ASTAM April 2025 Model Solutions

Question 1

The solution to this question is in the spreadsheet. It should be noted that as stated in the instructions for the exam, only work in the spreadsheet will be graded. Any work on paper is NOT graded for Excel problems.

ASTAM April 2025 Model Solutions

Question 2

(a) We have $N_1 \sim$ Poi(λ_1) and $N_2 \sim$ Poi(λ_2). Let $N = N_1 + N_2$.

$$P_N(z) = E[z^N] = E\left[z^{N_1+N_2}\right] = E\left[z^{N_1} \times z^{N_2}\right]$$
$$= E\left[z^{N_1}\right] E\left[z^{N_2}\right] \text{ by independence}$$
$$= P_{N_1}(z)\ P_{N_2}(z)$$
$$= \left(e^{\lambda_1(z-1)}\right)\left(e^{\lambda_2(z-1)}\right) \quad \text{(formula sheet)}$$
$$= e^{\left(\lambda_1+\lambda_2\right)(z-1)} \Rightarrow N \sim Poi\left(\lambda_1+\lambda_2\right)$$

Examiners' Notes:

1. *Read the question carefully. It asks to use PGF, not the probability distribution functions.*
2. *For full credit, candidates needed to provide a clear explanation of each step. Most candidates did not state that the second line was due to independence.*
3. *The underlining principle for this part is the one-on-one relationship between distributions and PGF functions. Hence you need prove that the PGF of (N1 + N2) is same as the PGF of a new Poisson distribution. So you lay out the PGF of (N1 + N2) using the definitive formula, then work it towards the distinctive expression of Poisson's PGF, and on the way you use (and say it clearly) the important condition of independence.*

(b) The aggregate claims from an individual Urban (U) policy, S_u are compound Poisson (Co Poi) distributed, with Poisson parameter $\lambda_u = 0.5$, and with severity random variable X_u. From properties of Co Poi distributions, we have for the mean :

$$E\left[S_u\right] = \lambda_u E\left[X_u\right] = 0.5 \times 1.5 = 0.75$$
where $E\left[X_u\right] = 0.6 \times 1 + 0.3 \times 2 + 0.1 \times 3 = 1.5$

And for the standard deviation, either:

$$V\left[S_u\right] = \lambda_u E\left[X_u^2\right]$$
$$E\left[X_u^2\right] = 0.6 \times 1 + 0.3 \times 4 + 0.1 \times 9 = 2.7$$
$$\Rightarrow V\left[S_u\right] = 0.5 \times 2.7 = 1.35 \Rightarrow SD\left[S_u\right] = 1.1619$$

Or:

$$
\begin{aligned}
&V[S_u]=E[V[S_u|N_u]]+V[E[S_u|N_u]]\\
&=E[N\times V[X_u]]+V[N\times E[X_u]]\\
&=\lambda_u V[X_u]+\lambda_u(E[X_u])^2\\
&V[X_u]=E[X_u^2]-(E[X_u])^2=2.7-1.5^2=0.45\\
&\Rightarrow V[S_u]=0.5\times 0.45+0.5\times 1.5^2=1.35\\
&\Rightarrow SD[S_u]=1.1619
\end{aligned}
$$

Examiners' Notes:

1. *Most candidates earned full credit for this part.*
2. *Read the question carefully. It is asking for results for one urban policy. It is not asking for the results for all four urban policies or for the entire portfolio.*

(c)

(i) The total number of claims from Urban drivers is $\lambda_U = 4(0.5) = 2.0$.
Similarly the total number of claims from Rural drivers is $\lambda_R = 6(0.3) = 1.8$.
So the total number of claims from all drivers is Poisson with $\lambda = 3.8$.

(ii) The probability function for the merged portfolio is a weighted average of the individual severity distribution, with λ_U and λ_R acting as weights. So

$$f_X(1) = \frac{2(0.60) + 1.8(0.10)}{3.8} = 0.36316$$

(iii) Similarly:

$$f_X(2) = \frac{2(0.30) + 1.8(0.40)}{3.8} = 0.34737$$

$$f_X(3) = \frac{2(0.10) + 1.8(0.50)}{3.8} = 0.28947$$

$$\left(\text{or } f_X(3) = 1 - f_X(1) - f_X(2) = 0.28947\right)$$

Examiners' Notes:

1. *Candidates did poorly on this part. Very few got part (a) correct with most determining that the $\lambda = 0.8$ which was the sum of the two λ. Part (i) is based on Part a), but from 4 U and 6 R polices.*
2. *To solve parts (ii) and (iii) correctly students need know that the severity of any individual claim in the portfolio is mixed distribution of the severities of U and R policies, with weights determined by the means of two original Poisson distributions that depict the frequency of Urban and Rural policies.*
3. *A common error is to use Poisson distribution to calculate the severity distribution.*

(d) (i) The probability is $\Pr\left[N=0\right] = e^{-3.8} = 0.022371$

(ii) Let $g(s)$ denote the aggregate claim probability function, and $f(y)$ denote the claim severity probability function. For the Poisson distribution, the a, b parameters are $a = 0, \quad b = \lambda$, so that the recursion formula is

$$g(s) = \frac{\lambda}{s}\sum_{j=1}^{s} j f_X(j)\; g(s-j)$$

$$g(0) = \Pr[N=0] = 0.02237$$

$$g(1) = \lambda f_x(1)\, g(0) = 0.03087$$

$$g(2) = \frac{\lambda}{2}\left(f_X(1)\, g(1) + 2f_X(2)\, g(0)\right) = 0.05083$$

Or

Let $p_k = \Pr[N=k] = \dfrac{\lambda^k}{k!} e^{-\lambda}$ denote the Poisson probability function.

Then

$$g(2) = p_1 f_X(2) + p_2\left(f_X(1)\right)^2$$

$$= 0.08501(0.34737) + 0.16152(0.36316)^2$$

$$= 0.05083$$

Examiners' Notes:

1. *Candidates did better on this part that on Part (c).*
2. *Most got Part (a) correct and about half earned most of all of the points for part (b).*

(e) The net premium for the stop loss is

$$P=\sum_{y=3}^{\infty}(y-3)\ g(y)=\sum_{y=3}^{\infty}y\,g(y)-3\sum_{y=3}^{\infty}g(y)$$

$$=\sum_{y=0}^{\infty}y\,g(y)-(g(1)+2g(2))-3\big(1-(g(0)+g(1)+g(2))\big)$$

$$=E[S]-E\big[S\wedge 3\big]$$

We have

$$E[S]=\lambda E[X]$$

$$E[X]=f_X(1)+2f_X(2)+3f_X(3)=1.92631$$

$$\Rightarrow E[S]=3.8\times 1.92631=7.32$$

Or

$$E[S]=6E\big[S_R\big]+4E\big[S_U\big]=4(0.75)+6(0.72)=7.32$$

$$E\big[S\wedge 3\big]=g(1)+2g(2)+3\big(1-(g(0)+g(1)+g(2))\big)$$

$$=0.03087+2\times 0.05083+3\times 0.89593=2.82031$$

$$\Rightarrow P=7.3200-2.8203=4.500$$

Examiners' Comment:

1. *Most candidates knew that* $P=E[S]-E[S\wedge 3]$.
2. *Candidates struggled to calculate either of the values* $E[S]$ *or* $E[S\wedge 3]$. *The whole concept was that the information needed to calculate these values was available from prior parts but most candidates did not use these values.*

ASTAM April 2025 Model Solutions

Question 3

(a)

- Insurer may not know the ground up loss distribution. Any losses below the deductible are generally unobserved by the insurer.
- Losses that are above the deductible, but by a relatively small amount may also be unobserved, as policyholders will not claim small amounts (to avoid a premium increase, or not worth the hassle, for example).
- Adverse selection impacts the distribution of losses by deductible. Higher risk drivers may select a lower deductible if they expect to be making claims.
- On the other hand, wealthier drivers are more inclined to self-insure, and are more likely to choose a higher deductible, even if they are high risk. The loss severity of wealthier drivers is likely to be different to less wealthy drivers.
- The premium saved by choosing a higher deductible may (in some cases) be greater than the difference in deductible, leading to an arbitrage pushing policyholders to choose the higher deductible.

Examiners' Comments: Candidates struggled with this part often not addressing the question that was asked.

(b)

(i) Let $Y \sim$ Pareto(α, θ) denote the ground-up loss random variable.
Given a deducible d, the claim severity random variable is $Y_d = Y - d \big| Y > d$, so that

$$\Pr\left[Y_d > y\right] = \Pr[Y - d > y | Y > d] = \frac{S_Y(y+d)}{S_Y(d)}$$

$$= \left(\frac{\theta}{\theta + y + d}\right)^{\alpha} \left(\frac{\theta + d}{\theta}\right)^{\alpha} = \left(\frac{\theta + d}{\theta + d + y}\right)^{\alpha}$$

$$\Rightarrow Y_d \sim \text{Pareto}(\alpha, \theta + d)$$

$$\Rightarrow E\left[Y_d\right] = \frac{\theta + d}{\alpha - 1}$$ Note: This is also given in the formula sheet.

$\Rightarrow$ the expected claim severity for class A is $\dfrac{\theta + d}{\alpha - 1} = \dfrac{2500 + 500}{2 - 1} = 3000$

$\Rightarrow$ the expected aggregate claim amount for Class A is

$$\lambda_A \times 3000 = (0.0625)(3000) = 187.50$$

(ii) The expected aggregate claims for a Class B policy are $0.0510(3500) = 178.50$.

The differential relative to Class A is $\dfrac{178.50}{187.50} = 0.9520$

(iii) The expected aggregate claims for a Class C policy are $0.0430(4000) = 172.00$.

(iv)

The differential relative to Class A is $\dfrac{172.00}{187.50} = 0.9173$

Examiners Comments:

1. *Candidates do not have to derive the Pareto result in (i) as the formula is given in the formula sheet.*
2. *Candidates did well on this part.*

(c)

(i) and (ii) Cost per unit of exposure for the three groups are:

$$A: \quad \frac{3,465,000}{18,500} = 187.297$$

$$B: \quad \frac{1,335,000}{8,000} = 166.875$$

$$C: \quad \frac{350,000}{2,200} = 159.091$$

So the differentials relative to group A are:

$$B: \quad \frac{166.875}{187.297} = 0.8910$$

$$C: \quad \frac{159.091}{187.297} = 0.8494$$

(iii) The off-balance factor is the ratio of the new average differential to the old average differential, weighted by exposure, i.e.

$$\frac{18,500(1) + 8000(0.8910) + 2200(0.8494)}{18,500(1) + 8000(0.9520) + 2200(0.9173)} = 0.9771$$

Examiners' Notes: Overall candidates did reasonably good on this question. However, many candidates over complicated the question and made the calculations more difficult than necessary. The part was very straight forward.

(d)

(i) Using the same exposure, given a new base rate of *B*, and the differentials in (c), for a 10% overall rate increase we have:

$$B\{18,500(1) + 8000(0.8910) + 2200(0.8494)\}$$

$$= 1.1(200)\{18,500(1) + 8000(0.9520) + 2200(0.9173)\}$$

$$\Rightarrow B = \frac{1.1(200)}{0.9771} = 225.16$$

$$\Rightarrow \text{increase is } \frac{225.16}{200.00} - 1 = 12.58\%$$

(ii) Similarly, the increase for the group C policies is

$$\frac{225.16(0.8494)}{200(0.9173)} - 1 = 4.25\%$$

(iii) The actual claims from 2024 indicate that the premiums were high side for the higher deductible policies. Charging an across-the-board increase could encourage higher deductible policyholders to move to other insurers.

Examiners' Comments: Overall, candidates did a good job on this part.

ASTAM April 2025 Model Solutions

Question 4

(a)

(i) We have samples of the ground up loss <u>conditional</u> on the loss exceeding the deductible, 1000. Let $n = 20$. Then

$$L(\alpha) = \prod_{i=1}^{n} \frac{f(x_i)}{S(d)} = \prod_{i=1}^{n} \left(\frac{\alpha\theta^{\alpha}}{\left(\theta + x_i\right)^{\alpha+1}} \right) \left(\frac{\theta + d}{\theta} \right)^{\alpha} = \prod_{i=1}^{n} \frac{\alpha(\theta + d)^{\alpha}}{\left(\theta + x_i\right)^{\alpha+1}}$$

$$= \alpha^{n} (\theta + d)^{\alpha n} \prod_{i=1}^{n} \left(\theta + x_i\right)^{-(\alpha+1)}$$

$$\Rightarrow l(\alpha) = n \log \alpha + \alpha n \log(\theta + d) - (\alpha + 1) \sum_{i=1}^{n} \log\left(\theta + x_i\right)$$

(ii)

$$l'(\alpha) = \frac{n}{\alpha} + n \log(\theta + d) - \sum_{i=1}^{n} \log\left(\theta + x_i\right)$$

Set equal to 0 for

$$\hat{\alpha} = \frac{n}{\sum_{i=1}^{n} \log\left(\theta + x_i\right) - n \log(\theta + d)} = \frac{20}{181.66 - 20 \log(6000)} = 2.60766$$

Examiners' Comments: Candidates did very well. Most candidates understand that for left-truncated reported losses, conditional density should be used for the likelihood function. Common mistakes include ignoring the truncation or treating the reported losses as net losses after applying the deductible (although the question states that they are ground-up losses).

(b) Take the second derivative of the log likelihood:

$$l''(\alpha) = -\frac{n}{\alpha^2} \Rightarrow Var[\hat{\alpha}] \approx \frac{\hat{\alpha}^2}{n} = 0.3400 = 0.5831^2$$

$\Rightarrow 95\%$ CI is approximately $(2.608 - 1.96(0.5831)\,,\ 2.608 + 1.96(0.5831)) = (1.46\ ;\ 3.75)$

Examiners' Comments: Candidates did well in this part.

(c)

Method 1:

$$Y + (X \wedge d) = X$$

$$E[Y] = E[X] - E[X \wedge d] = \frac{\theta}{\alpha - 1} - \frac{\theta}{\alpha - 1}\left(1 - \left(\frac{\theta}{\theta + d}\right)^{\alpha - 1}\right)$$

.

$$= \frac{\theta}{\alpha - 1}\left(\frac{\theta}{\theta + d}\right)^{\alpha - 1} = \frac{\theta^{\alpha}}{(\alpha - 1)(\theta + d)^{\alpha - 1}}$$

$$\Rightarrow \text{MLE of } E[Y] = \frac{\theta^{\hat{\alpha}}}{(\hat{\alpha} - 1)(\theta + d)^{\hat{\alpha} - 1}} = 2319.9$$

Method 2:

$$E[Y] = E[X - d | X > d]\Pr[X > d]$$

$(X - d | X > d) \sim \text{Pareto}(\alpha, \theta + d)$ from the tables

$$\Rightarrow E[Y] = \frac{\theta + d}{\alpha - 1}\left(\frac{\theta}{\theta + d}\right)^{\alpha} = \frac{\theta^{\alpha}}{(\alpha - 1)(\theta + d)^{\alpha - 1}}$$

$$\Rightarrow \text{MLE of} E[Y] = \frac{\theta^{\hat{\alpha}}}{(\hat{\alpha} - 1)(\theta + d)^{\hat{\alpha} - 1}} = 2319.9$$

Method 3:

$$E[Y]=\int_d^\infty (x-d)f(x)\,dx=\int_d^\infty (x-d)\,\frac{\alpha\theta^\alpha}{(\theta+x)^{\alpha+1}}dx$$

$$=\int_0^\infty z\frac{\alpha\theta^\alpha}{(\theta+d+z)^{\alpha+1}}dx=\frac{\theta^\alpha}{(\theta+d)^\alpha}\int_0^\infty z\frac{\alpha(\theta+d)^\alpha}{(\theta+d+z)^{\alpha+1}}dx$$

$$=\frac{\theta^\alpha}{(\theta+d)^\alpha}\left(\frac{\theta+d}{\alpha-1}\right)=\frac{\theta^\alpha}{(\alpha-1)(\theta+d)^{\alpha-1}}$$

$$\Rightarrow \text{MLE of } E[Y]=\frac{\theta^{\hat{\alpha}}}{(\hat{\alpha}-1)(\theta+d)^{\hat{\alpha}-1}}=2319.9$$

Examiners' Comments: Candidates did well. A few candidates calculate the MLE for the expected loss per payment instead of the expected loss per loss requested in question. Some candidates didn't apply deductible correctly and mistakenly calculate $E\left[(X-d)^+\right]$ *as* $E[X]-d$.

.

(d)

$$LER=\frac{E[X\wedge d]}{E[X]}$$

$$E[X]=\frac{\theta}{\hat{\alpha}-1}=3110.1$$

$$E[X\wedge d]=E[X]-E[Y]=3110.1-2319.9=790.16$$

$$\Rightarrow LER=25.406\%$$

Examiners' Comments: Candidates did excellently in this part.

(e)

- Deductibles tend to eliminate small claims, reducing expense ratios. In this case, around 28% of small losses would be eliminated with a deductible of 1000. Using co-insurance, every loss may be submitted for partial reimbursement, even very small ones.
- A major motivation for auto insurance is to cover the rare large losses, particularly (for mandatory insurance) the liability losses. Coinsurance fails to provide adequate cover, as 75% of a very large loss is still a very large loss, leading to a large potential increase in defaults on policyholder share of third party payouts.
- Regulators may not accept co-insurance for mandatory cover as it would likely lead to loss of cover for third parties.

Examiners' Comments: Overall, most candidates explained well why insurers may prefer deductible to a coinsurance design when the expected losses are similar. Some candidates try to explain the advantages of deducible, but fail to logically explain why it is preferrable, compared with coinsurance. Some candidates discussed the question from the policyholder perspective rather than the insurer perspective.

ASTAM April 2025 Model Solutions

Question 5

(a) The outstanding claims for AY 3 are .

$$\hat{C}_{3,3} = C_{3,0}(f_0 \times f_1 \times f_2)$$

$$\left(f_0 \times f_1 \times f_2\right) = \lambda_0 = \left(\frac{1155+1304+1568}{660+792+825}\right)\left(\frac{1403+1650}{1155+1304}\right)\left(\frac{1568}{1650}\right)$$

$$= (1.7686)(1.2416)(1.1176) = 2.4540$$

$$\widehat{C}_{3,3} - C_{3,0} = 941(2.4540 - 1) = 1368.2$$

Examiners' Comments: Part (a) done very well. The most common mistake was to calculate the estimated ultimate claims but fail to subtract out the claims that have already been paid in order to get to the OCR.

(b)

(i) We have that $E\left[C_{i,j+1}|C_{i,j}\right] = C_{i,j} + \left(\beta_{j+1} - \beta_j\right)\mu_i$. So

$$E\left[C_{i,J}|C_{i,J-1}\right] = C_{i,J-1} + \left(\beta_J - \beta_{J-1}\right)\mu_i$$

$$= C_{i,J-1} + \left(1 - \beta_{J-1}\right)\mu_i$$

$$\Rightarrow E\left[C_{i,J}|C_{i,J-2}\right] = E\left[E\left[C_{i,J}|C_{i,J-1}\right]|C_{i,J-2}\right]$$

$$= E\left[C_{i,J-1}|C_{i,J-2}\right] + \left(1 - \beta_{J-1}\right)\mu_i$$

$$= C_{i,J-2} + \left(\beta_{J-1} - \beta_{J-2}\right)\mu_i + \left(1 - \beta_{J-1}\right)\mu_i$$

$$= C_{i,J-2} + \left(1 - \beta_{J-2}\right)\mu_i$$

Continuing similarly…

$$\Rightarrow E[C_{i,J}|C_{i,j}] = E[C_{i,j+1}|C_{i,j}] + (1 - \beta_{j+1})\mu_i$$

$$= C_{i,j} + (\beta_{j+1} - \beta_j)\mu_i + (1 - \beta_{j+1})\mu_i$$

$$= C_{i,j} + (1 - \beta_j)\mu_i$$

Alternative solution: (iterating forwards instead of backwards)

$E[C_{i,j+1}|C_{i,j}] = C_{i,j} + (\beta_{j+1} - \beta_j)\mu_i$ (from BF (1))

$E[C_{i,j+2}|C_{i,j+1}] = C_{i,j+1} + (\beta_{j+2} - \beta_{j+1})\mu_i$ (also from BF (1))

$$E[C_{i,j+2}|C_{i,j}] = E[E[C_{i,j+2}|C_{i,j+1}]|C_{i,j}]$$

$$= E[C_{i,j+1} + (\beta_{j+2} - \beta_{j+1})\mu_i|C_{i,j}]$$

$$= E[C_{i,j+1}|C_{i,j}] + (\beta_{j+2} - \beta_{j+1})\mu_i$$

$$= C_{i,j} + (\beta_{j+1} - \beta_j)\mu_i + (\beta_{j+2} - \beta_{j+1})\mu_i$$

$$= C_{i,j} + (\beta_{j+2} - \beta_j)\mu_i$$

And applying this idea iteratively yields

$$E[C_{i,J}|C_{i,j}] = C_{i,j} + (\beta_J - \beta_j)\mu_i$$

$= C_{i,j} + (1 - \beta_j)\mu_i$ since we are given that $\beta_J = 1$

(ii) By induction:

We have that the result is true for $j = 0$ from BF assumption (1).
We are required to show that if the result is true for j, it must also be true for $j + 1$.

$$\text{Let } E[C_{i,j}] = \beta_j \mu_i \qquad \text{(inductive assumption)}$$
$$\Rightarrow E[C_{i,j+1}] = E\left[E[C_{i,j+1}|C_{i,j}]\right] = E[C_{i,j} + (\beta_{j+1} - \beta_j)\mu_i]$$
$$= E[C_{i,j}] + (\beta_{j+1} - \beta_j)\mu_i$$
$$= \beta_j\ \mu_i + (\beta_{j+1} - \beta_j)\mu_i$$
$$= \beta_{j+1}\mu_i \qquad \text{as required}$$

Examiners' Comments: Many candidates made little or no attempt to do part (b). However, the candidates who attempted this part were generally able to get much of the credit. The most common error to show the iteration used. In part (b)(ii), many candidates attempted a proof by induction, but failed to give the base case and/or induction steps.

(c)

(i) We have

$$\tilde{C}_{i,J} = E[C_{i,J}|C_{i,j}] = C_{i,j} + (1 - \beta_j)\mu_i$$
$$\hat{C}_{i,J} = \hat{\lambda}_j\ C_{i,j} \Rightarrow C_{i,j} = \beta_j \hat{C}_{i,J}$$
$$\Rightarrow \tilde{C}_{i,J} = \beta_j \hat{C}_{i,J} + (1 - \beta_j)\mu_i$$

(ii) A credibility estimate is a weighted average of an estimate based on data directly from the risk, and an estimate based on external sources.

The BF is just such a linear weighted average:

$\hat{C}_{i,J}$, is the estimate based on the data alone;

μ_i, which is the estimate based on external sources (i.e. loss ratio); and

β_j, is the credibility factor.

Examiners' Comments: Part (c) was done fairly well. For (c)(i), candidates seemed to generally understand the concepts, and the most common error was to fail to justify the steps used. Candidates did a good job of explaining the credibility formula in (c)(ii), though many candidates lost some credit due to incomplete or inadequate explanations.

(d) We have $\mu_3 = (0.9)(2400) = 2160$.

The BF estimate of outstanding claims for AY 3 is $\tilde{C}_{3,3} - C_{3,0}$,

$$\tilde{C}_{3,3} = C_{3,0} + (1 - \hat{\beta}_0)\mu_3$$
$$\Rightarrow \tilde{C}_{3,3} - C_{3,0} = (1 - \hat{\beta}_0)\mu_3 = \left(1 - \frac{1}{\lambda_0}\right)\mu_3$$
$$= \left(1 - \frac{1}{2.4540}\right) 2160 = 1279.8$$

Alternative solution: Using the "credibility" formula from the previous part, the estimated ultimate claims are:

$$\tilde{C}_{3,3} = \hat{\beta}_0 \hat{C}_{3,3} + (1 - \hat{\beta}_0)\mu_3$$

$$\tilde{C}_{3,3} = \frac{1}{2.4540}(2309.21) + \left(1 - \frac{1}{2.4540}\right) 2160 = 2220.8$$

so that the estimated outstanding claims reserve is

$$\tilde{C}_{3,3} - C_{3,0} = 2220.84 - 941 = 1279.8$$

Examiners' Comments: Part (d) was generally done well; candidates used both approaches in roughly equal numbers; either was fine. A large number of candidates (perhaps as many as half) calculated the estimated ultimate claims instead of the OCR. On a related note, some candidates used the credibility formula incorrectly, mixing ultimate claims for one method with OCR for the other method.

(e)

Advantage: The Chain Ladder bases the outstanding claims estimate for AY3 entirely on the data from the first development year; here, that's about 40% of the total. If the claims payment system is not stable, the Chain Ladder approach may be misleading – e.g. if the claims are processed faster in AY3, the Chain Ladder estimate will overstate the outstanding claims for that AY in the early development years.

Some other advantages from the Brown and Lennox text (mostly related to the above; any of these would get full credit):

- BF method is more stable than CL.
- BF allows the inclusion of other data sources.

Disadvantage: The BF estimate is highly dependent on the external estimate of losses, in this case based on a 90% loss ratio. If the premiums are too low, then the loss ratio will be too low, leading to an underestimate of outstanding claims in the early development years.

Examiners' Comments: Part (e) was well done, with most candidates correctly identifying the main points. Some of the explanations were imprecise (such as describing the estimate derived from the expected loss ratio method as "arbitrary"), which led to minor deductions.

ASTAM April 2025 Model Solutions

Question 6

(a) Let $n = 100$.

$$\pi_{\theta|\mathrm{x}}(\theta) \propto L(\theta)\,\pi(\theta) = \left(\prod_{i=1}^{n} \frac{e^{-x_i/\theta}}{\theta}\right)\left(\frac{\beta^{\alpha} e^{-\beta/\theta}}{\theta^{\alpha+1}\,\Gamma(\alpha)}\right)$$

$$= \frac{e^{-\frac{1}{\theta}\left(\sum_{i=1}^{n} x_i\right)} \beta^{\alpha}\, e^{-\beta/\theta}}{\theta^{n+\alpha+1}\,\Gamma(\alpha)}$$

$$\propto \frac{e^{-\frac{1}{\theta}\left(\sum_{i=1}^{n} x_i + \beta\right)}}{\theta^{n+\alpha+1}} \quad \text{(ignoring terms not involving } \theta\text{)}$$

$$\Rightarrow \theta|\mathrm{x} \sim \mathrm{IG}\left(\alpha + n,\ \beta + \sum_{i=1}^{n} x_i\right) \qquad \text{as required.}$$

Examiners' Comments: If a question says "show that" in the directions, a detailed proof is expected. Many candidates cited this as a known result without providing a detailed proof.

(b) Let $\alpha^* = \alpha + 100 = 130$; $\quad \beta^* = \beta + \sum_{i=1}^{100} x_i = 60,000.$

$$E[X] = E\big[E[X|\theta]\big] = E[\theta] = = \frac{\beta^*\ \Gamma(\alpha^* - 1)}{\Gamma(\alpha^*)} = \frac{\beta^*}{\alpha^* - 1} = \frac{60,000}{130 - 1} = 465.12$$

Examiners' Comments: The question asked for a predictive mean. Many students gave the posterior mean and didn't indicate why the posterior and predictive means are the same in this case.

(c) The hypothetical parameters are calculated using the prior distribution for θ.

(i)

$$\mu(\theta) = E[X|\theta] = \theta$$

$$\mu = E\left[\mu(\theta)\right] = E[\theta] = \frac{\beta}{\alpha - 1} = \frac{10,000}{29} = 344.83$$

(ii)

$$v(\theta) = Var[X|\theta] = \theta^2$$

$$v = E[v(\theta)] = E[\theta^2] = \frac{\beta^2 \Gamma(\alpha - 2)}{\Gamma(\alpha)} = \frac{\beta^2}{(\alpha - 1)(\alpha - 2)} = 123,152.7$$

(iii)

$$a = Var[\mu(\theta)] = Var[\theta] = E\left[\theta^2\right] - E[\theta]^2 = 4246.65$$

Examiners' Comments: None.

(d) (i) The credibility factor is

$$Z = \frac{n}{n + {}^{v}/_{a}} = \frac{100}{100 + 29} = 0.7752$$

(ii)

$$P = Z\overline{X} + (1 - Z)\mu$$

$$\overline{X} = \frac{50,000}{100} = 500$$

$$\Rightarrow P = 0.7752(500) + 0.2248(344.83) = 465.12$$

Examiners' Comments: Some candidates misidentified n=1 rather than n=100. There was also some difficulty in calculating x-bar. It seemed that some candidates knew the formula but didn't understand what the inputs in the formula meant.

(e) The Buhlmann premium is the nearest linear combination of prior and posterior estimates to the Bayesian premium.

In this case, the Bayesian estimate is itself a linear function of the prior and posterior means so the Bühlmann and Bayesian are the same.

Examiners' Comments: Not all conjugate prior pairs have exact credibility, and even in the case where there are linear exponential families and conjugate priors, there isn't always exact credibility. The question was looking for an understanding of Bayesian and Buhlmann credibility and their relationship. The idea that the Buhlmann credibility is the best linear estimate is a fundamental one, and understanding that concept is very important.