static allocation in equity,

Strategy B: Asset Transfer Program.

- Highest reduction on guaranteed cost and hedge loss than the other strategies
  - Actively reallocate the fund (from equity to cash) when the portfolio becomes in-the-money at the defined trigger level.
  - More active risk-control than capped volatility and VIX-Indexed fee strategies.

        - As volatility spikes and equity value falls, the strategy is heavily invested in cash, leading the volatility level to be near expectation and stabilizing cash flow despite market fluctuation

Strategy D: Capped Volatility Fund Strategy

- Mild reduction on guaranteed cost (vs. 100% static allocation in equity)
    - Only activate when the equity volatility exceeds the cap level.
    - Given the cap at 60% (vs. the current at 20%), the strategy is expected to eliminate only a small portion of volatility cost.
- Lower hedge loss between D and E.
    - The volatility cap creates a protection against the "tail spike" in volatility, which can reduce the frequency and severity of the ultra-large returns, mitigating the hedge breakage

Strategy E: VIX-Indexed Fee Strategy

- Mild reduction on guaranteed cost and hedge loss (vs. 100% static allocation in equity)
    - The allocation in equity remains at 100%
    - The rider fee increases with the level of volatility, providing some offsets to guaranteed cost and hedge loss; however, given the fee level, the magnitude is expected to be small.
    - No protection against the "tail spike" in volatility and thus less effective than Asset Transfer Program or Capped Volatility in reducing the hedge loss.

(d)

(i) Calculate the Vega under each of the 3 volatility management strategies (Hint: use finite difference approximation).

(ii) Explain how low Vega can benefit the hedge program.

(iii) Propose a volatility management strategy from the insurer's perspective based on the results in part (c) and (d) (i).

Commentary on Question:

For part i), many candidates could correctly calculate the Vega given the provided data.

For part ii), most candidates knew that Vega was the sensitivity to the change in volatility, but failed to demonstrate their understanding on how low Vega could benefit the hedge program.

For part iii), only some candidates recognized that Asset Transfer Program is the strategy that best addresses the insurer's principal objectives in manufacturing an equity-based guarantee.

(i)

<u>Asset Transfer Program:</u>

Guarantee Cost (S0=100, $\sigma_{s,0}$= 10%) = 4.38

Guarantee Cost (S0=100, $\sigma_{s,0}$= 40%) = 10.35

Vega = (10.35 – 4.38)/(40%-10%) = 20.23

<u>Capped Volatility Fund:</u>

Guarantee Cost (S0=100, $\sigma_{s,0}$= 10%) = 7.55

Guarantee Cost (S0=100, $\sigma_{s,0}$= 40%) = 20.91

Vega = (20.91 – 7.55)/(40%-10%) = 44.53

<u>VIX-Indexed Fee:</u>

Guarantee Cost (S0=100, σs,0= 10%) = 7.35

Guarantee Cost (S0=100, σs,0= 40%) = 21.5

Vega = (21.5 – 7.35)/(40%-10%) = 47.17

(ii)

Vega is the rate of change in value of the portfolio with respect to the volatility of the underlying asset. Low Vega can stabilize the performance of hedge program.

(iii)

Given the result in d) -i) and c), Asset Transfer Program has the lowest guaranteed cost, the lowest hedge loss, and the smallest Vega when the volatility increases from 10% to 40%.

→ Best fit the objective of writing profitable business, as well as stabilizing ALM and hedging performance.

(e) Critique whether Joe's proposal meets the needs of the clients in the target market.

Commentary on Question:

Most candidates could recognize that Asset Transfer Program had the lowest equity allocation over time but failed to assess Joe's proposal based on the other two metrics.

The target clients value the upside investment potential and are willing to pay extra fees for it.

The three metrics used to measure the upside investment potential are:

i) Return and volatility profile
   - Higher return relative to realized volatility is preferred
   - Volatility management strategies do not alter the overall investment proposition much from a static 100% equity allocation strategy

ii) Equity allocation over time
   - Higher allocation in equity has better "upside investment potential"

iii) Cumulative fee paid
   - Additional fee paid for the volatility management strategy could reduce account value accumulation or decrease the guaranteed value.

Asset Transfer Program is the most active risk-control strategy among the three, rebalancing with cash based on the in-the-moneyness of the fund.

For i):

- Asset Transfer Program is expected to have the return and volatility profile changed the most from a static 100% equity allocation fund.
- VIX-Indexed Fee and Capped volatility fund likely offer a more similar return and volatility profile as a static 100% equity allocation fund. (The former has 100% allocation in equity, and for the later, rebalancing is only activated when equity volatility exceeds the cap at 60%.)

For ii):

- Asset Transfer Program is expected to have the lowest equity allocation over time, due to the active risk control.
- VIX-Indexed Fee is expected to have the highest equity allocation over time.

For iii):

- VIX-Indexed Fee is the only strategy that would incur extra rider fee.
- Given the result in b) (volatility spikes up to 60%), the extra rider fee doesn't have material impact to the account value accumulation over the rider term (2 years).
- The target clients are willing to pay extra fees for the upside potential. Therefore, the fee saving of Asset Transfer Program over VIX-Indexed Fee may not add much value to the target clients.

➔ Asset Transfer Program doesn't meet the client's need. VIX-Indexed Fee better fits the need of target clients.

QFI QF Spring 2023 Question 11

Learning Outcomes:

f) Identify and evaluate embedded options in liabilities (e.g., indexed annuity, structured product based variable annuity, and variable annuity guarantee riders including GMxB, etc.)

g) Demonstrate and understand target volatility funds and hedging for embedded options

Source References:

- INV201-106-25: Variable Annuity Volatility Management: An Era of Risk-Control
- INV201-108-25: Mitigating Interest Rate Risk in Variable Annuities: An Analysis of Hedging Effectiveness under Model Risk

Commentary on Question:

The majority of candidates performed poorly on this question. Many candidates either did not attempt the question, or only attempted a limited part of the question.

Solution:

(a) Explain the considerations when using each of the approaches above.

Commentary on Question:

Most candidates answered this part adequately. Candidates were generally able to explain the differences and considerations between the three different approaches for calibrating the instantaneous variance process. No credit was awarded for only providing definitions.

(i) Since VAs have long term maturities, extracting appropriate implied volatilities will often involve unsound extrapolation.
Using implied volatilities relates to the fact that two models that are well calibrated to the implied volatility vanilla option surface may lead to very different prices and hedge ratios for exotic options.

(ii) The VIX index is constructed in a model free way, i.e. does not rely on the B-S model, therefore does not suffer from model risk. However, VIX is generally an upward biased forecast.

(iii) Historical volatility yields stable estimates over time. However would not reflect any forward-looking market expectations.

(b) Show that the insurer's expected present value of prospective rider fees becomes:

$$Y_t = \alpha G \left(\frac{1 - e^{-r(T-t)}}{r} \right) {}_{T-t}p_{x+t}$$

Commentary on Question:

Candidates' performances on this part varied greatly. Candidates needed to show steps to their derivation to receive credit. Candidates that did attempt the question performed well, but many candidates either did not attempt or did not show steps to their derivation.

Derive the value of prospective fees from first-principles:

$$L_t = \left(\int_t^T e^{-r(s-t)} \alpha G ds \right) {}_{T-t}p_{x+t}$$

$$= \alpha G \left(\frac{1 - e^{-r(T-t)}}{r} \right) {}_{T-t}p_{x+t}$$

(c) Explain whether the following has increased, decreased, or remained the same after this change, from the insurer's perspective.

(i) Delta of the liability net of rider fees.

(ii) Vega of the liability net of rider fees.

Commentary on Question:

Candidates performed poorly on this part. Many candidates did not correctly understand the impact of the change in rider fee to the delta and vega of the VA net liability. Credit was not awarded for just providing the answer without any rationale.

(i) Net liability delta has increased. Previous rider fee delta was positive and so it contributes to the negative delta from the GMMB put option. New rider fee no longer a function of the account value, so its delta is 0.

(ii) Net liability vega has not changed. Previous rider fee was 0. New rider fee is also not a function of volatility, so its vega is 0.

(d) Show that the fair value of prospective fees at time t, as defined as the risk-neutral expected present value of fees that will be collected by the insurer before the contract's maturity at time T, is:

$$L_t = G\left[(m+\lambda\theta)\left(\frac{1-e^{-r(T-t)}}{r}\right) + \lambda(v_t-\theta)\left(\frac{1-e^{-(r+\kappa)(T-t)}}{r+\kappa}\right)\right]{}_{T-t}p_{x+t}$$

Commentary on Question:

Candidates performed very poorly on this part, with many candidates skipping this question. Candidates needed to show steps to their derivation to receive credit. For those that did attempt the question, many either did not correctly integrate the stochastic variance term or did not provide any steps in their derivation.

Derive the value of prospective fees from first-principles:

$$\begin{aligned}
L_t &= \int_t^T e^{-r(s-t)}(m+\lambda E^{\mathbb{Q}}[v_s])G\,ds\ {}_{T-t}p_{x+t} \\
&= \left[\int_t^T e^{-r(s-t)}mG\,ds + \int_t^T e^{-r(s-t)}\lambda G E^{\mathbb{Q}}[v_s]\,ds\right]{}_{T-t}p_{x+t} \\
&= \left[\int_t^T G(m+\lambda\theta)e^{-r(s-t)}ds + \int_t^T \lambda G(v_t-\theta)e^{-(r+\kappa)(s-t)}ds\right]{}_{T-t}p_{x+t} \\
&= G\left[(m+\lambda\theta)\left(\frac{1-e^{-r(T-t)}}{r}\right) + \lambda(v_t-\theta)\left(\frac{1-e^{-(r+\kappa)(T-t)}}{r+\kappa}\right)\right]{}_{T-t}p_{x+t}
\end{aligned}$$

(e) Explain whether you agree or disagree with the following statements made by your analyst.

(i) "The new rider fee is not a function of A_t, therefore it is not sensitive to changes in the account value."

(ii) "The new rider fee has a positive Vega."

Commentary on Question:

Candidates performed poorly in this part. For those that attempted the question, most candidates did not recognize that the account value and the rider fee are correlated.

(i) Disagree. As account value decreases, v_t will tend to increase due to $\rho < 0$, and therefore rider fee will increase.

(ii) Agree. The vega of the expected PV of the new rider fee is positive. As $\sqrt{v_t}$ increases, the rider fee increases.

QFI QF Fall 2023 Question 7

Learning Outcomes:

e) Understand how hedge strategies may fail

Source References:

- INV201-104-25: Which Free Lunch Would You Like Today, Sir?

Solution:

(a) List the pros and the cons of hedging with implied volatility and actual volatility.

Commentary on Question:

Candidates generally did well on this part of the question.

Pros of hedging with implied volatility:

- No local fluctuations in profit and loss (continually making a profit)
- Only need to be on the right side of the trade to profit (buy when actual is going to be higher than implied and sell if lower)

- The number that goes into the delta is implied volatility, which is easy to observe
- The profit each day is deterministic

Cons of hedging with implied volatility:

- You don't know how much money you will make, only that it is positive. The present value of the total profit at expiration is path dependent

Pros of hedging with actual/realized volatility:

- Profit at expiration is known when hedging with actual volatility

Cons of hedging with actual/realized volatility:

- Subject to profit and loss fluctuations during the life of the option, which can be less appealing from a local risk management perspective
- Unlikely to be totally confident in your volatility forecast (the number put into the delta formula)

(b) Choose the most appropriate volatility for hedging under each of the following two constraints.

(i) Mark to model

(ii) Mark to market

Commentary on Question:

Candidates generally did well on this question.

Under the constraint of "Mark to model" where you are not concerned about the day-to-day fluctuations in the mark-to-market profit and loss, it is better to hedge with actual volatility if you are confident about estimating the actual volatilities. Its expected total profit is not far from the optimal payoff under hedging with implied vol and its standard deviation of final profit is zero.

Under the constraint of "Mark to Market" where you must worry about the short-term fluctuations of profit and loss, it is more appropriate to hedge with implied volatility under which you continuously make profit without much short-term fluctuation and annoyance from risk management despite the final profit is path dependent.

(c) Design a volatility arbitrage to make money assuming that your prediction is correct and that you hedge with actual volatility.

Commentary on Question:

Most candidates noted *<u>why</u>* *you should buy the call option, but not all did (i.e., the call was undervalued since actual volatility is higher than implied). Most candidates correctly wrote to buy the call option and sell the stock, although not all mentioned that the number of shares is determined by delta (N(d1)). Many candidates missed the last piece of the volatility arbitrage strategy – to invest the cash earning the risk-free rate or borrow paying the risk-free rate – and many candidates missed the fact that the volatility arbitrage needs to be rebalanced frequently.*

Because the predicted actual volatility is higher than the implied volatility, the call option is under-valued.

Thus, the volatility arbitrage strategy is to:

(a) Buy the call option
(b) Sell the stock XYZ by shares determined by the Delta N(d1) where d1 is calculated using actual volatility
(c) Invest the cash earning the risk-free rate or borrow paying the risk-free rate

The strategy needs to be executed and the delta hedge to be rebalanced as frequently as possible (e.g., daily)

(d) Calculate the final profit from the arbitrage executed in part (c).

Commentary on Question:

Many candidates did well here.

S=100, K=100, r=0%, T=1

σ(actual) = 30%; σ(implied) = 20%

$$F(S_t, t) = S_t N(d_1) - Ke^{-r(T-t)} N(d_2)$$

$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{\ln(S/K) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

Plug in all the values, the Black-Scholes formula for the call option can be simplified because of r=0, d=0, T=1, and S/K=1 c = 2*100*[Norm(0.15) – Norm(0.1)] = 3.958

$$F(S_t, t) = S_t N(d_1) - K e^{-r(T-t)} N(d_2)$$

$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{\ln(S/K) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

Value *x*	Standard Normal *N(x)*
0.10	0.5398
0.15	0.5596
0.20	0.5793
0.25	0.5987
0.30	0.6179

Simplied calculation

0.1500	0.1000
-0.1500	-0.1000
0.5596	0.5398
0.4404	0.4602

Call (actual vol)	Call (implied Vol)
11.9235	7.9656

Profit (hedging w/ actual vol)

	3.9580	3.958

QFI QF Fall 2023 Question 9

Learning Outcomes:

c) Understand delta hedging, and the interplay between hedging assumptions and hedging outcomes

e) Understand how hedge strategies may fail

Source References:

- *The Volatility Smile*, Derman, Emanuel and Miller, Michael, 2016, Chapter 3

Commentary on Question:

The majority of candidates performed poorly on this question. Particularly for parts (b) and (c), many candidates either performed poorly on or entirely skipped those questions. For those that did attempt the question, the most common mistakes were not being able to derive the Greeks for the Asian call option.

Solution:

(a)

(i) Identify the type of options which should be purchased.

(ii) Calculate the values in the table below, (assuming a Black-Scholes framework):

Commentary on Question:

Candidates performed adequately for this part. Most were able to identify that a geometric mean Asian option needed to be purchased.

RB = 1,000,000*(1- exp(-0.03)) = 29,554.4664515

Amount of ZCBs to buy = (1,000,000 – RB)/1000 = 970.4455335 (assuming each bond has a notional of $1,000)

The risk budget should be invested in long ATM geometric mean Asian call options with maturity of 1 year.

Using the BS framework, the value of a vanilla European call option is:

C_0^E = N(d1) $S_0 e^{-qt}$ – N(d2)*K*e^{-rt} where q= continuous dividend rate and d1 = $\frac{\ln\left(\frac{S_t}{k}\right)+\left(r-q+\frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}}$

Substituting in q = $-\frac{1}{2}\left(r-\frac{\sigma_a^2}{2}\right)+r$ and $\sigma_a = \frac{\sigma}{\sqrt{3}}$ gives:

C_0^a = N(d1) $S_0 e^{\left(\frac{1}{2}\left(r-\frac{\sigma_a^2}{2}\right)-r\right)t}$ – N(d2)*K*e^{-rt} and

d1 = $\frac{\ln\left(\frac{S_t}{k}\right)+\left(r-\left(\frac{-1}{2}\left(r-\frac{\sigma_a^2}{2}\right)+r\right)+\frac{\sigma_a^2}{2}\right)t}{\sigma_a\sqrt{t}}$ = $\frac{\ln\left(\frac{S_t}{k}\right)+\left(\frac{1}{2}\left(r-\frac{\sigma_a^2}{2}\right)+\frac{\sigma_a^2}{2}\right)t}{\sigma_a\sqrt{t}}$ = $\frac{\ln(1)+\left(\frac{1}{2}\left(0.03-\frac{\frac{0.2^2}{3}}{2}\right)+\frac{\frac{0.2^2}{3}}{2}\right)}{\frac{0.2}{\sqrt{3}}}$ = 0.158771324

d2 = d1 - $\sigma_a\sqrt{t}$ = 0. 0.158771324 – $\frac{0.2}{\sqrt{3}}$ = 0.04330127

$C_0^a = \mathrm{N}(0.158771324) * 100e^{\left(\frac{1}{2}\left(0.03-\frac{\frac{0.2^2}{3}}{2}\right)-0.03\right)}$ – N(0.04330127)*100*$e^{-0.03}$

= $0.563075 * 100e^{\left(\frac{1}{2}\left(0.03-\frac{\frac{0.2^2}{3}}{2}\right)-0.03\right)}$ – 0.517269*100$e^{-0.03}$

= 5.086478857

# of call options to purchase = RB/C = 29554.4664515/5.086478857 = 5810.398

$p_{dpa} = \frac{RB}{\frac{1,000,000C_0^a}{100}}$ = 29554.4664515/(10^4*5.086478857) = 0.581039798

(b)

(i) Determine the Vega of the Asian call options above.

(ii) Explain the value of the above Vega in relation to the Vega of a European call option and why this relation intuitively makes sense.

Commentary on Question:

Candidates performed poorly in this part. A large majority of candidates did not correctly derive the expression for the Vega of an Asian call option necessary for part (i). In part (ii), most candidates were able to correctly explain that the Vega of an Asian call is less than that of a European call.

C_0^a = N(d1) $S_0 e^{\left(\frac{1}{2}\left(r-\frac{\sigma_a^2}{2}\right)-r\right)t}$ – N(d2)*K*e^{-rt} = $S_0 e^{-rt}[\mathrm{N(d1)}e^{\frac{1}{2}\left(r-\frac{\sigma^2}{6}\right)} - N(d2)]$

By the product rule and chain rule:

$\frac{\partial C_0^a}{\partial \sigma} = S_0 e^{-rt}[\frac{-2\sigma}{12} N(d1) e^{\frac{1}{2}\left(r-\frac{\sigma^2}{6}\right)} + e^{\frac{1}{2}\left(r-\frac{\sigma^2}{6}\right)} n(d1) * \frac{\partial}{\partial \sigma} d1 - n(d2) * \frac{\partial}{\partial \sigma} d2]$

$$\frac{\partial}{\partial \sigma} d1 = \frac{\partial}{\partial \sigma} \frac{\left(\frac{1}{2}\left(0.03 - \frac{\frac{\sigma^2}{3}}{2}\right) + \frac{\frac{\sigma^2}{3}}{2}\right)}{\frac{\sigma}{\sqrt{3}}} = \frac{\partial}{\partial \sigma} \frac{\left(0.015 + \frac{\sigma^2}{12}\right)}{\frac{\sigma}{\sqrt{3}}} = \frac{-0.015 * \sqrt{3}}{\sigma^2} + \frac{\sqrt{3}}{12} = -0.50518$$

Note: d2 = d1 - $\frac{\sigma}{\sqrt{3}}$ -> $\frac{\partial}{\partial \sigma} d2 = \frac{\partial}{\partial \sigma} d1 - \frac{1}{\sqrt{3}} = -1.08253$

From Part A, N(d1) = N(0.158771324) = 0.563075

$\frac{\partial C_0^a}{\partial \sigma}$ = $100e^{-0.03}(\frac{-0.2}{6}$ *1.011735* 0.563075 + $(e^{\frac{1}{2}\left(r-\frac{\sigma^2}{6}\right)} n(d1) * -0.50518) - (n(d2) * -1.08253))$

n(d1) = $\frac{e^{\frac{-(0.158771324^2)}{2}}}{\sqrt{2\pi}}$= 0.393946, n(d2) = $\frac{e^{\frac{-(0.04330127^2)}{2}}}{\sqrt{2\pi}}$= 0.398568

$\frac{\partial C_0^a}{\partial \sigma}$ = $100e^{-0.03}$(-0.018899 + 1.011735 * 0.393946 * -0.50518 - 0.398568*-1.08253) = 20.48845

➔ Total Option Vega = 20.48845* 1,000,000/100 = 204,884.5

European call option Vega = $S_0 n(d1)$, d1 = $\frac{(r+\frac{\sigma^2}{2})}{\sigma}$=$\frac{(0.03+\frac{0.04}{2})}{0.2}$ = 0.25

Vega = 100*$\frac{e^{\frac{-0.25^2}{2}}}{\sqrt{2\pi}}$=38.66681168

The Vega of the European call option is greater than the Vega of the Asian option.

This relation makes sense since Asian options sample the underlying asset price across the entire option period rather than simply the final price, resulting in a shorter average duration for the impact of the volatility. Since volatility and its impact on option prices scales with time this results in a lower Vega.

Note: award ½ point for recognizing the European option Vega is larger if appropriate value or logic is given. Award second half point as long as the candidate references stock prices being sampled across the period rather than just the final price, and this resulting in a lower sensitivity to volatility.

(c) Determine an initial Delta-Vega hedge position using an ATM 1-year European call option and the underlying stocks.

Commentary on Question:

Candidates performed very poorly in this part. Many candidates skipped this question. For those that attempted the question, they were not able to correctly calculate the Greeks for the Asian call option.

Need to solve for position such that delta and Vega = 0

Vega of European Call Option = S_0 *N'(d1) * Sqrt(T-t)

d1 = $\frac{\ln\left(\frac{S_t}{k}\right)+\left(r-q+\frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}} = \frac{0+\left(0.03+\frac{0.2^2}{2}\right)}{0.2}$ = 0.25

N'(d1) = 0.386668

Vega = 100 * 0.386668 =38.6668

- ➔ Need Vega of option to equal 204,884.5
- ➔ Need to buy 204,884.5/38.6668 = 5,298.718798 ATM 1 year euro call options

Delta of European Call Option = N(d1) = 0.598706

Delta of Asian option:

C_0^a = N(d1) $Se^{\left(\frac{1}{2}\left(r-\frac{\sigma_a^2}{2}\right)-r\right)t}$ – N(d2)*K*e^{-rt}

= $e^{-rt}[S\mathrm{N(d1)}e^{\frac{1}{2}\left(r-\frac{\sigma^2}{6}\right)} - KN(d2)]$

By the product rule and chain rule:

$$\frac{\partial C_0^a}{\partial S} = e^{-rt}[\mathrm{N(d1)}e^{\frac{1}{2}\left(r-\frac{\sigma^2}{6}\right)} + Se^{\frac{1}{2}\left(r-\frac{\sigma^2}{6}\right)}n(d1) * \frac{\partial}{\partial S}d1 - K\,n(d2) * \frac{\partial}{\partial S}d2]$$

$$\frac{\partial}{\partial S}d1 = \frac{\ln(S) - \ln(k) - f(\mathrm{r},\sigma)}{\sigma_a} = \frac{1}{\sigma_a S}$$

$$\frac{\partial}{\partial S}d2 = \frac{\partial}{\partial S}(d1 - \sigma) = \frac{1}{\sigma_a S}$$

Note: since k =S_0

$$\frac{C_0^a}{S} = e^{-rt}[\mathrm{N(d1)}e^{\frac{1}{2}\left(r-\frac{\sigma^2}{6}\right)} + \frac{\mathrm{S}e^{\frac{1}{2}\left(r-\frac{\sigma^2}{6}\right)}n(d1) - S_0 n(d2)}{\sigma_a S}]$$

$$\frac{\partial C_0^a}{\partial S} = e^{-0.03}\left[0.563075 * 1.011735 + \frac{1.011735 * n(d1) - n(d2)}{0.2}\right]$$

$$= e^{-0.03}\left[0.563075 * 1.011735 + \frac{1.011735 * 0.393946 - 0.398568}{0.2}\right]$$

= 0.970446*[0.569683163 + 0] = 0.552846

Portfolio Delta = Asían Option # *Asían Option Delta – European Option Delta

=0.552846 *1,000,000/100 - 5,298.718798 *0.598706= 2356.09007

Delta of Stock = 1

Need to sell 2356.09007 shares of stock.

QFI QF Fall 2023 Question 10

Learning Outcomes:

f) Identify and evaluate embedded options in liabilities (e.g., indexed annuity, structured product based variable annuity, and variable annuity guarantee riders including GMxB, etc.)

g) Demonstrate an understanding of hedging for embedded option in liabilities

Source References:

- INV201-105-25: An Introduction to Computational Risk Management of Equity-Linked Insurance, Feng, 2018 (sections 1.2-1.3, 4.7 & 6.2-6.3)
- INV201-108-25: Mitigating Interest Rate Risk in Variable Annuities: An Analysis of Hedging Effectiveness under Model Risk

Commentary on Question:

This question is testing the candidates' ability to recognize embedded option in a variable annuity contract with a GMDB rider and derive a delta-rho hedge for it. In addition, it tests the candidates' knowledge of how the difference between the model and actual outcomes affect the hedging results for this product. Overall, the attempt rate for this question was low, especially for parts a) and b) which involve calculations.

Solution:

(a) Derive the no-arbitrage value of the net liability L_t at time *t*.

Commentary on Question:

Most candidates did not attempt this part of the question. To earn points for this question, candidates needed to manipulate the given equation for $\mathrm{L_t}$ and derive the equation for the expected value. Partial points were awarded to the candidates who successfully took the calculation further than copying down the given equation, mostly for recognizing that $E\left[e^{-r(s-t)}\max(G-F_s,0)\right]$ is a put option and writing down the value.

Net Liability = Expected PV of benefits – Expected PV of fee income, which is given:

$$
\begin{aligned}
L_t = {}_tp_x(\Omega_t - \Upsilon_t) - {}_tp_xE^{\mathbb{Q}}\left[\int_t^T mF_se^{-r(s-t)}{}_{s-t}p_{x+t}ds\right] \\
= {}_tp_xE^{\mathbb{Q}}\left[\int_t^T e^{-r(T-t)}\max(G - F_s, 0)_{s-t}p_{x+t}\mu_{x+s}ds\right] \\
- {}_tp_xE^{\mathbb{Q}}\left[\int_t^T mF_se^{-r(s-t)}{}_{s-t}p_{x+t}ds\right]
\end{aligned}
$$

The no-arbitrage value of the net liability is the expected value with respect to the risk neutral measure. Due to independence of mortality and equity return, the first term can be written as

$$
\begin{aligned}
{}_tp_xE^{\mathbb{Q}}\left[\int_t^T e^{-r(T-t)}\max(G - F_s, 0)_{s-t}p_{x+t}\mu_{x+s}ds\right] \\
= \int_t^T E^{\mathbb{Q}}[e^{-r(T-t)}\max(G - F_s, 0)]\ {}_tp_{x\,s-t}p_{x+t}\mu_{x+s}ds \\
= \int_t^T E^{\mathbb{Q}}[e^{-r(T-t)}\max(G - F_s, 0)]\ {}_sp_x\mu_{x+s}ds
\end{aligned}
$$

Since $F_t = S_te^{-mt}$

$$
\begin{aligned}
dF_t = -me^{-mt}S_tdt + e^{-mt}dS_t = -me^{-mt}S_tdt + re^{-mt}S_tdt + \sigma e^{-mt}S_tdW_t \\
= e^{-mt}S_t(r - m)dt + \sigma e^{-mt}S_tdW_t = F_t(r - m)dt + \sigma F_tdW_t
\end{aligned}
$$

$E\left[e^{-r(s-t)}\max(G - F_s, 0)\right]$ is the no-arbitrage price of a put option, thus

$$
Value\ of\ benefits = \int_t^T \left[Ge^{-r(s-t)}N(-d_2) - S_te^{-m(s-t)}N(-d_1)\right]\ {}_sp_x\mu_{x+s}ds
$$

where $d_1 = \frac{ln\frac{S_t}{G} + (s-t)(r-m+\frac{\sigma^2}{2})}{\sigma\sqrt{s-t}}$ and $d_2 = d_1 - \sigma\sqrt{s-t}$.

The second term is the PV of fees:

$$
{}_tp_xE^{\mathbb{Q}}\left[\int_t^T mF_se^{-r(s-t)}{}_{s-t}p_{x+t}ds\right] = E\left[\int_t^T e^{-r(s-t)}mF_s\ {}_sp_xds\right] = \int_t^T E\left[e^{-r(s-t)}mF_s\ {}_sp_x\right]ds = \int_t^T E\left[e^{-r(s-t)}S_s\right]e^{-ms}m\ {}_sp_xds = \int_t^T S_te^{-ms}m\ {}_sp_xds = mF_t\int_t^T e^{-m(s-t)}\ {}_sp_xds
$$

(b) Derive the positions of stock, zero-coupon bond and money market account for a portfolio Π_t that hedges the Delta and Rho of the net liability in part (a).

Commentary on Question:

A lot of candidates did not attempt this part of the question. To earn points for this part, candidates needed to correctly describe the positions in underlying, bond, and money market account to set up the hedge, and derive the equation for the positions, especially for ρ_t and B_t. Partial points were awarded for describing the hedge, although most candidates did not finish the derivation of the positions.

To hedge delta and rho of L_t, invest in

- Δ_t share of the underlying S_t and $\Delta_t = \frac{\partial L_t}{\partial S_t}$, which is given
- ρ_t unit in the zero-coupon bond and $\rho_t = \frac{\partial L_t / \partial r}{\partial P_t / \partial r}$
- B_t in the money market account
- $\Pi_t = \Delta_t S_t + \rho_t P_t + B_t = L_t$

$$\rho_t = \frac{\partial L_t / \partial r}{\partial P_t / \partial r} = \frac{-\int_t^T G(s-t)e^{-r(s-t)}N(-d_2)\ {}_sp_x\mu_{x+s}ds}{-(T-t)e^{-r(T-t)}} = $$
$$= \frac{\int_t^T G(s-t)e^{-r(s-t)}N(-d_2)\ {}_sp_x\mu_{x+s}ds}{(T-t)e^{-r(T-t)}}$$

And

$$\begin{aligned} B_t &= L_t - \Delta_t S_t - \rho_t P_t \\ &= \int_t^T \left[Ge^{-r(s-t)}N(-d_2) - S_t e^{-m(s-t)}N(-d_1)\right]\ {}_sp_x\mu_{x+s}ds \\ &\quad - mF_t\int_t^T e^{-m(s-t)}\ {}_sp_x ds - S_t\int_t^T Ge^{-m(s-t)}[N(d_1)-1]\ {}_sp_x\mu_{x+s}ds \\ &\quad + mS_t\int_t^T e^{-ms}\ {}_sp_x ds - \frac{\int_t^T G(s-t)e^{-r(s-t)}N(-d_2)\ {}_sp_x\mu_{x+s}ds}{(T-t)e^{-r(T-t)}}e^{-r(T-t)} \end{aligned}$$

$$mS_t\int_t^T e^{-ms}\ {}_sp_x ds = mF_t e^{mt}\int_t^T e^{-ms}\ {}_sp_x ds = mF_t\int_t^T e^{-m(s-t)}\ {}_sp_x ds$$

Thus

$$B_t = \int_t^T \Big[Ge^{-r(s-t)}N(-d_2) - S_t e^{-m(s-t)}N(-d_1) - S_t G e^{-m(s-t)}[N(d_1) - 1] - \frac{s-t}{T-t} Ge^{-r(s-t)}N(-d_2) \Big] \ {}_s p_x \mu_{x+s} ds$$

$$B_t = \int_t^T \left[\frac{T-s}{T-t} Ge^{-r(s-t)}N(-d_2) - S_t e^{-m(s-t)}N(-d_1)(1-G) \right] \ {}_s p_x \mu_{x+s} ds$$

(c) Describe the hedging effectiveness you expect to observe under each of the 3 models of simulating interest rates (specified in the table above). Explain your reasoning.

Commentary on Question:

Many of the candidates who attempted this question did relatively well, as they were able to correctly order the three models for their hedging effectiveness and describe reasoning for their response. But some candidates did not properly understand the question and directly compared the pros and cons of the three models for interest rate hedging, which did not earn points.

For the control where interest rate is not simulated stochastically, and follows the deterministic path r_t, the hedging is expected to be effective, since the hedging model is the same as the simulation model used for the assessment. Hedging gain/loss at time T should be small.

For interest rate model option 1, interest rates are simulated stochastically, instead of the deterministic interest r_t which is used to develop the hedge. Higher hedging errors are expected at maturity T due the model risk that deterministic assumption r_t does not capture all the variabilities in the simulated interest scenarios.

For interest rate model option 2, additional difference between the simulation model and the hedging model is introduced due to the additional factors in the simulating model, which allows yield curve to take on different shapes. Thus, the hedging error is expected to be higher than option 1.

(d) Describe changes in hedging effectiveness in comparison to part (c) for the Interest rate model 1 and the Interest rate model 2.

Commentary on Question:

Some candidates did well on this part, though many did not properly understand the question and made general comparison of the two models for interest rate hedging, which did not earn points.

For interest rate model option 1, using the one-factor Vasicek model to develop the hedge should improve the hedging effectiveness when compared to using deterministic r_t. GMDB is more impacted by the long-term trend in the interest rate, which is relatively well captured by the one-factor Vasicek model compared to the simulation model of CIR. Thus, using a stochatic model for developing the hedge reduces the model risk vs. the simulation model and improves the hedging results.

For interest rate model option 2, using the one-factor Vasicek model to develop the hedge may not have significant improvement on the hedging results. As the simulation model has a lot more flexibility with three factors and can produce simulations with more variability in the shape of the term structure, using a one-factor stochastic model to develop the hedge does not significantly reduce the model risk vs. the simulation model, when compared to a deterministic r_t.

QFI QF Fall 2024 Question 8

Learning Outcomes:

c) Understand delta hedging, and the interplay between hedging assumptions and hedging outcomes

d) Understand the concepts of realized versus implied volatility.

e) Understand derivatives mishaps

Source References:

- *The Volatility Smile*, Derman, Emanuel and Miller, Michael, 2016, Chapters 3, 5-7

Commentary on Question:

This question tests the candidates' understanding of the various kinds of volatilities and the interplay between the hedging assumptions vs. outcomes under a theoretical delta hedge construct. To do well on this question, the candidates need to demonstrate understanding of both the mathematical derivations of the hedging results with realized and implied volatilities, as well as the conclusions and implications under different paths of the underlying asset that could materialize.

Solution:

(a) Calculate the gain or loss of the hedged portfolio dV_t over an infinitesimal period dt.

Commentary on Question:

Many candidates showed partial understanding of how to derive dV_t leveraging Taylor's expansion and Black-Schole Equation, though only some are able to complete all the steps. Partial marks are given in these cases.

The delta hedged portfolio has value $V_t = C_t - \Delta S_t$ at time t, where $\Delta = \frac{\partial C_t}{\partial S_t}$, thus

$$dV_t = dC_t - \Delta dS_t - rV_t dt$$

where the last term represents the borrowing cost of the hedge. Using Taylor's expansion of the call price:

$$dC_t = \frac{\partial C_t}{\partial t}dt + \frac{\partial C_t}{\partial S_t}dS_t + \frac{1}{2}\frac{\partial^2 C_t}{\partial S_t^2}dS_t^2$$

Thus,

$$\begin{aligned} dV_t = dC_t - \frac{\partial C_t}{\partial S_t}dS_t - rV_t dt &= \frac{\partial C_t}{\partial t}dt + \frac{1}{2}\frac{\partial^2 C_t}{\partial S_t^2}dS_t^2 - rV_t dt \\ &= \frac{\partial C_t}{\partial t}dt + \frac{1}{2}\frac{\partial^2 C_t}{\partial S_t^2}\sigma_R^2 S_t^2 dt - rV_t dt \end{aligned}$$

Based on the Black-Schole Equation, value of the call C_t should satisfy the following equation with the implied volatility Σ:

$$\frac{\partial C_t}{\partial t} + rS_t\frac{\partial C_t}{\partial S_t} + \frac{1}{2}\frac{\partial^2 C_t}{\partial S_t^2}\Sigma^2 S_t^2 = rC_t$$

Thus

$$\begin{aligned} dV_t &= \left(rC_t - rS_t\frac{\partial C_t}{\partial S_t} - \frac{1}{2}\frac{\partial^2 C_t}{\partial S_t^2}\Sigma^2 S_t^2\right)dt + \frac{1}{2}\frac{\partial^2 C_t}{\partial S_t^2}\sigma_R^2 S_t^2 dt - rV_t dt \\ &= \frac{1}{2}\frac{\partial^2 C_t}{\partial S_t^2}(\sigma_R^2 - \Sigma^2)S_t^2 dt + \left(C_t - S_t\frac{\partial C_t}{\partial S_t}\right)rdt - rV_t dt \\ &= \frac{1}{2}\frac{\partial^2 C_t}{\partial S_t^2}(\sigma_R^2 - \Sigma^2)S_t^2 dt \end{aligned}$$

(b)

(i) Prove that the gain or loss of the hedged portfolio dV_t over an infinitesimal period dt is $dV_t = e^{rt} d[e^{-rt}(C_t - C_t^R)]$

(ii) Derive the present value of the total gain or loss to maturity $\int_t^T e^{r(s-t)} dV_s$.

Commentary on Question:

Many candidates are able to partially solve the question with partial marks awarded. For part b), a minus sign is missing in the exponent (i.e. present value of total gain or loss should be $\int_t^T e^{-r(s-t)} dV_s$). Points are awarded if the candidates either followed the equation given or used the correct sign in their solutions themselves.

i) The delta hedged portfolio has value $V_t = C_t - \Delta_R S_t$ at time t, where Δ_R is computed using realized volatility.

$$dV_t = dC_t - \Delta_R dS_t - rV_t dt = dC_t - \Delta_R dS_t - r(C_t - \Delta_R S_t)dt$$
$$= dC_t - rC_t dt - \Delta_R(dS_t - rS_t dt)$$

If C_t is replaced by C_t^R in the above equation, then the hedged portfolio becomes risk-less, and

$$dC_t^R - rC_t^R dt - \Delta_R(dS_t - rS_t dt) = 0$$

$$dC_t^R - rC_t^R dt = \Delta_R(dS_t - rS_t dt)$$

Substituting back into the equation for dP_t, then

$$dV_t = dC_t - rC_t dt - dC_t^R + rC_t^R dt$$

Apply product rule on the right hand side of the given equation

$$e^{rt} d[e^{-rt}(C_t - C_t^R)] = e^{rt}[-re^{-rt} dt(C_t - C_t^R) + e^{-rt} d(C_t - C_t^R)]$$
$$= -r(C_t - C_t^R)dt + d(C_t - C_t^R) =] = dC_t - rC_t dt - dC_t^R + rC_t^R dt$$

Thus $dV_t = e^{rt} d[e^{-rt}(C_t - C_t^R)]$.

ii)

$$\int_t^T e^{-(s-t)r} dV_s = \int_t^T e^{-(s-t)r} e^{rs} d[e^{-rs}(C_s - C_s^R)] = \int_t^T e^{rt} d[e^{-rs}(C_s - C_s^R)]$$
$$= e^{rt}[e^{-rT}(C_T - C_T^R) - e^{-rt}(C_t - C_t^R)]$$

At maturity, value of the option is equal to the instrinsic value, i.e. $C_T = C_T^R = \max[S_T - K, 0]$, thus

$$\int_t^T e^{-(s-t)r} dV_s = C_t^R - C_t$$

(c) Compare $\int_0^{100} e^{r(s-t)} dV_s$ between the two paths if they materialize respectively, assuming

(i) The portfolio is delta-hedged based on σ_R.

(ii) The portfolio is delta-hedged based on Σ.

Commentary on Question:

Many candidates remembered the conclusions of how the hedged portfolios would behave if delta-hedged based on σ_R vs. Σ. However, many could not apply the textbook knowledge to the given construct, and misunderstood the question as that the graphs given are paths of the hedged portfolios instead of the underlying asset. Partial marks are still given for the correct knowledge from the textbook.

From part b), if the portfolio is delta-hedged based on σ_R, $\int_0^{100} e^{-(s-t)r} dV_s = C_0^R - C_0$. Values of C_0^R and C_0 are independent of the path of S_t that materializes, and thus the value is the same between the two paths.

From part a), if the portfolio is delta-hedged based on Σ,

$$dV_t = \frac{1}{2} \frac{\partial^2 C_t}{\partial S_t^2} (\sigma_R^2 - \Sigma^2) S_t^2 dt$$

The infinitesimal gain or loss on the hedged portfolio is proportional to $(\sigma_R^2 - \Sigma^2)$ by the ratio of $\frac{1}{2} \frac{\partial^2 C_t}{\partial S_t^2} S_t^2$, which is dependent on the level of S_t, thus in this case, $\int_0^{100} e^{-(s-t)r} dV_s$ would be different between the two paths. Gamma of call options is the highest when S_t is close to the strick price, and decrease as S_t moves further into or out of money. The level of S_t is also lower for path 1. Thus $\int_0^{100} e^{-(s-t)r} dV_s$ should be lower for path 1 than path 2 in this case.

QFI QF Fall 2024 Question 12

Learning Outcomes:

f) Identify and evaluate embedded options in liabilities (e.g., indexed annuity, structured product based variable annuity, and variable annuity guarantee riders including GMxB, etc.)

g) Demonstrate an understanding of hedging for embedded option in liabilities

Source References:

- INV201-105-25: An Introduction to Computational Risk Management of Equity-Linked Insurance, Feng, 2018 (sections 1.2-1.3, 4.7 & 6.2-6.3)

Commentary on Question:

This question tests the candidates' ability to apply theories in quantitative finance to the valuation and risk management of a variable annuity with GMAB option. Specifically, a candidate needs to apply the properties of an equity return process following a Geometric Brownian Motion to derive the guarantee and cap rate under given contexts, price a GMAB option with cliquet feature, and critique on probabilistic statements based on differences in the risk neutral and real-world measures. Many candidates did not make an attempt for this question.

Solution:

(a) Show that the guarantee rate is 1.25%.

Commentary on Question:

The candidates performed below average on this section. While many candidates were able to derive the return process under the participation feature, few candidates correctly derived the expression of the price expectation.

From the given stock price process, $S(T)^\alpha$ also follows a Geometric Brownian Motion with the drift and volatility terms scaled by a factor of α. Therefore, we have

$$\mathbb{E}[S(T)^\alpha] = \mathbb{E}\left[e^{\alpha\left(r-\frac{\sigma^2}{2}\right)T+\alpha\sigma W_T}\right]$$

$$= e^{\alpha\left(r-\frac{\sigma^2}{2}\right)T}\mathbb{E}[e^{\alpha\sigma W_T}]$$

$$= e^{\alpha\left(r-\frac{\sigma^2}{2}\right)T}e^{\frac{\alpha^2\sigma^2}{2}T}$$

$$= e^{\alpha\left(r+\frac{1}{2}(\alpha-1)\sigma^2\right)T}$$

Substituting in the parameters, we get

$$\mathbb{E}[S(T)^{\alpha}] = e^{0.5\left(.04+\frac{1}{2}(0.5-1)(0.2)^2\right)T} = e^{.015T}$$

Since the guaranteed rate is 25 bps lower than the expected return under the participation factor, we have:

Guaranteed rate = 1.5%-0.25%=1.25%

(b) Derive the $\mathbb{Q}$-probability that the EIA credits the guaranteed rate in a single year.

Commentary on Question:

The candidates did poorly on this section. Among the few reasonable attempts made on this question, common mistakes include having the inequality reversed and misinterpreting the definition of the guarantee rate.

Using the results from part (a), we have:

$$\begin{aligned}
\Pr(S(T)^{\alpha} \le e^{.0125T}) &= \Pr\left(e^{\alpha\left(r-\frac{\sigma^2}{2}\right)T+\alpha\sigma W_T} \le e^{.0125T}\right) \\
&= \Pr\left(\alpha\left(r-\frac{\sigma^2}{2}\right)T + \alpha\sigma W_T \le .0125T\right) \\
&= \Pr\left(W_T \le \frac{.0125T - \alpha\left(r-\frac{\sigma^2}{2}\right)T}{\alpha\sigma}\right) = \Phi\left(\frac{.0125T - \alpha\left(r-\frac{\sigma^2}{2}\right)T}{\alpha\sigma\sqrt{T}}\right) \\
&= \Phi\left(\frac{.0125 - 0.5\left(.04-\frac{0.2^2}{2}\right)}{0.5(0.2)}\right) = \Phi(0.025) \approx 51\%
\end{aligned}$$

(c) Derive the appropriate cap rate.

Commentary on Question:

The candidates performed poorly on this section. Few candidates correctly established the required probability expression.

The question asks for the value of cap rate *c* such that

$$\Pr(S(T)^{\alpha} > e^{cT}) = 1 - \Pr(S(T)^{\alpha} \le e^{cT}) = 0.10$$

Similar to the steps in the solution to part (b), we have

$$\Pr(S(T)^\alpha \le e^{cT}) = 0.90 \Rightarrow \Phi\left(\frac{c - 0.5\left(.04 - \frac{0.2^2}{2}\right)}{0.5(0.2)}\right) = 0.9$$

$$\Rightarrow 10c - 0.1 = 1.28$$

$$\Rightarrow c = 13.8\%$$

(d) Calculate the risk-neutral price for a 5-year cliquet EIA.

Commentary on Question:

The candidates performed poorly on this section. Most of the candidates left it unanswered.

Let g denote the guarantee rate, we have

$$Price_{1-yr\ EIA} = e^{-r}\left\{\mathbb{E}\left[e^g\ \mathbb{I}_{\left\{\ln S(1) \le \frac{g}{\alpha}\right\}} + S(1)^\alpha \mathbb{I}_{\left\{\frac{g}{\alpha} < \ln S(1) \le \frac{c}{\alpha}\right\}} + e^c \mathbb{I}_{\left\{\ln S(1) > \frac{c}{\alpha}\right\}}\right]\right\}$$

From part (c) we know that:

$$\Pr(S(T)^\alpha \le e^{cT}) = \Phi(10c - 0.1) = \Phi(1.4 - 0.1) = \Phi(1.3) = 0.9032$$

$$\Rightarrow \Pr(S(T)^\alpha > e^{cT}) = 0.0968$$

From part (b) we know that:

$$\Pr(S(T)^\alpha \le e^{gT}) = 0.51$$

In addition, from the given stock price process, we know that:

$$S(1)^\alpha \sim LN\left(\alpha\left(r - \frac{\sigma^2}{2}\right), \alpha\sigma\right)$$

Hence:

$$\mathbb{E}\left[S(1)^{0.5}\mathbb{I}_{\{.025 < \ln S(1) \le .280\}}\right]$$

$$= e^{\alpha\left(r - \frac{\sigma^2}{2}\right) + \frac{\alpha^2\sigma^2}{2}}\left[\Phi\left(\frac{0.14 - 0.5\left(0.04 - \frac{0.2^2}{2}\right)}{0.5(0.2)} - (0.5)(0.2)\right) - \Phi\left(\frac{0.0125 - 0.5\left(0.04 - \frac{0.2^2}{2}\right)}{0.5(0.2)} - (0.5)(0.2)\right)\right]$$

Combining all these results together, we have:

$$Price_{1-yr\ EIA} = 1.007728$$

The price of a 5-year cliquet is thus given by:

$$Price_{5-yr\ cliquet\ EIA} = \left(Price_{1-yr\ EIA}\right)^5 = (1.007728)^5 = 1.039242506$$

(e) Critique the following statement made by your analyst:

"By setting the cap rate such that the probability that $S(T)^\alpha$ exceeds cap rate is no more than 10% in a single year, we should expect to pay the cap rate approximately once every ten years."

Commentary on Question:

The candidates performed poorly on this section. While many candidates pointed out that the statement is incorrect, very few were able to give the proper rationale based on the different nature of real-world vs. risk neutral measures.

Disagree with the statement. The cap rate was set such that the risk-neutral probability of the single year return hitting the cap is 10%. Actual observation will abide by the real-world measure, which should be higher than 10% given an appropriate risk premium.