GUIDED EXAMPLES

INV 201 – Quantitative Finance

Important Information:

- These guided examples are intended to enhance specific curriculum resources where additional examples, practice calculations, and/or application of material could benefit candidates. These are not part of the required syllabus but are intended to make the required syllabus topics easier to master. These examples may be longer, more in depth, and/or include more calculation than would likely be used in an assessment environment.
- These guided examples are presented in two formats – a version where candidates can attempt to navigate the problem/situation independently, and a narrated version where a solution is presented along with assistance to explain the steps involved.
- These guided examples present one method of arriving at a solution; there could be equally appropriate alternative solutions.
- These guided examples are not intended to approximate a course assessment, and candidates should not use them as proxies for assessment items. For examples of assessment items, we recommend referencing the curated past exam questions for this course.
- Candidates should note that this course has additional guided examples in a companion Excel file.
- These guided examples have been developed by Coaching Actuaries, with review and modifications by course curriculum committee volunteers and SOA staff. We will continue to refine and expand this example set over time; candidates who would like to recommend source material that could benefit from additional guided examples should reach out to: **education@soa.org**.

Version 2025-1.1 Updated: July 9, 2025

Version history:

Version 1 published July 8, 2025

Version 1.1 published July 9, 2025 – typographical correction made to second payoff domain for Question 1.

Question 1 – The Black Scholes Merton Model

Resource:

- Hull Chapter 15 (Sections 15.7 – 15.9)

Learning Objectives:

2. The candidate will understand the principles and techniques for the valuation of derivatives.

Learning Outcomes:

2e) The candidate will be able to understand and apply the concepts of risk-neutral measure, forward measure, normalization, and the market price of risk.

2f) The candidate will be able to understand option pricing techniques including calculating an expectation by solving a PDE.

Question:

Assume the Black-Scholes-Merton framework. You are given:

- $S(t)$ is the time-t price of the stock. $S(0) = 100$.
- The stock pays no dividends.
- The volatility of the stock is 30% per annum.
- The continuously compounded risk-free interest rate is 7%.

A derivative security has the following payoff at time 1:

$$\text{Payoff} = \begin{cases} 90 - S(1) & \text{if } S(1) \leq 90 \\ 0 & \text{if } 90 < S(1) \leq 110 \\ S(1) - 110 & \text{if } S(1) > 110 \end{cases}$$

Calculate the time-0 price of the derivative security.

Solution:

The payoff of the derivative security can be decomposed into two components:

$$\begin{aligned}\text{Payoff} &= \begin{cases} 90 - S(1), & S(1) \le 90 \\ 0, & S(1) > 90 \end{cases} + \begin{cases} 0, & S(1) \le 110 \\ S(1) - 110, & S(1) > 110 \end{cases} \\ &= \max[0, 90 - S(1)] + \max[0, S(1) - 110]\end{aligned}$$

Notice that the payoff of the derivative security is equivalent to the combined payoffs of:

- A 1-year 90-strike European put, and
- A 1-year 110-strike European call.

Using risk-neutral valuation, the time-0 price of the derivative security, denoted as f, is the risk-neutral expected payoff, discounted at the risk-free interest rate:

$$f = e^{-rT}\hat{E}[\text{Payoff}] = e^{-rT}\hat{E}[\max[0, 90 - S(1)] + \max[0, S(1) - 110]]$$

Notice that:

- $e^{-rT}\hat{E}[\max[0, 90 - S(1)]]$ represents the price of a 1-year 90-strike European put
- $e^{-rT}\hat{E}[\max[0, S(1) - 110]]$ represents the price of a 1-year 110-strike European call

In other words, to calculate the price of the derivative security, we can sum the prices of the European call and put.

- Calculate the price for 1-year 90-strike European put:

$$\begin{aligned} d_1 &= \frac{\ln\frac{100}{90} + \left(0.07 + \frac{1}{2}(0.30^2)\right)(1)}{0.30\sqrt{1}} = 0.73454 \\ d_2 &= 0.73454 - 0.30\sqrt{1} = 0.43454 \\ N(-d_1) &= 0.23131 \\ N(-d_2) &= 0.33195 \\ &\therefore p = 90e^{-0.07(1)}(0.33195) - 100(0.23131) = 4.725 \end{aligned}$$

- Calculate the price for 1-year 110-strike European call:

$$\begin{aligned} d_1 &= \frac{\ln\frac{100}{110} + \left(0.07 + \frac{1}{2}(0.30^2)\right)(1)}{0.30\sqrt{1}} = 0.06563 \\ d_2 &= 0.06563 - 0.30\sqrt{1} = -0.23437 \\ N(d_1) &= 0.52616 \\ N(d_2) &= 0.40735 \\ &\therefore c = 100(0.52616) - 110e^{-0.07(1)}(0.40735) = 10.837 \end{aligned}$$

Finally, the time-0 price of the derivative security is:

$$f = p + c = 4.725 + 10.837 = \boxed{15.56}$$

Question 2 – The Black Scholes Merton Model

Resource:

- Hull Chapter 15 (Section 15.12)

Learning Objectives:

2. The candidate will understand the principles and techniques for the valuation of derivatives.

Learning Outcomes:

2e) The candidate will be able to understand and apply the concepts of risk-neutral measure, forward measure, normalization, and the market price of risk.
2f) The candidate will be able to understand option pricing techniques including calculating an expectation.

<u>Question:</u>

You are tasked with pricing an American call option.

a. Briefly explain under what circumstances it is not optimal to exercise an American call option.

Assume the Black-Scholes-Merton framework. You are given:

- $S(t)$ is the time-t price of the stock. $S(0) = 80$.
- The stock pays a dividend of \$2 in six months and a dividend of \$5 eighteen months from now.
- The volatility of the stock is 25% per annum.
- The continuously compounded risk-free interest rate is 5%.

Consider a one-year American call option on the stock with a strike price of \$82.

b. Determine whether early exercise is optimal for this American call option.

c. Calculate the price of the American call option.

Solution:

Part a:

In the absence of dividends, American call options should never be exercised early.

If $D_n \leq K\left[1 - e^{-r(T-t_n)}\right]$, it cannot be optimal to exercise at time t_n.

Part b:

Since the dividend of $5 occurs 18 months from now, which is beyond the option's expiration, it does not affect the option price.

The relevant dividend is the $2 dividend paid at time $t = 0.5$.

To determine whether early exercise is optimal at $t = 0.5$, we use the early exercise criterion:

$$D_n \leq K\left[1 - e^{-r(T-t_n)}\right]$$

In this problem:

- $D_n = 2$
- $K\left[1 - e^{-r(T-t_n)}\right] = 82\left[1 - e^{-0.05(1-0.5)}\right] = 2.025$

Since $D_n \leq K\left[1 - e^{-r(T-t_n)}\right]$, early exercise is not optimal.

Part c:

Since early exercise is not optimal, the American call option should be priced as a European call option.

Step 1: Adjust Stock Price for Dividend

Under the Black-Scholes-Merton model, the stock price is reduced by the present value of all dividends paid during the option's life:

$$S_{\text{adjusted}} = 80 - 2e^{-0.05 \cdot 0.5} = 78.0494$$

Step 2: Compute Black-Scholes Parameters and the Call Option Price

$$d_1 = \frac{\ln \frac{78.0494}{82} + \left(0.05 + \frac{1}{2}(0.25^2)\right)(1)}{0.25\sqrt{1}} = 0.12749$$

$$d_2 = 0.12749 - 0.25\sqrt{1} = -0.12251$$

$$N(d_1) = 0.55072$$

$$N(d_2) = 0.45125$$

Finally, the price of the American call option is:

$$c = 78.04938(0.55072) - 82e^{-0.05(1)}(0.45125) = \boxed{7.7855}$$

Question 3 – The Greek Letters

Resource:

- Hull Chapter 19 (Section 19.4)

Learning Objectives:

3. The candidate will understand various applications and risks of derivatives.

Learning Outcomes:

3a) The candidate will be able to understand the Greeks of derivatives.

3b) The candidate will be able to understand static and dynamic hedging.

3c) The candidate will be able to understand delta hedging, and the interplay between hedging assumptions and hedging outcomes.

<u>Question:</u>

You are given the following information for call options on a stock:

Type	Strike Price	Day	Option Price	Delta
Call	50	0	\$6.04	0.55165
Call	55	0	\$4.17	0.43299
Call	50	1	\$7.80	0.62102
Call	55	1	\$5.58	0.50585

The stock price is \$50 on day 0 and \$53 on day 1. The continuously compounded risk-free interest rate is 8% per year.

A market-maker writes 1,000 units of a bull spread, each consisting of a long call option with a strike price of \$50 and a short call option with a strike price of \$55. The underlying stock does not pay dividends.

The market-maker delta-hedges the bull spread using shares of stock and finances any necessary funds by borrowing at the risk-free rate, ensuring that the initial net investment is zero. The hedge is adjusted daily, with interest payments made on any borrowed funds.

Assume there are 365 days in a year.

Calculate the investment required on day 1 to maintain the delta hedge.

Solution:

On Day 0:

The initial delta of a short position in 1,000 units of the bull spread is:

$$-1{,}000 \times (+0.55165 - 0.43299) = -118.66$$

To delta-hedge the position, we set the total portfolio delta to zero:

$$\begin{aligned} -118.66 + N_{\text{Stock}} \cdot \Delta_{\text{Stock}} &= 0 \\ -118.66 + N_{\text{Stock}} \cdot 1 &= 0 \\ N_{\text{Stock}} &= 118.66 \end{aligned}$$

Thus, the market-maker buys 118.66 shares of stock.

The initial net investment is:

$$1{,}000 \times (-6.04 + 4.17) + 118.66 \times 50 = +4{,}063$$

On Day 1:

The delta of a short position in the bull spread is updated to:

$$-1{,}000 \times (+0.62102 - 0.50585) = -115.17$$

Let x be the number of additional shares the market-maker must buy to maintain the delta hedge.

The new delta-hedged portfolio consists of:

- A short position in 1,000 bull spreads
- A long position in $(118.66 + x)$ shares of stock

Since a delta-hedged portfolio must have zero net delta, we set:

$$\begin{aligned} -115.17 + (118.66 + x) \cdot \Delta_{\text{Stock}} &= 0 \\ -115.17 + (118.66 + x) \cdot 1 &= 0 \\ x &= -3.49 \end{aligned}$$

Thus, to re-hedge on day 1, the market-maker must sell 3.49 shares.

The proceeds from selling the shares:

$$3.49 \times 53 = 184.97$$

The market-maker also pays interest on the previous day's investment of \$4,063 at a continuously compounded rate of 8% per year:

$$4{,}063 \times \left(e^{0.08/365} - 1\right) = 0.89$$

The total investment required on day 1 is:

$$-184.97 + 0.89 = \boxed{-184.08}$$

Thus, the market-maker **receives \$184.08 on day 1**.

Question 4 – The Greek Letters

Resource:

- Hull Chapter 19 (Section 19.4 – 19.7)

Learning Objectives:

3. The candidate will understand various applications and risks of derivatives.

Learning Outcomes:

3a) The candidate will be able to understand the Greeks of derivatives.

<u>Question:</u>

Consider two European call options, Call(30) and Call(35), with strike prices of 30 and 35, respectively, on a non-dividend-paying stock. You are given the following Greeks:

Greek	**Call(30)**	**Call(35)**
Δ	0.4125	0.6159
Γ	0.0422	0.0414

An investor has constructed a portfolio consisting of 1,000 long Call(30) options and 1,000 short Call(35) options.

The investor wants to delta-gamma hedge this position using a 30-strike European put option, Put(30), and the underlying stock. Put(30) is written on the same stock and has the same time to expiration as Call(30) and Call(35).

Determine the number of units of Put(30) and the underlying stock the investor should buy or sell in order to delta-gamma hedge the portfolio.

Solution:

Step 1: Determine the Greeks for Put(30)

Δ for Put(30) can be determined as follows:

$$\begin{aligned} \Delta_{\text{Call}(30)} - \Delta_{\text{Put}(30)} &= 1 \\ 0.4125 - \Delta_{\text{Put}(30)} &= 1 \\ \Delta_{\text{Put}(30)} &= -0.5875 \end{aligned}$$

Γ for Put(30) can be determined as follows:

$$\Gamma_{\text{Put}(30)} = \Gamma_{\text{Call}(30)} = 0.0422$$

Step 2: Set Up Delta and Gamma Hedge Equations

Define:

- x_1 = Number of units of Put(30) to buy.
- x_2 = Number of units of stock to buy.

To delta-hedge, we set the portfolio delta to 0:

$$\begin{aligned} +1000 \cdot \Delta_{\text{Call}(30)} - 1000 \cdot \Delta_{\text{Call}(35)} + x_1 \cdot \Delta_{\text{Put}(30)} + x_2 \cdot \Delta_{\text{Stock}} &= 0 \\ +1000 \cdot 0.4125 - 1000 \cdot 0.6159 + x_1 \cdot (-0.5875) + x_2 \cdot 1 &= 0 \quad \cdots \text{Eq (1)} \end{aligned}$$

To gamma-hedge, we set the portfolio gamma to 0:

$$\begin{aligned} +1000 \cdot \Gamma_{\text{Call}(30)} - 1000 \cdot \Gamma_{\text{Call}(35)} + x_1 \cdot \Gamma_{\text{Put}(30)} + x_2 \cdot \Gamma_{\text{Stock}} &= 0 \\ +1000 \cdot 0.0422 - 1000 \cdot 0.0414 + x_1 \cdot 0.0422 + x_2 \cdot 0 &= 0 \\ x_1 &= -18.9573 \end{aligned}$$

Substituting $x_1 = -18.9573$ into Eq (1) yields $x_2 = 192.2626$.

Thus, to delta-gamma hedge, the investor should **sell 18.96 units of Put(30) and buy 192.26 units of the underlying stock.**

Question 5 – Exotic Options

Resource:

- Hull Chapter 26 (Section 26.13)
- Hull Chapter 13 (Section 13.3)

Learning Objectives:

1. The candidate will understand key types of derivatives.
2. The candidate will understand the principles and techniques for the valuation of derivatives.

Learning Outcomes:

1c) The candidate will be able to compare European, American, Bermudan, Asian options, and various exotic options.

2f) The candidate will be able to understand option pricing techniques including calculating an expectation by solving a PDE.

<u>Question:</u>

For a two-period binomial tree with each period being 6 months, you are given:

- The current price of a non-dividend-paying stock is $50.
- The continuously compounded risk-free interest rate is 6%.
- The volatility of the stock is 30%.
- $u = 1.2740$
- $d = 0.8335$

Calculate the price of the following options:

a. A 1-year arithmetic average price Asian call with strike price $50

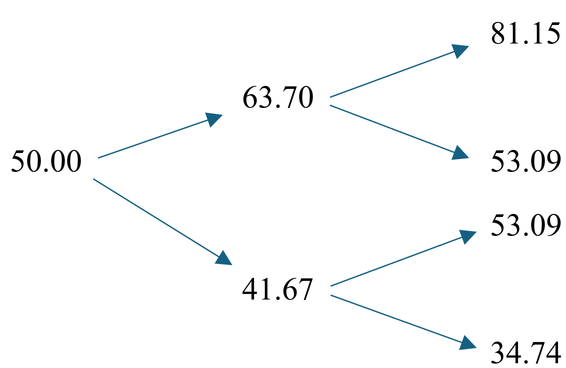
b. A 1-year geometric average strike Asian call

Solution:

The risk-neutral probability of an upward move is:

$$p = \frac{e^{r\Delta t} - d}{u - d} = \frac{e^{0.06(0.5)} - 0.8335}{1.2740 - 0.8335} = 0.4472$$

The binomial stock price tree is:



Part a

Calculate the arithmetic average, the call payoff, as well as the probability for each path:

Node	Arithmetic Average $A(S)$	Call Payoff $\max[0, A(S) - 50]$	Probability
uu	$\frac{63.70 + 81.15}{2} = 72.42$	22.42	$p \cdot p = 0.2000$
ud	$\frac{63.70 + 53.09}{2} = 58.39$	8.39	$p \cdot (1 - p) = 0.2472$
du	$\frac{41.67 + 53.09}{2} = 47.38$	0.00	$(1 - p) \cdot p = 0.2472$
dd	$\frac{41.67 + 34.74}{2} = 38.20$	0.00	$(1 - p)^2 = 0.3056$

Using risk-neutral valuation, the time-0 price of the option is the risk-neutral expected payoff, discounted at the risk-free interest rate:

$$\begin{aligned} V_0 &= e^{-rT} \cdot \hat{E}[\text{Payoff}] \\ &= e^{-0.06(1)}[22.42(0.2000) + 8.39(0.2472) + 0 + 0] \\ &= \boxed{6.18} \end{aligned}$$

Part b

Note that the strike is not fixed in this example. Calculate the geometric average strike, the call payoff, as well as the probability for each path:

Node	Geometric Average $G(S)$	Call Payoff $\max[0, S(1) - G(S)]$	Probability
uu	$(63.70 \cdot 81.15)^{\frac{1}{2}} = 71.90$	9.25	$p \cdot p = 0.2000$
ud	$(63.70 \cdot 53.09)^{\frac{1}{2}} = 58.15$	0.00	$p \cdot (1-p) = 0.2472$
du	$(41.67 \cdot 53.09)^{\frac{1}{2}} = 47.04$	6.05	$(1-p) \cdot p = 0.2472$
dd	$(41.67 \cdot 34.74)^{\frac{1}{2}} = 38.05$	0.00	$(1-p)^2 = 0.3056$

Using risk-neutral valuation, the time-0 price of the option is the risk-neutral expected payoff, discounted at the risk-free interest rate:

$$\begin{aligned} V_0 &= e^{-rT} \cdot \hat{E}[\text{Payoff}] \\ &= e^{-0.06(1)}[9.25(0.2000) + 0 + 6.05(0.2472) + 0] \\ &= \boxed{3.15} \end{aligned}$$

Question 6 – Exotic Options

Resource:

- Hull Chapter 26 (Section 26.9)
- Hull Chapter 13 (Section 13.3)

Learning Objectives:

1. The candidate will understand key types of derivatives.
2. The candidate will understand the principles and techniques for the valuation of derivatives.

Learning Outcomes:

1c) The candidate will be able to compare European, American, Bermudan, Asian options, and various exotic options.

2f) The candidate will be able to understand option pricing techniques including calculating an expectation by solving a PDE.

<u>Question:</u>

You are given the following information about a non-dividend-paying stock:

- The current price of a stock is \$100.
- The continuously compounded risk-free interest rate is 6% per annum.

Consider a two-year European down-and-in barrier put option with a strike price of \$90 and a barrier level of \$80.

To price the option, a two-period binomial tree model is used, with each period representing one year. The up factor is $u = 1.2$, and the down factor is $d = 0.7$.

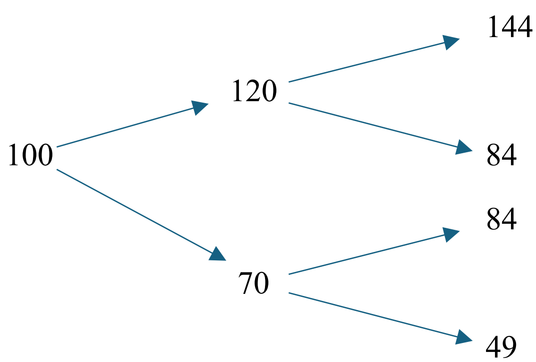
Calculate the price of the option.

Solution:

The risk-neutral probability of an upward move is:

$$p = \frac{e^{r\Delta t} - d}{u - d} = \frac{e^{0.06(1)} - 0.7}{1.2 - 0.7} = 0.72367$$

The binomial stock price tree is:



Next, calculate the put payoff and the probability for each path. Note that the knock-in option will only go into existence if the barrier of \$80 is reached over the life of the option. Since the barrier of \$80 is never reached in the first two paths, the option **will not go into existence** in those paths.

Node	In or Out?	Put Payoff $\max[0, 90 - S(2)]$	Probability
uu	Out	0	$p \cdot p = 0.52370$
ud	Out	0	$p \cdot (1-p) = 0.19997$
du	In	$\max[0, 90 - 84] = 6$	$(1-p) \cdot p = 0.19997$
dd	In	$\max[0, 90 - 49] = 41$	$(1-p)^2 = 0.07636$

Using risk-neutral valuation, the time-0 price of the option is the risk-neutral expected payoff, discounted at the risk-free interest rate:

$$\begin{aligned} V_0 &= e^{-rT} \cdot \hat{E}[\text{Payoff}] \\ &= e^{-0.06(2)}[0 + 0 + 6(0.19997) + 41(0.07636)] \\ &= \boxed{3.84} \end{aligned}$$

Question 7 – Binomial Trees

Resource:

- Hull Chapter 12 (Section 12.3)
- Hull Chapter 13 (Section 13.3)

Learning Objectives:

1. The candidate will understand key types of derivatives.
2. The candidate will understand the principles and techniques for the valuation of derivatives.

Learning Outcomes:

1e) Understand basic derivatives strategies including: call spreads, put spreads, collars, covered calls, butterfly spreads, straddles and strangles.

2f) The candidate will be able to understand option pricing techniques including calculating an expectation by solving a PDE.

<u>Question:</u>

You are given the following information for a non-dividend-paying stock:

- The current price of a stock is \$75.
- The continuously compounded risk-free interest rate is 4%.

A bear spread is constructed using European put options with strike prices of \$70 and \$80, both expiring in six months. These options are based on the same underlying stock.

a. Describe how to construct the 70-80 bear spread using European put options.

To price the bear spread, a two-period binomial tree model is used, with each period representing three months. The up factor is u = 1.2, and the down factor is d = 0.7.

b. Calculate the initial net cost of the bear spread.
c. Using the result from part b, determine the minimum and maximum profit at expiration. (Ignore the time value of money for this part).

Solution:

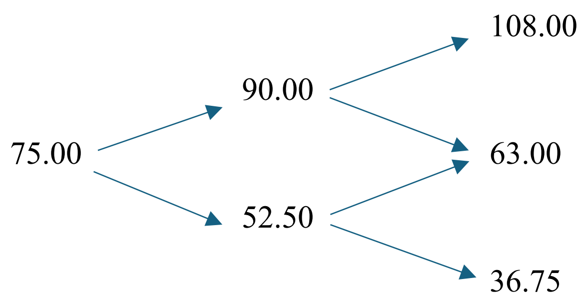
Part a

The 70-80 bear spread is constructed using European put options as follows:

- Short one European put with a strike price of \$70
- Long one European put with a strike price of \$80

Part b

The binomial stock price tree is:



The risk-neutral probability of an upward move is:

$$p = \frac{e^{r\Delta t} - d}{u - d} = \frac{e^{0.04\left(\frac{3}{12}\right)} - 0.7}{1.2 - 0.7} = 0.62010$$

The total payoff at expiration is the sum of the individual payoffs:

$$\text{Total Payoff} = -\max\left[0, 70 - S_{\frac{6}{12}}\right] + \max\left[0, 80 - S_{\frac{6}{12}}\right]$$

We evaluate this at each terminal node:

Node	$\boldsymbol{S_{\frac{6}{12}}}$	**Total Payoff**	**Probability**
uu	108.00	$-0 + 0 = 0$	$p \cdot p \ = \ 0.38452$
ud/du	63.00	$-(70 - 63) + (80 - 63) = 10$	$2 \cdot p \cdot \ (1 - p) \ = \ 0.47115$
dd	36.75	$-(70 - 36.75) + (80 - 36.75) = 10$	$(1 - p)^2 = 0.14432$

Using risk-neutral valuation, the initial net cost is the risk-neutral expected payoff, discounted at the risk-free interest rate:

$$
\begin{aligned}
V_0 &= e^{-rT} \cdot \hat{E}[\text{Payoff}] \\
&= e^{-0.04\left(\frac{6}{12}\right)}[0(0.38452) + 10(0.47115) + 10(0.14432)] \\
&= \boxed{6.033}
\end{aligned}
$$

Part c

The profit is calculated by subtracting the initial net cost from the payoff at expiration:

$$\text{Profit} = \text{Payoff} - \text{Initial Net Cost}$$

(Note: The Hull text ignores the time value of money when calculating the profit; it doesn't accumulate the initial net cost to expiration.)

Thus, to calculate the minimum and maximum profit (ignoring time value), focus on the minimum and maximum payoff.

For a 70-80 bear spread:

- The maximum payoff occurs when the stock price at expiration is less than the first (smaller) strike price.
- The minimum payoff occurs when the stock price at expiration is greater than the second (larger) strike price.

This can be seen by constructing the payoff table:

Stock Price Range	**Total Payoff**
$S_{\frac{6}{12}} \leq 70$	$-\left(70 - S_{\frac{6}{12}}\right) + \left(80 - S_{\frac{6}{12}}\right) = 10$
$70 < S_{\frac{6}{12}} \leq 80$	$0 + \left(80 - S_{\frac{6}{12}}\right) = 80 - S_{\frac{6}{12}}$
$S_{\frac{6}{12}} > 80$	$0 + 0 = 0$

Thus, the minimum payoff is 0, and the maximum payoff is 10. The initial net cost (from part b) is 6.033. Consequently, ignoring the time value of money, the minimum and maximum profits at expiration are:

- $\text{Minimum profit} = \text{Minimum payoff} - 6.033 = 0 - 6.033 = \boxed{-6.033}$
- $\text{Maximum profit} = \text{Maximum payoff} - 6.033 = 10 - 6.033 = \boxed{+3.967}$

Question 8 – The Process for a Stock Price

Resource:

- Hull Chapter 14 (Section 14.3, 14.7)

Learning Objectives:

2. The candidate will understand the principles and techniques for the valuation of derivatives.

Learning Outcomes:

2d) The candidate will be able to understand stochastic calculus theory and technique used in pricing.

2f) The candidate will be able to understand option pricing techniques including calculating an expectation by solving a PDE.

<u>Question:</u>

You are given the following information for a non-dividend-paying stock:

- The current time is time 0, and the current price of the stock is \$55.
- The stock price at time t is denoted by S_t. The evolution of the stock price follows:

$$\frac{dS_t}{S_t} = 0.05\,dt + 0.30\,dz$$

where $\{z\}$ is a standard Wiener process.

Determine:

a. The distribution and the parameters of $\ln\frac{S_3}{S_0}$
b. The distribution and the parameters of $\ln S_3$
c. The distribution and the parameters of S_3
d. An equation for S_3
e. $E[S_3]$ and $Var[S_3]$
f. The 80$^{\text{th}}$ percentile of S_3
g. The 95% prediction interval for S_3

Part a

Let μ be the expected rate of return on the stock and σ be the volatility of the stock. Then:

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma\, dz \Leftrightarrow \ln\frac{S_t}{S_0} \sim N\left[m = \left(\mu - \frac{\sigma^2}{2}\right)t, v^2 = \sigma^2 t\right]$$

where m and v^2 represent the mean and variance of the normal distribution, respectively.

In this example, $\mu = 0.05$ and $\sigma = 0.30$. Thus, we have:

$$\ln\frac{S_3}{S_0} \sim N\left[m = \left(0.05 - \frac{0.30^2}{2}\right)(3) = 0.015, v^2 = 0.30^2(3) = 0.27\right]$$

Thus, $\ln\frac{S_3}{S_0}$ is normally distributed with mean 0.015 and variance 0.27.

Part b

Since S_0 is known, $\ln S_0$ is a constant. Recall that the variance of a constant is 0; therefore, adding a constant has no impact on the variance. As a result, we can write:

$$\ln S_t - \ln S_0 \sim N\left[m = \left(\mu - \frac{\sigma^2}{2}\right)t, v^2 = \sigma^2 t\right]$$

$$\therefore \ln S_t \sim N\left[m = \left(\mu - \frac{\sigma^2}{2}\right)t + \ln S_0\,, v^2 = \sigma^2 t\right]$$

In this example:

$$\ln S_3 \sim N\left[m = \left(0.05 - \frac{0.30^2}{2}\right)(3) + \ln 55 = 4.0223, v^2 = 0.30^2(3) = 0.27\right]$$

Thus, $\ln S_3$ is normally distributed with mean 4.0223 and variance 0.27.

Part c

From part b, $\ln S_3 \sim N[m = 4.0223, v^2 = 0.27]$.

Recall that if $X \sim N(m, v^2)$, then $e^X \sim LogN(m, v^2)$. So:

$$e^{\ln S_3} \sim LogN(m = 4.0223, v^2 = 0.27)$$
$$S_3 \sim LogN(m = 4.0223, v^2 = 0.27)$$

Thus, S_3 is lognormally distributed with parameters $m = 4.0223$ and $v^2 = 0.27$.

Note that $m = 4.0223$ and $v^2 = 0.27$ are NOT the mean and variance of the lognormal random variable; they are merely the parameters. We will discuss how to calculate the mean and variance of a lognormal random variable shortly.

Part d

Recall that a normal random variable X can be expressed in terms of a standard normal random variable Z as follows:

$$Z = \frac{X - m}{v} \Rightarrow X = m + v \cdot Z$$

Similarly, since $\ln \frac{S_t}{S_0}$ is a normal random variable with mean $m = \left(\mu - \frac{\sigma^2}{2}\right)t$ and variance $v^2 = \sigma^2 t$, we can express it in terms of Z as:

$$\ln \frac{S_t}{S_0} = \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma\sqrt{t} \cdot Z$$

Exponentiating both sides gives:

$$\frac{S_t}{S_0} = e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma\sqrt{t} \cdot Z}$$

Rearranging, we obtain an equation for S_t:

$$S_t = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma\sqrt{t} \cdot Z}$$

In this example:

$$S_3 = 55e^{\left(0.05 - \frac{0.30^2}{2}\right)(3) + 0.30\sqrt{3} \cdot Z} = \boxed{55e^{0.015 + 0.5196 \cdot Z}}$$

Part e

Recall that if $X \sim N(m, v^2)$, then $Y = e^X \sim LogN(m, v^2)$, and we have:

$$E[Y] = e^{m+\frac{1}{2}v^2}$$
$$Var[Y] = (E[Y])^2(e^{v^2} - 1)$$

Thus, if $\ln\frac{S_t}{S_0}$ is normally distributed with mean $m = \left(\mu - \frac{\sigma^2}{2}\right)t$ and variance $v^2 = \sigma^2 t$, then $e^{\ln\frac{S_t}{S_0}}$ or $\frac{S_t}{S_0}$ is lognormally distributed with parameters m and v^2.

As a result, we can derive the mean of S_t as follows:

$$E\left[\frac{S_t}{S_0}\right] = e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \frac{1}{2}\sigma^2 t} = e^{\mu t}$$
$$\therefore E[S_t] = S_0 e^{\mu t}$$

In this example:

$$E[S_3] = 55e^{0.05(3)} = \boxed{63.9}$$

Similarly, we can derive the variance of S_t as follows:

$$Var\left[\frac{S_t}{S_0}\right] = \left(E\left[\frac{S_t}{S_0}\right]\right)^2 (e^{v^2} - 1)$$
$$Var[S_t] \cdot \frac{1}{S_0^2} = (E[S_t])^2 \cdot \frac{1}{S_0^2}(e^{v^2} - 1)$$
$$\therefore Var[S_t] = (E[S_t])^2(e^{\sigma^2 t} - 1)$$

In this example:

$$Var[S_3] = (E[S_3])^2\left(e^{\sigma^2(3)} - 1\right) = 63.9^2\left(e^{0.30^2(3)} - 1\right) = \boxed{1{,}265.7}$$

Part f

To find the 80th percentile of S_t, we first determine the 80th percentile of the standard normal random variable Z. The 80th percentile of Z is the value of $z_{0.80}$ such that:

$$\Pr[Z \leq z_{0.80}] = 0.80 \Rightarrow z_{0.80} = \Phi^{-1}(0.80) = 0.84162$$

Then, substitute the resulting value of Z into the equation for S_3 (from part d):

$$\text{80th percentile of } S_3 = 55e^{0.015+0.5196 \cdot z_{0.80}} = 55e^{0.015+0.5196 \cdot 0.84162} = \boxed{86.46}$$

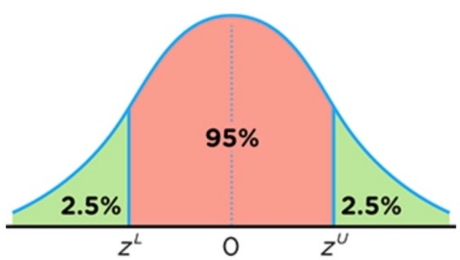
Part g

To determine the 95% prediction interval for S_3, we determine the lower and upper stock prices, S_3^L and S_3^U, such that:

$$\Pr[S_3 < S_3^L] = 0.025 \text{ and } \Pr[S_3 > S_3^U] = 0.025$$

This can be accomplished by first identifying the corresponding z-values, z^L and z^U, such that:

$$\Pr[z^L < Z < z^U] = 0.95$$



We have:

- $\Pr[Z < z^L] = 0.025 \Rightarrow z^L = \Phi^{-1}(0.025) = -1.96$
- $\Pr[Z < z^U] = 0.975 \Rightarrow z^U = \Phi^{-1}(0.975) = 1.96$

Note the lower and upper z-values are just opposite of each other.

Once z^L and z^U are determined, we substitute them into the equation for S_3 to obtain the corresponding stock prices:

- $S_3^L = 55e^{0.015+0.5196 \cdot z^L} = 20.16$
- $S_3^U = 55e^{0.015+0.5196 \cdot z^U} = 154.59$

Thus, the 95% prediction interval for S_3 is $\boxed{(20.16, 154.59)}$.

