QFI QF Model Solutions
Spring 2025

1. **Learning Objectives:**
1. The candidate will understand the foundations of quantitative finance.

Learning Outcomes:
(1d) Understand and apply Ito's Lemma.

Sources:
Neftci Ch. 10

Commentary on Question:
Most candidates did well on this question.

Solution:
(a) Verify using Ito's Lemma that

$$d\left[\left(X(t)+Y(t)\right)^2\right]$$
$$= 2X(t)dX(t) + 2Y(t)dY(t) + 2X(t)dY(t) + 2Y(t)dX(t) + \left(\sigma(t)+v(t)\right)^2 dt$$

Commentary on Question:
Most candidates did well on this part.

$$d\left(\left(X(t)+Y(t)\right)^2\right) = 2\left(X(t)+Y(t)\right)\left(dY(t)+dX(t)\right) + \left(\sigma(t)+v(t)\right)^2 dt$$

$$d\left(\left(X(t)+Y(t)\right)^2\right) = 2X(t)dX(t) + 2Y(t)dY(t) + 2X(t)dY(t) + 2Y(t)dX(t) + \left(\sigma(t)+v(t)\right)^2 dt$$

(b) Verify, using Ito's Lemma and part (a), that

$$d\left(X(t)Y(t)\right) = X(t)dY(t) + Y(t)dX(t) + \sigma(t)v(t)dt$$

Commentary on Question:
Most candidates verified this part by using Ito's lemma. Many did this without using part (a).

1. **Continued**

$X(t)Y(t) = \frac{1}{2}((\big(X(t) + Y(t)\big)^2 - Y^2(t) - X^2(t))$

Using Ito's Lemma

$d\big(X(t)Y(t)\big) = \frac{1}{2}(d\big(X(t) + dY(t)\big)^2 - d\big(Y^2(t)\big) - d(X^2(t))$ -- equation 1

$d\left(X^2(t)\right) = 2\,X(t)dX(t) + \sigma^2(t)dt$ -- Equation 2

$d\left(Y^2(t)\right) = 2\,Y(t)dY(t) + v^2(t)dt$ –- Equation 3

From part(a)

$d\left(\big(X(t) + Y(t)\big)^2\right) = 2X(t)dX(t) + 2Y(t)dY(t) + 2X(t)dY(t) + 2Y(t)dX(t) + \big(\sigma(t) + v(t)\big)^2 dt$ – Equation 4

Substitute equations 2, 3, and 4 into equation 1

$$d\big(X(t)Y(t)\big) = 0.5\left(2X(t)dX(t) + 2Y(t)dY(t) + +2X(t)dY(t) + 2Y(t)dX(t) + \big(\sigma(t) + v(t)\big)^2 dt\right) - 2\,X(t)dX(t) - \sigma^2(t)dt - 2\,Y(t)dY(t) - v^2(t)dt)$$

$$\mathrm{d}\big(\mathrm{X(t)Y(t)}\big) = X(t)dY(t) + Y(t)dX(t) + \sigma(t)v(t)dt$$

(c) Derive *A*, *B*, *C*, and *D* using part (b).

Commentary on Question:
Many candidates obtained correct values for the quantities.

$Let\ X(t) = B + C\int_0^t e^{Ds}\,dW(s)$
$dX(t) = Ce^{Dt}dW(t)$
$Let\ Y(t) = e^{-At}$
$dY(t) = -AY(t)dt$
$S(t) = Y(t)X(t)$
$dS(t) = d(Y(t)X(t))$

Applying the result of part (b)
$d\big(Y(t)X(t)\big) = -AX(t)Y(t)dt + Ce^{(D-A)t}dW(t) + 0$
$d\big(Y(t)X(t)\big) = -AS(t)dt + Ce^{(D-A)t}dW(t)$

1. **Continued**

Comparing the coefficient of the equation A = 10, D = 10, C = 8
B = *S (0) = 1*

<u>Alternatively:</u>

This is an Ornstein-Uhlenbeck SDE. Let $X(t) = S(t)\exp(10t)$
Then:

$$
\begin{aligned}
dX(t) &= 10X(t)dt + \exp(10t)\, dS(t) \\
&= 10S(t)\exp(10t) + \exp(10t)\left(-10S(t)dt + 8dW(t)\right) \\
&= 8\exp(10t)dW(t).
\end{aligned}
$$

We now have a driftless SDE. Therefore

$$X(t) = X(0) + 8\int_0^t \exp(10s)\, dW(s)$$

$$S(t) = \exp(-10t)\left[S(0) + 8\int_0^t \exp(10s)\, dW(s)\right]$$

2. **Learning Objectives:**

1. The candidate will understand the foundations of quantitative finance.

Learning Outcomes:

(1a) Understand and apply concepts of probability and statistics important in mathematical finance.

(1c) Understand Ito integral and stochastic differential equations.

(1h) Define and apply the concepts of martingale, market price of risk and measures in single and multiple state variable contexts.

Sources:

Neftci Ch. 8, 9, Chin page 57

Commentary on Question:

Many candidates got full points for part (a) and (b), as they are straightforward. Very few candidates got full points for part (c), as the variance calculation is quite difficult. But many of them got partial points for showing the steps of integral derivations. About part (d), a number of candidates got full points, but still many did not get any points due to lack of understanding of the definition of standard Brownian motion.

Solution:

(a) Verify that $Cov\ (B_s, B_t) = \min\{s, t\}$.

Since $B_t \sim N(0, t)$
And by definition
$Cov(B_s, B_t) = E(B_s\, B_t) - E(B_s)E(B_t) = E(B_s B_t)$

Let s ≤ t
$E(B_s B_t) = E(B_s(B_t - B_s) + B_s^2)$
$= E(B_s)E(B_t - B_s) + E(B_s^2)$
$= s$

Therefore, $Cov(B_s, B_t) = min(s, t)$

(b) Derive $E[B_4 - B_1|F_3]$.

We find E(B4 −B1|F3) = E(B4|F3)−E(B1|F3). Since $\{B_t$, t ≥ 0} is a martingale, we know E(B4|F3) = B3.
We know E(B1|F3) = B1. Combined, this gives
E(B4 − B1|F3) = B3 − B1.

2. Continued

(c) Derive the distribution of the Riemann integral $\int_0^1 e^s B_s ds$.

Using Integration by parts formula for Wiener integrals

$\int_0^1 e^s dB_s = e^1 B_1 - e^0 B_0 - \int_0^1 e^s B_s ds$

$\int_0^1 e^s dB_s = e^1 B_1 - \int_0^1 e^s B_s ds$

$eB_1 = \int_0^1 e dB_s$

$=> \int_0^1 e^s B_s ds = \int_0^1 e dB_s - \int_0^1 e^s dB_s = \int_0^1 (e^1 - e^s) dB_s$

The Distributional property of Wiener integrals implies that

$\int_0^1 (e^1 - e^s) dB_s \sim N(0, \int_0^1 (e^1 - e^s)^2 ds)$

$\int_0^1 (e^1 - e^s)^2 ds = \frac{4e - e^2 - 1}{2}$

$\int_0^1 e^s B_s ds \sim N(0, \frac{4e - e^2 - 1}{2})$

(d) Verify that $\{X_t, t \geq 0\}$ is not a standard Brownian motion.

If {Xt, t ≥ 0} were a standard Brownian motion, then it would necessarily be the case that Xt ~ N (0, t) for every t > 0.

However, since Bt and Wt are independent, we conclude that
$E[X_t^2] = E[B_t^2 W_t^2] = E[B_t^2]E[W_t^2] = t^2$

and so Var(Xt) ≠ t which proves that {Xt, t ≥ 0} is not a standard Brownian motion.

3. **Learning Objectives:**

1. The candidate will understand the foundations of quantitative finance.

Learning Outcomes:

(1b) Understand the importance of the no-arbitrage condition in asset pricing.

(1i) Demonstrate understanding of the differences and implications of real-world versus risk-neutral probability measures, and when the use of each is appropriate.

Sources:

Neftci Ch. 1, 3, 5

Commentary on Question:

The purpose of this question was to test the application of quantitative finance concepts to an exotic option, its payoff, and price under the Black-Scholes framework. Beyond a very basic grasp of GBM properties, indicator functions, and mathematical definition of expectations, candidates needed to perform some algebraic manipulations. Candidates received maximums points for showing the explicit steps taken in order to arrive at the desired results.

Solution:

(a)

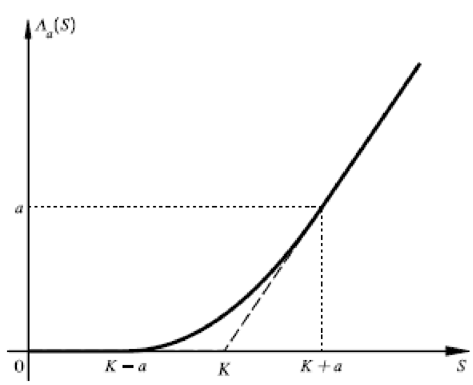
(i) Graph $P(S_T)$ as a function of S_T.

(ii) Compare $P(S_T)$ with the payoff of a vanilla European call option with the same underlying asset and strike price K.

Commentary on Question:

For (i) Most candidates successfully graphed the payoff of the exotic call. For (ii) most candidates were able to speak to the payoff of the exotic option being greater than or equal to the vanilla call option. Only some candidates recognized the convexity of the exotic option.

(i)



3. Continued

(ii)

- $P(S_T) \geq max\big((S_T - K), 0\big)$
- $P(S_T)$ decreases montonically to the standard call option as a drops to 0
- In contrast to the standard call $(S-K)^+$ where derivative with respect to S has a unit jump discontinuity at $S = K$, it has a continuous derivative for S
- The graph of $P(S_T)$ resembles the graph of the option value at time t<T.

(b) Verify that $E_t\left[S_T^\beta\right] = S_t^\beta e^{\beta\left(r+\frac{1}{2}\sigma^2(\beta-1)\right)\tau}$, where $\tau = T - t$ and β is a constant.

Commentary on Question:
Successful candidates needed to show the steps below for full points. Candidates also received full points for accurately applying Ito's Lemma and then taking the expectation. Candidates did not receive full points if they jumped to the final statement without showing interim work or if that work did not lead to the desired result.

$$S_T = S_t e^{\left(r-\frac{1}{2}\sigma^2\right)\tau+\sigma W_T}$$

$$E_t[{S_T}^\beta] = e^{\beta\left(r-\frac{1}{2}\sigma^2\right)\tau} E[{S_t}^\beta e^{\beta\sigma Z\sqrt{\tau}} | S_t]$$

$$= {S_t}^\beta e^{\beta\left(r-\frac{1}{2}\sigma^2\right)\tau} E_t[e^{\beta\sigma Z\sqrt{\tau}}]$$

$$= {S_t}^\beta e^{\beta\left(r-\frac{1}{2}\sigma^2\right)\tau} e^{\frac{1}{2}(\beta^2\sigma^2\tau)}$$

$$= {S_t}^\beta e^{\beta\left(r+\frac{1}{2}\sigma^2(\beta-1)\right)\tau}$$

(c) Verify that for any $A > 0$

$$E_t\left[\mathbb{I}_{S_T>A}\right] = N\left(d_-\left(\frac{S_t}{A}, \tau\right)\right)$$

where $d_-(x,\tau) = \frac{\ln(x)+\left(r-\frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}$.

Commentary on Question:
Candidates generally performed well on this part. Showing the steps below is enough for full credit.

3. Continued

$$\begin{aligned} E_t[\mathbb{I}_{S_T>A}] &= \mathbb{Q}_t[S_T > A] \\ &= \mathbb{Q}\left[S_t e^{\left(r-\frac{\sigma^2}{2}\right)\tau+\sigma Z\sqrt{\tau}} > A\right] \\ &= \mathbb{Q}\left[Z > \frac{-ln\left(\frac{S_t}{A}\right)-\left(r-\frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}\right] \\ &= N\left(d_-\left(\frac{S_t}{A},\tau\right)\right). \end{aligned}$$

by symmetry.

(d) Verify that

$$E_t[\,\mathbb{I}_{\,K-a<S_T\leq K+a}] = N\left(d_-\left(\frac{S_t}{K-a}\,,\tau\right)\right) - N\left(d_-\left(\frac{S_t}{K+a}\,,\tau\right)\right)$$

Commentary on Question:
Successful candidates were able to explicitly relate the expectation to the difference of two indicator functions or probabilities, and then directly apply the result of part (c).

$$\begin{aligned} E_t[\,\mathbb{I}_{\,K-a<S_T\leq K+a}] &= E_t[\mathbb{I}_{S_T>K-a}] - E_t[\mathbb{I}_{S_T>K+a}] \\ &= N\left(d_-\left(\frac{S_t}{K-a}\,,\tau\right)\right) - N\left(d_-\left(\frac{S_t}{K+a}\,,\tau\right)\right) \end{aligned}$$

Since from part (c), we know:

$$E_t[\mathbb{I}_{S_T>K-a}] = N\left(d_-\left(\frac{S_t}{K-a},\tau\right)\right)$$

and

$$E_t[\mathbb{I}_{S_T>K+a}] = N\left(d_-\left(\frac{S_t}{K+a},\tau\right)\right)$$

(e) Verify that for any $A > 0$,

$$E_t\left[S_T^\beta \mathbb{I}_{S_T>A}\right] = S_t^\beta e^{\beta\left(r+\frac{1}{2}\sigma^2(\beta-1)\right)\tau} N\left(d_-\left(\frac{S_t}{A},\tau\right)+\beta\sigma\sqrt{\tau}\right).$$

3. **Continued**

Commentary on Question:
Most candidates struggled with this question. Partial credit was given where candidates were able to recognize and attempt to develop and solve the needed integral below.

$$E_t[{S_T}^\beta \mathbb{I}_{S_T>A}] = {S_t}^\beta e^{\beta\left(r-\frac{\sigma^2}{2}\right)\tau} E_t\left[e^{\beta\sigma Z\sqrt{\tau}} \mathbb{I}_{Z>-d_-\left(\frac{S_t}{A},\tau\right)}\right]$$

The expectation on the right-hand side is equivalent to:

$$\int_{-d_-\left(\frac{S_t}{A},\tau\right)}^{\infty} e^{\beta\sigma z\sqrt{\tau}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = \int_{-d_-\left(\frac{S_t}{A},\tau\right)}^{\infty} e^{\frac{\beta^2\sigma^2\tau}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-\beta\sigma\sqrt{\tau})^2}{2}} dz$$
$$= e^{\frac{\beta^2\sigma^2\tau}{2}} \int_{-d_-\left(\frac{S_T}{A},\tau\right)-\beta\sigma\sqrt{\tau}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}} dw$$

after a change in variables.

Therefore:

$$E_t[{S_T}^\beta \mathbb{I}_{S_T>A}] = {S_t}^\beta e^{\beta\left(r-\frac{\sigma^2}{2}\right)\tau} e^{\frac{\beta^2\sigma^2\tau}{2}} N\left(d_-\left(\frac{S_T}{A},\tau\right) + \beta\sigma\sqrt{\tau}\right)$$
$$= {S_t}^\beta e^{\beta\left(r+\frac{\sigma^2}{2}(\beta-1)\right)\tau} N\left(d_-\left(\frac{S_T}{A},\tau\right) + \beta\sigma\sqrt{\tau}\right)$$

(f) Verify that the no-arbitrage pricing formula for this option $C(t,S;K,a)$ is

$$C(t,S;K,a) = e^{-r\tau}\left(\frac{1}{4a} E_t[{S_T}^2\, \mathbb{I}_{K-a<S_T\le K+a}] - \frac{(K-a)}{2a} E_t[S_T\, \mathbb{I}_{K-a<S_T\le K+a}] + \frac{(K-a)^2}{4a} E_t[\,\mathbb{I}_{K-a<S_T\le K+a}] + E_t[S_T\, \mathbb{I}_{S_T>K+a}] - K E_t[\mathbb{I}_{\{S_T>K+a)}]\right)$$

Commentary on Question:
Most candidates use a right approach this question. The key element is to recognize the no-arbitrage value is the present value of the expected payoffs under the risk-neutral measure. After that, the rest of the problem is largely simple algebra. No results from prior parts were needed to complete part (f). Points were received where candidates were successfully able to identify the payoff of the exotic option and appropriately apply the risk-neutral discount rate in their answer.

3. Continued

Due to no arbitrage, the value of the option is the risk-neutral conditional expectation of the payoffs

$$\begin{aligned} C(t,S;K,a) &= e^{-r\tau}E_t[P(S_T)\,] \\ &= e^{-r\tau}E_t\left[\frac{1}{4a}\left({S_T}^2 - 2(K-a)S_T + (K-a)^2\right)\mathbb{I}_{\,K-a<S\le K+a} \right. \\ &\left. + (S_T - K)\mathbb{I}_{S>K+a}\right]. \end{aligned}$$

By linearity of expectations:

$$\begin{aligned} C(t,S;K,a) = e^{-r\tau}&\left(\frac{1}{4a}E_t[{S_T}^2\mathbb{I}_{K-a<S\le K+a}] - \frac{(K-a)}{2a}E_t[S_T\mathbb{I}_{K-a<S\le K+a}] \right. \\ &\left. + \frac{(K-a)^2}{4a}E_t[\,\mathbb{I}_{\,K-a<S\le K+a}] + E_t[S_T\mathbb{I}_{S>K+a}] - KE_t[\mathbb{I}_{S>K+a}]\right) \end{aligned}$$

4. Learning Objectives:

1. The candidate will understand the foundations of quantitative finance.

2. The candidate will understand:
   - The Quantitative tools and techniques for modeling the term structure of interest rates.
   - The standard yield curve models.
   - The tools and techniques for managing interest rate risk.

3. The candidate will understand:
   - How to apply the standard models for pricing financial derivatives.
   - The implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory.
   - How to evaluate risk exposures and the issues in hedging them.

Learning Outcomes:

(1a) Understand and apply concepts of probability and statistics important in mathematical finance.

(1c) Understand Ito integral and stochastic differential equations.

(1d) Understand and apply Ito's Lemma.

(2a) Understand and apply the concepts of risk-neutral measure, forward measure, normalization, and the market price of risk, in the pricing of interest rate derivatives.

(3a) Demonstrate an understanding of option pricing techniques and theory for equity derivatives.

Sources:

Fixed Income Securities pp.522, 538, 711

Commentary on Question:

The question tested candidates' understanding of stochastic calculus and its applications on deriving and proving bond pricing formulas. Candidates were tested on their derivation and proofs of the formulas, rather than stating results directly. Candidates generally performed better in part (a) and part (b), but struggled with part (c).

Solution:

(a) Derive the bond price formula by finding $A(t,T)$ and $B(t,T)$ in the expression $V(t,r,T) = \exp(A(t,T) - B(t,T)\, r_t)$.

Hint: You could use Fubini's theorem in calculus such that

4. **Continued**

$$\int_S \left(\int_Q f(t,s)dW_t\right)ds = \int_Q \left(\int_S f(t,s)ds\right)dW_t$$

where $f(t,s)$ is an integrable function, $\{W_t: t \geq 0\}$ is a Wiener process and S, Q are time domains.

Commentary on Question:
For candidates who attempted this question, performance was adequate. Many candidates did not attempt to derive the bond price formula, but stated A(t, T) and B(t, T) directly from the Ho-Lee model – these candidates were given partial credit. Similarly, candidates who attempted to derive the formula often made smaller errors but were given partial credits for performing the intermediate steps (formula set-up and integration).

From the expression of the bond price

$$E\left(exp\left(-\int_t^T r_s\, ds\right)\right),$$

$$\int_t^s dr_v = r_s - r_t = \int_t^s \sigma dW_v,$$

$$r_s = r_t + \int_t^s \sigma dW_v.$$

Let $U = \int_t^T r_s\, ds$, then

$$U = \int_t^T r_t\, ds + \int_t^T \int_t^s \sigma dW_v\, ds.$$

Rearranging the second term using Fubini's rule of integral to apply Ito isometry,

$$\int_t^T \int_t^s \sigma dW_v\, ds = \sigma \int_t^T \int_v^T ds\, dW_v$$
$$where\ t < v < s < T.$$

Therefore the mean and variance of U are

$$E(U) = r_t(T-t) + E\left(\sigma \int_t^T (T-v)dW_v\right) = r_t(T-t),$$

$$Var(U) = E(U^2) - E(U)^2,$$

4. Continued

Using the Ito isometry and after arranging, the variance becomes

$$E\left[\left(\sigma\int_t^T (T-v)dW_v\right)^2\right]$$
$$= \sigma^2 E\left(\int_t^T (T-v)^2 dv\right)$$
$$= \frac{\sigma^2(T-t)^3}{3},$$
$$Var(U) = \frac{\sigma^2(T-t)^3}{3}.$$

Since $\int_t^T r_s\, ds$ is normally distributed, from the charateristic function of normal distribution shows

$$E\left(exp\left(-\int_t^T r_s\, ds\right)\right) = E\big(exp(-U)\big)$$
$$= exp\left(-E(U) + \frac{1}{2}Var(U)\right)$$

Finally the bond price is

$$V(t,T) = \exp\big(A(t,T) - B(t,T) * r(t)\big)$$

$$= exp\left(\frac{\sigma^2(T-t)^3}{6} - (T-t) * r(t)\right),$$
$$A(t,T) = \frac{\sigma^2(T-t)^3}{6},$$
$$B(t,T) = (T-t).$$

* Alternative to Fubini's theorem:

$$\int_t^T \int_t^s \sigma dW_v\, ds = \sigma \int_t^T (W_s - W_t)\, ds,$$
$$=\sigma\left(\int_t^T W_s ds - \int_t^T W_t ds\right)$$
$$= \sigma\left(\int_t^T W_s ds - W_t(T-t)\right),$$
$$\int_t^T W_s ds = \int_t^T d(sW_s) - \int_t^T s dW_s = TW_T - tW_t - \int_t^T s dW_s$$

from integrating by parts. So,

$$\sigma\left(\int_t^T W_s ds - W_t(T-t)\right) = \sigma\left(TW_T - tW_t - \int_t^T s dW_s - W_t(T-t)\right)$$
$$= \sigma\left(TW_T - TW_t - \int_t^T s dW_s\right) = \sigma\left(T(W_T - W_t) - \int_t^T s dW_s\right) = \sigma\left(\int_t^T T dW_s - \int_t^T s dW_s\right) =$$
$$\sigma\left(\int_t^T (T-s) dW_s\right).$$

4. Continued

Hence the Ito isometry can be applied and it follows same derivations of mean and variance as above.

(b) Verify that the bond pricing function $V(t,r,T)$ from part (a) satisfies the fundamental pricing PDE:

$$rV = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial r}\mu(r,t) + \frac{1}{2}\frac{\partial^2 V}{\partial r^2}\sigma^2.$$

Commentary on Question:
Candidates who attempted this question performed well. Candidates were generally able to identify the arbitrage free bond price and derive each component of the left-hand side of the equation. Partial credits were awarded for deriving each part of the equation.

From the fundamental pricing theory, the arbitrage free bond price should satisfy

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial r}\mu(r,t) + \frac{1}{2}\frac{\partial^2 V}{\partial r^2}\sigma^2 = rV$$

With

$$dr_t = \mu(r,t)dt + \sigma dW_t.$$

By finding each component of left hand side of the equation,

$$\frac{\partial V}{\partial t} = \left(\frac{\partial A}{\partial t} - \frac{\partial B}{\partial t}r\right)V = \left(-\frac{\sigma^2(T-t)^2}{2} + r\right)V,$$
$$\frac{\partial V}{\partial r}\mu(r,t) = \frac{\partial V}{\partial r}0 = 0,$$
$$\frac{1}{2}\frac{\partial^2 V}{\partial r^2}\sigma^2 = \frac{1}{2}B^2 V\sigma^2 = \frac{\sigma^2(T-t)^2}{2}V,$$

Hence the sum is the right hand side of the fundamantal pricing equation.

$$\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial r^2}\sigma^2 = rV$$

(c) Derive the mean and the variance of r_T under T-forward risk-neutral measure.

Commentary on Question:
Candidates performed poorly on this question. Many candidates did not attempt this question, and for the candidates that did, most candidates were not able to determine the final mean and variance of r_T under T-forward risk-neutral measure. Partial credits were awarded for defining the interest rate process and attempting to perform the steps to derive the mean and variance.

4. Continued

From the forward risk neutral measure, the interest rate process is defined as

$$dr_t = \left(\mu(r,t) + \sigma_{V(t,T)}\sigma\right)dt + \sigma dW_t.$$

Since $\mu(r,t) = 0$,

$$dr_t = \sigma_{V(t,T)}\sigma dt + \sigma dW_t$$

where

$$\sigma_{V(t,T)} = volatility\ of\ \frac{dV}{V}$$
$$= \frac{1}{V}\frac{\partial V}{\partial r}\sigma = -(T - t)\sigma$$

Hence under the forward risk neutral measure,

$$r_T - r_t = \int_t^T dr_t = \sigma \int_t^T -(T - t)\sigma\, dt + \int_t^T \sigma\, dW_t,$$
$$E(r_T) = r_t + \sigma \int_t^T -(T - t)\sigma\, dt$$
$$= r_t + \sigma^2 \left[\frac{(T - t)^2}{2}\right]_t^T$$

$$= r_t - \sigma^2 \frac{(T - t)^2}{2}$$
$$Var(r_T) = E({r_T}^2) - E(r_T)^2$$
$$= E\left[\left(r_t - \sigma^2 \frac{(T - t)^2}{2}\right)^2 + 2 \times \left(r_t - \sigma^2 \frac{(T - t)^2}{2}\right) \times \int_t^T \sigma\, dW_t + \left(\int_t^T \sigma\, dW_t\right)^2\right]$$
$$- E\left(r_t - \sigma^2 \frac{(T - t)^2}{2}\right)^2$$
$$= E\left(\int_t^T \sigma\, dW_t\right)^2$$
$$= \sigma^2(T - t),$$

since

$$E\left[2 \times \left(r_t - \sigma^2 \frac{(T - t)^2}{2}\right) \times \int_t^T \sigma\, dW_t\right] = 0.$$

Therefore

$$E(r_T) = r_t - \frac{\sigma^2(T - t)^2}{2}$$
$$Var(r_T) = \sigma^2(T - t).$$

5. **Learning Objectives:**

2. The candidate will understand:

- The Quantitative tools and techniques for modeling the term structure of interest rates.
- The standard yield curve models.
- The tools and techniques for managing interest rate risk.

Learning Outcomes:

(2a) Understand and apply the concepts of risk-neutral measure, forward measure, normalization, and the market price of risk, in the pricing of interest rate derivatives.

(2f) Understand and be able to apply various model calibration techniques under both risk-neutral and real-world measures

Sources:

Fixed Income Securities: Valuation, Risk, and Risk Management, Veronesi, Pietro, 2010 (pg. 551-552)

Commentary on Question:

Separate comments provided for each part.

Solution:

(a) Identify the real-world model parameters and risk-neutral world parameters. Justify your choice.

Commentary on Question:

Candidates did well on this portion. For avoidance of doubt, candidates should restate the parameters, identify if they are real-world vs. risk-neutral, and justify their choice.

- Daily overnight rates reflect historical real-world data. Parameters based on this data are thus real-world parameters ($\bar{r} = 0.0513, \gamma = 0.67$, and $\alpha = 0.049$)
- STRIPS are sets of zero-coupon bonds reflecting current bond prices. Calibrating a model using bond prices are thus risk neutral parameters ($\bar{r} = 0.0624, \gamma = 0.57$, and $\alpha = 0.049$)

(b) Calculate the stationary mean and variance of the model in the real world.

Commentary on Question:

Candidates who retrieved the correct formula from the formula sheet generally did well. Some Candidates did not apply the concept of stationary mean / variance, and so attempted to evaluate the formula directly (with t = 1), rather than taking the limit.

5. **Continued**

From the formula sheet (15.67) and (15.68):

$E[r_t|r_0] = \bar{r} + (r_0 - \bar{r}\}) \exp(-\gamma t)$
$Var[r_t|r_0] = r_0 \frac{\alpha}{\gamma}(\exp(-\gamma t) - \exp(-2\gamma t)) + \frac{\bar{r}\alpha}{2\gamma}(1 - \exp(-\gamma t))^2$

Stationary means and variances are calculated by taking the limit (e.g. $\lim_{t\to\infty} E[r_t|r_0]$), so:

$\lim_{t\to\infty} E[r_t|r_0] = \bar{r}$
$\lim_{t\to\infty} Var[r_t|r_0] = \frac{\bar{r}\alpha}{2\gamma}$

Stationary mean is 5.13% and

the stationary variance $\frac{0.0513*0.049}{2(0.67)} = 0.0018759$

(c) Identify the stationary distribution.

Commentary on Question:
Most candidates received partial credit on this question. Some candidates did not apply the concept of stationary distribution through taking the limit. Other candidates did not justify their response.

$f(r_t|r_0) = c_t\chi^2(c_t r_t, v, \lambda_t)$

$\text{Lim}_{t\to\infty} c_t = \lim_{t\to\infty} \frac{4\gamma}{\alpha(1-\exp(-\gamma t))} = \frac{4\gamma}{\alpha}$
$v = \frac{4\gamma\bar{r}}{\alpha}$
$\lim_{t\to\infty}\lambda_t = \lim_{t\to\infty} c_t r_0 \exp(-\gamma t) = \left(\lim_{t\to\infty} c_t\right)\left(\lim_{t\to\infty} \exp(-\gamma t)\right) = 0$

Therefore $\lim_{t\to\infty} f(r_t|r_0) = \frac{4\gamma}{\alpha}\chi^2\left(\frac{4\gamma}{\alpha} r_\infty, v, 0\right)$ it is chi-square distribution with v degree of freedom.

(d) Calculate the price of a 10-year 5% semi-annual coupon bond.

Commentary on Question:
Candidates did well on this portion. Those who used the real-world parameters were given partial credit.

See attached Excel

6. **Learning Objectives:**

3. The candidate will understand:
   - How to apply the standard models for pricing financial derivatives.
   - The implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory.
   - How to evaluate risk exposures and the issues in hedging them.

Learning Outcomes:

(3a) Demonstrate an understanding of option pricing techniques and theory for equity derivatives.

(3e) Analyze the Greeks of common option strategies.

(3i) Define and explain the concept of volatility smile and some arguments for its existence.

Sources:

The Volatility Smile, Derman, Emanuel and Miller, Michael B., 2016

QFIQ-120-19: Chapters 6 and 7 of Pricing and Hedging Financial Derivatives, Marroni, Leonardo and Perdomo, Irene, 2014

QFIQ-115-17: Which Free Lunch Would You Like Today, Sir?: Delta Hedging, Volatility Arbitrage and Optimal Portfolios

Commentary on Question:

Candidates performed as expected for this question.

Solution:

(a) Explain why there should be an upper-bound on the slope of the volatility smile from call options.

Commentary on Question:
Candidates performed as expected for part (a) and (b).

According to Black-Scholes-Merton, for a given strike, a call option price increases as implied volatility increases, now suppose implied volatility varies with strikes. If implied volatility were to increase too quickly, its effect on the call price might more than offset the decline in the call price due to increase in strike, leading to an increase in the call price at a higher strike. But this cannot be right because a call with a higher-strike cannot be worth more than a call with a lower-strike. Therefore, there must be an upper bound at which implied volatility can increase with strikes.

6. **Continued**

The following explanations are also acceptable:

1) The price of a call option, as a function of the strike price, must be **non-increasing** (or Monotonicity), meaning that

$$C(K_1) \geq C(K_2) \text{ for } K_1 < K_2$$

because options with lower strike prices provide higher intrinsic value.
If the slope of the volatility smile is too steep, the implied prices of higher-strike calls may exceed those of lower-strike calls. This would violate the monotonicity condition that call prices must decrease as strike prices increase.

2) A steep slope in the volatility smile corresponds to higher implied volatilities for calls at higher strikes. This could lead to pricing inconsistencies between adjacent strikes.
Traders could exploit such inconsistencies by constructing **vertical spreads** (buying a call at one strike and selling a call at a higher strike) to lock in risk-free profits. This arbitrage opportunity must be avoided in a no-arbitrage market.

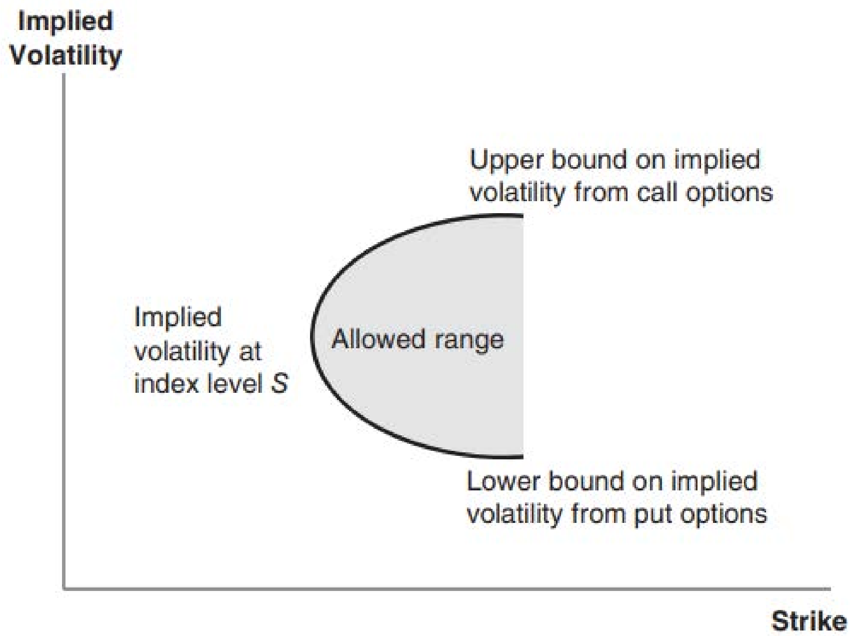
(b) Explain why there should be a lower-bound on the slope of volatility smile from put options.

According to Black-Scholes-Merton, for a given strike, a put option price increases as implied volatility increases, now suppose implied volatility varies with strikes. If implied volatility were to decrease too quickly, its effect on the put price might more than offset the decrease in the put price due to increase in strike, leading to a decrease in the put price at a higher strike. But this cannot be right because a put with a higher-strike cannot be worth less than a put with a lower-strike. Therefore, there must be a lower bound at which implied volatility can decrease with strikes.

If the slope of the implied volatility smile (or skew) were too steep, it could lead to violations of the no-arbitrage conditions. Specifically:

- A steep downward slope in implied volatility with respect to strike price would imply disproportionately low prices for higher-strike puts relative to lower-strike puts.
- Such discrepancies could result in arbitrage opportunities, as traders could construct synthetic positions (e.g., through vertical spreads or butterfly spreads) that guarantee risk-free profits.

6. **Continued**



(c)

(i) Show that $-\sqrt{\frac{\pi}{2\tau}}\frac{1}{K}$ is approximately a no-arbitrage lower-bound on the slope of volatility smile from put option.

(ii) Show that $\sqrt{\frac{\pi}{2\tau}}\frac{1}{K}$ is approximately a no-arbitrage upper-bound on the slope of volatility smile from call options.

Commentary on Question:
Candidates did not perform well for this question. Only a few candidates provided correct formula.

For a European call on a non-dividend-paying underlying, because a call with a higher-strike cannot be worth more than a call with a lower-strike, we have:

$$\frac{\partial C}{\partial K} \leq 0$$

Using the Black-Scholes-Merton (BSM) parameterization, $C(S, t, K, T) \equiv C_{\mathrm{BSM}}(S, t, K, T, \Sigma)$ where the implied volatility $\Sigma = \Sigma(K, T)$ varies with strikes. we have

$$\frac{\partial C}{\partial K} = \frac{\partial C_{\mathrm{BSM}}}{\partial K} + \frac{\partial C_{\mathrm{BSM}}}{\partial \Sigma}\frac{\partial \Sigma}{\partial K} \leq 0$$

6. **Continued**

Rearranging the terms, we have

$$\frac{\partial \Sigma}{\partial K} \le \frac{-\dfrac{\partial C_{\mathrm{BSM}}}{\partial K}}{\dfrac{\partial C_{\mathrm{BSM}}}{\partial \Sigma}}$$

Plugging in BSM Greeks, we have

$$\frac{\partial \Sigma}{\partial K} \le \frac{e^{-r\tau} \mathrm{N}(d_2)}{e^{-r\tau} K\sqrt{\tau}\mathrm{N}'(d_2)} = \frac{\mathrm{N}(d_2)}{K\sqrt{\tau}\mathrm{N}'(d_2)}$$

Because volatilities are assumed small and strike price is at-the-money forward, therefore,

$d_2 \approx 0, \mathrm{N}(d_2) \approx 0.5 \text{ and } \mathrm{N}'(d_2) \approx 1/\sqrt{2\pi}$

For small changes in $\boldsymbol{dK}$, then

$$d\Sigma \le \sqrt{\frac{\pi}{2\tau}} \frac{dK}{K}$$

$$\frac{d\Sigma}{dK} \le \sqrt{\frac{\pi}{2\tau}} \frac{1}{K}$$

For a European put on a non-dividend-paying underlying, because a put with a higher-strike cannot be worth less than a put with a lower-strike, we have:

$$\frac{\partial P}{\partial K} \ge 0$$

Using the Black-Scholes-Merton Greeks, we have

$$\frac{\partial \Sigma}{\partial K} \ge \frac{-e^{-r\tau} \mathrm{N}(-d_2)}{e^{-r\tau} K\sqrt{\tau}\mathrm{N}'(d_2)} = \frac{-\mathrm{N}(-d_2)}{K\sqrt{\tau}\mathrm{N}'(d_2)}$$

For small volatilities, at-the-money forward strike, $-d_2$ is approximately zero, giving an approximate lower-bound of

$$-\sqrt{\frac{\pi}{2\tau}} \frac{dK}{K}$$

Thus, we have

$$\frac{d\Sigma}{dK} \ge -\sqrt{\frac{\pi}{2\tau}} \frac{1}{K}$$

(d) Estimate the upper bound for implied volatility for three-month European calls with a strike of 5,050.

6. **Continued**

Commentary on Question:
Candidates did not perform very well for this question. Only a few candidates used the right formula for this question.

According to the derivation (or given) in part (c)

$$d\Sigma \le \sqrt{\frac{\pi}{2\tau}}\frac{dK}{K}$$

$$\le \sqrt{\frac{\pi}{2 * 0.25}}\frac{5{,}050 - 5{,}000}{5{,}000}$$

$$\le 0.025$$

Therefore, the upper-bound of the implied volatility is 17.5% = 15% + 2.5%

7. Learning Objectives:

3. The candidate will understand:
   - How to apply the standard models for pricing financial derivatives.
   - The implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory.
   - How to evaluate risk exposures and the issues in hedging them.

Learning Outcomes:

(3a) Demonstrate an understanding of option pricing techniques and theory for equity derivatives.

(3b) Identify limitations of the Black-Scholes-Merton pricing formula.

(3c) Demonstrate an understating of the different approaches to hedging – static and dynamic.

(3e) Analyze the Greeks of common option strategies.

Sources:

The Volatility Smile, Derman, Emanuel and Miller, Michael B., 2016 Ch. 1, 2, Ch.3, Ch. 5, Ch.9

Commentary on Question:

This question tests the candidate's understanding of varying investment products with caps and downside protection mechanisms such as minimum guarantees and buffers from losses. It assesses the candidate's knowledge of how to price and calculate Greeks for the replicating option portfolios of such products.

Solution:

(a)

(i) Determine, for each equity option, the type (call/put), position (long/short), units, strike price, and the total current value.

(ii) Calculate the present value of the zero-coupon bond.

(iii) Determine the amount of the cash position, if any.

Commentary on Question:

Candidates generally performed well on this question. For part (i), partial credit was awarded if the equity options, types, positions, and units were correct, but the strike prices were incorrect. Many candidates who did not receive full credit did not implement the Black-Scholes formula appropriately. For parts (ii) and (iii), full credit was only awarded if both the calculations and final answers were correct.

Answered in Excel

7. Continued

(b)

(i) Calculate PHX's cash position at inception, if any.

(ii) Determine the equity delta of the portfolio at inception.

Commentary on Question:
Candidates generally performed well on this question. For part (i), partial credit was awarded if the equity options, types, positions, and units were correct, but the strike prices were incorrect. For part (ii), some candidates correctly determined the deltas for the component vanilla options but incorrectly applied the aggregation to get the portfolio delta.

Answered in Excel

(c) Analyze whether the premium covers the cost of a buy-and-hold strategy for the 10% cap and 10% buffer under the current market conditions.

Commentary on Question:
Candidates performed well on this question. Compared to parts (a) and (b) of this question, more candidates were able to correctly identify the strike prices for the 3 component vanilla options. If there was no explicit conclusion for whether the premium was sufficient, based on the final cash position, full credit was not awarded.

Answered in Excel

(d) Explain how this new design can be appealing to the customers.

Commentary on Question:
Many candidates only received partial credit for this question. To receive full credit, candidates were expected to state both the appeal of more upside potential and the willingness to take more downside risk.

The product can be appealing to the customers that are feeling more optimistic about a rising market and thus willing to take downside risk in exchange for more upward potential (10%, as opposed to 5% cap in the existing design).

8. Learning Objectives:

3. The candidate will understand:
   - How to apply the standard models for pricing financial derivatives.
   - The implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory.
   - How to evaluate risk exposures and the issues in hedging them.

Learning Outcomes:

(3e) Analyze the Greeks of common option strategies.

(3i) Define and explain the concept of volatility smile and some arguments for its existence.

(3j) Compare and contrast "floating" and "sticky" smiles.

Sources:

QFIQ-120-19: Chapters 6 and 7 of *Pricing and Hedging Financial Derivatives*, Marroni, Leonardo and Perdomo, Irene, 2014

Commentary on Question:

This question tests candidates' understanding of the volatility smile and the related trading strategies.

Solution:

(a) Your boss asks why options with lower strikes have higher implied volatility than those with higher strikes.

List two reasons that explain why.

Commentary on Question:

Candidates performed below expectations. Partial credit was awarded for each correct reason that was identified by candidates.

Any of the following helps explain the vol curve

- The implied vol reflects the typical market belief that a falling stock market is likely to be more volatile than a rising market
- Investors are willing to pay more for downside protections
- Market tends to price in fat left tails, expecting a larger downside gap
- General market supply/demand on certain options (such as demand for insurance/hedging)

8. Continued

(b) (i) Describe the construction of the strategy; specify the long/short position.

(ii) Plot the Vega as a function of the underlying index price using V2 volatilities.

Commentary on Question:
Candidates performed as expected on this part.

Straddle is the simultaneous purchase or sale of a call and a put at the same strike price and the same maturity.

The Vega plot can be drawn with K= 100, σ = 13%, r = 2%, t = 0.5

$$d1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}}$$

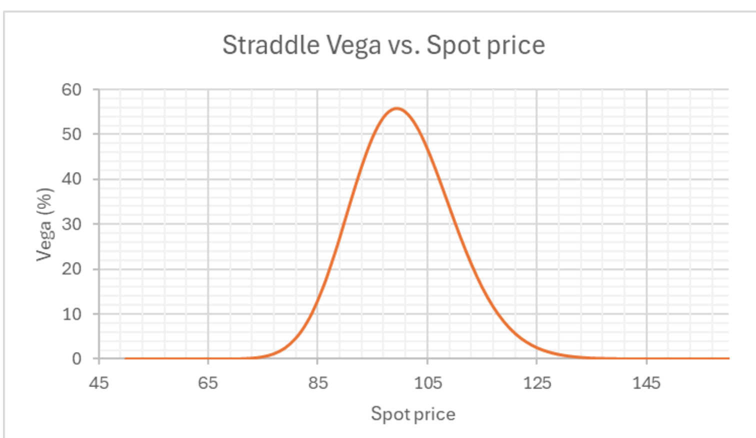
$$\varphi(d1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{d1^2}{2}}$$

$$Vega = S * \varphi(d1) * \sqrt{t}$$

Straddle Vega = 2 * Vega

S = 50, 51, 52, … 170 as given in the Excel Sheet.

QFIQF_Solution_to_Q8.xlsx



8. Continued

(c) (i) Describe the construction of the strategy; specify the long/short position.

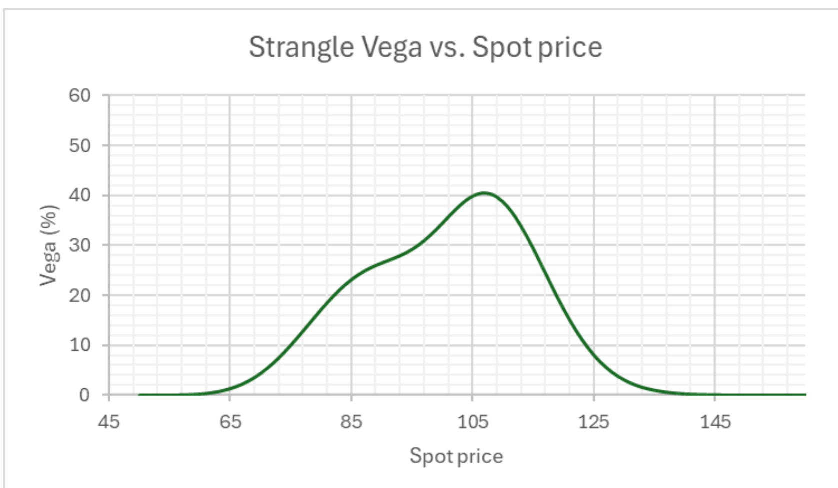
(ii) Plot the Vega as a function of the underlying index price using V2 volatilities.

Commentary on Question:
Candidates performed as expected on this part.

Strangle is the simultaneous purchase or sale of a call at higher strike 110 and a put at lower strike 90 with the same maturity.

The Vega plot can be drawn as
Vega @ K= 90, σ = 19%
Vega @ K = 110, σ = 11%
The formula for Vega is same as in part (b) and is not repeated here.

Strangle Vega = [Vega @ K= 90, σ = 19%] + [Vega @ K= 110, σ = 11%]



9. Learning Objectives:

4. The candidate will learn how to apply the techniques of quantitative finance to applied business contexts.

Learning Outcomes:

(4a) Identify and evaluate embedded options in liabilities, e.g., indexed annuity, structured product based variable annuity, and variable annuity guarantee riders including GMxB, etc.

(4b) Demonstrate an understanding of embedded guarantee risk including market, insurance, policyholder behavior, and basis risk factors

Sources:

Neftci "An Introduction to the Mathematics of Financial Derivatives", Hirsa, Ali and Neftci, Salih N., 3rd Edition 2nd Printing, 2014, Page 296

QFIQ 134-22, An Introduction to Computational Risk Management of Equity-Linked Insurance (Chapters 1.2-1.3, 4.7-4.8 (background), 6.2-6.3)

Commentary on Question:

Most candidates received partial credits from this question. Candidates can usually get the full credits if the formulas are provided in the Excel spreadsheet for sub questions. It is observed that there are a handful of candidates who did not attempt to solve the questions, therefore received no credit.

Solution:

(a) Identify three underlying assumptions necessary for the above pricing formula to be valid.

Commentary on Question:
The candidates perform well on this question. Partial credits are given to candidates who provided less than 3 assumptions below.

Full credits are given to candidates who provide at least 3 of the following:

- No-arbitrage (p 151 of QFIQ 134-22)
- Risk-free rate is constant (Neftci p 296)
- No transaction costs (Neftci p296) or "No Friction cost" (p 158 of QFIQ 134-22)
- Mortality is independent of equity returns (p 152 of QFIQ 134-22)
- Other acceptable include
  - Unlimited borrowing and shorting at risk -free rate
  - Fractional amounts in trading

9. Continued

(b)

(i) Calculate the prices of the point-to-point option for participation rates α of 60% and 120%.

(ii) Estimate the participation rate of the point-to-point option such that the price is $98.

Commentary on Question:
The majority of candidates performed well on this question. Most of those who attempted it received full credits. Only a few earned partial credit due to incorrect application of the formula, while others received no credit for not attempting the question.

(i) According to the provided formula

The point-to-point option = $S_0 * (Y_1 + Y_2)$, with Y1 and Y2 defined below:

$$Y_1 = e^{\left((\alpha-1)r+\frac{1}{2}\alpha(\alpha-1)\sigma^2\right)T}\Phi\left(\frac{\left(r-\frac{1}{2}\sigma^2+\alpha\sigma^2\right)T-\frac{g}{\alpha}T}{\sigma\sqrt{T}}\right)$$

$$Y_2 = e^{(g-r)T}\Phi\left(\frac{\frac{g}{\alpha}T-\left(r-\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)$$

Participation rate	Y1	Y2	**PtP Option Price**
60%	0.5224	0.4431	**96.5549**
120%	0.8114	0.3794	**119.0792**

(ii) The participation rate can be solved by using the Goal-Seek function is Excel. The answer is 64.81%

(c) Calculate Y, the participation rate 2 (α_2), such that the price of the double threshold design is equal to $98, using the *DTD(T)* formula above.

Commentary on Question:
Most candidates performed relatively well on this question. Most of those who attempted it received full credit.

Similar to question b(ii), the participation rate 2 (α_2), can be backed solved by the formula provided in the Excel spreadsheet with Goal-Seek function. The solved result is 80.26%.

9. Continued

(d)

(i) Verify that the prices for both options are equal.

(ii) Calculate the payoff at maturity for both options for the following annualized index returns:

Annualized Returns	PtP	Double Threshold
3%		
8%		
13%		
18%		
23%		
28%		
33%		

Commentary on Question:
Many candidates received full credits for part (d)(i). However, only a small portion of candidates received full credits for part (d)(ii). Most common mistakes are not applying the conditions in the formula of Barrier options. See Excel for formula details.

(i) Given the parameters for PtP options and double threshold options respectively, Both options both equals to $94.72. See Excel solutions for more details.

(ii) The results are shown below. See the Excel solutions for more details.

Annualized Returns	PtP	Barrier	Difference
3%	110.52	110.52	-
8%	124.36	110.52	(13.84)
13%	142.51	151.93	9.42
18%	163.31	190.31	27.00
23%	187.15	227.56	40.41
28%	214.47	272.10	57.63
33%	245.78	325.36	79.58

10. **Learning Objectives:**

2. The candidate will understand:
   - The Quantitative tools and techniques for modeling the term structure of interest rates.
   - The standard yield curve models.
   - The tools and techniques for managing interest rate risk.

Learning Outcomes:

(2b) Understand and be able to apply various one-factor interest rate models and various simulation techniques including Euler-Maruyama discretization and transition density methods

(2c) Understand and be able to apply multifactor interest rate models

Sources:

Fixed Income Securities;Veronesi, Pietro

Study Note on Interest Rate Calibration

Commentary on Question:

This question tests candidates' understanding of multi-factor modeling and interest rate calibration.

Solution:

(a) Describe how this model incorporates dependency between the short rate r_t and the long-term yield rate $r_t(\tau)$ for suitable τ.

Commentary on Question:

Many candidates cited only the given correlation between the two factors in the Vasicek model and failed to address dependency between the short rate and long-term yield rate.

Better prepared candidates were able to provide more in-depth analysis, identifying the formulas for the short rate and long-term yield rates, pointing out that both are dependent on the same factors.

Zero coupon bond prices with the two-factor Vasicek model is given by

$Z(\phi_{1,t}, \phi_{2,t}, t; T) = \exp\left(A(t;T) - \phi_{1,t}B_1(t;T) - \phi_{2,t}B_2(t;T)\right)$

$A(t;T) = A(0;(T-t)) = A(T-t)$ and $B_i(t;T) = B_i(0;T-t) = B_i(T-t)$

The short-term yield is given by $r_t = \phi_{1,t} + \phi_{2,t}$

The long-term yield at time t, $r_t(\tau)$, implied in the model can be obtained by solving

$$Z(\phi_{1,t}, \phi_{2,t}, t; \tau + t) = \exp\left(-\tau r_t(\tau)\right)$$

to get

10. Continued

$$r_t(\tau) = -\frac{A(\tau)}{\tau} + \phi_{1,t}\frac{B_1(\tau)}{\tau} + \phi_{2,t}\frac{B_2(\tau)}{\tau}$$

Therefore, the two are dependent due to the inclusion of
$\phi_{1,t}$ and $\phi_{2,t}$ in both of the formulas.

(b) Describe how to simulate short rate paths using the transition density method.

Commentary on Question:
Many candidates omitted this question, most likely due to time constraints. Most candidates who attempted this question were able to correctly identify that the general process of the transition density method was an exact process that utilized a normal distribution to calculate short rate transition paths to give the next estimated value of r_t.

Only the most prepared candidates were able to correctly identify that the method utilized a bivariate normal distribution, and that under a two-factor Vasicek framework, the transition density method seeks to estimate $\phi_{1,t}$ and $\phi_{2,t}$ rather than r_t.

Solution
First, we choose an initial short-term rate r_0 and initial long-term rate $r_0(\tau)$ for a suitable value of τ. From this, we can solve the system of equations for $\phi_{1,0}$ and $\phi_{2,0}$:

$$\phi_{1,0} = \frac{B_2(\tau)r_0 - \tau r_0(\tau) - A(\tau)}{B_2(\tau) - B_1(\tau)}$$
$$\phi_{2,0} = \frac{\tau r_0(\tau) + A(\tau) - B_1(\tau)r_0}{B_2(\tau) - B_1(\tau)}$$

We then simulate two dependent normal variables Z_1, Z_2 by simulating two independent variables X_1, X_2 and transforming them using correlation ρ:

$Z_1 = X_1$
$Z_2 = \rho(s) \cdot X_1 + \sqrt{1-\rho(s)^2} \cdot X_2$

Then, $\phi_{1,1}$ and $\phi_{2,1}$ are calculated using Z_1, Z_2:

$$\phi_{i,t+s} = \phi_{i,t} \cdot \exp(-\gamma_i s) + \vec{\phi}_i(1 - \exp(-\gamma_i s)) + \sqrt{\frac{\sigma_i^2}{2\gamma_i}(1 - \exp(-2\gamma_i s))} \cdot Z_i\ ,$$
$$i = 1,2$$

and

10. **Continued**

$r_{t+s} = \phi_{1,t+s} + \phi_{2,t+s}$

The last steps are repeated as necessary.

(c) Explain the procedure used in calibrating the model.

Commentary on Question:
Many candidates omitted this question, most likely due to time constraints.
Many candidates who attempted this question provided only a conceptual overview of the process.
Most candidates who attempted this question were able to correctly identify that a non-linear regression function is used to minimize the sum of squared differences between zero-coupon bond rates from the model and the corresponding market rates from the dataset.
Only the well prepared candidates were able to identify the specific steps executed in the procedure.

Solution
Standard deviations are calculated for the short rate and the 5-year zero-coupon yield, as is the correlation coefficient between them.
These standard deviations are converted to an annual basis, whereas the correlation coefficient is the same on a daily or annual basis.
Using initial guesses of $\gamma_1^*, \vec{\phi}_1$, and γ_2^* , a function is used to solve for σ_1, σ_2 and ρ.
Values of $\gamma_1^*, \vec{\phi}_1, \sigma_1, \gamma_2^*, \sigma_2$ and ρ can then be used to calculate theoretical prices for the zero-coupon bonds, where
$Z(\phi_{1,t}, \phi_{2,t}, t; T) = \exp(A(t;T) - B_1(t;T)\phi_{1,t} - B_2(t;T)\phi_{2,t})$

Differences between market rates of zero-coupon bonds from the dataset and those determined via the model are determined.
The sum of the squares of these differences is minimized using a non-linear least-squares method.
The resulting parameters minimize the objective function in

$$J(\gamma^*, \vec{\phi}^*, \sigma^*) = \sum_{i=1}^{n} (Z^{two-factor\ Vasicek}(r_0, t_0, T_i) - Z^{Data}(0, T_i))^2$$

(d) Estimate the parameters $\gamma_1^*, \vec{\phi}_1, \sigma_1, \gamma_2^*, \sigma_2$ and ρ.

10. Continued

Commentary on Question:
Some candidates omitted this part, most likely due to time constraints.
Most candidates who attempted this question were able to correctly identify the parameters $\gamma_1^*, \vec{\phi}_1$, and γ_2^* from the procedure output.
Better prepared candidates were able to identify σ_1, σ_2 and ρ.
Some candidates attempted to calculate σ_1, σ_2 and ρ $from\ first\ principles$.

Solution

$$\gamma_1^* = 1.74802$$
$$\vec{\phi}_1 = 0.28457$$
$$\sigma_1 = 0.020998$$
$$\gamma_2^* = -0.04090$$
$$\sigma_2 = 0.011652$$
$$\rho = -0.272098$$

(e) Assess the adequacy of the fit.

Commentary on Question:
Some candidates omitted this part, most likely due to time constraints.
Most candidates were able to correctly assess that the fit was adequate with reasonable evidence.
No candidate was able to comment that a residual plot should be examined for full model validation.

Solution
The fit of the model is good due to the p-values presented in the function output. However, a residual plot should be analyzed in order to give full commentary on adequacy.