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Managing Investment and Liquidity Risks Within Nonfundamental Market Sell-off and Volatility Feedback Loops: A Market Impact Perspective

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Market impact is an illustration of market inefficiency. Theories of efficient markets typically expect that investors buy and sell assets based on assessments of their intrinsic value, in contrast with large derivatives players who often act based on market price movements, which may not be linked to fundamentals. Market impact risk actually refers to the degree to which large transactions can be carried out in a timely fashion with minimal impact on prices. As a result, managing investment and liquidity risks for large players requires introducing an explicit market impact function; its application to derivatives significantly depends on whether or not there is significant delta hedging activity. In the case of no significant delta hedging activity, risk appetite has significant influence on the optimal execution strategy. With significant delta hedging activity, the optimal trading involves feedback hedging effects, translating into a modified Black-Scholes hedging strategy.

Soaring market volatility necessitates updated hedging strategies. In the last six years, we have had more short-lived but sharp transitions from low volatility to high volatility with no well-known fundamental catalysts than in the prior two

CONTINUED ON PAGE 4

decades[1]—growing evidence that we are in a new volatility regime. Low liquidity and low conviction environments lead to this becoming increasingly more common.

Although fears about growth or sovereign debt sustainability are valid explanations for the significant volatility spikes experienced during the May 2006, May 2010, August 2011, August 2015, June 2016 or February 2018 market sell-offs, they do not fully explain either the extreme magnitude of the shocks or the repeated occurrence at the close in European and U.S. markets. Also, the Volatility Index averaged 11 through 2017—the lowest since 1990, in the context of easy monetary policy, share buybacks and solid fundamental factors such as continued global growth, solid-to-positive earnings and falling unemployment. (There was a short VIX futures strategy profit and loss at +150 percent in 2017, making money every single month that year.)

Derivatives activity by large players might have exacerbated the acuity of such volatility spikes from the illiquidity premium in option markets since the late 1990s[2] stemming from a structural imbalance between supply and demand in derivatives, as illustrated by the 70 percent more put options outstanding than outstanding call positions[1], or the growing hedging of U.S. variable annuities, Asian structured products and U.S. mortgages convexity. Such imbalance in the derivatives markets is at the source of hedging inefficiencies, where market makers tend to sell more as the market drops or buy more as the market rallies, independent of fundamentals.

As a result, the cost of placing one large order to close a position becomes far greater than the sum of infinitely small orders differed in time. For this reason, an explicit modeling is required through a market impact function, the influence of which the agent will try to minimize. The optimal execution turns out to be the sequence of small trades over the course of several days that optimizes a target, (e.g., minimizes the mean cost of trading). In this article, we consider the optimal execution price and strategies of options when market impact is a driver of the option price, which depends on whether the options' delta hedging is significant or not.

- **No or insignificant delta hedging** (like for a life insurance company aiming to minimize the cost of buying a large quantity of put options to hedge liabilities). The optimal execution turns out to be strongly dependent on the risk appetite. Within a mean cost minimization objective, as the maturity approaches, the agent must make faster acquisitions as time passes; in contrast, within a mean-variance risk appetite (where the dispersion of revenues is also taken into account), the agent tends to liquidate her position at the beginning to reduce the P&L variance.
- **Significant delta hedging.** The optimal execution strategy is determined by a no arbitrage framework that incorporates the specific impact of the large trader's hedging activity (**hedging feedback effects**), which translates into a fully nonlinear modified Black-Scholes delta hedging strategy.

In this article, the most observed types of market impact on the investment and liquidity risks within derivatives strategies is illustrated and analyzed from a qualitative perspective. We then examine the optimal strategies in derivatives based on appropriate modeling of the market impact, depending on whether there is significant associated delta hedging activity or not.

EMPIRICAL MARKET IMPACT OF DERIVATIVES STRATEGIES

Hedging Financial Risk of Life Insurance Liabilities

Insurance companies utilize derivatives in a variety of ways to manage and mitigate risks inherent in their liability portfolios, which can be characterized by three main features: medium long-term duration, large volumes and significant market risk exposure. Specifically, guaranteed minimum income and withdrawal benefits greatly increase insurers' risk exposure to market volatility, while pension and other post-retirement benefits could be hurt if equity returns fall short of expected long-term rates of return.

Given the persistent low interest rate environment across the curve since the 2008 financial crisis, these large players need to hedge their liabilities even more, as illustrated by the significant increase in notionals from $786 billion as of fiscal year 2010 to $1,885 billion as of FY 2014. As the guarantees embedded within those liabilities hold a convex risk profile with respect to the underlying stock, traders need to buy some convex equity hedge assets such as options (in contrast to linear instruments like futures) in order to match the liability risk profile to improve hedge effectiveness. In that respect, the use of downside protection options is appropriate, such as put options, which accounted for 44 percent of the transactions (versus 24 percent for the call options) with 90 percent of them purchased, implying the growing cost of hedging.[3]

As equity derivatives are highly sensitive to supply/demand balance, buying large hedge portfolios requires taking into account the transaction size explicitly, which is not considered by traditional models.

Large Derivatives Imbalances

Large derivatives imbalances are likely to imply net short positions in options by market makers, thus synthetic replication with significant delta hedging activity is likely to exacerbate market moves through hedging feedback effects.

Effects on Short-dated Vanilla Options

Investors typically buy index put options as downside protection (with little or no hedging), thus market makers short put options, which they delta hedge by selling futures to be market neutral. If the market suddenly drops, they would need to sell further to adjust, which amplifies the down market move and volatility.

Effects on Long-dated Exotic Options

Autocallables are upside (capped) participation with capital guaranteed (floor) and an embedded-up and knock-out barrier that can cause their gamma to reverse across very small movements as spot rallies toward the barrier while the expiry approaches. This requires the seller to sell large amounts, which tends to prevent the spot market from actually hitting the barrier.

Despite this selling of spot and gamma, the barrier may at some point break, where the option disappears, and the trader is left only with his hedge (i.e., a naked position), which he has to cover by buying back spot and gamma. Delta hedging tends to exaggerate spot moves even more (higher spot → needs to buy → drives spot higher; lower spot → needs to sell → drives spot lower), which will cause the spot market to become more liable to choppy trading and can cause the market to gap higher. Because of leverage in barrier options, the delta amounts grow to multiples of the size of the original option.

The hedging of variable annuities can also be a major driver of such market feedback loops given those embedded life insurance guarantees are upside (capped) participation with capital guaranteed (floor), while their positions tend to leave the variable annuities players "the same way around"—either buying or selling particular types of hedging instruments. As such, insurers buy volatility when it rises and vice versa, exaggerating any move. While the impact on the derivatives markets gamma is still under control given most hedge assets (futures, options, varswaps) are short dated, the vega hedging needs are huge as a result of the very long dated life insurance policies.

OPTIMAL DERIVATIVES STRATEGIES

No Significant Delta Hedging Activity

Here we consider that delta hedging is either nil or negligible in terms of market impact, which is consistent with practice on the main market indices as their exchanged volumes are far larger than for the options contracts. The average shares traded per day for the S&P 500 has grown from 2.3 million to 4.1 billion, with Oct. 10, 2008, the busiest trading day ever for the S&P 500 when a phenomenal 11,456,230,400 shares changed hands. Options contracts exchanged volumes are significantly lower.[4] See Figure 1.

Figure 1
S&P 500 Historical Volume Data
(Jan. 2, 1951, to March 31, 2012)

	Total Shares	Avg. Shares	Correlation	R^2
1950s	5,777,550,000	2,298,150	0.66	0.44
1960s	19,072,060,000	7,656,387	0.73	0.53
1970s	57,655,100,000	22,833,703	0.48	0.23
1980s	306,188,530,000	121,118,881	0.76	0.58
1990s	1,195,610,210,000	473,134,234	0.93	0.86
2000s	7,091,918,888,000	2,819,848,464	(0.07)	0.01
2010s	1,274,419,730,000	4,058,661,561	(0.32)	0.10
Total	**9,950,642,068,000**		**0.72**	**0.52**

Source: Yahoo Finance, CFA Institute

As a result, an agent who is willing to trade a large quantity of options will see the impact as an important dilemma, as the cost of placing one large order to close his position will be far greater than the sum of infinitely small orders differed in time. In practice, the orders are usually broken up into smaller ones and executed over the course of several days.[5] Only 20 percent of the market value of the trades split in their set of data are completed within a day, and 53 percent are spread over four trading days or more.

For this reason, an explicit modeling is made through a market impact function, the influence of which the agent will try to minimize. The market impact function depends on the **temporary impact strength** proportional to the main empirically observed drivers, such as the speed of option trading (i.e., the number of options per unit of time), the equity stock level and the option sensitivity to the equity stock. The optimal execution turns out to be the sequence of trades that optimizes the target (e.g., minimizes the mean cost of trading over a fixed period) or the mean-variance criterion if the volatility of revenues is taken into account.

Market Impact Function, Resulting Option Price

The model is inspired from Leland's option replication with transaction costs[6] incorporated into the option price as an additional variable within the volatility function:

$$\tilde{\sigma}^2 = \sigma^2 + f(t, \dot{x}_t, x_t, \sigma)$$

where σ is the asset volatility and f is the market impact function (dependent on time, volatility, inventory and trading speed).

In terms of market impact function, we follow the approach by Almgren[7] where the price impact is a combination of two components: a permanent component that reflects the information transmitted to the market by the buy/sell imbalance, and a temporary component that reflects the price concession needed to attract counterparties within a specified short time interval. We adapt such approach to derivatives through the **enlarged volatility** expression as follows:

$$\tilde{\sigma}_t^2 = \sigma^2 + (\tilde{\eta}\dot{x}_t + \tilde{\gamma}(x_t - x_0))\sqrt{\hat{T} - t}\sigma$$

where $\tilde{\eta} = \eta\sqrt{\frac{8}{h\pi}}$ and $\tilde{\gamma} = \gamma\sqrt{\frac{8}{h\pi}}$.

And η and γ are constants. The number of shares is x(t) while $\dot{x}_t$, its derivative with regards to time, corresponds to the speed of trading of the security. The term $\eta\dot{x}_t$ corresponds to the temporary or instantaneous impact of trading $\dot{x}_t dt$ shares at time t and only affects this current order. The term $\gamma(x_t - x_0)$ is the permanent price impact that was accumulated by all transactions until time t.

The option effective price is then expressed through a Black-Scholes-like partial differential equation with such modified enlarged volatility in order to compensate for the market impact cost, where buying the option will typically lead to increasing its price. The higher the trading speed and quantity, the higher the volatility and thus the option price:

$$\begin{cases} \partial_u \tilde{P}(u,S) + \frac{1}{2}\tilde{\sigma}_t^2 S^2 \partial_{SS}\tilde{P}(u,S) = 0, & (u,S) \in [t,\hat{T}[\times]0,\infty] \\ \tilde{P}(\hat{T},s) = (K-s)^+. \end{cases}$$

Using a simple Taylor approximation to the first order, we can rewrite the expression as a sum of the Black-Scholes option price and an additional term corresponding to the option market impact:

$$\begin{aligned} \tilde{P}(t,S_t) &\approx P(t,S_t) + (\tilde{\sigma}_t^2 - \sigma^2)\partial_V P(t,S_t) \\ &\approx P(t,S_t) + \frac{1}{2}\{\tilde{\eta}\dot{x}_t + \tilde{\gamma}(x_t - x_0)\}\sqrt{\hat{T} - t}\nu(t,S_t) \end{aligned}$$

where $\nu(t,S_t) = \partial_\sigma P$ is the Black-Scholes vega of the option:

$$\nu(t,S_t) = \sqrt{\hat{T} - t}S_t N'(d_1) = \sqrt{\hat{T} - t}KN'(d_2)$$

$$N'(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2},$$

$$d_1 = \frac{\log\frac{S_t}{K} + \frac{1}{2}\sigma^2(\hat{T} - t)}{\sigma\sqrt{\hat{T} - t}},$$

$$d_2 = \frac{\log\frac{S_t}{K} - \frac{1}{2}\sigma^2(\hat{T} - t)}{\sigma\sqrt{\hat{T} - t}} = d_1 - \sigma\sqrt{\hat{T} - t}.$$

We will next develop the framework under the Black-Scholes case as a temporary market impact only, with permanent impact excluded (i.e., $\tilde{\gamma} = 0$). In that case, the effective price is given by:

$$\tilde{P}_t = P_t + \frac{1}{2}\tilde{\eta}\dot{x}_t\sigma S_t^2(\hat{T} - t)^{3/2}\Gamma(t,S_t).$$

Optimal Execution Problem

The optimal execution is a strategy that unfolds over the course of several days [0,T] by means of a dynamic order execution strategy that ought to adapt to changing market conditions. The end user's purpose is to hedge the risk of a complex product (structured product, variable annuity, etc. . . .) indexed on an underlying asset, by acquiring vanilla put options on that same underlying asset.

Let us consider a buying trade execution strategy x(t) in which an amount of options X with fixed strike K and maturity T needs to be bought by a fixed time horizon [0,T] with the conditions x(0) = X and x(T) = 0.

Let $(\Omega,\mathcal{F},\mathbb{P})$ be the usual probability space on the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ satisfying the usual assumptions. In the absence of market impact and under a null risk-free rate, the no-arbitrage price of a put option is defined by $P_t = \mathbb{E}_{\mathbb{Q}}[(K - S_{\hat{T}})^+|\mathcal{F}_t]$ under the risk-neutral probability measure Q in which the asset price is a martingale. At each time t, $\dot{x}_t dt$ options are bought at price $\tilde{P}_t$ which is the option impact price defined by the price equation above. Thus, the cost arising from the strategy x is $\mathcal{C}(x) := \int_0^T \tilde{P}_t\dot{x}_t dt$:

$$\mathcal{C}(x) = \int_0^T P_t\dot{x}_t dt + \frac{1}{2}\tilde{\eta}\int_0^T \dot{x}_t^2\sigma S_t^2(\hat{T} - t)^{3/2}\Gamma(t,S_t)dt,$$

$$\mathcal{C}(x) = -XP_0 - \int_0^T \sigma x_t S_t\Delta(t,S_t)dW_t + \frac{1}{2}\tilde{\eta}\sigma\int_0^T \dot{x}_t^2 S_t^2(\hat{T} - t)^{3/2}\Gamma(t,S_t)dt,$$

where Δ is the Black-Scholes delta of the option.

The agent's objective is then to minimize a certain objective function, which takes into account his risk aversion, and may involve both cost and risk terms. Here we will consider two risk appetite cases:

- The mean cost $\mathbb{E}[\mathcal{C}(x)]$
- A risk/reward criterion, the mean-variance cost case $\mathbb{E}[\mathcal{C}(x)] + \lambda\mathbb{V}ar[\mathcal{C}(x)]$ (which includes the mean case with λ = 0), where λ > 0 is the variance penalty.

The mean cost is usually used for an agent who does not monitor the risk of his strategy:

$$\mathbb{E}[\mathcal{C}(x)] = -XP_0 + \frac{1}{2}\tilde{\eta}\mathbb{E}\left[\int_0^T \dot{x}_t^2 S_t^2(\hat{T} - t)^{3/2}\Gamma(t,S_t)dt\right].$$

Theorem 1. The optimal strategy x^* resulting in minimizing the mean cost under the Black-Scholes framework is characterized by [8]

$$\dot{x}^*(t) = \frac{K_1}{(\hat{T}-t)^{3/2}},$$
$$x^*(t) = \frac{K_1}{(\hat{T}-t)^{1/2}} + K_2$$

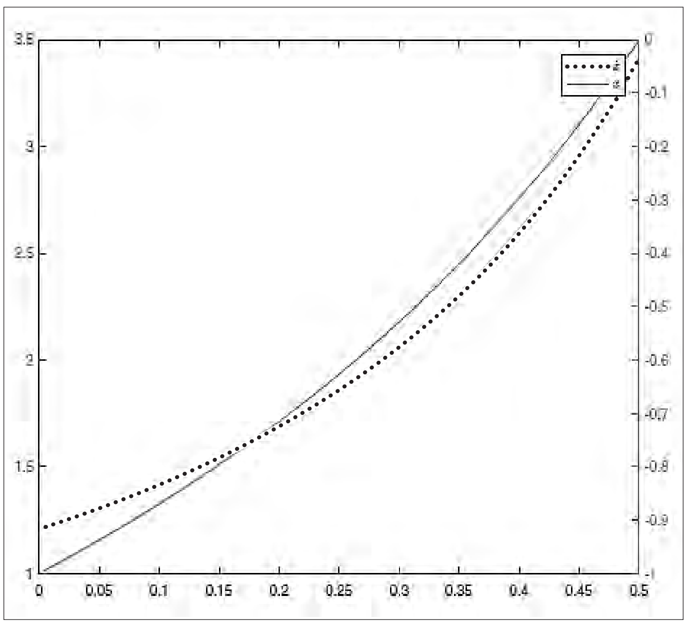
where

$$K_1 = \frac{X}{2(\hat{T}^{-\frac{1}{2}} - (\hat{T}-T)^{-\frac{1}{2}})}$$

$$K_2 = -2K_1(\hat{T}-T)^{-1/2}.$$

The theorem is illustrated in Figure 2 for t = 1, T = 0.5, X –1:

Figure 2
Optimal Execution Trade Quantity and Speed Depending Residual Time



In summary, the optimal execution strategy to minimize the mean cost provides a rather stable pace of trading. The pace is rather constant at the beginning and then gradually increases as it gets close to maturity, which is intuitive given the fixed quantity to buy within a fixed time period, implying the insurer must acquire at a faster rate as time passes.

We will now develop the optimal execution framework under the mean-variance case, where the optimal strategy turns out to be more sensitive to the underlying price evolution.

Optimal Execution Strategy Depends on Risk Appetite

Investors usually take into account their risk aversion using risk/reward criterion.

For the mean cost case, we are interested in the price impact formulation with temporary impact only. That is, we can easily deduce that the mean-variance objective function can be approximated as:

$$E[C(x)] + \lambda Var[C(x)] \approx E\left[\int_0^T \frac{1}{2}\tilde{\eta}\sigma \dot{x}_t^2 S_t^2 (\hat{T}-t)^{\frac{3}{2}} \Gamma(t,S_t)dt + \tilde{\lambda}\int_0^T x_t^2 \sigma^2 S_t^2 \Delta^2(t,S_t)dt\right]$$

We then set up the dynamic programming problem where we parameterize as before the strategies x by their trading speed or **trading rate** α defined as $-\dot{x}_t : x_t^\alpha := X - \int_0^t \alpha_s ds, \quad 0 \le t \le T.$

We restrict our framework to a Markovian trading rate (i.e., the agent's optimal trading speed at time t is completely determined by the current state). Using the standard procedure of deriving the Hamilton-Jacobi-Bellman equation in stochastic control problems, the solution to the reduced optimization problem solves the following PDE:

$$\partial_t U + \frac{1}{2}\sigma^2 S^2 \partial_{SS} U + \tilde{\lambda} x^2 S^2 \Delta^2(t,S) + \inf_{\alpha \in \mathbb{R}} \left\{ \alpha^2 S^2 (\hat{T}-t)^{3/2} \Gamma(t,S) - \alpha \partial_x U \right\} = 0$$

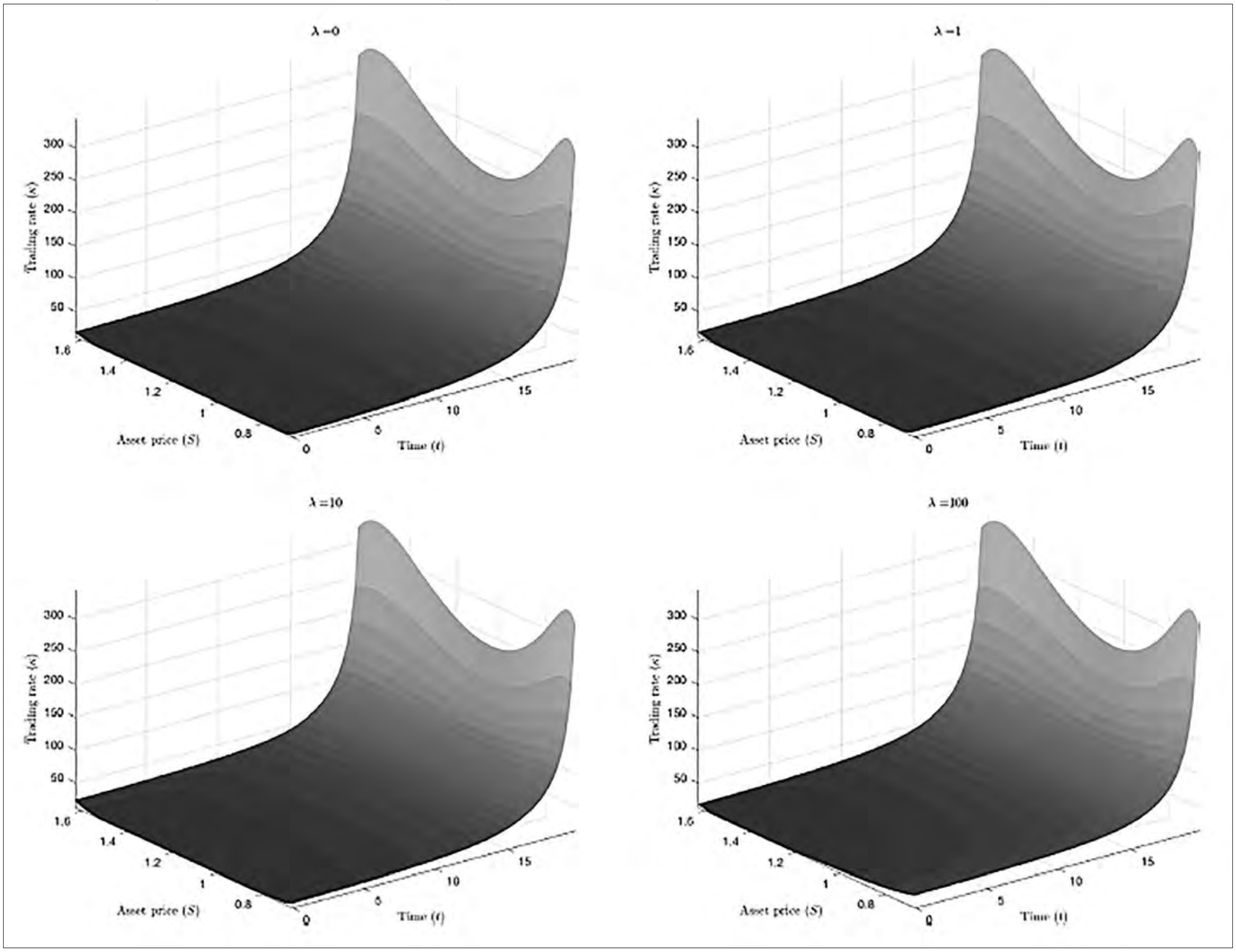
combined with the so-called finite-fuel constraint (i.e., $\int_0^T \alpha_t dt = X$).

Although this minimization problem does not admit a closed-form solution, the quasi-linear PDE can be solved numerically using finite differences methods[8]. Table 1 shows the results for a long position on at-the-money put options.

Table 1
Long Position on ATM Put Options

Parameter	Value
σ	30%
T (the strategy horizon)	1/12 (years)
$\hat{T}$ (the option maturity)	1 (years)
μ	0
r	0
S_0	1
K	S_0
Action	Buy
x_0	–1
$\tilde{\eta}$	0.05
Trading frequency	4 trades per day
λ	0, 1, 10, 100

Figure 3
Rate of Trading as a Function of Underlying Price S, Time t for Different Values of λ



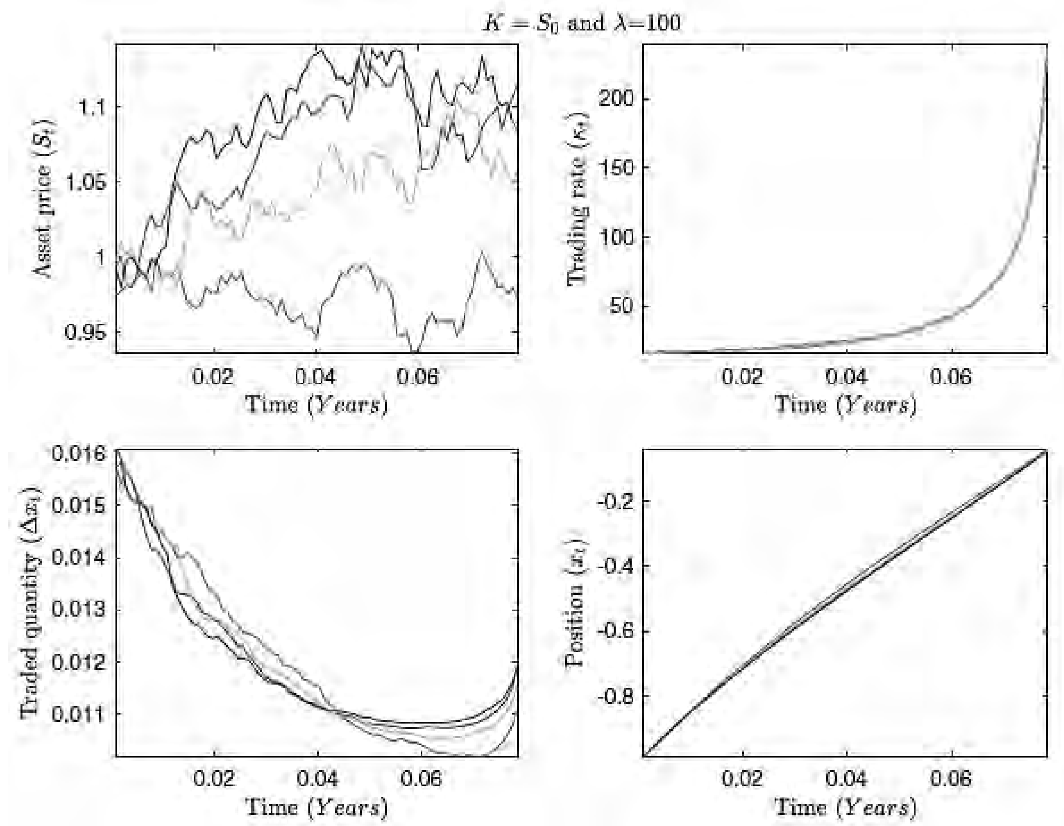
Note: Mean objective (λ = 0, top left) or mean-variance (λ = 1, top right; 10, bottom left; 100, bottom right)

Figure 3 illustrates the optimal execution strategy through the rate of trading as a function of the underlying price S and time t. The strategy hardly depends on the trader inventory position. However, as time increases, the trading rate increases (convex in time). As the maturity approaches, the agent must acquire faster as time passes, with a shape of an inverse function of time.

The mean case ($\lambda = 0$) is the least affected by the spot variation. In contrast, this representation allows seeing that the mean-variance (i.e., $\lambda \neq 0$) with a high-risk aversion is most sensitive to price movements. The agent tends to liquidate her position at the beginning to reduce the P&L variance that plays a non-negligible role in her choice. To gain additional insight, in Figure 4 (Pg. 9), we plot four paths of the underlying price together with the rate of trading, the inventory and quantity to be traded, where adding the variance pushes the agent to adapt the strategy to the underlying level. As the risk aversion parameter increases, the traded quantity tends to be larger at the beginning.

In contrast, within the mean-variance case where the dispersion of the profit and loss becomes an additional driver of the risk appetite, the optimal execution strategy significantly depends on the stock path, with a faster pace when the stock level is low compared to when the stock level is high. Indeed, as the stock decreases, the cost of the put option increases. This prevents the insurance company from waiting until maturity to trade a large quantity, and instead favors a decreasing trading pace as time passes.

Figure 4
Sample Paths of Evolution of Price, Rate, Inventory and Quantity



Significant Delta Hedging Activity

We consider here the interaction of one "large trader" whose action affects prices and many price-takers or "small traders"; the usual no arbitrage condition[9] doesn't apply. We use a continuous time version of Jarrow's no market manipulation strategies[10], which requires additional but relevant required assumptions:

- The asset price is independent of the large trader's past holdings
- Real wealth (as if the holdings were liquidated)
- Synchronous markets condition
- Prices adjust instantaneously across underlying and derivatives
- Absence of corners

Effects on Option Prices

Large dealers are net writers of options and thus need to neutralize the risk by synthetically replicating options. As a result, an additional process—the number of underlying assets held by the large trader—needs to be introduced, which gives rise to nonlinear feedback effects.[11]

Actually, traded options exist only for well established markets and relatively short maturities. For very long dated options, dynamic replication is the only way for market makers to hedge a short-put position. They do this by taking an offsetting position in the underlying asset; the required size changes with the price of the underlying asset.

More precisely, to compensate for an increase in the price sensitivity of a call option, a hedge position in the underlying asset must be made larger as well, in return affecting its price process. If the transaction size in the underlying asset becomes very significant, thus implying market impact, this mechanism generates the potential for positive feedback in price dynamics because the hedge adjustment is to buy (sell) the underlying asset after its price rises (falls), as the transactions could introduce further upward (downward) pressure on prices after an initial upward (downward) shock to asset prices.

The underlying asset price dynamics can be modeled as

$$d\tilde{S}_t = \sigma_t \tilde{S}_t dW_t + \rho_t \lambda \left(\tilde{S}_t\right) \tilde{S}_t d\alpha_t$$

where λ is a continuous function called **market liquidity profile**, used to retrieve a particular shape of the implied volatility smile, while ρ represents the intensity of the liquidity impact. A possible choice is the ratio of change in the price of the underlying to the quantity traded, which is observable given an order book. So $\frac{1}{\rho\lambda_t\left(\tilde{S}_t\right)\tilde{S}_t}$ represents the market depth at time t, (i.e., the order flow required to move prices by one unit).

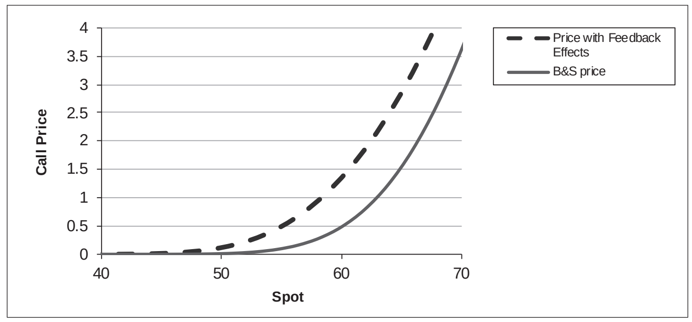
If we now apply the Black-Scholes methodology, under a zero risk-free interest rate (for simplicity of notation), we obtain a modified Black-Scholes PDE:[12]

$$\begin{cases} u_t\left(t,\tilde{S},\gamma\right)+\frac{1}{2}\frac{1}{\left(1+\rho\lambda\left(\tilde{S}\right)\tilde{S}u_{\tilde{S}\tilde{S}}\left(t,\tilde{S},\gamma\right)\right)^2}\sigma^2\tilde{S}u_{\tilde{S}\tilde{S}}\left(t,\tilde{S},\gamma\right)=0 \\ u\left(T,\tilde{S},\gamma\right)=nh\left(\tilde{S}\right) \end{cases}.$$

This modified Black-Scholes equation is a fully nonlinear parabolic PDE, requiring specific numerical implementation ensuring accuracy, flexibility and stability.[13]

Actually, as the large trader sells European calls, she has to buy a large amount of the underlying assets to hedge synthetically, which makes the underlying asset price rise, thus the short delta decreases, implying a short gamma, so the feedback volatility rises. Consequently, the option unit price turns out to be higher than the usual price-taker Black-Scholes price. This can be seen in Figure 5.

Figure 5
European Call Price With Feedback Effects vs. Black-Scholes Price



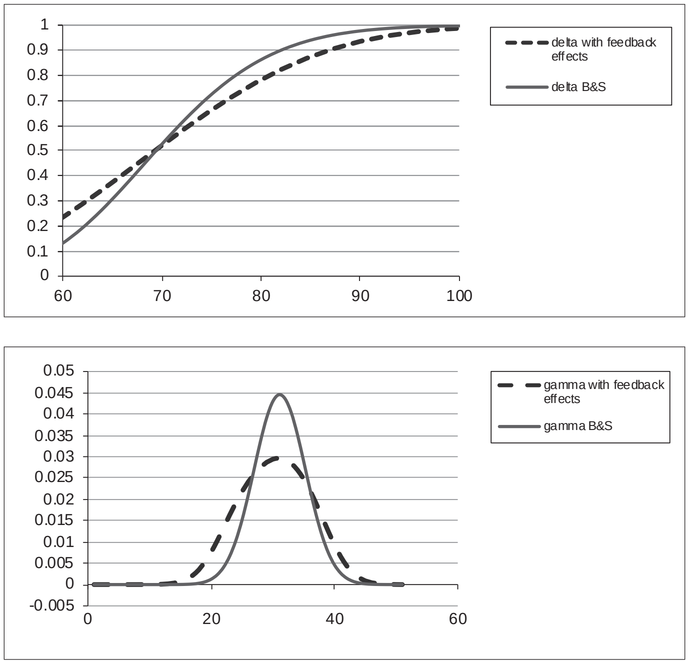
An apparent paradox arises in empirically observed markets in regard to large traders' transactions: Selling a large amount of calls causes the price to rise. In fact, when a large amount of options is used in such trading strategies, the market dynamics may be affected by the trading strategy itself, leading to potentially destabilizing price paths.

Illiquidity appears as an endogenous trading cost compensating for the sharing of risks measured here by the spot market volatility. By buying with rising prices, the large trader's demand is procyclic. Therefore, the apparent paradox is just a consequence of the positive feedback effect induced by the dynamic hedging of the large trader through its portfolio insurance strategy, designed to protect the capital during a market downturn by replicating option positions. In fact, this positive feedback effect stems from the absence of sufficient natural counterparts to meet the demand for puts and calls, where large dealers can meet the demand by selling puts and calls. In doing so, they become short the option. They can neutralize their net risk exposure by synthetically replicating long option positions, which requires selling as the market falls and buying as it rises. This ensures the hedge position is sufficient to cover the option rising exposure, which introduces transactions large enough to amplify the initial price shock. It generates precisely the kind of vicious feedback loop that destabilizes markets.

Effects Impact on Greeks

The gap caused by the hedging feedback effect (tracking error) is always positive, so the Black-Scholes delta hedging strategy always implies a loss, directly linked to the difference of volatilities, while growing with the gamma (i.e., the large trader hedging activity) and with lower liquidity (higher ρ).

Figure 6
Optimal Delta and Gamma Greeks under market impact vs. Delta and Gamma Greeks under Black-Scholes model



In terms of delta hedging, we distinguish three effects:

- **A positive moneyness effect.** The large trader buys more underlying assets for in-the-money calls (more likely to be exercised).
- **A negative volatility effect.** For in-the-money calls, a higher volatility implies a higher probability to leave out of the money, which reduces the delta.
- **A negative time to maturity effect.** As residual time to maturity decreases, the optimal quantity to hedge is more predictable, which reduces the delta.

CONCLUSION

Market impact risk refers to the degree to which large size transactions can be carried out in a timely fashion with minimal impact on prices. As a result, managing investment and liquidity risks for large players requires introducing an explicit market impact function, and applying to derivatives significantly depends on whether or not there is significant delta hedging activity. In the case of no significant delta hedging activity, the risk appetite has significant influence on the optimal execution strategy. In the case of significant delta hedging activity, the optimal trading involves feedback hedging effects translating into a modified Black-Scholes hedging strategy. ■

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REFERENCES

Gatheral, Jim, and Alexander Schied. 2011. Optimal trade execution under geometric Brownian motion in the Almgren and Chriss framework. *International Journal of Theoretical and Applied Finance* 14, no. 3:353–68.

Kalife, Aymeric, Xiaolu Tan, and Lihang Wong. 2012. Dynamic hedging by a large player: From theory to practical implementation. Neural, Parallel, and Scientific Computations 20, no. 2:191–214.

Kalife, Aymeric, Saad Mouti, and Xiaolu Tan. 2015. Minimizing market impact of hedging insurance liabilities within risk appetite constraints. *Insurance Markets and Companies: Analyses and Actuarial Computations* 6, no. 2:58–64.

Kalife, Aymeric, and Saad Mouti. 2016. On optimal options book execution strategies with market impact. *Market Microstructure and Liquidity* 2, no. 3,4.

Leland, Hayne E. 1985. Option pricing and replication with transaction costs. *The Journal of Finance* 40, no. 5:1283–1301.

Lozenz, Julian, and Robert Almgren. 2011. Mean-variance optimal adaptive execution. *Applied Mathematical Finance* 18, no. 11:311–23.

Tse, S. T., P. A. Forsyth, J. S. Kennedy, and H. Windcliff. 2013. Comparison between the mean-variance optimal and the mean-quadratic-variation optimal trading strategies. *Applied Mathematical Finance* 20, no. 5:415–49.

ENDNOTES

1 Bloomberg Finance LP.

2 NAIC *https://www.naic.org/capital_markets_archive/150826.htm*.

3 See Christoffersen, Peter, Ruslan Goyenko, Kris Jacobs, and Mehdi Karoui. 2014. Illiquidity premia in the equity options market.

4 Yahoo Finance, CFA institute.

5 See Chan, Louis K.C., and Josef Lakonishok. 1995. The behavior of stock prices around institutional trades. *Journal of Finance* 50, no. 4:1,147–74; and Keim, Donald B., and Ananth Madhavan. 1995. Execution costs and investment performance: An empirical analysis of institutional equity trades. Working paper, School of Business Administration, University of Southern California.

6 See Leland, Hayne E. 1985.

7 Almgren, Robert F. 2003. Optimal execution with nonlinear impact functions and trading enhanced risk. *Applied Mathematical Finance* 10, no. 1:1–18.

8 See Kalife, Aymeric, and Saad Mouti. 2016.

9 F. Delbaen, W. Schachermayer, A General Version of the Fundamental Theorem of Asset Pricing, Math. Annalen 300 (1994), 463–520.

10 Jarrow, Robert A. 1994. Derivative security markets, market manipulation, and option pricing theory. *Journal of Financial and Quantitative Analysis* 29, no. 2:241–61.

11 Jarrow, Robert A. 1994. Derivative security markets, market manipulation, and option pricing theory. Journal of Financial and Quantitative Analysis 29, no. 2:241–61.

12 Frey, Rudiger. 1998. Perfect option hedging for a large trader. Finance and Stochastics 2, no. 2:115–41.

13 See Kalife, Aymeric, Xiaolu Tan, and Lihang Wong. 2012