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Wavelet-Based Equity VaR Estimation

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Abstract

Economic risk analysis has two dimensions: time and frequency. Asset return varies by time because of economic cycles and economic structural changes. Risk is higher during economic recessions and smaller during economic expansions. In addition, different economic structures may exist at different time scales. Risk measures calculated based on daily, weekly, monthly and yearly historical data can be very different. The appropriate frequency depends on the time horizon of the analysis. Therefore, it is important to measure economic risk at both the time level and the frequency level. However, time series analysis and statistical analysis that have been widely adopted in economic risk analysis focus on only one of the two dimensions.

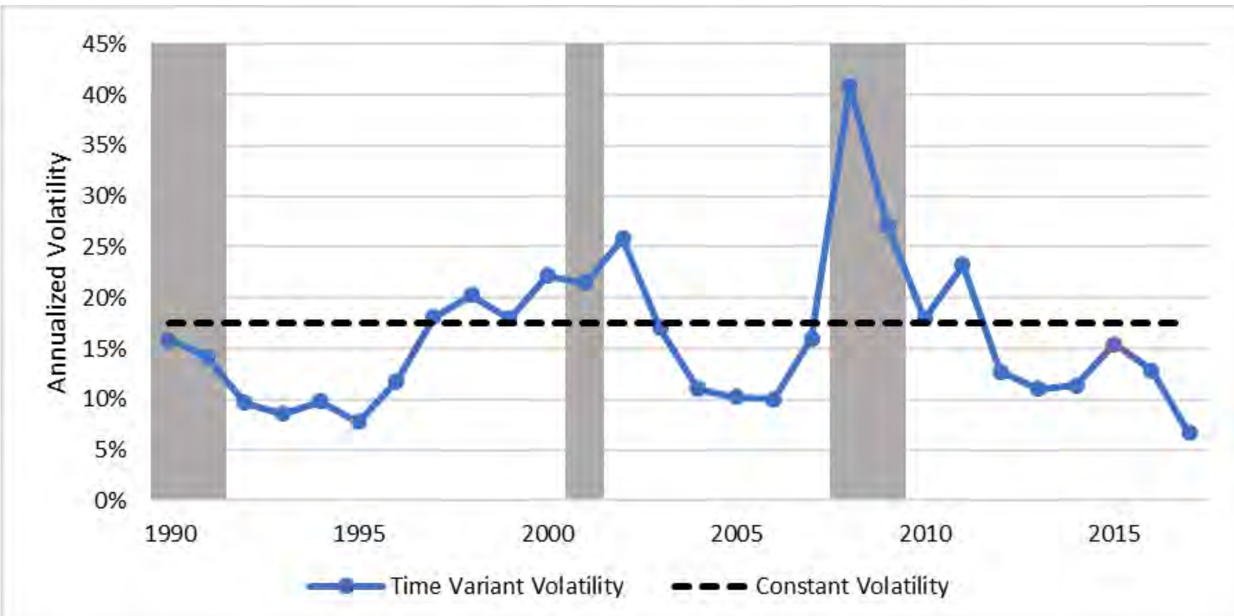
In this paper, wavelet models are used to enhance both time-invariant analysis and time-variant analysis. Wavelet models can systematically analyze risk by time and frequency and provide richer information than time series models. Using wavelet models, risk measurement can easily be adjusted based on time horizon in a consistent way. Equity VaR estimation is selected to demonstrate the application of wavelet analysis in risk management.

1. Introduction

Economic risk is an important risk for insurers offering long-term products with guaranteed benefits. It originates from the economic system and can be reflected as either market risk or credit risk.

When estimating the magnitude of economic risk, historical data are usually used. For example, to estimate the volatility of the equity market, one can simply calculate the standard deviation of historical equity returns. However, an implicit assumption of this method is that the risk is time invariant. In reality, equity market volatility varies by time. It is caused by either economic cycles or economic structural changes. Figure 1 shows the annualized volatility using daily S&P 500 index return from 1990 to 2017. Assuming a time-invariant (constant) volatility, the annualized volatility is 17.7%. If calculating the annualized volatility on a yearly basis, the volatility could go above 40%, as evidenced during the 2008 financial crisis. It is important to acknowledge the temporal impact in economic risk analysis.

Figure 1. S&P 500 Index Return Annualized Volatility (1990–2017)



Another complication of economic risk analysis is the frequency of historical data to be used. The annualized volatility calculated based on different frequencies varies a great deal. Table 1 shows the annualized volatility and empirical value at risk (VaR) of S&P 500 equity index return using daily, monthly and yearly data from 1990 to 2017. For simplicity, the calculation assumes that the volatility and VaR are time invariant and that the equity index follows a geometric Brownian motion, a common assumption used when generating stochastic equity return scenarios. In this paper, VaR with a confidence level of p is defined as $inf[x \in \mathbb{R}: Pr(X + x < 0) \leq 1 - p]$, the smallest number of the probability that X exceeds −VaR is at least p. It is the opposite of the negative return value in the left tail.

Table 1. S&P 500 Index Return Volatility and VaR by Frequency

Frequency	Time-Invariant Volatility	Annualized Volatility[1]	99.5% Empirical VaR	Annualized Empirical VaR[2]
Daily	1.1%	**17.5%**	3.9%	**69.3%**
Monthly	4.2%	**14.5%**	19.3%	**75.3%**
Quarterly[3]	7.9%	**15.5%**	26.9%	**64.2%**
Yearly	17.7%	**17.5%**	43.5%	**43.5%**

Notes:

1. Annualized volatility = Time-invariant volatility $\times \sqrt{n}$, where *n* equals 250/12/4/1 for daily/monthly/quarterly/yearly frequency.
2. Annualized empirical VaR = (99.5% Empirical VaR – Mean return) $\times \sqrt{n}$ – Mean return $\times\, n$.
3. Minimum value of quarterly and yearly return is used for 0.5% empirical VaR because the number of data points is less than 200.

Historical equity index returns exhibit different risk levels by frequency, especially for empirical VaR. Annualized empirical VaR based on high frequency data (daily and monthly) is higher than the VaR based on low frequency data (quarterly and yearly). This phenomenon indicates the need to analyze the economic risk at different frequencies to get a holistic view.

This paper uses wavelet models to analyze equity risk at both the time level and the frequency level. Section 2 discusses time series models and Fourier transform that examine equity risk by time and frequency, respectively. Section 3 introduces wavelet models that can systematically analyze equity risk by time and frequency at the same time, while Section 4 constructs the VaR measure in a wavelet model. Section 5 compares the performance of different models for estimating VaR, and Section 6 discusses the possible applications of wavelet models in risk management. Section 7 concludes the paper.

2. Time Series Model

Time series models, such as generalized autoregressive conditional heteroskedasticity (GARCH) and autoregressive moving average (ARMA), can be used to capture the time variant feature of equity volatility. An ARMA-GARCH model is used to analyze historical S&P 500 index daily returns.

$$ARMA(p,q) \sim r_t = c + \varepsilon_t + \sum_{i=1}^{p} \varphi_i r_{t-i} + \sum_{j=1}^{q} \theta_j \varepsilon_{t-j}$$

$$\varepsilon_t = z_t \sigma_t$$

$$GARCH(p,q) \sim \sigma_t^2 = \omega + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2$$

Where

r_t = S&P 500 index daily return. It is calculated as $log\left({S_t}/{S_{t-1}}\right)$.

z_t = iid with zero mean and unit variance.

The distribution of z_t should be chosen according to the experience data. Table 1 shows that the empirical VaR based on daily data materially exceeds the value implied from a normal distribution.[1] Therefore, a distribution that can more flexibly capture skewness and heavy tails is preferred. In this example, z_t is assumed to follow the skewed generalized error distribution (SGED). It has the following probability density function:

$$f_{SGED}(x;\mu,\sigma,\lambda,p) = \frac{pe^{-\left\{\frac{|x-\mu+m|}{v\sigma[1+\lambda sign(x-\mu+m)]}\right\}^p}}{2v\sigma\Gamma(1/p)}$$

Where

μ = location parameter. It is zero for z_t.

σ = scale parameter. It is one for z_t.

λ = skewness parameter.

p = shape parameter.

$m = \frac{2^{\frac{2}{p}} v\sigma\lambda\Gamma\left(0.5+\frac{1}{p}\right)}{\sqrt{\pi}}$ if the mean of variable x equals μ.

$v = \sqrt{\frac{\pi\Gamma\left(\frac{1}{p}\right)}{\pi(1+3\lambda^2)\Gamma\left(\frac{3}{p}\right) - 16^{\frac{1}{p}}\lambda^2\Gamma\left(0.5+\frac{1}{p}\right)^2\Gamma\left(\frac{1}{p}\right)}}$ if the volatility of variable x equals σ.

ARMA(3,3) and GARCH(2,2) with the SGED are used to analyze historical S&P 500 daily index returns from 1990 to 2017. The orders (*p* and *q*) are chosen based on Akaike information criterion (AIC). Table 2 lists the parameters of the fitted model. It also shows the results of the t test for each parameter.

[1] Annualized volatility based on daily returns is 17.5%. Based on the assumption of normal distribution, 99.5% VaR = 17.5% × 2.576 = 45.1%. The annualized empirical VaR equals 69.3%, which means a heavier left tail than the normal distribution.

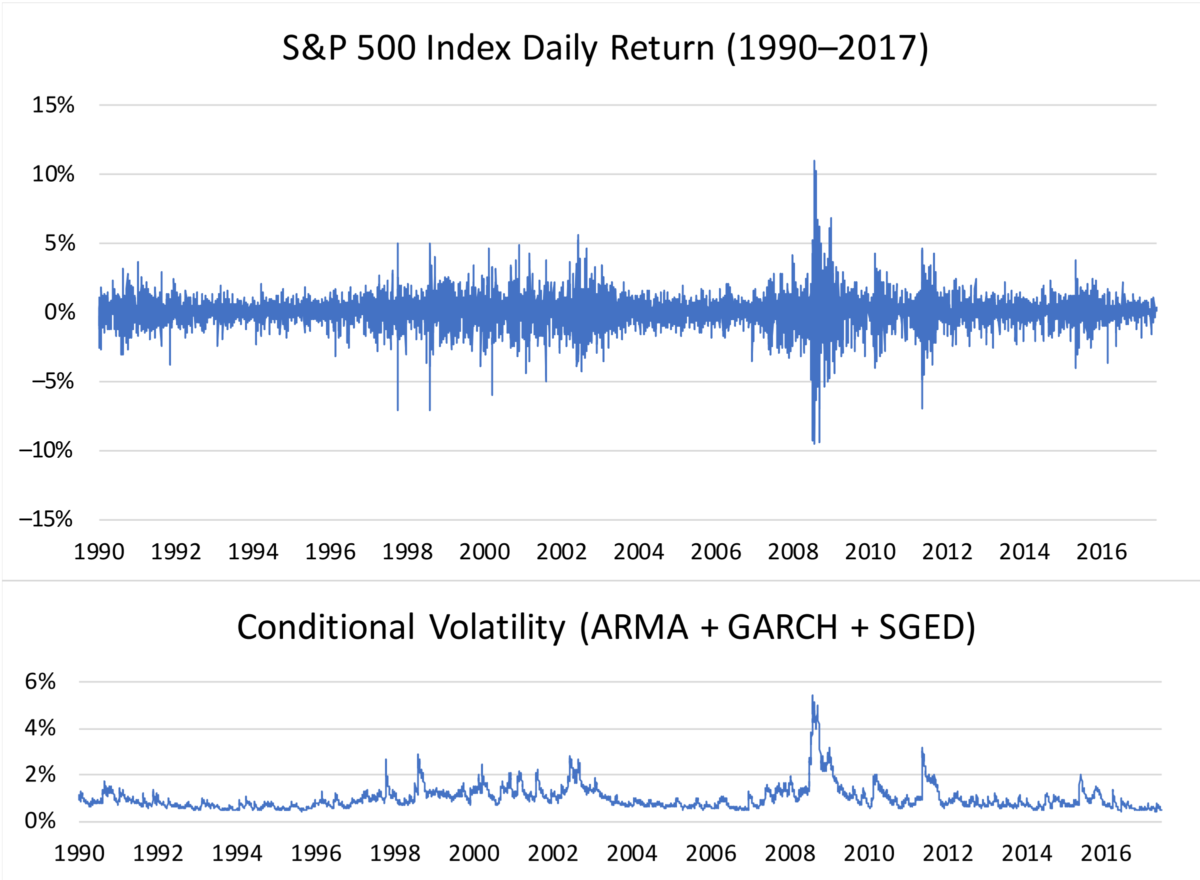
Table 2. ARMA + GARCH Model Parameters

Parameter	Value	t Value	***p*** **value**
c	0.0003	3.450	0.0006
φ_1	–0.0924	–1.140	0.2545
φ_2	–0.2317	–4.095	0.0000
φ_3	0.7326	12.021	0.0000
θ_1	0.0421	0.560	0.5757
θ_2	0.1810	3.592	0.0003
θ_3	–0.7897	–13.687	0.0000
ω	0.0000	4.330	0.0000
α_1	0.0376	3.253	0.0011
α_2	0.0914	5.985	0.0000
β_1	0.3132	1.694	0.0903
β_2	0.5446	3.132	0.0017
μ	0	N/A	N/A
σ	1	N/A	N/A
λ	0.2825	41.414	0.0000
p	0.9014	64.615	0.0000

Note: μ and σ are kept fixed in model calibration so that z_t has zero mean and unit variance.

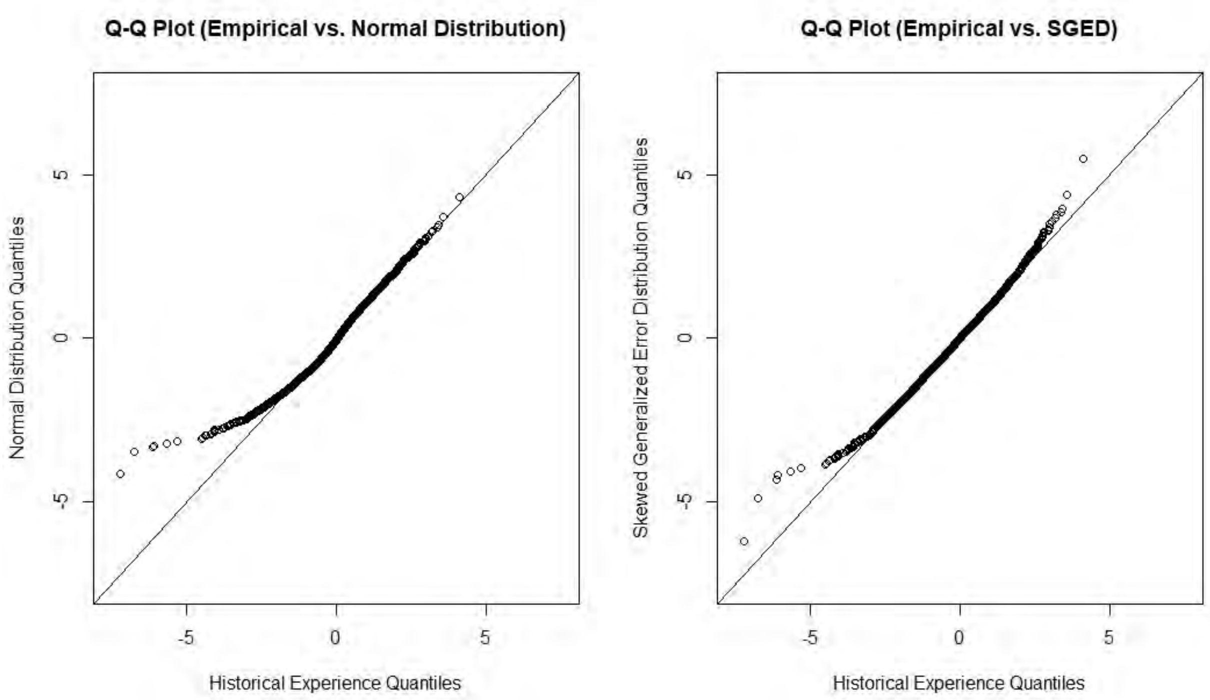
Figure 2 shows the daily return and the conditional volatility σ_t based on the ARMA-GARCH model. The conditional volatility varies greatly, with the highest value observed during the 2008 financial crisis.

Figure 2. S&P 500 Index Daily Return and Conditional Volatility



Standardized residuals are compared to standard normal distribution and fitted SGED to understand how well heavy tails have been captured by using the SGED. Figure 3 draws the quantile-quantile (Q-Q) plots between empirical distribution and theoretical distributions. The SGED does a better job capturing the left heavy tails than the normal distribution.

Figure 3. Q-Q Plot for Standardized Residuals



With the fitted model, future daily VaR can be predicted. Traditionally, daily VaR can be estimated with the following steps:

Step 1. Estimate the expected daily return l days after T, the ending date of the historical data.

$$\mathbb{E}(r_{T+l}) = c + \sum_{i=1}^{p} \varphi_i r_{T+l-i} + \sum_{j=1}^{q} \theta_j \varepsilon_{T+l-j}$$

$$\varepsilon_{T+l-j} = 0 \ \ if \ \ l-j > 0$$

Step 2. Estimate the expected conditional variance.

$$\mathbb{E}(\sigma_{T+l}^2) = \omega + \sum_{i=1}^{q} \alpha_i \mathbb{E}(\varepsilon_{T+l-i}^2) + \sum_{j=1}^{p} \beta_j \mathbb{E}(\sigma_{T+l-j}^2)$$

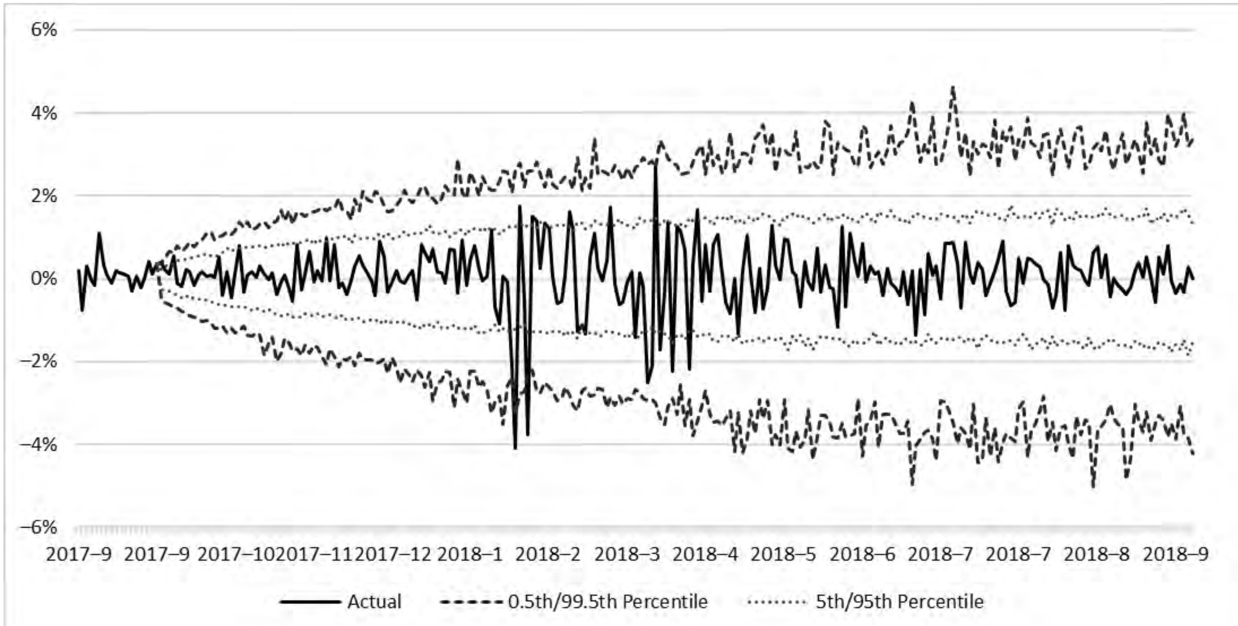
$$\mathbb{E}(\varepsilon_{T+l-i}^2) = \mathbb{E}(\sigma_{T+l-j}^2) \ \ if \ \ l-i > 0$$

Step 3. Estimate the VaR with a confidence level of p.

$$VaR_{T+l} = -[\mathbb{E}(r_{T+l}) + \sigma_{T+l} SGED^{-1}(1-p)]$$

However, this approach is not suitable for estimating a longer term VaR such as the annual one. Rather than deriving the VaR using the formula in Step 3, VaR can be estimated using simulation. Error term ε_{T+l} can be simulated as $\sigma_{T+l} z_{T+l}$, where z_{T+l} follows the calibrated SGED. With the simulated error terms, daily returns and volatilities can be projected concurrently. Figure 4 shows the results based on 1,000 simulations for 251 trading days from October 2017 to September 2018. Actual daily returns are compared with the projected ranges. While 10.4% of actual returns falls out of the middle 90% range (5th percentile to 95th percentile), 1.6% of actual returns falls out of the middle 99% range (0.5th percentile to 99.5th percentile). Although the SGED generates a better range prediction than the normal distribution, it still underestimates the probability of extreme returns for the projection period.

Figure 4. S&P 500 Index Daily Return Range Estimation



With simulated daily returns, annual returns can be calculated for each simulation. Annual VaR is estimated based on the calculated annual returns, as shown in Table 3. In this example, the SGED has a heavier left tail than the normal distribution.

Table 3. S&P 500 Index Return Annual VaR Estimation

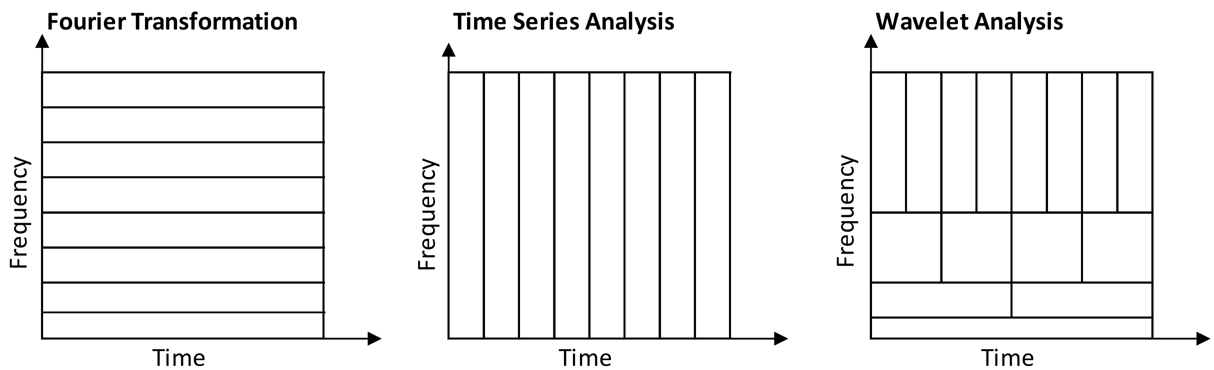
	95% VaR	**99.5% VaR**
SGED	4.6%	24.2%
Normal distribution	5.0%	14.1%

3. Wavelet Analysis

If the evolving of risk is driven by a few forces with different frequencies, a pure time series model may not be able to capture all the different patterns. In the equity risk example in Section 2, when predicting the return and conditional volatility, the ARMA-GARCH model reflects only the direct impact of returns and volatilities in the past three days. The model cannot effectively capture the impacts for medium- and long-term patterns. People may argue that less frequent (such as annual) data can be used to estimate annual VaR. However, historical data may not be sufficient for a credible estimate, and valuable information in high frequency data is lost.

To address the shortcoming of a pure time series analysis, wavelet analysis can be used to analyze the historical data from two dimensions (time and frequency) at the same time. Wavelet analysis can be considered a combination of time series analysis and Fourier transform. Fourier transform analyzes the data purely from the frequency domain, assuming that patterns are time invariant. Fourier transform itself is not very useful for economic risk analysis because of economic structural changes and length and magnitude variation of economic cycles. As shown in Figure 5, wavelet analysis keeps more time information for high frequency data and less time information for low frequency data.

Figure 5. Wavelet Analysis Concept



3.1 Maximal Overlap Discrete Wavelet Transform

In this paper, maximal overlap discrete wavelet transform (MODWT) is used to illustrate enhanced risk analysis based on wavelets. The MODWT is chosen over many other wavelets because its decomposition at different scales can easily be compared with original time series. The MODWT is also less sensitive than other wavelet transforms to the starting point of a time series. This is helpful to understand the patterns at different frequencies: short term, medium term or long term. In addition to signal processing, MODWT has been applied to the finance area as well. Conlon and Cotter (2012) used the MODWT to calculate the appropriate dynamic minimum-variance hedging ratio for various time horizons. Khalfaoui et al. (2015) studied individual stock market returns and their comovements at different frequencies using the MODWT and showed that stock returns have long memory dynamics. Risk is more concentrated at high frequency, and dependencies are more concentrated at low frequency.

Following the definition of Percival and Walden (2000), the MODWT of a time series $X_t, t = 1,2, \dots, N$ to the jth level works as the following:

Wavelet coefficient $\widetilde{W}_{j,t} = \sum_{l=0}^{L_j-1} \tilde{h}_{j,l} X_{t-l \; MOD \; N}$

Scale coefficient $\tilde{V}_{j,t} = \sum_{l=0}^{L_j-1} \tilde{g}_{j,l} X_{t-l \; MOD \; N}$

Where

$\tilde{h}_{j,l}$ = wavelet filter constructed by convolving *j* filters composed of $\tilde{g}_l$ and $\tilde{h}_l$. It suffices the following conditions:

$$\sum_{l=0}^{L-1} \tilde{h}_l = 0 \qquad \sum_{l=0}^{L-1} \tilde{h}_l^2 = \frac{1}{2} \qquad \sum_{l=-\infty}^{\infty} \tilde{h}_l \tilde{h}_{l+2n} = 0 \text{ for all integers } n > 0$$

$\tilde{g}_{j,l}$ = scale filter constructed by convolving *j* filters composed of $\tilde{g}_l$. It suffices the following conditions:

$$\sum_{l=0}^{L-1} \tilde{g}_l = 1 \qquad \sum_{l=0}^{L-1} \tilde{g}_l^2 = \frac{1}{2} \qquad \sum_{l=-\infty}^{\infty} \tilde{g}_l \tilde{g}_{l+2n} = 0 \text{ for all integers } n > 0$$

$$\sum_{l=-\infty}^{\infty} \tilde{g}_l \tilde{h}_{l+2n} = 0 \text{ for all integers } n$$

$L_j = (2^j - 1)(L - 1) + 1$. L is the width of the base level filter.

The maximum number of levels depends on the available data points. For example, 6,992 data points are used when analyzing S&P 500 index daily returns from January 1990 to September 2017. The maximum level of decomposition is the integer part of log(6,992)/log(2). Therefore, a maximum of 12 levels are feasible in this example. Table 4 lists the frequency of the first eight levels.

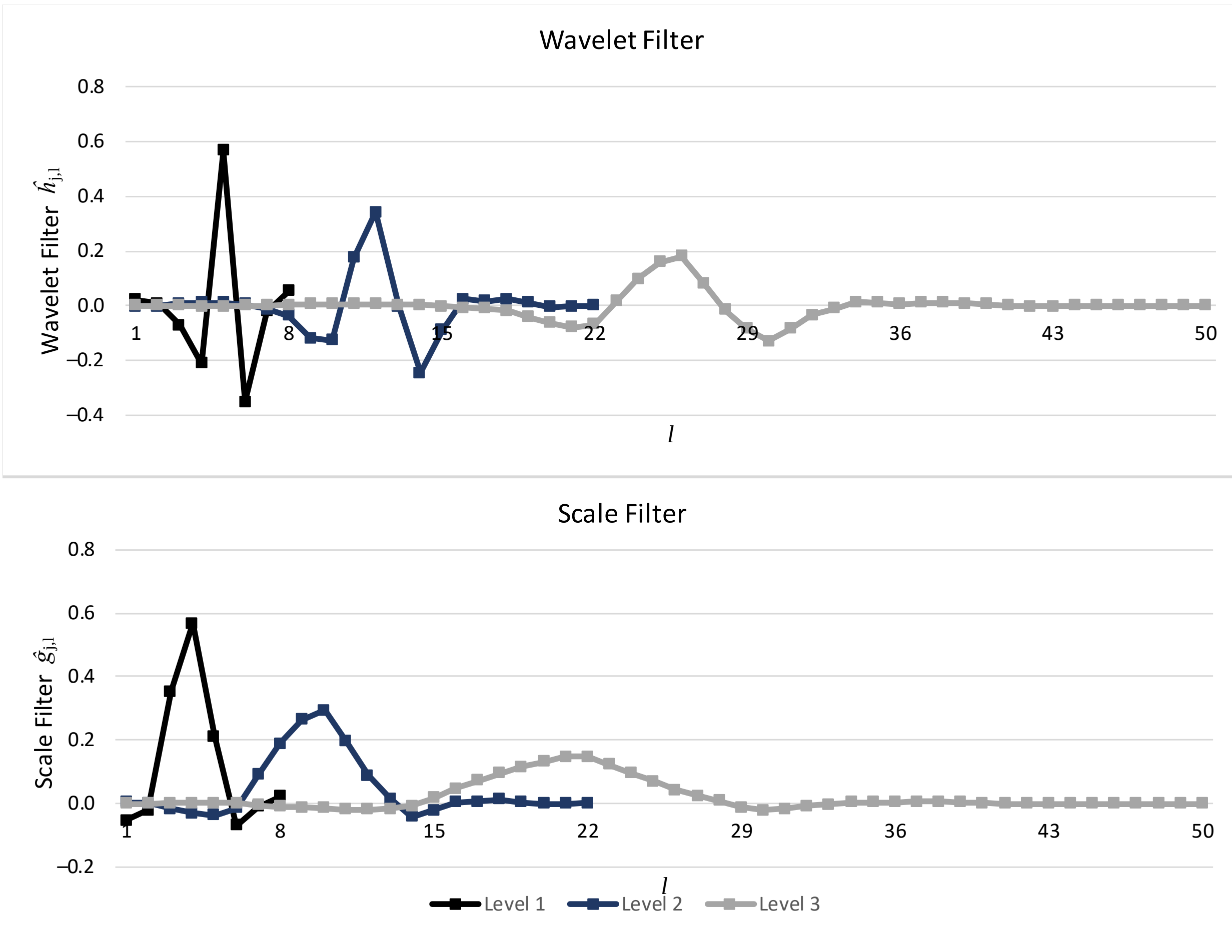
Table 4. Frequency of Decomposition Levels

Level (*j*)	Frequency	Scale (1/Frequency)
1	[1/4,1/2]	2–4 days
2	[1/8,1/4]	4–8 days
3	[1/16,1/8]	8–16 days
4	[1/32,1/16]	16–32 days
5	[1/64,1/32]	32–64 days
6	[1/128,1/64]	64–128 days
7	[1/256,1/128]	128–256 days
8	[1/512,1/256]	256–512 days

Note: The scale is measured in business days.

To analyze the equity risk, LA(8) (Daubechies least asymmetric filter with L = 8) is used to define $\tilde{h}_{j,l}$ and $\tilde{g}_{j,l}$. Figure 6 shows the wavelet filters $\tilde{h}_{j,l}$ and scale filters $\tilde{g}_{j,l}$ for the first three levels. The wavelet dampens out with larger width as the level goes up. The same pattern applies when the level goes higher than level 3.

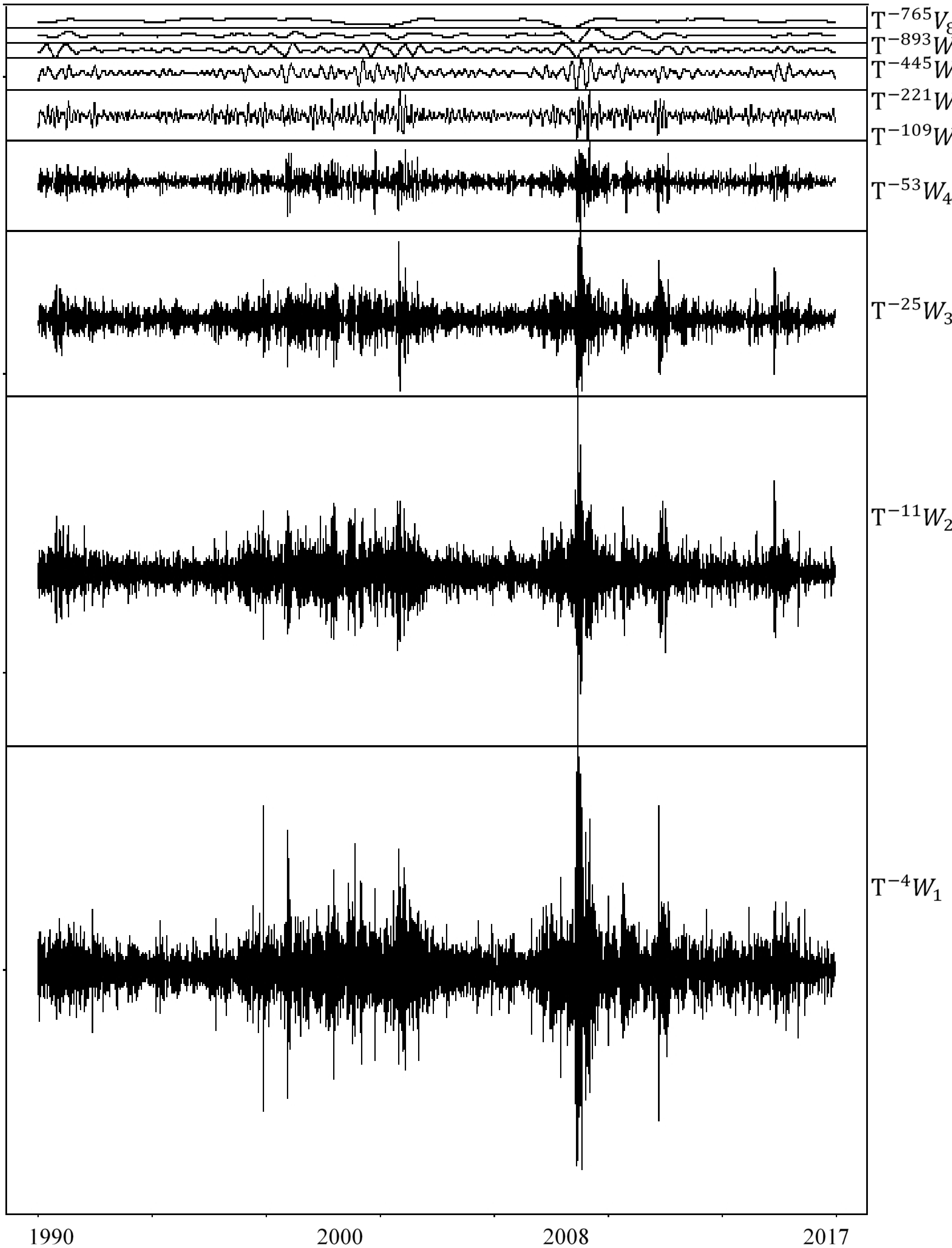
Figure 6. LA(8) Wavelet and Scale Filters for MODWT



LA filters with different L, and other filters can be used as well. They are not covered here because the focus of this paper is not on choosing the best filter. Also, the impact on the results is immaterial.

With all the assumptions set up for wavelet analysis, the original time series (S&P 500 index daily return) is decomposed into eight levels. Although a maximum of 12 levels can be used, eight levels are enough to represent the original time series with a maximum difference of less than 0.06%. Figure 7 shows the wavelet coefficients ($\widetilde{W}_{j,t}$) for all eight levels and the scale coefficients ($\tilde{V}_{j,t}$) for the eighth level. Coefficients are shifted so they coincide with the original time series. The wavelet coefficients are smoother at a higher level, representing longer term volatility. The scale coefficients at the highest level represent the volatility that is not explained by wavelet coefficients.

Figure 7. MODWT Wavelet Coefficients and Scaling Coefficients



Note: T^{-i} means that the series of the coefficients is shifted by i positions backward so that all series are on the same timeline.

3.2 Time-Invariant Risk Analysis

Before considering time-dependent risk estimation, wavelet analysis can be used to attribute the total volatility to different levels. The total variance can be calculated as the sum of the variances at each level:

$$\sigma_X^2 = \sum_{J=1}^{J_M} \sigma_X^2(j)$$

Where

σ_X^2 = total variance of the original time series.

$\sigma_X^2(j)$ = variance of the decomposition at level j.

J_M = number of levels used in wavelet analysis.

Also, $\sigma_X^2(j)$ has an unbiased estimator:

$$\hat{\sigma}_X^2(j) = \frac{1}{M_j} \sum_{t=L_j-1}^{N-1} \widetilde{W}_{j,t}^2$$

Where

$M_j = N - L_j + 1$

Skewness and kurtosis of each level can be estimated as well:

$$\textbf{Skewness} \quad \hat{S}_X(j) = \frac{\frac{1}{M_j}\sum_{t=L_j-1}^{N-1} \widetilde{W}_{j,t}^3}{\hat{\sigma}_X^3(j)}$$

$$\textbf{Kurtosis} \quad \widehat{K}_X(j) = \frac{\frac{1}{M_j}\sum_{t=L_j-1}^{N-1} \widetilde{W}_{j,t}^4}{\hat{\sigma}_X^4(j)}$$

Table 5 lists the mean, variance, skewness and kurtosis for each decomposition level and the original time series. The sum of the variance (volatility^2) of the eight levels explains more than 99.9% of the variance in the original time series. Low levels (high frequency/short term) contributes most of the variance of the original return series. Skewness and kurtosis are quite different among the eight levels, which indicates that the patterns at different frequencies are different, and it may be beneficial to model them separately.

Table 5. Descriptive Statistics at Different Decomposition Levels

	Mean	Volatility	Variance Contribution	Skewness	Kurtosis	99.5% Empirical VaR	99.5% VaR (Normal)
Level 1	0.0000%	0.8%	53.5%	0.3	12.7	3.0%	2.1%
Level 2	0.0000%	0.6%	24.9%	0.2	11.3	2.0%	1.4%
Level 3	–0.0001%	0.4%	12.3%	0.1	7.6	1.2%	1.0%
Level 4	0.0000%	0.2%	5.0%	–0.1	6.3	0.9%	0.6%
Level 5	–0.0001%	0.2%	2.3%	0.1	5.5	0.5%	0.4%
Level 6	–0.0002%	0.1%	1.2%	0.03	5.2	0.4%	0.3%
Level 7	0.0001%	0.1%	0.4%	–0.2	3.7	0.2%	0.2%
Level 8	–0.0001%	0.1%	0.3%	–0.3	6.4	0.2%	0.2%
Original	0.0274%	1.1%	—	–0.2	11.9	3.93%	2.84%

The value of skewness and kurtosis indicates the non-normality of the original time series and wavelet coefficients. The 99.5% empirical VaR is much larger than when assuming a normal distribution at most levels. The empirical VaR of the original time series can be approximated by aggregating the VaR at each decomposition level as follows:

$$VaR_{Agg} = \sqrt{\sum_{j=1}^{J_M} VaR_j^2}$$

Where

VaR_{Agg} = aggregated VaR.

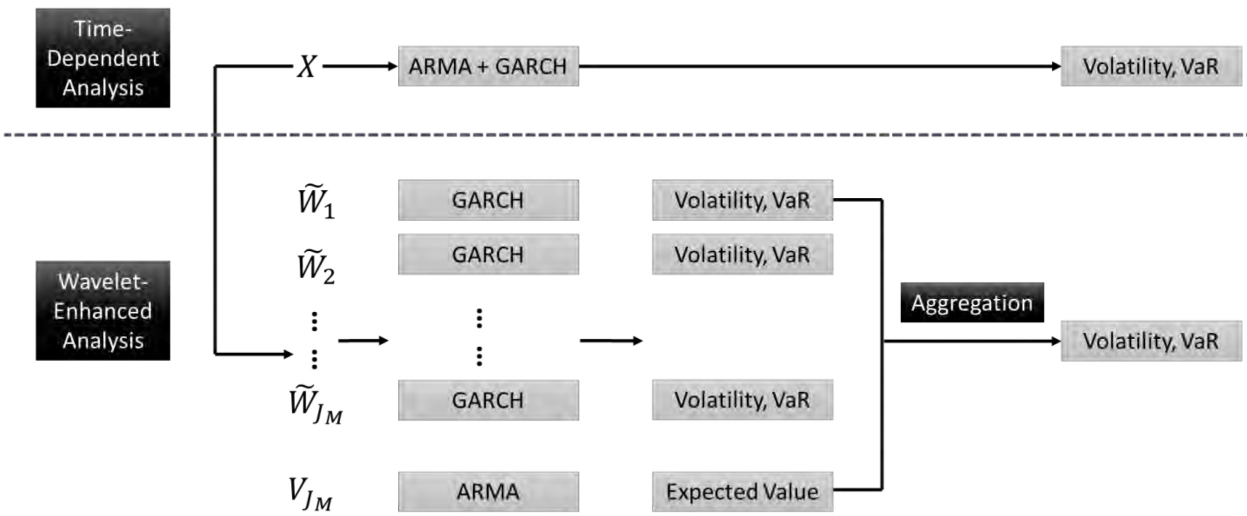
VaR_j = VaR at level j.

In this example, aggregated empirical VaR is 3.94%, compared to 3.93% calculated directly from the original time series. The non-normality of the original time series is preserved well by the wavelet coefficients in this example.

3.3 Time-Variant Risk Analysis

The wavelet analysis in the previous section assumes a constant volatility. Time-variant risk analysis can be enhanced with wavelet analysis as well to reflect different patterns at each wavelet decomposition level. This subsection builds on the ARMA-GARCH example in Section 2 to include analysis at each decomposition level. As shown in Figure 8, instead of modeling the original time series with one model, wavelet-enhanced time-dependant analysis studies wavelet coefficients at each level separately to understand the risk in different ranges of frequency. Wavelet coefficients are fitted into a GARCH model to get the volatility and VaR information. Scale coefficients at the highest level are fitted into ARMA and GARCH models to understand the trend of the time series. They are aggregated to get the predicted return, total volatility and VaR. In this way, different patterns (skewness and tail heaviness) at each decomposition level can be reflected independently.

Figure 8. Wavelet-Enhanced Time-Dependent Analysis Structure



Continuing with the example of S&P 500 index return, the GARCH model is calibrated to wavelet coefficients at each level. The orders p and q of this model are chosen based on the AIC. The error term is assumed to follow an SGED to reflect skewness and heavy tails. Table 6 lists the parameters of the fitted models.

Table 6. GARCH Model Parameters for Wavelet Coefficients

Level	Model	c	ω	α_1	α_2	β_1	β_2	μ	σ	λ	p
1	GARCH(1,2)	–7E-13	3E-06	0.38		0.23	0.37	0.00	1.00	0.01	1.81
2	GARCH(2,2)	–4E-12	2E-06	0.21	0.45	0.14	0.17	0.00	1.00	–0.03	2.07
3	GARCH(1,2)	–3E-12	5E-07	0.63		0.00	0.35	0.00	1.00	–0.08	3.34
4	GARCH(1,1)	3E-12	1E-07	0.72		0.00	0.20	0.00	1.00	0.04	5.72
5	GARCH(1,2)	3E-12	1E-07	0.64		0.02		0.00	1.00	0.05	5.79
6	GARCH(1,2)	7E-12	1E-08	0.52		0.00	0.14	0.00	1.00	–0.07	10.00
7	ARCH(1)	3E-12	6E-09	1.00				0.00	1.00	0.90	1.28
8	ARCH(1)	5E-12	4E-10	0.50				0.00	1.00	–0.06	10.00

As expected, the mean of wavelet coefficients at each level is negligible because wavelet coefficients capture the volatility instead of the trend. The t test for each parameter has a p value less than 0.005. With the volatility reflected in wavelet coefficients, the smoothed trend of the original time series needs to be incorporated as well. The scale coefficients at the highest decomposition level are fitted to an ARMA model to predict the future returns. In the example of S&P 500 index return, an ARMA(10,10) is chosen, given its low AIC. Table 7 lists the calibrated parameters. The magnitude of autocorrelation is higher, which indicates the significance of the trend. The volatility of the error term is small as most of the volatility has been captured by the wavelet coefficients.

Table 7. Scale Coefficients AMRA Model Parameters

AR Parameters		MA Parameters	
φ_1	1.48	θ_1	1.58
φ_2	–0.29	θ_2	1.45
φ_3	0.16	θ_3	1.36
φ_4	–1.02	θ_4	1.37
φ_5	1.14	θ_5	1.36
φ_6	–0.32	θ_6	1.35
φ_7	0.00	θ_7	1.34
φ_8	–0.49	θ_8	0.35
φ_9	0.59	θ_9	–0.23
φ_{10}	–0.24	θ_{10}	–0.09
c	0.0003	Volatility(ε_t)	8.9E-08

The ARMA model for the scale coefficients at the highest level can be used to predict the expected returns in the future. Following the simulation method used in Section 2 to simulate future equity returns, wavelet coefficients can be simulated at each decomposition level. Conditional volatility and VaR can be projected for each level according to the calibrated GARCH model. They can be aggregated to predict the total VaR:

$$VaR_{Agg,T+l} = \sqrt{\sum_{j=1}^{J_M} VaR_{j,T+l}^2} - \mathbb{E}(r_{T+l})$$

$$VaR_{j,T+l} = -\sigma_{j,T+l} SGED_j^{-1}(1-p)$$

Where

$VaR_{Agg,T+l}$ = aggregated daily VaR at $T + l$, l periods ahead of T.

$VaR_{j,T+l}$ = daily VaR at $T + l$ at decomposition level j. The expected value of wavelet coefficients is zero and therefore is not included in the formula.

$\sigma_{j,T+l}$ = projected conditional volatility of level j wavelet coefficient at $T + l$.

$SGED_j^{-1}(1-p)$ = the [100 × (1 – p)]th percentile of fitted SGED for level j wavelet coefficients.

Figure 9 shows the daily return range prediction based on 1,000 simulations for 250 trading days from the beginning of October 2017. Actual daily returns till September 2018 are compared with the projected ranges. While 10.2% of actual returns falls out of the middle 90% range (5th percentile to 95th percentile), 0.7% of actual returns falls out of the middle 99% range (0.5th percentile to 99.5th percentile). Compared to a pure time-dependent prediction, as in Figure 4, wavelet-enhanced prediction has a wider predicted range for extreme returns (0.5th percentile and 99.5th percentile).

Figure 9. Wavelet-Based S&P 500 Index Daily Return Range Estimation

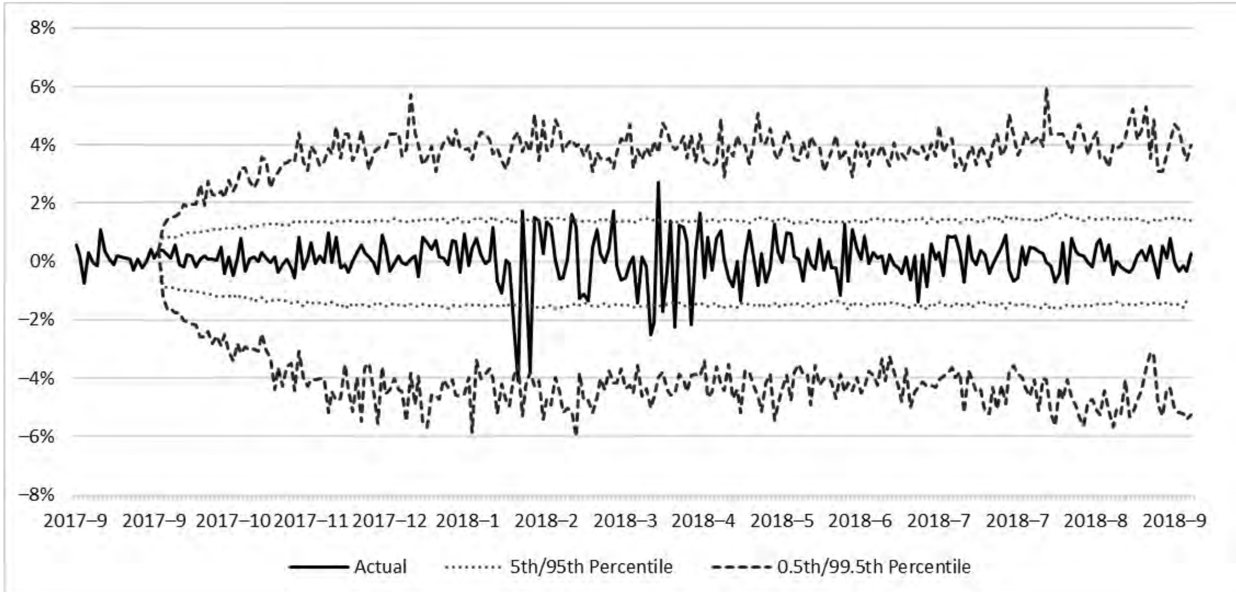
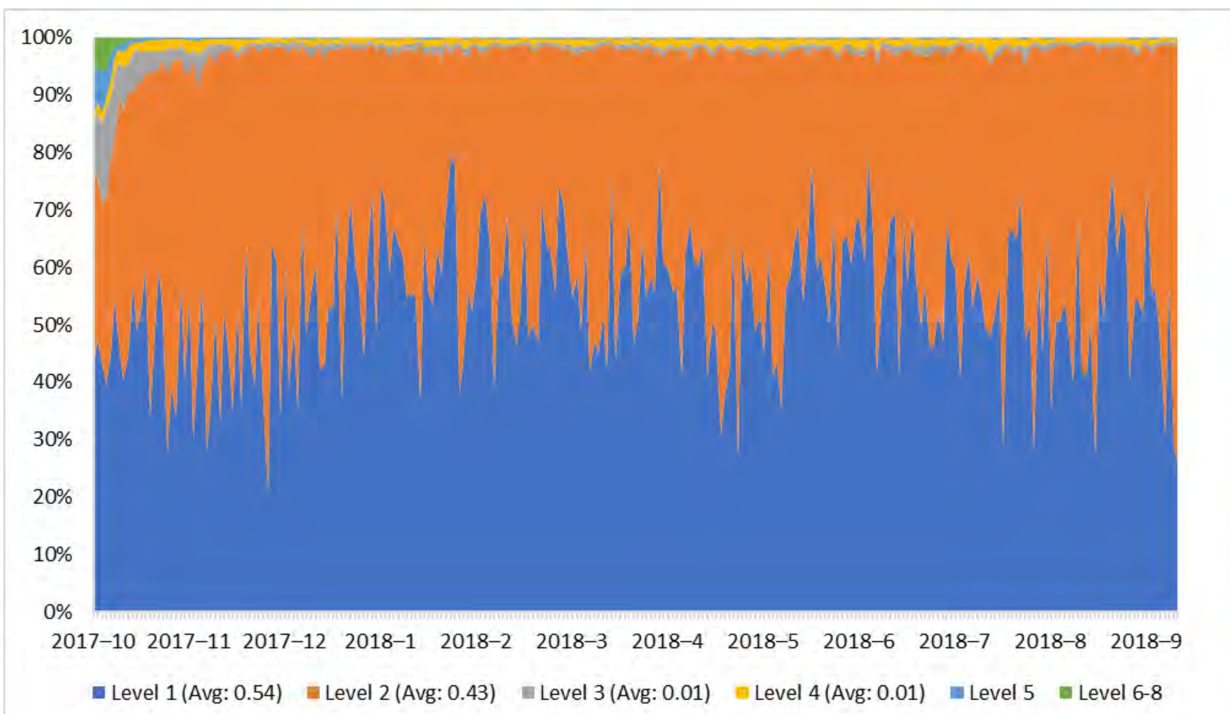


Figure 10 shows the contribution to total variance of daily equity index returns by frequency. On average, level 1 (2 to 4 days) contributes 54% of total variance in this time-dependent analysis.

Figure 10. Variance Contribution by Wavelet Coefficients



Neither the predicted return percentiles nor the variance contribution mix are smooth across time. Only 1,000 simulations are used in the analysis. More simulations will be able to smooth the results, but the trend is expected to be similar.

For decision makers with a longer time horizon, annual VaR is a better measure than daily VaR for risk assessment. Although daily VaR can be calculated directly from wavelet coefficients and scale coefficients, annual VaR is difficult to estimate because daily returns cannot be reconstructed directly from these coefficients. Wavelet coefficients and scale coefficients of MODWT need to be adjusted so that daily returns can be calculated as the sum of the transformed coefficients. Multiresolution analysis (MRA) based on MODWT can be used to construct daily returns from transformed coefficients that preserve the autocorrelation of daily returns. Annual returns are then calculated based on simulated daily returns. Table 8 compares the annual VaR derived by different methods for the period from October 2017 to September 2018. Wavelet-enhanced time-dependent analysis provides a much higher annual VaR than a pure time-dependent analysis given a low volatility environment in September 2017. It is obvious that wavelet analysis has a longer memory and helps preserve the long-term pattern much better than the time-dependent analysis in this example. Wavelet-enhanced time-dependent analysis also reflects current market conditions to predict the future risk in a given time horizon.

Table 8. S&P 500 Index Return Annual VaR Estimation

	Projection Type	Model	95% VaR	99.5% VaR
Time-dependent analysis	Conditional	ARMA + GARCH	4.6%	24.2%
Wavelet-enhanced time-dependent analysis	Conditional	MODWT + MRA	17.6%	39.9%
Empirical analysis (Jan. 1990–Sept. 2017)	Unconditional	Statistical analysis	26.9%	43.5%

4. Back-Testing

It is helpful to understand the models' performance regarding the equity return VaR prediction. Three models are compared:

1. Time-invariant analysis. It estimates the daily VaR based on the historical data, assuming the volatility is constant across time.
2. Time-dependent analysis. ARMA and GARCH models described in Section 2 are used to estimate the daily VaR.
3. Wavelet-enhanced time-dependent analysis. Time-dependent analysis at different frequencies using the MODWT model is used to estimate the daily VaR, as explained in Section 3.3.

To be comprehensive, both economic expansion and economic recession are considered in the back-testing. The accuracy of prediction is assessed in the following scenarios:

1. Starting with a medium volatility level at the end of 2007, the VaR estimation is compared to the actual daily returns in 2008, a high volatility period with an average conditional volatility of 2%.
2. Starting from a low volatility level at the end of September 2017, the VaR estimation is

compared to the actual daily returns from October 2017 to September 2018, a low volatility period with an average conditional volatility of 0.9%.

3. Starting from a high volatility level at the end of September 2008, the VaR estimation is compared to the actual daily returns from October 2008 to September 2009, a very volatile period with an average conditional volatility of 2.4%.

Models are recalibrated for each scenario based on different study periods. Table 9 lists the back-testing results. The assessment is conducted at four levels:

1. **Left Tail 0.5%.** The probability that the actual daily return is lower than the 0.5th percentile predicted by the model.
2. **Left Tail 5%.** The probability that the actual daily return is lower than the 5th percentile predicted by the model.
3. **Right Tail 5%.** The probability that the actual daily return is higher than the 95th percentile predicted by the model.
4. **Right Tail 0.5%.** The probability that the actual daily return is higher than the 99.5th percentile predicted by the model.

Table 9. Back-Testing Result

Threshold	**Analysis Method**	**Starting Period**		
		Medium Volatility (2007 Year End: 1.1%)	**Low Volatility (Sept. 2017: 0.5%)**	**High Volatility (Sept. 2008: 2.6%)**
		Testing Period		
		High Volatility (2008: 2% on Average)	**Low Volatility (Oct. 2017–Sept. 2018: 0.9% on Average)**	**High Volatility (Oct. 2008–Sept. 2009: 2.4% on Average)**
		Empirical Probability		
Left tail 0.5%	Time-invariant analysis	6.0%	0.8%	7.2%
	Time-dependent analysis	6.4%	0.8%	0%
	Wavelet-enhanced time-dependent analysis	**5.2%**	**0.4%**	**0%**
Left tail 5%	Time-invariant analysis	19.6%	2.8%	20.8%
	Time-dependent analysis	20.0%	4.4%	8.4%
	Wavelet-enhanced time-dependent analysis	**19.2%**	**3.2%**	**13.6%**
Right tail 5%	Time-invariant analysis	15.6%	0%	18.8%
	Time-dependent analysis	15.6%	6.0%	8.4%
	Wavelet-enhanced time-dependent analysis	**15.6%**	**2.8%**	**14.4%**

Right tail 0.5%	Time-invariant analysis	5.6%	0.4%	6.0%
	Time-dependent analysis	7.6%	0.8%	1.6%
	Wavelet-enhanced time-dependent analysis	**4.0%**	**0%**	**0%**

Ideally, the empirical probability should be consistent with the confidence level for perfect VaR estimation. For example, for the left tail 0.5%, a probability close to 0.5% is preferred. As expected, time-invariant analysis tends to underestimate the risk in low-volatility situations and overestimate the risk in high-volatility situations. In this example, wavelet-enhanced time-dependent analysis is better than the other two approaches for capturing the heavy tails (left tail 0.5% and right tail 0.5%). Like the other two approaches, wavelet-enhanced time-dependent analysis may underestimate VaR at a lower confidence level (left tail 5% and right tail 5%).

For VaR estimation at a high confidence level, wavelet-enhanced time-dependent analysis is the best option based on the back-testing results. In addition, this type of analysis can adjust itself based on new information in a timely manner. With the new information in the first three quarters of 2008, its prediction of extreme returns moved to the range seen in a financial crisis.

5. Application

As shown in Section 3, wavelet analysis can be used to understand the contribution of different frequencies to the total risk. In the example of S&P 500 index return, wavelet analysis identifies that short-term (2 to 4 days) volatility contributes more than half of the total volatility. It is valuable information for risk analysis when time horizon matters. For example, for hedging equity risk exposure, a daily hedging is quite different from a weekly or monthly hedging. Wavelet analysis help us understand the amount of volatility at different frequencies. A longer time horizon would expect a lower volatility.

Wavelet analysis can also enhance time-dependent analysis if different patterns exist for different frequencies. Modeling the time-dependent pattern separately by frequency can better capture the tail risk at different levels. Wavelet analysis can be applied to risks where high frequency data are available. Economic risks such as interest rate risk, credit risk and asset market risk are areas where wavelet analysis is expected to provide richer information than a pure time-dependent analysis.

While not shown in this paper, wavelet analysis can be applied to multiple time series at the same time. Correlations among time series can be calculated at different frequency levels. Long-, medium- and short-term correlation coefficients can be calculated separately and selected based on the time horizon of risk analysis as well.

6. Conclusion

Unlike time series analysis, wavelet analysis can be used to systematically analyze historical time series data by time and frequency at the same time. Wavelet analysis provides a decomposition of the total risk and can tell whether short-, medium- or long-term risk is dominating. Combined with time-dependent analysis, it can better capture different patterns at different frequency levels to improve risk estimation. Risk measures such as volatility and VaR can be calculated directly using wavelet models.

Wavelet analysis is especially useful when time horizon has a significant impact on risk analysis. It can help refine assumptions such as volatility, tail heaviness and correlation according to the time horizon of risk analysis.

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