QFIQF Model Solutions
Spring 2022

1. **Learning Objectives:**

1. The candidate will understand the foundations of quantitative finance.

Learning Outcomes:

(1c) Understand Ito integral and stochastic differential equations.

(1h) Define and apply the concepts of martingale, market price of risk and measures in single and multiple state variable contexts.

(1i) Demonstrate understanding of the differences and implications of real-world versus risk-neutral probability measures, and when the use of each is appropriate.

(1j) Understand and apply Girsanov's theorem in changing measures.

Sources:

Chin, Ch. 2,3,4; Neftci Ch. 8, 10, 14

Commentary on Question:

In general, candidates found most parts of this question challenging, in particular, parts (b), (d), (e), and (f). This question tested basic application of single and multivariate Ito's Lemma, in addition to principles of stochastic integration, properties of Brownian Motion and Martingales, and option pricing theory. Although not required, candidates could have also applied change of numeraire in part (f) to arrive at the answer.

Solution:

(a) Compute $E^{\mathbb{Q}}\big(Z(t)\big)$ and $Var^{\mathbb{Q}}\big(Z(t)\big)$.

Commentary on Question:

Candidates did well, as expected, on this part. A good number were awarded full marks. Most candidates earned points for at least the expectation. Some did not attempt the variance and some arrived at an answer in terms of rho and t that was not correct.

We compute mean and variance:

$E\big(Z(t)\big) = \frac{1}{\sqrt{1-\rho^2}}\Big(E\big(W_1(t)\big) - \rho E\big(W_2(t)\big)\Big) = 0$

$Var\big(Z(t)\big) = \frac{1}{1-\rho^2}\Big(Var\big(W_1(t)\big) + \rho^2 Var\big(W_2(t)\big) -$

$2\rho Cov\big(W_1(t), W_2(t)\big)\Big) = \frac{1}{1-\rho^2}(t + \rho^2 t - 2\rho^2 t) = t.$

1. Continued

(b) Show that $E^{\mathbb{Q}}\left(e^{sZ(t)}\right) = e^{\frac{s^2}{2}t}$ using Ito's Lemma.

Commentary on Question:
This question could have been answered using MGFs, but the point was to show it using Ito's Lemma, as explicitly stated in the question. About half of the candidates attempted this question.

Ito's Lemma yields the following equation:

$$d\left(e^{sZ(t)}\right) = e^{sZ(t)}s\,dZ(t) + \frac{1}{2}e^{sZ(t)}s^2\left(dZ(t)\right)^2 = \frac{1}{2}e^{sZ(t)}s^2dt + e^{sZ(t)}s\,dZ(t)$$

therefore

$$e^{sZ(t)} - 1 = \frac{s^2}{2}\int_0^t e^{sZ(u)}du + \frac{s}{\sqrt{1-\rho^2}}\int_0^t e^{sZ(u)}dW_1(u) - \frac{\rho s}{\sqrt{1-\rho^2}}\int_0^t e^{sZ(u)}dW_2(u).$$

Taking expectations, we get

$$E\left(e^{sZ(t)}\right) = 1 + E\left(\frac{s^2}{2}\int_0^t e^{sZ(u)}du\right) = 1 + \frac{s^2}{2}\int_0^t E\left(e^{sZ(u)}\right)du.$$

Denote $f(t) = \int_0^t E\left(e^{sZ(u)}\right)du$, then

$$f'(t) = 1 + \frac{s^2}{2}f(t).$$

Let $1 + \frac{s^2}{2}f(t) = e^{\frac{s^2}{2}t} \Rightarrow f(t) = \frac{2}{s^2}\left(e^{\frac{s^2}{2}t} - 1\right)$, then the differential equation is satisfied and

$$E\left(e^{sZ(t)}\right) = f'(t) = e^{\frac{s^2}{2}t}.$$

*Note, full marks were also awarded if candidates evaluated the integrand $E\left(e^{sZ(u)}\right)$ using MGFs the expectation of a lognormal random variable.

(c) Show that $Z(t)$ is uncorrelated with $W_2(t)$.

Commentary on Question:
Most candidates earned full marks

Compute $dZ(t)\,dW_2(t) = \frac{1}{\sqrt{1-\rho^2}}(\rho dt - \rho dt) = 0$

(d) Derive the dynamics of $Y(t)$ in terms of $Z(t)$ and $W_2(t)$.

Commentary on Question:
Candidates performed below expectation on this question. A large number did not know how to successfully apply multivariate Ito's Lemma. Of those who attempted this question, about half earned full marks or close to full marks but a fair number of candidates made computational errors or left their answer in terms of $W_1(t)$.

1. Continued

Using the multivariate Ito's lemma to evaluate $dY(t)$:

$$= \frac{\partial Y}{\partial S_2} dS_2 + \frac{\partial Y}{\partial S_1} dS_1 + \frac{\partial^2 Y}{\partial S_1 \partial S_2} dS_1 dS_2 + \frac{1}{2}\frac{\partial^2 Y}{\partial S^2{}_1}(dS_1)^2 + \frac{1}{2}\frac{\partial^2 Y}{\partial S^2{}_2}(dS_2)^2$$

$$= -\frac{S_1(t)}{S_2(t)^2}\left(rS_2(t)dt + \sigma_2 S_2(t)dW_2(t)\right)$$
$$+\frac{1}{S_2(t)}\left(rS_1(t)dt + \sigma_1 S_1(t)\left(\sqrt{1-\rho^2}dZ(t) + \rho dW_2(t)\right)\right) - \frac{1}{S_2(t)^2} dS_1 dS_2 + 0$$
$$+\frac{S_1(t)}{S_2(t)^3}(dS_2)^2$$
$$= \frac{S_1(t)}{S_2(t)}\left((\rho\sigma_1 - \sigma_2)dW_2(t) + \sigma_1\sqrt{1-\rho^2}dZ(t) + ({\sigma_2}^2 - \rho\sigma_1\sigma_2)dt\right)$$

(e) Derive the Radon-Nikodym derivative $\frac{d\mathbb{P}}{d\mathbb{Q}}$.

Commentary on Question:
Candidates performed below expectation on this question. A good number of candidates did not know what the Radon-Nikodym. derivative was. A handful of candidates earned full marks on this question and some provided a well-reasoned derivation using Girsanov's theorem. Candidates did not need to provide the full set of solutions for γ_2 and most set γ_2=0 which earned full marks.

Rewrite the SDE of Y(t) as follows:

$$dY(t) = Y(t)({\sigma_2}^2 - \rho\sigma_1\sigma_2)dt + \sigma Y(t)dW(t) = \sigma Y(t)\left(\frac{{\sigma_2}^2 - \rho\sigma_1\sigma_2}{\sigma}dt + dW(t)\right).$$

$$\frac{d\mathbb{P}}{d\mathbb{Q}} = \exp\left(\int_0^T(\gamma_1 dW + \gamma_2 dX_2) - \frac{1}{2}\int_0^T(\gamma_1^2 + \gamma_2^2)dt\right)$$
$$= \exp\left(\gamma_1 W - \frac{\gamma_1^2}{2}T\right)\exp\left(\int_0^T \gamma_2 dX_2 - \frac{1}{2}\int_0^T \gamma_2^2 dt\right)$$

where γ_2 is any function of t and

$$\gamma_1 = \frac{\rho\sigma_1\sigma_2 - {\sigma_2}^2}{\sigma}, \qquad X_2 = \frac{\sigma_2 - \rho\sigma_1}{\sigma}Z + \frac{\sigma_1\sqrt{1-\rho^2}}{\sigma}W_2.$$

In particular, when $\gamma_2 = \frac{\sigma_1\sigma_2\sqrt{1-\rho^2}}{\sigma}$, measure $\mathbb{P}$ gives rise to the specific pricing measure $\widetilde{\mathbb{P}}$ with S_2 as numeraire. X (X_1, X_2) is a standard two-dimensional Brownian motion under $\mathbb{Q}$

1. Continued

(f) Derive today's price of the exchange option using your knowledge of the Black-Scholes formula and the probability measure $\mathbb{P}$.

Commentary on Question:
This question was challenging, and candidates performed as expected on this section. No one earned full marks. Partial marks were given for candidates who successfully manipulated an expression for the price of the security and the expectation and recognized it as a call option on $S_1(0)$ *with strike* $S_2(0)$*). A surprising number of candidates worked in terms of time t when the question asked explicitly for today's price (i.e., time 0).*

The key was to recognize that Y(T) was a martingale under $\mathbb{P}$ *and to arrive at a familiar expression in the operand of the expectation that could be evaluated using the Black formula. Candidates could have started with the risk-neutral definition of the price of the security and apply a change of numeraire to* $\mathbb{P}$ *with* S_2 *numeraire (what is denoted as* $\widetilde{\mathbb{P}}$ *) , or, more simply, they could have started directly under* $\mathbb{P}$

$$c_T = max\{S_1(T) - S_2(T), 0\} = S_2(T)\max\{\frac{S_1(T)}{S_2(T)} - 1,0\}.$$

From

$$\frac{c_T}{S_2(T)} = \max(Y(T) - 1, 0)$$

we have

$$\mathbb{E}_{\widetilde{\mathbb{P}}}\left[\frac{c_T}{S_2(T)}\right] = \mathbb{E}_{\widetilde{\mathbb{P}}}[\max(Y(T) - 1, 0)].$$

Since $Y(T)$ is independent of the choice of γ_2, we have

$$\mathbb{E}_{\mathbb{P}}[\max(Y(T) - 1, 0)] = \mathbb{E}_{\widetilde{\mathbb{P}}}[\max(Y(T) - 1, 0)] = \mathbb{E}_{\widetilde{\mathbb{P}}}\left[\frac{c_T}{S_2(T)}\right] = \frac{c_0}{S_2(0)}.$$

Thus

$$c_0 = S_2(0) * \mathbb{E}_{\mathbb{P}}[\max(Y(T) - 1, 0)].$$

therefore

$$c_0 = S_2(0)(Y(0)N(d_1) - N(d_2)) = S_1(0)N(d_1) - S_2(0)N(d_2),$$

where

$$d_1 = \frac{ln\left(\frac{S_1(0)}{S_2(0)}\right) + 0.5\sigma^2 T}{\sigma\sqrt{T}},\ d_2 = d_1 - \sigma\sqrt{T}.$$

2. Learning Objectives:

1. The candidate will understand the foundations of quantitative finance.

4. The candidate will understand:
   - How to apply the standard models for pricing financial derivatives.
   - The implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory.
   - How to evaluate risk exposures and the issues in hedging them.

Learning Outcomes:

(1a) Understand and apply concepts of probability and statistics important in mathematical finance.

(1b) Understand the importance of the no-arbitrage condition in asset pricing.

(1d) Understand and apply Ito's Lemma.

(1h) Define and apply the concepts of martingale, market price of risk and measures in single and multiple state variable contexts.

(1i) Demonstrate understanding of the differences and implications of real-world versus risk-neutral probability measures, and when the use of each is appropriate.

(4a) Demonstrate an understanding of option pricing techniques and theory for equity derivatives.

(4d) Demonstrate an understanding of how to delta hedge, and the interplay between hedging assumptions and hedging outcomes.

Sources:

An Introduction to the Mathematics of Financial Derivatives, Hirsa, Ali and Neftci, Salih N., 3rd Edition 2nd Printing, 2014 (Ch. 1, 5, 6, 8, 11)

Problems and Solutions in Mathematical Finance: Stochastic Calculus, Chin, Eric, Nel, Dian and Olafsson, Sverrir, 2014 (page 52)

QFIQ-120-19: Chapters 6 and 7 of Pricing and Hedging Financial Derivatives, Marroni, Leonardo and Perdomo, Irene, 2014 (Ch. 6)

Commentary on Question:

This question attempts to test candidates' understanding of martingales and the valuation of non-standard options. Candidates' performance was uneven.

2. Continued

Solution:

(a) Prove that the discounted stock price $e^{-rt}S_t$ is a martingale.

Commentary on Question:
Candidates generally performed well on this question. Either approach is straightforward.

Under the risk-neutral measure,

$$
\begin{aligned}
E_t^{\mathbb{Q}}[e^{-rT}S_T] = \; & e^{-rT}E_t^{\mathbb{Q}}\left[S_t e^{\left(r-\frac{1}{2}\sigma^2\right)(T-t)+\sigma(W_T-W_t)}\right] \\
&= e^{-rT}S_t e^{\left(r-\frac{1}{2}\sigma^2\right)(T-t)}E_t^{\mathbb{Q}}\left[e^{\sigma(W_T-W_t)}\right] \\
&= e^{-rT}S_t e^{\left(r-\frac{1}{2}\sigma^2\right)(T-t)}e^{\frac{1}{2}\sigma^2(T-t)} \\
&= e^{-rT}S_t e^{r(T-t)} \\
&= e^{-rt}S_t
\end{aligned}
$$

Since the expectation of the future value is the current value of the process, it is a martingale (the other properties are obvious).

Alternative Solution:
Apply Ito's Lemma to the discounted stock price process:

$$
\begin{aligned}
d(e^{-rt}S_t) &= -re^{-rt}S_t dt + \; e^{-rt}dS_t \\
&= -re^{-rt}S_t dt + \; e^{-rt}(rS_t dt + \sigma S_t dW_t) \\
&= \sigma S_t dW_t
\end{aligned}
$$

which is driftless, and hence, a martingale.

(b) Show that:

(i) $V_5 = S_5\mathbb{I}_{\{S_3 \geq S_5\}} + S_3\mathbb{I}_{\{S_3 < S_5\}}$ where $\mathbb{I}_{\{A\}} = \begin{cases} 1 \text{ if } A \text{ is true} \\ 0 \text{ if } A \text{ is false} \end{cases}$.

(ii) $P[S_3 < S_5] = 0.583$ under $\mathbb{Q}$ measure.

Commentary on Question:
Most candidates did not receive full credit for part (i). Many simply re-stated the premise of the problem. Some mistakenly stated the indicator function was equivalent to a probability. To receive full credit, the indicator function needed to be explicitly incorporated within the proof.

2. Continued

Candidates who attempted part (ii) generally performed as expected. To receive full credit, candidates needed to demonstrate an understanding of the distribution of S_t. Credit was not given for correct final answers provided without justification.

(i) A straightforward calculation:

$$V_5 = min\{S_3, S_5\}$$

$$= \begin{cases} S_5 \text{ if } S_3 \geq S_5 \\ S_3 \text{ if } S_3 < S_5 \end{cases}$$

$$= \begin{cases} S_5 \text{ if } S_3 \geq S_5 \\ 0 \quad \text{if } S_3 < S_5 \end{cases} + \begin{cases} 0 \quad \text{if } S_3 \geq S_5 \\ S_3 \text{ if } S_3 < S_5 \end{cases}$$

$$= S_5 \mathbb{I}_{\{S_3 \geq S_5\}} + S_3 \mathbb{I}_{\{S_3 < S_5\}}.$$

(ii) Under the risk-neutral measure $\mathbb{Q}$, S_t follows a GBM with a drift equal to the risk-free rate. This is expressed in terms of the SDE $dS_t = rS_t dt + \sigma S_t dW_t$, which has the solution:

$$S_t = S_0 e^{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W_t}.$$

Therefore,

$$\begin{aligned} S_3 < S_5 &\Leftrightarrow {}^{S_5}/_{S_3} > 1 \\ &\Leftrightarrow e^{\left(0.02 - \frac{1}{2}(0.1)^2\right)(5-3) + 0.1(W_5 - W_3)} > 1 \\ &\Leftrightarrow (0.015)(2) + 0.1(W_5 - W_3) > 0 \\ &\Leftrightarrow W_5 - W_3 > -0.3 \end{aligned}$$

Given that $W_5 - W_3 \sim N(0,2)$, we obtain:

$$\mathbb{Q}[S_3 < S_5] = 1 - \Phi(-0.3/\sqrt{2}) = \Phi(0.21) = 0.583.$$

(c) Show that:

(i) $E_t\left[S_3 \mathbb{I}_{\{S_3 < S_5\}}\right] = 0.619\, e^{-0.02t} S_t.$

(ii) $E_t\left[S_5 \mathbb{I}_{\{S_3 \geq S_5\}}\right] = 1.03\, E_t[S_3] E\left[e^{\sqrt{0.02}Z} \mathbb{I}_{\{Z \leq -0.21\}}\right]$ with Z a standard normal random variable.

2. Continued

Commentary on Question:
Candidates who attempted part (i) did well. A key element of the solution is recognizing that the expectation of the indicator function is the probability of the indicated event.

Candidates performed poorly on part (ii). Most did not attempt a solution or wrote very minimal work that earned no credit. As implied by the statement candidates were asked to show, candidates needed to relate S_3 and S_5, similarly to the work expected in (b)(ii). In fact, much of the elements of a full credit response parallel that of the prior question.

(i)

$$\begin{aligned} E_t\left[S_3 \mathbb{I}_{\{S_3 < S_5\}}\right] &= E_t[S_3]\, E_t\left[\mathbb{I}_{\{S_3 < S_5\}}\right] \\ &= e^{0.02(3-t)} S_t E_t\left[\mathbb{I}_{\{S_3 < S_5\}}\right] \\ &= e^{0.06-0.02t}(0.583) \\ &= 0.619 e^{-0.02t} S_t \end{aligned}$$

since the expectation of an indicator function over a probability distribution is simply the probability of the indicated event, which was found in part (b)(ii).

(ii)

$$\begin{aligned} E_t\left[S_5 \mathbb{I}_{\{S_3 \geq S_5\}}\right] &= E_t\left[S_3 e^{\left(r-\frac{1}{2}\sigma^2\right)(5-3)+\sigma(W_5-W_3)} \mathbb{I}_{\{S_3 \geq S_5\}}\right] \\ &= e^{\left(0.02-\frac{1}{2}(0.1)^2\right)(2)} E_t\left[S_3 e^{0.1(W_5-W_3)} \mathbb{I}_{\{S_3 \geq S_5\}}\right] \end{aligned}$$

Using the fact that $S_3 < S_5 \Leftrightarrow Z > -0.21 \Rightarrow S_3 \geq S_5 \Leftrightarrow Z \leq -0.21$, the above is equivalent to

$$= 1.03 E_t[S_3] E[e^{\sqrt{0.02}Z} \mathbb{I}_{\{Z \leq -0.21\}}]$$

since $0.1(W_5 - W_3) \sim N\left(0, (0.1)^2(5-3)\right)$, i.e. $N(0, 0.02)$, and period from 3 to 5 years is independent from period t to 3 years.

(d) Calculate V_t and its Delta.

Commentary on Question:
Candidates performed reasonably well. A common mistake was not to include an appropriate discount factor in calculating V_t, thereby providing the expected payoff rather than the price. Candidates still received credit for their delta response if it was consistent with their answer for V_t.

2. Continued

V_t follows from part (c) and the statement, after discounting to time t. More specifically,

$$
\begin{aligned}
V_t &= e^{-0.02(5-t)}[0.619\, e^{-0.02t} S_t + 0.401\, e^{-0.02t} S_t] \\
&= e^{-0.1} S_t (0.619 + 0.401) \\
&= 0.92 S_t
\end{aligned}
$$

The delta of an option is the first partial derivative of the price with respect to the underlying stock price, i.e. $\frac{\partial V_t}{\partial S}$.

Thus, $\frac{\partial V_t}{\partial S} = \frac{\partial}{\partial S}(0.92 S_t) = 0.92$,

which remains static over the period $t < 3$.

(e) Your coworker claims that the special European-style option considered above can be Delta- and Gamma-hedged till its expiration by using a suitable short position in the underlying asset only.

Critique your coworker's claim.

Commentary on Question:
Candidates performed poorly on this part. To receive full credit, responses needed to highlight that the nature of the Greeks of this option changes once S_3 is known and fixed. Candidates needed to understand that the responses in parts (c) and (d) assumed $t < 3$.

My coworker is wrong. During the period $t < 3$, the delta of the option is constant and therefore, gamma is 0. After $t = 3$, S_3 is fixed, the delta of the option will depend on S_t, and the gamma will be non-zero. Since the underlying has a gamma of 0, a position in the underlying asset only will not allow for gamma-hedging until expiration.

3. Learning Objectives:

1. The candidate will understand the foundations of quantitative finance.

Learning Outcomes:

(1c) Understand Ito integral and stochastic differential equations.

(1d) Understand and apply Ito's Lemma.

(1h) Define and apply the concepts of martingale, market price of risk and measures in single and multiple state variable contexts.

Sources:

Hirsa and Neftci Chapter 6 & 9

Commentary on Question:

This question attempts to test candidates' understanding of the foundations of Ito's integral. Candidates performed below average on several parts of this question.

Solution:

(a) Define the Ito integral $\int_0^T B_t \, dB_t$ as the mean square limit of a suitable finite sum.

Commentary on Question:

Many candidates were able to write down the finite sum. However, only a few candidates wrote the finite sum correctly by using the left endpoint.

We divide the interval $[0,\ T]$ into n equal subintervals, each of which has a length of h, i.e., $T = nh$. Let the dividing points be $t_0 = 0, t_1 = h, \cdots, t_n = nh = T$. Denote $B_k := B_{t_k} = B_{kh}$ for $k = 0,1, \cdots, n$.

Then the Ito integral is defined as the mean square limit of

$$\sum_{k=1}^{n} B_{k-1}(B_k - B_{k-1})$$

as $n \to \infty$.

(b)

(i) Define the quadratic variation $[B]_T$ of $\{B_t: 0 \leq t \leq T\}$ as the mean square limit of a suitable finite sum.

(ii) Show that $[B]_T = T$ using the definition in part (b) (i).

(iii) Interpret $(dB_t)^2 = dt$.

3. Continued

Commentary on Question:
In general, most candidates did poorly on this part, especially (iii). For (ii), most candidates failed to show the limit of the variance of the finite sum is zero. For (iii), most candidates failed to interpret the equation in terms of an Ito integral.

(i) As before, divide the interval $[0,\ T]$ into n equal intervals, each of which has a length of h, i.e., $T = nh$. Let the dividing points be $t_0 = 0, t_1 = h, \cdots, t_n = nh = T$.

The quadratic variation $[B]_T$ is defined as the mean square limit of

$$\sum_{k=1}^{n} (B_k - B_{k-1})^2$$

as $n \to \infty$.

(ii) We will show that:

- $E(\sum_{k=1}^{n} (B_k - B_{k-1})^2) = T$
- $\lim\limits_{n\to\infty} Var(\sum_{k=1}^{n} (B_k - B_{k-1})^2) = 0$

We already know that $E((B_k - B_{k-1})^2) = t_k - t_{k-1} = h$; then use independence of Wiener process increments to get the first moment.

On the other hand, using equation 9.62 of Neftci (4th moment of a Wiener process increment), we have:

$$Var((B_k - B_{k-1})^2) = E((B_k - B_{k-1})^4) - [E((B_k - B_{k-1})^2)]^2 = 3h^2 - h^2 = 2h^2$$

therefore $Var(\sum_{k=1}^{n} (B_k - B_{k-1})^2) = 2Th \to 0$ as $n \to \infty$.
The conclusion now follows.

(iii) First, $\int_0^T (dB_t)^2$ is the mean square limit of $\sum_{k=1}^{n} (B_k - B_{k-1})^2$, therefore:

$$\lim_{n\to\infty} E\left[\sum_{k=1}^{n} (B_k - B_{k-1})^2 - \int_0^T (dB_t)^2\right]^2 = 0$$

At the same time, we just showed that $\sum_{k=1}^{n} (B_k - B_{k-1})^2$ is the quadratic variation $[B]_T$ which equals T, i.e.

$$\lim_{n\to\infty} E\left[\sum_{k=1}^{n} (B_k - B_{k-1})^2 - T\right]^2$$
$$= \lim_{n\to\infty} E\left[\sum_{k=1}^{n} (B_k - B_{k-1})^2 - \int_0^T dt\right]^2 = 0$$

Putting these two equalities together, we obtain the desired expression.

3. Continued

(c) Compute $\int_0^T B_t\, dB_t$ using the definition in part (a).

Commentary on Question:
Many candidates failed to compute the Ito integral by the definition. Instead, they used Ito's lemma to compute the Ito integral.

Observe that

$$B_{k-1}(B_k - B_{k-1}) = \frac{1}{2}\left({B_k}^2 - {B_{k-1}}^2\right) - \frac{1}{2}(B_k - B_{k-1})^2$$

and therefore

$$\sum_{k=1}^{n} B_{k-1}(B_k - B_{k-1}) = \frac{1}{2}{B_n}^2 - \frac{1}{2}\sum_{k=1}^{n}(B_k - B_{k-1})^2.$$

In the mean square limit, the left-hand side becomes $\int_0^T B_t\, dB_t$
while the right-hand side equals $\frac{1}{2}{B_T}^2 - \frac{1}{2}[B]_T = \frac{1}{2}{B_T}^2 - \frac{1}{2}T$.

(d) Describe the behavior in terms of Brownian Motion trajectories of:

(i) The first-order variation.

(ii) Variations of order higher than two.

Commentary on Question:
Candidates didn't perform as expected on this part. Only a few candidates were able to properly describe the behavior of the trajectories.

Neftci lists the following variation properties for Brownian trajectories:

1. The first order variation of Brownian trajectories will converge to infinity in some probabilistic sense and the continuous martingale will behave very irregularly
2. All high-order variations will vanish in some probabilistic sense. This means that high-order variations do not contain much information as the first and second order variations.

4. **Learning Objectives:**

2. The candidate will understand the fundamentals of fixed income markets and traded securities.

Learning Outcomes:

(2c) Understand measures of interest rate risk including duration, convexity, slope, and curvature.

Sources:

Fixed Income Securities (Chapters 4)

Commentary on Question:

In this question, the candidates have to demonstrate their understanding of the concept of duration, convexity and their application in interest rate risk hedging. They also have to comprehend barbell-bullet bond portfolio and the relationship between convexity and time decay

Candidates generally performed well in part a-e. Comments will be made in the appropriate section as to some more frequent mistakes that were made as candidates do not know the appropriate formula to estimate profit and loss with duration and convexities.

Solution:

Your company has a portfolio with two sets of installment payments receivable:

- Installment payments 1: 2-year annual payment of $200,000 each
- Installment payments 2: 5-year annual payment of $300,000 each

(a) Calculate the duration and the convexity of the portfolio, assuming the term structure of interest rates currently is flat at a continuously compounded rate of 2%.

Commentary on Question:

Most candidates can apply duration and convexity formula correctly, common mistakes of this question include misunderstanding of cash flows, portfolio duration/convexities are calculated as the sum of each receivable rather than the weighted average, leading to partial credits

$P_c(0,T_n) = \sum_{i=1}^{n-1} \mathrm{c} * P_z(0,T_i)$
$2\ year\ annuity\ value\ P_1(0,T_2) = 388{,}198$
$5\ year\ annuity\ value\ P_2(0,T_5) = 1{,}413{,}212$

4. Continued

$w_i = \frac{\frac{c}{2} * P_z(0,T_i)}{P_c(0,T_n)}$ for i = 1, …, n-1

$w_n = \frac{(1+\frac{c}{2}) * P_z(0,T_n)}{P_c(0,T_n)}$

$D_c = \sum_{i=1}^{n} w_i D_{z,T_i} = \sum_{i=1}^{n} w_i T_i$

Convexity $C_c = \sum_{i=1}^{n} w_i * T_i^2$

For the 2 year bond:
$D_c = 1.4950$
$C_c = 2.4850$
Weight = 388,198/(388,198 + 1,413,212) = 22%

For the 5 year bond:
$D_c = 2.96$
$C_c = 10.7606$
Weight = 1 – 22% = 78%
Portfolio duration = (1.495*.0.22 + 2.96*.78) = 2.6443
Portfolio convexity = (2.4850*.0.22 + 10.7606*.78) = 8.9772

Your company's CFO suggests a delta hedging strategy (duration matching) of using 5-year zero-coupon bond to against changes of the portfolio from changes in interest rates.

(b) Construct a hedging portfolio based on the CFO's suggestion.

Commentary on Question:
Common mistakes of this question are not including five-year bond price in the calculation, and/or not specifying the short position, leading to partial credits

$D_{Z(0,T)} = T, C_{Z(0,T)} = T^2$: the durations and convexities of two zero-coupon bonds
PV of $P = P_1(0,T_2) + P_2(0,T_5) = 388{,}198 + 1{,}413{,}212 = 1{,}801{,}410$
$Z(0,5) = 0.90484$
$D_{Z(0,5)} = 5$
k = $\frac{-1{,}801{,}410}{0.90484} * \frac{2.6443}{5} = -1{,}052{,}890$
So short 1,052,890 of the 5-year zero-coupon bond.

(c) Calculate the estimated portfolio value changes using the duration-convexity approximation t with and without the delta hedging when interest rate increases 10bps, 50bps and 100bps respectively.

4. Continued

Commentary on Question:
Common mistakes are: ignoring convexity impact on the hedging instrument; calculating portfolio convexity as the sum of convexities, not recognizing the opposite signs of convexity of the 5 year bond and receivables. Full credits were awarded when candidates recalculated PVs with each interest rate changes rather than using duration-convexity approximation.

$dP = P * (-D * r + \frac{C}{2} * r^2)$
$dZ(0,5) = Z(0,5) * (-D_2 * r + \frac{C_2}{2} * r^2)$
$dV = dP + k * dZ(0,5)$

Interest change	dP	k *dZ(0,5)	dV
0.1%	-4,755.39	4,751.56	-3.82
0.5%	-23,615.22	23,519.65	-95.57
1%	-46,826.14	46,443.86	-382.28

You recommend an alternative delta-gamma hedging strategy (duration-convexity matching) using two zero-coupon bonds (a 2-year zero-coupon bond and a 5-year zero-coupon bond)

(d) Construct this alternative hedging portfolio using two zero-coupon bonds.

Commentary on Question:
Common mistakes are not knowing the hedge ratios formula, missing bond price adjustment term in the formula

The hedging portfolio would be :

$$V = P + k_1 * Z(0,2) + k_2 * Z(0,5)$$

Where

P : Present Value (PV) of the GIC (in millions)
$Z(0,2)$: PV of 2-year zero-coupon bond (in millions)
$Z(0,5)$: PV of 5-year zero-coupon bond (in millions)

$$k_1 = -\frac{P}{Z(0,2)} * \left(\frac{(D_P * C_{Z(0,5)} - C_P * D_{Z(0,5)})}{(D_{Z(0,2)} * C_{Z(0,5)} - C_{Z(0,2)} * D_{Z(0,5)})}\right)$$

$$k_2 = -\frac{P}{Z(0,5)} * \left(\frac{(D_P * C_{Z(0,2)} - C_P * D_{Z(0,2)})}{(D_{Z(0,5)} * C_{Z(0,2)} - C_{Z(0,5)} * D_{Z(0,2)})}\right)$$

Where

$D_{Z(0,T)} = T, C_{Z(0,T)} = T^2$: the durations and convexities of two zero-coupon bonds

PV of $P = \sum_{i=1}^{n} CF_i * Z(0, T_i) = 1{,}801{,}410$

$$Z(0,2) = 0.96079, Z(0,5) = 0.90484$$

$$D_{Z(0,2)} = 2, C_{Z(0,2)} = 2^2 = 4$$

4. Continued

$D_{Z(0,5)} = 5, C_{Z(0,5)} = 5^2 = 25$

$$k_1 = -\frac{1{,}801{,}410}{0.96079} * \left(\frac{(2.6443*25-8.9772*5)}{(2*25-4*5)}\right) = -1{,}326{,}285$$

$$k_2 = -\frac{1{,}801{,}410}{0.90484} * \left(\frac{(2.6443*4-8.9772*2)}{(5*4-25*2)}\right) = -489{,}571$$

So short 1,326,285 of the 2-year zero-coupon bond and 489,571 of the 5-year zero-coupon bond.

(e) Calculate the estimated portfolio value changes with the alternative hedging strategy you have recommended when interest rate increases 10bps, 50bps and 100bps respectively.

Commentary on Question:
Common mistakes are: not knowing the P&L calculation with duration and convexity, not recognizing 0 P&L after hedge, or recorded 0 P&L without providing explanations.

$$dV = -(D_p * P + k_1 * D_{Z(0,2)} * Z(0,2) + k_2 * D_{Z(0,5)} * Z(0,5)) * dr$$
$$+\frac{1}{2} * (C_p * P + k_1 * C_{Z(0,2)} * Z(0,2) + k_2 * C_{Z(0,5)} * Z(0,5)) * dr^2$$

$dV = -(2.6443 * 1{,}801{,}410 - 1{,}326{,}285 * 2 * 0.96079 - 489{,}571 * 5 * 0.90484) *$
$(-0.0010) +$
$\frac{1}{2} * (8.9772 * 1{,}801{,}410 - 1{,}326{,}285 * 4 * 0.96079 - 489{,}571 * 25 * 0.90484) *$
$(-0.0010)^2 = 0$ (small number without rounding)
Similarly, we have the follwing results:

Interest change	dP	Hedging value change	dV
0.1%	-4,755.39	4,755.39	0.00
0.5%	-23,615.22	23,615.22	0.00
1%	-46,826.14	46,826.14	0.00

One analyst recommends a barbell-bullet bond portfolio to achieve positive portfolio returns. This analyst also claims that this barbell-bullet bond portfolio represents a short-term arbitrage opportunity if interest rates do not move significantly over time.

4. **Continued**

(f) Explain how to construct a barbell-bullet bond portfolio.

Commentary on Question:
Some candidates did not note that portfolio duration is close to 0 (or 0), leading to partial credit

A barbell-bullet bond portfolio consists of a barbell bond portfolio which is long both long-dated bonds and short-dated bonds, and a bullet bond portfolio which is short medium-dated bonds.

The duration from the long-dated bonds and short-dated bonds is largely offset by the duration from the medium-dated bonds, resulting in an overall portfolio duration close to 0 (or zero). The overall convexity of the portfolio is positive.

(g) Explain whether the barbell-bullet bond portfolio can always achieve a positive portfolio return when the small parallel shifts of a flat term structure of interest rates.

Commentary on Question:
Some candidates failed to understand the positive convexity and duration neutral position of the barbell-bullet portfolio

By construct, this portfolio is duration hedged but not convexity hedged.

Pulling together the terms in dr and dr^2 we obtain

$$dV = -(D \times P + k \times D_z \times P_z) \times dr + \frac{1}{2} \times (P \times C + k \times P_z \times C) \times dr^2 \qquad (4.16)$$

The first parenthesis is zero while the second parenthesis is positive.

This shows that the portfolio can always achieve a positive portfolio return when the moves (dr) of interest rates are small and random. (Small is required because otherwise the formula 4.16 would not valid).

(h) Critique the analyst's claims.

Commentary on Question:
Some candidates did not understand the offset from time decay here.

4. Continued

The barbell-bullet hedging strategy does not represent an arbitrage opportunity

The gain in value from higher convexity is offset by a lower gain due to passage of time.

The formula 4.16 has one more component missing – the passage of time (Theta), which captures the changes in value in the long-dated, short-dated bonds and the medium-dated zero bonds due to passage of time exactly offset the convexity gain

5. Learning Objectives:

2. The candidate will understand the fundamentals of fixed income markets and traded securities.

Learning Outcomes:

(2c) Understand measures of interest rate risk including duration, convexity, slope, and curvature.

(2d) Understand the characteristics and uses of interest rate forwards, swaps, futures, and options.

Sources:

Fixed Income Securities: Valuation, Risk, and Risk Management, Veronesi, Pietro, 2010. Ch 3,5.

Commentary on Question:

In this question, the candidates have to demonstrate their understanding of the concept of a duration mismatch between the total assets of a financial institution and its liabilities, and to compute the dollar duration of the firm's equity. They also have to establish a strategy using a fixed-for-floating swap in order to manage the interest rate risk. It is important that they distinguish between the traditional duration measure (D) and the dollar duration measure ($D) to arrive to the desired result.

The candidates performed well. Comments will be made in the appropriate section as to some more frequent mistakes that were made in particular in the use of the correct duration measure and the calculation of the duration of the interest rate swap.

Solution:

(a) Calculate the dollar duration of the firm's equity.

Commentary on Question:

The easiest and fastest way to calculate the dollar duration of the equity is to use the formula of the model solution.

Some have calculated first the traditional duration of both the assets and the liabilities as the weighted duration of each of their component and then applied the formula:
$D_E = [A/(A-L)] * D_A - [L/(A-L)] * D_L$ that should then be multiplied by the dollar value of the equity.

One mistake is that some have directly subtracted $D_A – D_L$ multiplied by (A-L) what was not the correct answer.

5. Continued

Assets			
Item	*Amount*	*Duration*	*Dollar Duration*
Cash	*150*	*0*	*0*
S.T. Loans	*350*	*0.9*	*315*
M.T. Loans	*580*	*3.5*	*2030*
L.T. Loans	*620*	*11*	*6820*
Total	*1700*		*9165*

Liabilities			
Item	*Amount*	*Duration*	*Dollar Duration*
Deposits	*550*	*0*	
S.T. Debt	*380*	*0.4*	*152*
M.T. Debt	*320*	*4.5*	*1440*
L.T. Debt	*150*	*9*	*1350*
Total	*1400*		*2942*

Dollar Duration of Equity = $9165 - $2942 = $6223 (1GP) i.e. dollar duration of $6.223 billion.

(b) Explain implications of the duration mismatch for this firm.

The implication of this mismatch is that:

- a parallel upward shift in interest rates of 1% generates a decline in assets far greater than in liabilities,

- implying an equity decline of $62.23 million. In percentage, this corresponds to a 21% decline in market value of equity.

(c) List two advantages of using swaps to protect against a decline in the value of the firm's equity.

(1) Initial cost of a fixed-for-floating swap is zero.

(2) it can dramatically change the duration of a portfolio.

5. Continued

(d) Construct a hedging portfolio using a 5-year semi-annual swap.

Commentary on Question:
This part had an important weight on the grading points for the question. The results are good even if some candidates only arrived at the exact solution and got full credit. The reasons are a combination of the following factors:

- Not having the correct duration of the floating coupon bond (0.5);
- It was easy to get the value of the duration of the fixed coupon bond as the $\sum W_i T_i$ but some have introduced the discount factors in the calculation;
- Not converting correctly, the durations to dollar durations;
- Having some items missing or not of the exact value in the final formula:
    - $D_E^{\$} = D_A^{\$} - D_L^{\$} + ND^{\&}_{Swap} = 0$

Recognizing that a swap can be considered a long-short portfolio (long a floating rate bond and short a fixed rate bond, both valued at par $100), i.e.

$$D^{\$}_{swap} = D^{\$}_{floating} - D^{\$}_{fixed}$$

Therefore, one needs to calculate the dollar duration of the corresponding floating coupon bond and fixed coupon bound.

Recognizing that the duration of floating coupon bond is equal to Ti – t, where t denotes current time, Ti denotes the next coupon date,
Hence in this case, $D^{\$}_{floating} = T_{i+1} - T_i = 0.5$

$D^{\$}_{floating}$ = *\$100 x 0.5 = \$50.00*

$D_{fixed} = \sum_{i=1}^{10} w_i \times T_i = 4.7829$

$D^{\$}_{fixed}$ = *\$100 x 4.7829 = \$478.29*

$D^{\$}_{swap}$ = *\$50 - \$478.29 = - \$428.29 per \$100 notional.*

In order to hedge the dollar duration of the equity position, we need to purchase N notional units of 10 year semi-annual swap with negative duration (pays fixed, receive float), where:

$$D_E^{\$} = D_A^{\$} - D_L^{\$} + ND^{\$}_{swap} = 0$$

From part (a), dollar duration of equity to be hedged is $6.223 billion

5. Continued

So Per \$100 notional of swap dollar duration should be $D^{\$}_{\text{swap}}$ = \$428.29

N = 1.453 billion 5 year semi-annual float-for-fixed swap.

(e) Calculate the swap rate C for 2-year semi-annual swap given the information provided above.

Commentary on Question:
The calculation was straightforward. A more common error was to make the summation of the Z (0, T_i) in the denominator starting at T_i = 0, instead of T_i = 0.5.

T_i	6-month LIBOR	$Z(0,T_i)$
0	2.32%	1.0000
0.5	1.48%	0.9885
1	1.06%	0.9813
1.5	2.49%	0.9761
2	1.80%	0.9641

$$c = n \times \left(\frac{1 - Z(0, T_M)}{\sum_{j=1}^{M} Z(0, T_j)} \right)$$

$$c = 2 \times \left(\frac{1 - Z(0,2)}{Z(0,0.5)+Z(0,1)+Z(0,1.5)+Z(0,2)}\right) = 1.8366\%$$

6. **Learning Objectives:**

3. The candidate will understand:
   - The Quantitative tools and techniques for modeling the term structure of interest rates.
   - The standard yield curve models.
   - The tools and techniques for managing interest rate risk.

Learning Outcomes:

(3a) Understand and apply the concepts of risk-neutral measure, forward measure, normalization, and the market price of risk, in the pricing of interest rate derivatives.

(3f) Apply the models to price common interest sensitive instruments including: callable bonds, bond options, caps, floors, and swaptions.

(3j) Understand and apply the Heath-Jarrow-Morton approach including the LIBOR Market Model.

Sources:

Fixed Income Securities: Valuation, Risk, and Risk Management, Veronesi, Pietro, 2010 (Ch.20, Ch. 21)

Commentary on Question:

Candidates performed fairly on this question. Most candidates only attempted part (a) and (b)(i). About 20% of candidates did not attempt this question.

Solution:

(a) Calculate the corresponding forward rate volatilities $S_i, i = 1,2, \dots ,4$.

Commentary on Question:

Candidates performed well on this part. Most candidates reached the correct numbers using one of the approaches below.

$S_1 = 0.2$

$$S_2 = \sqrt{\frac{0.22^2 * 0.5 - 0.2^2 * 0.25}{0.25}} = 0.23833$$

$$S_3 = \sqrt{\frac{0.26^2 * 0.75 - 0.2^2 * 0.25 - 0.23833^2 * 0.25}{0.25}} = 0.32558$$

6. Continued

$$S_4 = \sqrt{\frac{0.29^2 * 1 - 0.2^2 * 0.25 - 0.23833^2 * 0.25 - 0.32558^2 * 0.25}{0.25}}$$
$$= 0.36551$$

Alternatively,

$S_1 = 0.2$

$$S_2 = \sqrt{\frac{0.22^2 * 0.5 - 0.2^2 * 0.25}{0.25}} = 0.23833$$

$$S_3 = \sqrt{\frac{0.26^2 * 0.75 - 0.22^2 * 0.5}{0.25}} = 0.32558$$

$$S_4 = \sqrt{\frac{0.29^2 * 1 - 0.26^2 * 0.75}{0.25}} = 0.36551$$

(b)

(i) Show that the price V at time 0 of the security is given by:

$$V = Z(0,T)N\,\Delta\,\{2f_n(0,\tau,T) - r_K\}.$$

Consider now the case in which the same payoff $N\,\Delta\,(2r_n(\tau,T) - r_K)$ is paid at τ instead at time T.

(ii) Show that, if the payoff is paid at time τ, the price V at time 0 of the security is given by:

$$V = Z(0,T)N\,\Delta\,\{2f_n(0,\tau,T) - r_K - f_n(0,\tau,T)\,r_K\Delta + 2E_f^*[r_n(\tau,T)^2]\Delta\},$$

where

- $Z(\tau,T)$ is the value of zero-coupon bond at time τ with maturity T.
- $f_n\,(0,\tau\,,T)$ is the n-times compounded forward rate at 0 for an investment at τ and maturity T.
- σ_f is the volatility of the forward rate implied from caplet prices.
- $E_f^*[r_n(\tau,T)]$ is the expected value of $r_n(\tau,T)$ with respect to T-forward dynamics.

6. Continued

Commentary on Question:
Candidates did well on part (i). To receive full credits, candidates have to show all the steps listed below.

Candidates performed poorly on part (ii). About half of the candidates did not attempt this part. Many candidates were not able to show that $V = Z(0,T)\ E_f^*\ [\frac{G(r_n(\tau,T))}{Z(\tau,T)}]$.

(i)

$V = Z(0,T)\ E_f^*\ [G(r_n(\tau,T)]$

$= Z(0,T)\ N\ \Delta\ E_f^*[2r_n(\tau,T) - r_K]$

$= Z(0,T) N\ \Delta\ \{2f_n(0,\tau,T) - r_K\}$

as $[E_f^* r_n(\tau,T)] = f_n(0,\tau,T)$

(ii)

$$V = Z(0,T)\ E_f^*\ [\frac{G(r_n(\tau,T))}{Z(\tau,T)}]$$

as $Z(\tau,T) = \frac{1}{(1+r_n(\tau,T)\Delta)}$

$V = Z(0,T)\ N\ \Delta\ E_f^*[(2r_n(\tau,T) - r_K)(1 + r_n(\tau,T)\Delta)]$

$V = Z(0,T)\ N\ \Delta\ E_f^*[2r_n(\tau,T) - r_K - r_K r_n(\tau,T)\Delta + 2r_n(\tau,T)^2\Delta]$

as $[E_f^* r_n(\tau,T)] = f_n(0,\tau,T)$

$= Z(0,T) N\ \Delta\ \{2f_n(0,\tau,T) - r_K - f_n(0,\tau,T)\ r_K\Delta + 2E_f^*[r_n(\tau,T)^2]\Delta\}$

(c) Calculate the value of the security using the formula in part (b) (ii) and assuming the LMM.

Commentary on Question:
Candidates did fairly in this part. Many candidates were able to receive partial credit even if the final answers were inaccurate.

Below are common mistakes made by candidates:

- *Some candidates used continuous forward rate formula to solve for* $f_4(0, 0.5, 0.75)$*. Partial credit was given.*
- *Most candidates were not able to solve for* $E_f^*[r_n(\tau,T)^2]$

6. **Continued**

$$V = Z(0,T)N\,\Delta\,\{2f_n(0,\tau,T) - r_K - f_n(0,\tau,T)\,r_K\Delta + 2E_f^*[r_n(\tau,T)^2]\Delta\}$$

$$f_4(0,0.5,0.75) = \left[\frac{Z(0{,}0.5)}{Z(0{,}0.75)} - 1\right] * 4 = 0.0285132$$

$$E_f^*[r_n(\tau,T)^2] = f_n(0,\tau,T)^2 e^{\sigma_f(T)^2\, x\, \tau} = 0.0285132^2\ e^{0.22^2\ 0.5} = 0.000832919$$

$V = 0.982\ x\ \$100\ million\ x\ 0.25\ x$ {2 x 0.0285132-0.02- 0.0285132 x 0.02 x 0.25+ 2 x $0.0285132^2\ e^{0.22^2\ 0.5}$x 0.25}

V= $ 0.915724 million

7. Learning Objectives:

3. The candidate will understand:
   - The Quantitative tools and techniques for modeling the term structure of interest rates.
   - The standard yield curve models.
   - The tools and techniques for managing interest rate risk.

Learning Outcomes:

(3b) Understand and apply various one-factor interest rate models.

(3d) Describe the practical issues related to calibration, including yield curve fitting.

(3f) Apply the models to price common interest sensitive instruments including: callable bonds, bond options, caps, floors, and swaptions.

(3k) Understand and apply multifactor interest rate models.

Sources:

Fixed Income Securities: Valuation, Risk, and Risk Management, Chapter 15, 22

Commentary on Question:

This question tests the candidates' understanding of both the one-factor and two-factor Vasicek models. It tests the candidates' ability to work with the SDEs to come up with the basic bond price formulas implied by the short rate dynamics, to apply the models and simulate the behaviors of the bond yields, and to compare and contrast the two versions of the models.

Solution:

(a) Derive a formula for r_t in terms of W_{1t} and W_{2t}.

Commentary on Question:

Candidates did relatively well on this part. Many candidates knew the basic techniques of using an integrating factor and were able to obtain most points. However, there are still some candidates who did not know how to proceed with deriving r_t from the given SDE.

For each of ϕ_{1t} and ϕ_{2t}, use integrating factors $e^{\gamma_1 t}$, $e^{\gamma_2 t}$, respectively

$$d(e^{\gamma_1 t}\phi_{1t}) = e^{\gamma_1 t}d\phi_{1t} + \gamma_1\phi_{1t}e^{\gamma_1 t}dt = e^{\gamma_1 t}\gamma_1(\overline{\phi_1} - \phi_{1t})dt + \sigma_1 e^{\gamma_1 t}dW_{1t} + \gamma_1\phi_{1t}e^{\gamma_1 t}dt$$
$$= e^{\gamma_1 t}\gamma_1\overline{\phi_1}dt + \sigma_1 e^{\gamma_1 t}dW_{1t}$$

Alternatively,

$$d\phi_{1t} + \gamma_1\phi_{1t}dt = \gamma_1\overline{\phi_1}dt + \sigma_1 dW_{1t}$$

Multiply both sides with integrating factor $e^{\gamma_1 t}$

$$e^{\gamma_1 t}d\phi_{1t} + e^{\gamma_1 t}\gamma_1\phi_{1t}dt = e^{\gamma_1 t}\gamma_1\overline{\phi_1}dt + \sigma_1 e^{\gamma_1 t}dW_{1t}$$
$$d(e^{\gamma_1 t}\phi_{1t}) = e^{\gamma_1 t}\gamma_1\overline{\phi_1}dt + \sigma_1 e^{\gamma_1 t}dW_{1t}$$

7. Continued

Integrating both sides,

$$e^{\gamma_1 t}\phi_{1t} - \phi_{10} = \int_{s=0}^{t} e^{\gamma_1 s}\gamma_1 \overline{\phi_1} ds + \int_{s=0}^{t} \sigma_1 e^{\gamma_1 s} dW_{1s} = \overline{\phi_1}(e^{\gamma_1 t} - 1) + \int_{s=0}^{t} \sigma_1 e^{\gamma_1 s} dW_{1s}$$

$$\phi_{1t} = \phi_{10}e^{-\gamma_1 t} + \overline{\phi_1}(1 - e^{-\gamma_1 t}) + \sigma_1 e^{-\gamma_1 t}\int_{s=0}^{t} e^{\gamma_1 s} dW_{1s}$$

Similarly,

$$\phi_{2t} = \phi_{20}e^{-\gamma_2 t} + \overline{\phi_2}(1 - e^{-\gamma_2 t}) + \sigma_2 e^{-\gamma_2 t}\int_{s=0}^{t} e^{\gamma_2 s} dW_{2s}$$

Thus

$$r_t = \phi_{1t} + \phi_{2t} = \phi_{10}e^{-\gamma_1 t} + \phi_{20}e^{-\gamma_2 t} + \overline{\phi_1}(1 - e^{-\gamma_1 t}) + \overline{\phi_2}(1 - e^{-\gamma_2 t}) + \sigma_1 e^{-\gamma_1 t}\int_{s=0}^{t} e^{\gamma_1 s} dW_{1s} + \sigma_2 e^{-\gamma_2 t}\int_{s=0}^{t} e^{\gamma_2 s} dW_{2s}$$

(b) Derive $Z(t,T)$ under the two-factor Vasicek model.

Commentary on Question:
Only a small number of candidates were able to do well on this part. No point was awarded if the candidates simplified copied the bond price formula from the formula sheet, or if the candidates only mentioned that ϕ_1 and ϕ_2 are independent but did not show how that would lead to the resulting bond price formula. Partial points were awarded if the candidates showed understanding of how bond is priced under a stochastic interest rate model.

$$Z(t,T) = E[e^{-\int_t^T r_s ds}]$$

From part a), for either the one-factor or two-factor Vasicek models, r_t is normally distributed, thus $e^{\int_t^T r_s ds}$ is log-normally distributed.
Denote $m = E(\int_t^T r_s ds)$ and $v = Var(\int_t^T r_s ds)$, then $Z(t,T) = e^{m+\frac{1}{2}v}$.
For the two-factor model,

$$\int_t^T r_s\, ds = \int_t^T \phi_{1s}\, ds + \int_t^T \phi_{2s}\, ds$$

Since W_{1t} and W_{2t} are independent, $\int_t^T \phi_{1s}\, ds$ and $\int_0^T \phi_{2s}\, ds$ are also independent. Thus,

$$E\left(\int_t^T r_s\, ds\right) = E\left(\int_t^T \phi_{1s}\, ds\right) + E\left(\int_t^T \phi_{2s}\, ds\right)$$

$$Var\left(\int_t^T r_s\, ds\right) = Var\left(\int_t^T \phi_{1s}\, ds\right) + Var\left(\int_t^T \phi_{2s}\, ds\right)$$

Given the bond price equation under the one-factor model, we know

$$-\left[E\left(\int_t^T \phi_{1s}\, ds\right) - \frac{1}{2}Var\left(\int_t^T \phi_{1s}\, ds\right)\right] = A(T) - B_1(T)\phi_{1t}$$

7. Continued

And the same can be applied to ϕ_{2t}, thus the bond price under the two-factor model should be

$$Z(t,T) = e^{A'(t,T) - B_1(t,T)\phi_{1t} - B_2(t,T)\phi_{2t}}$$

where $B_1(t,T)$ is as defined in the question above, and

$$B_2(t,T) = \frac{1}{\gamma_2}(1 - e^{-\gamma_2(T-t)})$$

$$A'(t,T) = (B_1(t,T) - (T-t))\left(\overline{\phi_1} - \frac{\sigma_1^2}{2\gamma_1^2}\right) - \frac{\sigma_1^2 B_1(t,T)^2}{4\gamma_1}$$
$$+ (B_2(t,T) - (T-t))\left(\overline{\phi_2} - \frac{\sigma_2^2}{2\gamma_2^2}\right) - \frac{\sigma_2^2 B_2(t,T)^2}{4\gamma_2}$$

Alternatively,

$$Z(t,T) = E\left[e^{-\int_t^T r_s ds}\right] = E\left[e^{-\int_t^T (\phi_{1t}+\phi_{2t})ds}\right] = E\left[e^{-\int_t^T \phi_{1t}ds} e^{-\int_t^T \phi_{2t}ds}\right]$$

Due to independence of W_{1t} and W_{2t}, ϕ_{1t} and ϕ_{2t} are also independent, thus

$$Z(t,T) = E\left[e^{-\int_t^T \phi_{1t}ds}\right] E\left[e^{-\int_t^T \phi_{2t}ds}\right] = e^{A_1(t,T)+A_2(t,T)-B_1(t,T)\phi_{1t}-B_2(t,T)\phi_{2t}}$$

Bond price under the two-factor model is the product of the two bond prices under the one-factor models with ϕ_{1t} and ϕ_{2t} representing the short rate respectively.

(c) Perform each of the following calculations respectively for the one-factor and the two-factor Vasicek models.

(i) Derive the $Correlation(r_t(T-t), r_t)$ where $r_t(T-t) = -\frac{\log(Z(t,T))}{T-t}$.

You are given the following parameter values for the two models for part (ii):

	Two Factor Model		One Factor Model
	$i = 1$	$i = 2$	$i = 1$
γ_i	0.6	-0.1	0.2522
$\overline{\phi_i}$	0.01	0	0.04
σ_i	0.02	0.01	0.0224
ϕ_{i0}	-1%	1.5%	0.5%

(ii) Calculate $Z(15)$ at current t = 0.

7. Continued

You are given the following two scenarios for the values of ϕ_{i0} for part (iii)

	Two Factor Model		One Factor Model
	i = 1	i = 2	i = 1
ϕ_{i0} scenario 1	-1%	1.5%	0.5%
ϕ_{i0} scenario 2	0%	1.5%	1.5%

(iii) Graph annualized yields of zero-coupon bonds under the two scenarios in the same graph against maturity for $T \leq 20$.

(iv) Describe one advantage of the two-factor Vasicek model over the one-factor version observed in parts (c)(i) and (c)(iii).

Commentary on Question:
Part (c)(i) had low attempt rate, and many of the candidates who attempted were not successful with carrying the calculation forward. A few candidates were able to derive the covariance but did not properly derive the correlation. Candidates did relatively well on part (c)(ii), and partial points were awarded for incorrect final answers if intermediate calculations were available in the excel and correct (e.g. values of A(t,T) and B(t,T)). Many candidates received partial points for part (c)(iii), but only a small number of candidates were able to graph the yields completely correctly. Many were either not able to complete the calculations correctly or copy the results over for the graphs properly. Most candidates were aware of the two-factor model's advantage of de-correlating different points on the yield curve from the textbook, but did not relate this to the results from part (i) or (iii), in which case partial points were awarded.

(c) (i) With the bond price derived above, it's easy to see that:

Under the one-factor model

$$r_t(t,T) = -\frac{\log\big(Z(t,T)\big)}{T-t} = -\frac{A(t,T)}{T-t} + \frac{B_1(t,T)r_t}{T-t}$$

$$dr_t(t,T) = \frac{B_1(t,T)}{T-t} dr_t$$

$$Corr(r_t(T), r_t) = \frac{Cov(r_t(t,T), r_t)}{\sqrt{Var(r_t(T))Var(r_t)}} = \frac{\frac{B_1(t,T)}{T-t} Var(r_t)}{\sqrt{\left(\frac{B_1(t,T)}{T-t}\right)^2 Var(r_t)^2}} = 1$$

The longer-tem yield is perfectly correlated with the short rate.

7. Continued

Under the two-factor model

$$r_t(t,T) = -\frac{\log\big(Z(t,T)\big)}{T-t} = -\frac{A'(t,T)}{T-t} + \frac{B_1(t,T)\phi_{1t}}{T-t} + \frac{B_2(t,T)\phi_{2t}}{T-t}$$

Since ϕ_{1t} and ϕ_{2t} are independent due to independence of W_{1t} and W_{2t},

$$Corr(r_t(t,T), r_t) = \frac{Cov(\frac{B_1(t,T)\phi_{1t}}{T-t} + \frac{B_2(t,T)\phi_{2t}}{T-t}, \phi_{1t} + \phi_{2t})}{\sqrt{Var\big(r_t(t,T)\big)Var(r_t)}}$$

$$= \frac{\frac{B_1(t,T)}{T-t}V_1 + \frac{B_2(t,T)}{T-t}V_2}{\sqrt{\left[\left(\frac{B_1(t,T)}{T-t}\right)^2 V_1 + \left(\frac{B_2(t,T)}{T-t}\right)^2 V_2\right](V_1+V_2)}}$$

Where $V_1 = Var(\phi_{1t})$ and $V_2 = Var(\phi_{2t})$

(c) (ii) Plug the given values into the formulas of bond prices under the one-factor and two-factor models respectively.

Two factor model:

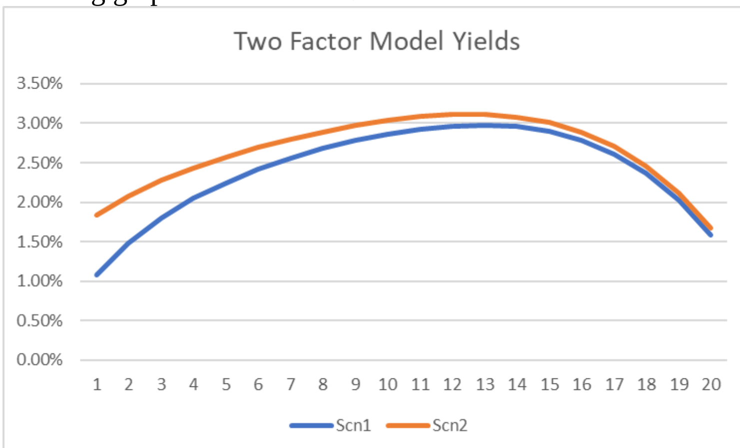
$$A'(15) = 0.07758$$
$$B_1(15) = 1.6665$$
$$B_2(15) = 34.817$$
$$Z(15) = e^{0.07758 - 1.6665\times(-1\%) + 34.817\times 1.5\%} = 0.6518$$

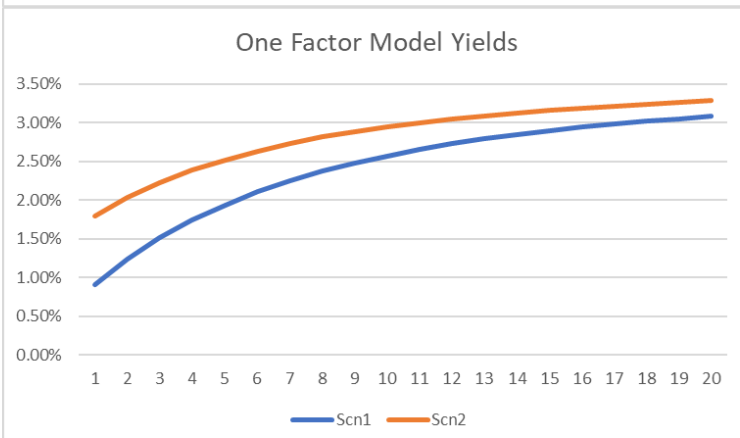
One factor model:

$$A(15) = -0.4086$$
$$B_1(15) = 3.8749$$
$$Z(15) = e^{-0.4086 - 3.8749\times 0.5\%} = 0.6518$$

7. Continued

(c) (iii) Follow instructions in the excel, use the formulas of bond prices under the one-factor and two-factor models respectively to calculate values of $A(t)$ and $B_i(15)$, and copy over the resulting yields to the designated placeholders for the graphs. The resulting graphs are as follow:





(c) (iv) One important issue with the one-factor Vasicek model is that all yields are perfectly correlated with each other. But the two factor model partly decouples the long-term yield from the short-term rate r_t. In part c) i), the correlation between long-term and short-term rate is 1 under the one factor model, but is different from 1 in the two factor version. In part c) iii), it's demonstrated that the long term yield responses readily to changes in the short term yield under the one factor model, but the relationship is not as close under the two factor model. The additional parameters under the two factor model allows the ability to add more curvature to the yield curve.

7. Continued

(d) Describe one additional advantage of the two-factor Vasicek model over the one-factor version regarding the volatility of $dr_t(T-t)$.

Commentary on Question:
Candidates did not do well on this part. Many candidates confused this with part (c) (iv) and gave similar or vague answers. Some candidates said the volatility is constant under the one-factor model due to the presence of the constant σ in the SDE, which is not correct for $dr_t(T-t)$. Only a few candidates mentioned the advantage regarding the term structure of volatility.

Using results from part c) i), under the one-factor model,

$$r_t(T-t) = -\frac{\log\big(Z(t,T)\big)}{T-t} = -\frac{A(\mathrm{t},T)}{T-t} + \frac{B_1(t,T)r_t}{T-t}$$

Volatility of dr_t is σ_1, thus volatility of $dr_t(T-t)$ is $\frac{B_1(t,T)}{T-t}\sigma_1 = \frac{1-e^{-\gamma_1(T-t)}}{\gamma_1(T-t)}\sigma_1$. This quantity decreases quickly as T increases in years.

Under the two-factor model,

$$r_t(T-t) = -\frac{\log\big(Z(t,T)\big)}{T-t} = -\frac{A'(t,T)}{T-t} + \frac{B_1(t,T)\phi_{1t}}{T-t} + \frac{B_2(t,T)\phi_{2t}}{T-t}$$

Volatility of $dr_t(T-t)$ is $\sqrt{\left(\frac{1-e^{-\gamma_1(T-t)}}{\gamma_1(T-t)}\right)^2\sigma_1^2 + \left(\frac{1-e^{-\gamma_2(T-t)}}{\gamma_2(T-t)}\right)^2\sigma_2^2}$.

Due to the existence of two volatility parameters σ_1 and σ_2, it allows the model to fit the short end and long end in the term structure of volatility, and closer to the actual shape of the term structure than the one-factor version.

8. **Learning Objectives:**

3. The candidate will understand:
   - The Quantitative tools and techniques for modeling the term structure of interest rates.
   - The standard yield curve models.
   - The tools and techniques for managing interest rate risk.

Learning Outcomes:

(3i) Understand the implications of replacing LIBOR with alternatives reference rates.

(3l) Demonstrate an understanding of the issues and approaches to building models that admit negative interest rates.

Sources:

QFIQ-129-21: Negative Interest Rates and Their Technical Consequences, AAE, 12/2016

QFIQ-131-21: Beyond LIBOR: A Primer on the New Reference Rates, BIS, March 2019

Commentary on Question:

This is one of the few written questions in the exam. Over 10% candidates skipped the whole question. For the ones who attempted the question, many skipped some parts. No candidates fully understand the implications of the negative interest rates as well as what tools could be used for dealing with it.

Solution:

(a) Explain whether you agree or disagree with the Chief Actuary's concern.

Commentary on Question:

Most candidates answered this part correctly by disagreeing with the Chief Actuaries concerns. Most candidates were able to explain reasons for why adding a shift parameter will not complicate the calibration process.

Disagree.

The shift parameter does not have to be calibrated along with the other model parameters at each valuation date but can instead be considered a meta-parameter not necessarily updated for each calibration.

The meta-parameter is determined outside the regular calibration process, probably after carrying out intensive traditional calibration processes.

8. Continued

(b) You recommended two shifted LMM models:

- Displaced Diffusion LMM (DD-LMM), and
- LMM+

Identify a key difference between these two models.

Commentary on Question:

Many candidates recognized that LMM+ had a stochastic variance process. To earn full credit, candidates needed to fully describe the variance process as a mean-reverting CIR type process.

Just shifting the forward rate diffusion as in LMM /a stochastic variance process is added in LMM +.
Instead of considering a time-dependent volatility function , a stochastic mean-reversion type Cox-Ingersoll-Ross (CIR) process is used in LMM +.

LMM+ stochastic vol is a mean reverting process.

(c) Explain the implication of a low shift parameter to the interest rate model.

Commentary on Question:

Candidates performed poorly on this part. Many candidates only answered that a low shift parameter means less negative interest rates. But most candidates did not explain any other implications to earn full credit.

The lower the Θ shift the less negative interest rates can become.

Replication of initial implied volatilities surfaces may be hindered by adopting small values of Θ.

High volatiilty together with a low shift value can produce explosive interest rate scenarios which may not be accepted by ALM models

(d) To prepare for the market transition away from LIBOR benchmark, you were also asked to review the construction of term benchmark rate from overnight risk-free rates (RFRs) using the following two methodologies.

- Backward-looking
- Forward-looking

Compare and contrast the two methodologies.

8. Continued

Commentary on Question:
Most candidates were able to correctly define backward-looking and forward-looking methodologies. However, many candidates did not compare and contrast other aspects of between the two methodologies.

Backward-looking term

- Is the simplier way to obtain a term rate even in
- the absence of underlying transactions in term instruments or in derivatives. (advantage)
- is constructed from past realisations of O/N rates based on "compounded in arrears" methodolgy.
- is less prone to quarter- or year-end volatility due to its construction as a geometric average
- daily rate (Advantage)
- Will be sluggish to respond to actual developments in O/N market interest rates (disadvantage)

Forward-looking term

- Are known at the beginning of the period to which they apply and are not based on mechanical compounding of O/N rates.
- are an outcome of a market-based price formation process, they embed market participants' expectations about future interest rates and market conditions. (advantage)
- Provide certainty for budgeting, cash flow and risk management purposes (advantage)

(e) Explain how the term structure constructed from derivatives linked to the new risk-free rates would be different from the ones derived from LIBOR.

Commentary on Question:

Most candidates answered poorly for this part or skipped this part. Candidates could not explain the different between the term structures constructed from new RFR vs LIBOR.

RFRs term rates based on derivatives reflect the market implied expected path of future O/N rates over the term of the contract do not embed premia for term funding risk, whereas LIBOR would reflect fluctuations in financial intermediaries' marginal term funding costs when based on unsecured funding instruments

Derivatives markets linked to new RFRs are still in their infancy but LIBOR have been use in the pricing of a wide array of financial contracts for decades.

8. Continued

(f) Your company is trying to develop an asset liability management strategy using assets benchmarked to the new overnight risk-free rate benchmarks.

Explain the shortcoming of this strategy and recommend a solution for this.

Commentary on Question:
Many candidates skipped this part. Few candidates understood the connection between asset-liability risk and various benchmarks

Since RFRs do not have the funding costs embedded in it, developing an ALM strategy using the RFRs-linked assets could exposed companies to basis risk in periods when their marginal funding costs diverge from interest rates earned on their assets benchmarked to the new RFRs, resulting in a margin squeeze.

To manage asset-liability risk, financial intermediaries may use multiply benchmarks that have a variety of characteristics to fulfil differing purposes and market needs.

9. Learning Objectives:

3. The candidate will understand:
   - The Quantitative tools and techniques for modeling the term structure of interest rates.
   - The standard yield curve models.
   - The tools and techniques for managing interest rate risk.

Learning Outcomes:

(3j) Understand and apply the Heath-Jarrow-Morton approach including the LIBOR Market Model.

Sources:

Fixed Income Securities: Valuation, Risk, and Risk Management, Veronesi, Piertro, 2010, p. 708, 731

Commentary on Question:

Candidates are expected to apply the HJM approach to the specified case instead of repeating what was stated in the textbook. Full points are awarded when the candidates appropriately apply the concepts to solve the questions.

Solution:

(a) Show that the short rate $r^D(t)$ is normally distributed when $\sigma^D(t,T)$ is deterministic.

Commentary on Question:

Candidates did not perform very well. Partial points were awarded if candidates indicated the deterministic part and showed the logic to the normal distribution.

$f^D(t,T) = f^D(0,T) + \int_0^t \alpha^D(\tau,T)d\tau + \int_0^t \sigma^D(\tau,T)dW^D(\tau) = f^D(0,T) + \int_0^t \sigma^D(\tau,T) \int_\tau^T \sigma^D(\tau,v)\, dv d\tau + \int_0^t \sigma^D(\tau,T)dW^D(\tau)$

Denote $m(t,T) \equiv f^D(0,T) + \int_0^t \sigma^D(\tau,T) \int_\tau^T \sigma^D(\tau,v)\, dv d\tau$.

$f^D(0,T)$ is observable and $\int_0^t \sigma^D(\tau,T) \int_\tau^T \sigma^D(\tau,v)\, dv d\tau$ is deterministic since $\sigma^D(t,T)$ is deterministic. The only stochastic part is therefore $\int_0^t \sigma^D(\tau,T)dW^D(\tau)$.

$$\therefore f(t,T) \sim N\left(m(t,T), \int_0^t (\sigma^D(\tau,T))^2 d\tau\right)$$

Since $r(t) = f(t,t), r(t)$ is also normally distributed.

9. Continued

(b) Show that under the domestic martingale measure the foreign forward rate drift is:

$$\tilde{\alpha}^F(t,T) = \sigma^F(t,T)\left(\int_t^T \sigma^F(t,s)ds - \sigma_X(t)\right)$$

Commentary on Question:
Candidates did not perform very well. Only a few candidates demonstrated the logic, and partial points were awarded when the logic was valid.

$df^F(t,T) = \alpha^F(t,T)dt + \sigma^F(t,T)dW^F(t)$, where $\alpha^F(t,T) = \mu^F(t,T) - \sigma^F(t,T) \cdot \lambda^F(t)$, and $\mu^F(t,T)$ is a drift term under the real probability. Then the drift term can be expressed as follows from the provided condition:

$$\alpha^F(t,T) = \mu^F(t,T) - \sigma^F(t,T) \cdot \lambda^F(t) = \mu^F(t,T) - \sigma^F(t,T) \cdot [\lambda^D(t) - \sigma_X(t)]$$

Let $\tilde{\alpha}^F(t,T)$ be the drift of the foreign forward rate process under the domestic martingale measure.

$$\begin{aligned} \therefore \alpha^F(t,T) &= \mu^F(t,T) - \sigma^F(t,T) \cdot \lambda^D(t) + \sigma^F(t,T) \cdot \sigma_X(t) \\ &= \tilde{\alpha}^F(t,T) + \sigma^F(t,T) \cdot \sigma_X(t) \end{aligned}$$

Since the coefficients of the foreign martingale measure satisfy the HJM drift condition, we have:

$$\alpha^F(t,T) = \sigma^F(t,T) \int_t^T \sigma^F(t,s)ds$$

From the previous equation,

$$\tilde{\alpha}^F(t,T) + \sigma^F(t,T) \cdot \sigma_X(t) = \sigma^F(t,T) \int_t^T \sigma^F(t,s)ds$$

$$\therefore \tilde{\alpha}^F(t,T) = \sigma^F(t,T)\left(\int_t^T \sigma^F(t,s)ds - \sigma_X(t)\right)$$

(c) Determine the condition under which domestic and foreign martingale measures are equivalent.

Commentary on Question:
Some candidates stated the exchange rate is deterministic. Many candidates failed to state it.

For domestic and foreign martingale measures to be equivalent, , $\tilde{\alpha}^F(t,T) = \alpha^F(t,T)$.
$\Leftrightarrow \sigma_X(t) = 0$ i.e., the exchange rate is deterministic.

10. **Learning Objectives:**

4. The candidate will understand:
   - How to apply the standard models for pricing financial derivatives.
   - The implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory.
   - How to evaluate risk exposures and the issues in hedging them.

Learning Outcomes:

(4c) Demonstrate an understating of the different approaches to hedging – static and dynamic.

(4e) Analyze the Greeks of common option strategies.

Sources:

The Volatility Smile p. 77-80

Commentary on Question:

Most candidates did not do well on this question. Most commonly, candidates referred to other formulas in the source reading which were not applicable to selected situation of a variance swap replication using a finite number of options.

Solution:

(a)

(i) Write down the piecewise-linear replication function to approximate the payoff at expiration.

(ii) Explain the key parameters used in the function.

The piecewise-linear replication function can be represented as follows:

$$V(t) = \cdots + \left(\lambda_1^P - \lambda_0^P\right) P\left(K_P^1\right) + \lambda_0^P P(K_0) + \lambda_0^C C\left(K_0\right) + \left(\lambda_1^C - \lambda_0^C\right) C\left(K_C^1\right) + \cdots \tag{4.42}$$

The function uses all the calls and puts in the market available at the available strike prices

Each lambda λ represents the magnitude of the slope of the line segment to approximate the payoff curve of the one-year variance swap.

Each P represents a discrete put option price, while each C represents a discrete call option price. K represents the strike price.

10. **Continued**

(b) Describe the key steps in the piecewise-linear replication strategy.

The piecewise-linear replication strategy is to use a finite number of puts and calls available in the market to approximate the market price of a variance swap. The key steps of the process are:

1) Calculate the π(i) of the replicating portfolio at each of the available strike prices. K(0) = S(0)

$$\pi\left(S_T, S_0, T, T\right) = \frac{2}{T}\left[\left(\frac{S_T - S_0}{S_0}\right) - \ln\left(\frac{S_T}{S_0}\right)\right] \qquad (4.41)$$

2) Calculate the slope of piecewise linear function $[\pi(K_i) - \pi(K_{i-1})]/(K_i - K_{i-1})$

3) Use the absolute value of the slope to calculate the weight for each of the options.

4) Multiply the weights by the option prices. Use the puts that are below <u>and</u> at the current price. Use the calls that are above <u>and</u> at the current price.

5) Sum up the weighted option prices for the variance. Prices for variance swaps are typically quoted in terms of volatility.

(c) Estimate the market price of the one-year variance swap on the S&P 500.

Commentary on Question:
For the candidates who attempted this question, many recognized that a weighting scheme would need to be applied against the option costs. A common mistake was only including one of the at-the-money put and call.

Following the methodology prescribed in part b):

K(i)	Pi(Ki)	Lambda(i)	W(i)	C(i)	P(i)	w(i)*O(i)
3000	0.075					
3200	0.046	0.000145	0.000039	847.44	47.44	0.0019
3400	0.025	0.000106	0.000035	686.45	86.45	0.0030
3600	0.011	0.000072	0.000031	543.56	143.56	0.0044
3800	0.003	0.000041	0.000028	420.78	220.78	0.0061
4000	0.000	0.000013	0.000001	318.62	318.62	0.0080
4200	0.002	0.000012	0.000023	236.22	436.22	0.0054
4400	0.009	0.000035	0.000021	171.68	571.68	0.0036
4600	0.020	0.000055	0.000019	122.48	722.48	0.0023
4800	0.035	0.000074	0.000017	85.89	885.89	0.0015
5000	0.054	0.000092	0.000092			
					variance	0.0361
					vol	0.1900

(d) Assess without calculations whether the piecewise linear approximation underestimates or overestimates the value of the variance swap.

Commentary on Question:
Candidates were awarded points for recognizing the approximation was an underestimate. There are several ways to recognize this, and candidates were awarded full points for any justifiable answer (one such sample included below).

For the piecewise linear replication strategy to have an accurate estimate of the market price of a variance swap, a wide range of calls and puts are required. With the strikes extending just +/-25% of the current market price, the range of options used in the replicating portfolio is too narrow, and therefore the approximation could cause us to underestimate the true value of a variance swap.

For the question, the option prices are calculated using a fixed implied volatility of 20%, while the estimated volatility is 19%. This confirms the narrow range does underestimate the market price of the variance swap.

11. **Learning Objectives:**

4. The candidate will understand:
   - How to apply the standard models for pricing financial derivatives.
   - The implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory.
   - How to evaluate risk exposures and the issues in hedging them.

Learning Outcomes:

(4d) Demonstrate an understanding of how to delta hedge, and the interplay between hedging assumptions and hedging outcomes.

(4e) Analyze the Greeks of common option strategies.

(4f) Appreciate how hedge strategies may go awry.

(4g) Describe and explain some approaches for relaxing the assumptions used in the Black-Scholes-Merton formula.

Sources:

The Volatility Smile, Derman Miller Park, Ch 6 and Ch 7

Commentary on Question:

Commentary listed underneath question component.

Solution:

(a) Explain which Strategy is associated with Figure 1 and Figure 2, respectively.

Commentary on Question:
Candidates did well on this part of the question. Most candidates were able to identify the correct strategies.

Hedging discretely rather than continuously at the correct realized volatility introduces uncertainty in the hedging outcome but does not bias the final P&L — the expected value is zero. The hedging error decreases as we increase the number of times that we re-hedge the portfolio (i.e., as we measure the volatility more accurately), but only with the square root of n. In order to halve the hedging error, we need to quadruple the number of re-hedgings.

Since Strategy 1 and Strategy 3 are based on realized volatility, and Strategy 1 has higher rebalancing frequency than Strategy 3, we have:

Figure 1 = Strategy 1 (Relative P&L is narrowly around 0)
Figure 2 = Strategy 3 (Relative P&L is widely around 0)

11. Continued

(b) Explain why Figure 3 looks similar to Figure 4.

Commentary on Question:
Candidates did well on this part of the question. Most candidates were able to explain the common point between both figures.

Unless we rebalance an option at the realized volatility, increasing the frequency of replication will not significantly diminish the replication error in the P&L. The reason is evident from Chapter 5: If the option is not hedged at the realized volatility, the incremental P&L dP&L(I, R) in Equation 5.35 of Chapter 5 contains a term proportional to $(\Delta I - \Delta R)\, dS$. This dependence on dS introduces a random noise into the P&L whose standard deviation does not diminish with more frequent hedging.

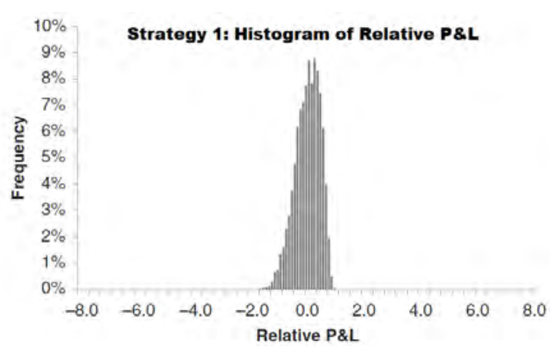
We now know that Figure 3 and Figure 4 are associated with Strategy 2 and 4 because none of the two strategies is based on realized volatility. Therefore, the standard deviation of daily hedging histogram could be similar to that of weekly hedging histogram, thus Figure 3 and Figure 4 look similar to each other.

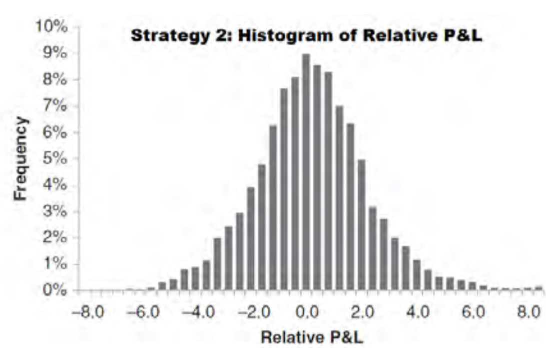
(c)

(i) Sketch the histograms of relative P&L for Strategy 1 and Strategy 3, respectively. Note: You need not mark any values on your *x*-axis and *y*-axis. The key is to show the shape or contour of the histogram.

(ii) Explain the key drivers for the differences in the histogram.

Commentary on Question:
Many candidates were able to identify key drivers for the differences between the strategies





When introducing transaction costs, increasing the hedging frequency can lower the variance of relative P&L, but will also increase the hedging cost at the same time. Transaction costs will shift the mean of both distributions below 0.

11. Continued

(d) Compare m_1 vs. m_3 vs. 0. Justify your ranking.

Commentary on Question:
Candidates performed fairly well on this question. Most candidates got the correct ranking.

The more you rebalance, the more of your profit you give away in transaction costs, so that the mean of the P&L distribution decreases. Hence:

m1 < m3 < 0 (Strategy 1 has higher expected loss than Strategy 3)

(e) Compare s_1 vs. s_3. Justify your ranking.

Commentary on Question:
Most candidates identified the less volatile strategy.

The more frequently you rebalance, the more accurately you replicate the option and the smaller the standard deviation (SD) of the profit and loss (P&L) histogram. The less you rebalance, the less profit you relinquish, but the less certain that profit is. Hence:

s1 < s3 (Strategy 1 has lower standard deviation than Strategy 3)

12. Learning Objectives:

4. The candidate will understand:
   - How to apply the standard models for pricing financial derivatives.
   - The implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory.
   - How to evaluate risk exposures and the issues in hedging them.

Learning Outcomes:

(4a) Demonstrate an understanding of option pricing techniques and theory for equity derivatives.

(4c) Demonstrate an understating of the different approaches to hedging – static and dynamic.

(4e) Analyze the Greeks of common option strategies.

(4g) Describe and explain some approaches for relaxing the assumptions used in the Black-Scholes-Merton formula.

Sources:

The Volatility Smile, Derman, Emanuel and Miller, Michael B., 2016

QFIQ-120-19: Chapters 6 and 7 of *Pricing and Hedging Financial Derivatives*, Marroni, Leonardo and Perdomo, Irene, 2014

QFIQ-114-17: Chapter 2, pp. 162-173 and 223-225, of *Frequently Asked Questions in QuantitativeFinance*, Wilmott, Paul, 2nd Edition, 2009

Commentary on Question:

Overall, candidates performed well on this question. For part a, to receive full credit, candidates must also show the derivation of the first derivative. For part e, to receive full credit, candidates must provide appropriate critique that is consistent with the source material.

Solution:

(a) Show that the Gamma of the European call is:

$$\text{Gamma} = N'(d_1)\frac{1}{S\sigma\sqrt{T}}$$

12. **Continued**

Calculate the first derivative:

$$\begin{aligned}
\frac{\partial C}{\partial S} &= \frac{\partial[SN(d_1) - Ke^{-rT}N(d_2)]}{\partial S} \\
&= N(d_1) + S\frac{\partial N(d_1)}{\partial S} - Ke^{-rT}\frac{\partial N(d_2)}{\partial S} \\
&= N(d_1) + S\frac{\partial N(d_1)}{\partial d_1}\frac{\partial d_1}{\partial S} - Ke^{-rT}\frac{\partial N(d_2)}{\partial d_2}\frac{\partial d_2}{\partial S} \\
&= N(d_1) + SN'(d_1)\frac{1}{S\sigma\sqrt{T}} - Ke^{-rT}N'(d_2)\frac{1}{S\sigma\sqrt{T}} \\
&= N(d_1) + \frac{1}{S\sigma\sqrt{T}}[SN'(d_1) - Ke^{-rT}N'(d_2)] \\
&= N(d_1) + \frac{1}{S\sigma\sqrt{T}}[0] \\
&= N(d_1)
\end{aligned}$$

Calculate the second derivative:

$$\Gamma = \frac{\partial^2 C}{\partial S^2} = \frac{\partial}{\partial S}[N(d_1)] = \frac{\partial N(d_1)}{\partial d_1} \cdot \frac{\partial d_1}{\partial S} = N'(d_1) \cdot \frac{1}{S\sigma\sqrt{T}}$$

(b) Prove that the Gamma of a European call is equal to the Gamma of an otherwise equivalent European put.

Apply the put-call parity equation:

$$C - P = S - Ke^{-rT}$$

$$\frac{\partial C}{\partial S} - \frac{\partial P}{\partial S} = 1$$

Show that the second partial derivative for call is equal to that for put:

$$\frac{\partial^2 C}{\partial S^2} - \frac{\partial^2 P}{\partial S^2} = 0$$

$$\frac{\partial^2 C}{\partial S^2} = \frac{\partial^2 P}{\partial S^2} \Rightarrow \Gamma_{Call} = \Gamma_{Put}$$

12. **Continued**

(c) Identify whether each of the following statements is true or false. Briefly justify your answer.

(i) Gamma approaches 0 for deep-in-the money calls.

(ii) Gamma approaches 1 for deep-out-of-the-money puts.

(iii) For an out-of-the-money option with an underlying asset price that is exhibiting low volatility, Gamma is expected to be relatively low.

(iv) For an option that happens to be right at-the-money very near to the expiry date, a stable Gamma is likely to be observed.

(i) True. For deep-in-the-money calls, the delta has to stay close to +1. The delta will not change much irrespective of the change in the price of the underlying, and thus the rate of change (i.e., gamma) must be close to 0.

(ii) False. For deep out-of-the-money puts, the delta has to stay close to 0. Changes in delta will be strictly limited and so gamma must be close to 0.

(iii) True. Due to low volatility, the probability that the price of the underlying will cross the strike price before the expiry date is relatively low, so we should not expect a strong sensitivity of the delta of the option to changes in the price of the underlying asset.

(iv) False. For small increases or decreases in the price of the underlying, the option delta will quickly converge to 1 or 0 for call or to -1 to 0 for put, so gamma is very unstable.

(d) Estimate the change in the value of your company's portfolio by using the Taylor series expansion of the option prices if the price of the underlying stock decreases by 10 after 5 days.

The Taylor series expansion is:

$$V_h = V_0 + \Delta ds + \frac{1}{2}\Gamma(ds)^2 + \theta d$$

Calculate the delta, gamma, and theta of the portfolio:

$\Delta_{Port} = 100(0.7890) - 200(0.6795) = -57$

$\Gamma_{Port} = 100(0.0180) - 200(0.0222) = -2.64$

$\theta_{Port} = 100(-0.0118) - 200(-0.0129) = 1.4$

12. Continued

Calculate the change in the portfolio value:

$$V_5 - V_0 = -57(-10) + \frac{1}{2}(-2.64)(-10)^2 + 1.4(5) = 445$$

(e) Your co-worker comments that the B-S model is not robust because one of the assumptions underpinning the model is that hedging is continuous and thus the B-S model does not apply if hedging is discrete.

Critique your co-worker's comment.

If you hedge discretely it turns out that the B-S is right on average. Sometimes you lose because of discrete hedging, sometimes you win, but on average you break even, and B-S still applies.

13. **Learning Objectives:**

4. The candidate will understand:
   - How to apply the standard models for pricing financial derivatives.
   - The implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory.
   - How to evaluate risk exposures and the issues in hedging them.

Learning Outcomes:

(4h) Compare and contrast the various kinds of volatility, e.gl, actual, realized, implied and forward, etc.

(4k) Describe and contrast several approaches for modeling smiles, including: stochastic volatility, local-volatility, jump-diffusions, variance-gamma, and mixture models.

Sources:

QFIQ-120-19: Chapters 6 and 7 of *Pricing and Hedging Financial Derivatives*, Marroni, Leonardo and Perdomo, Irene, 2014

The Volatility Smile, Derman, Emanuel and Miller, Michael B., 2016

Commentary on Question:

Candidates performed fairly well overall on this question, although many skipped part (a) and provided incomplete solutions or reasoning for parts (b) through (d).

Solution:

(a) Calculate the Monte Carlo step volatility for each of the three days of the barrier option.

Commentary on Question:

Many candidates skipped this part or used incorrect formulas.

For Day 1, the Monte Carlo step volatility matches the market volatility of 19%.

For Day 2, use the formula

$volmkt^2(2d) * 2d = vol^2(1)* dt + vol^2(2) * dt$

$(.20)^2(2)=(.19)^2*1+ vol^2(2)*1$

$vol(2)=20.95\%$

For Day 3, use

$vol^2(3) * dt = voLmkt^2(3d) * 3d - voLmkt^2(2d) * 2d$

$vol^2(3) * 1 = (.18)^2 * 3 – (.20)^2 * 2=13.11\%$

13. **Continued**

Your colleague is considering replacing the current model with a stochastic volatility model.

(b) Describe three limitations to stochastic volatility models.

Commentary on Question:
Most candidates received partial credit on this part by identifying the difficult in calibrating the model. Most candidates did not discuss exotic versus vanilla options and the tradeoffs there. Very few candidates identified the model limitation in reproduce European option prices.

1. European option prices cannot be reproduced perfectly, only approximately. Stochastic volatility models might be appropriate for exotic trades but they may not be appropriate for vanilla instruments.
2. The calibration of such models can be unstable, resulting in jumps in mark-to-market profit and loss.
3. If one calibrates such models using vanilla option prices, they could still give prices for exotics that are not in line with prices observed in the market. Conversely, if one calibrates models using prices for exotics, the models might not be able to get near to the price of the vanillas.

To overcome the limitations of the stochastic volatility models, a colleague suggests modeling using a local stochastic volatility model.

(c) Describe three limitations to the local stochastic volatility models.

Commentary on Question:
As with part (b), most candidates performed fairly on this part, identifying at least one or two limitations to stochastic volatility models. Most candidates mentioned difficulty calibrating the model but did not specify that the model is unstable or not good for options of different maturities.

1. In many markets there is not enough liquid data on exotic trades to calibrate reliably an LSV model.
2. The calibration of an LSV model might also be unstable, making the use of the model difficult for anything more than initial pricing.
3. It is often difficult to calibrate a model so that the prices of options with wildly different maturities are all reconstructed correctly.

Your colleague decides to model volatility V using the following stochastic process:

$$\frac{dV}{V} = \alpha\, dt + \xi\, dW$$

where α and ξ are positive constants and dW is a standard Brownian motion.

13. Continued

(d) Critique this choice of model for volatility and, if appropriate, suggest a better model.

Commentary on Question:
Most candidates performed well on this part, identifying an appropriate model with justification.

In this model, diffusion is unconstrained, so that over time volatility will tend to move farther and farther away from its initial level (not mean-reverting).

Suggestion: Mean reverting process for volatility, such as Ornstein-Uhlenbeck.

Assume that volatility can be described by the following mean-reverting discrete time series model:

$$\Delta\sigma_t = \sigma_{t+1} - \sigma_t = 0.3(15\% - \sigma_t) + \epsilon_t$$

where ϵ_t is a random variable with zero mean, $\epsilon_0 = 3\%$ and $\epsilon_1 = -2\%$. The initial volatility σ_0 is 16%.

(e) Determine the value of σ_2.

Commentary on Question:
Most candidates performed well on this part and were able to provide the correct value.

With σ_0 = 16% and ε_0 =+3%,
$\sigma_1 = \sigma_0 + 0.3(15\%-\sigma_0) + \varepsilon_0$
= 16%+0.3(15%−16%) + 3%= 16%-0.3%+3%
= 18.7%.

With σ_1 = 18.7% and ε_1 =-2%,
$\sigma_2 = \sigma_1 + 0.3(15\%-\sigma_1) + \varepsilon_1$
= 18.7%+0.3(15%−18.7%) -2%= 18.7%-1.11%-2%
= 15.59%.

14. Learning Objectives:

5. The candidate will learn how to apply the techniques of quantitative finance to applied business contexts.

Learning Outcomes:

(5a) Identify and evaluate embedded options in liabilities, specifically indexed annuity and variable annuity guarantee riders (GMAB, GMDB, GMWB and GMIB).

(5b) Demonstrate an understanding of embedded guarantee risk including: market, insurance, policyholder behavior, and basis risk.

(5c) Demonstrate an understanding of dynamic and static hedging for embedded guarantees, including:
- (i) Risks that can be hedged, including equity, interest rate, volatility and cross Greeks.
- (ii) Risks that can only be partially hedged or cannot be hedged including policyholder behavior, mortality and lapse, basis risk, counterparty exposure, foreign bonds and equities, correlation and operation failures

(5d) Demonstrate an understanding of target volatility funds and their effect on guarantee cost and risk control.

Sources:

QFIQ-124-20: Variable Annuity Volatility Management: An Era of Risk-Control

QFIQ-132-21: Investment Instruments with Volatility Target Mechanism, Albeverio, Steblovskaya, and Wallbaum, 2013

QFIQ-135-22: Structured Product Based Variable Annuities, Deng, Dulaney, Husson, McCann (sections 2 & 3)

Commentary on Question:

Commentary listed underneath question component.

Solution:

(a) Calculate the resulting target volatility fund prices X and Y in Table 2, assuming a continuously compounded risk-free rate of 3%, a target volatility of 15% and a maximum equity % of 200%.

Commentary on Question:

Most candidates received at least partial credit. Some candidates did not use the correct formula to calculate X and Y and thus did not get the correct answers.

14. **Continued**

	t=0	t=1	t=2	t=3	T=4
Equity %	0.75	0.375	1.5	0.5	0.75
Bond %	0.25	0.625	-0.5	0.5	0.25
Equity Price	100	88	105	110	93
Bond Index Price	100	103.0455	106.1837	109.4174	112.7497
Target Vol Fund Price	100	91.7614	100.1554	105.7843	99.2209

(See formula in QFI 132-21, page 1520.)

X = 91.76
Y = 105.78

(b) Compare the relative performance of the target volatility fund, capped volatility fund, and underlying asset under the scenarios in Table 3, where the target volatility = σ_T and cap volatility = σ_C and $\sigma_T < \sigma_C$.

Commentary on Question:
Most candidates did well on this part. Candidates need to justify their answers to receive full credit. Some candidates only compare the performance of two out of the three returns and thus received partial credit.

Scenario 1:
Since the volatility level is above the cap (and thus above the target volatility), the equity allocation for both the target and capped volatility fund will be less than 1. Because the target volatility is below the cap, the target fund will have a lower equity allocation than the capped fund. Since the market returns are negative, this will result in:

Target fund return > capped fund return > underlying stock return

Scenario 2:
When the market volatility equals the target volatility, all 3 funds will have an equity allocation = 1, thus:

Target fund return = capped fund return = underlying stock return

(Note: There was a typo in the Excel spreadsheet provided at the exam. Candidates who answered correctly based on Scenario 2 in the Excel spreadsheet received full credit.)

14. Continued

Scenario 3:
Since the volatility level is below target volatility (and thus below the cap volatility), the capped fund will have an equity allocation of 100% while the target volatility fund will have an allocation >100%, so:

Underlying stock return = capped fund return > target fund return

(c) Explain whether the following statements are True or False:

(i) Call options on a target volatility fund should be cheaper than or equal to the equivalent call options on the underlying risky-asset.

(ii) Call options on a capped volatility fund should be cheaper than or equal to the equivalent call options on the underlying risky-asset.

Commentary on Question:
Candidates need to justify their answers to receive full credit.

(i) False. While call option prices increase with volatility, prices of call options on target volatility funds are only cheaper if the target vol is less than the market vol.

(ii) True. Call option prices increase with volatility, and the volatility on a capped vol fund is always equal to or less than the vol of the underlying asset.

(d) Compare the impact on the company's market risk of offering a capped volatility fund versus the target volatility fund in the product design.

Commentary on Question:
Many candidates received partial credit. There were alternative answers that were accepted.

Switching to a capped volatility fund has no impact on market risk during periods of low volatility. However, it lowers the volatility of the asset underlying the VA during periods of high volatility. This lowers the cost of embedded guarantees but does not change the pay-off structure of the product, so the company still has exposure to losses on the capped volatility fund. The cap structure changes the market risk profile by having the product self-hedge through giving up equity upside to fund the floor on fund value.

14. **Continued**

(e)

(i) Calculate the risk budget of the EIA product.

(ii) Calculate the number of at-the-money call options that may be purchased using the risk budget in part (e)(i).

(iii) Calculate the break-even participation rate that can be funded with the risk budget in part (e)(i).

Commentary on Question:
Candidates generally did well on this part. Some candidates did not use the correct formulas.

(i) Risk budget = 10000*(1- exp(-0.02)) = 198.01

(ii) The risk budget should be invested in long ATM call options with maturity of 1 year.

Using the BS framework, the value of the ATM call option is:

C = N(d1)S_t – N(d2)*K*e^{-rt}

$$d1 = \frac{\ln\left(\frac{S_t}{k}\right)+\left(r+\frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}}$$

$$= \frac{\ln\left(\frac{100}{100}\right)+\left(0.02+\frac{0.25^2}{2}\right)}{0.25} = 0.205$$

d2 = d1 - σ$\sqrt{t}$ = 0.20833 – 0.25 = -0.045

C = N(.205)*100 – N(-0.045) * 100 * $e^{-.02}$ = 10.87056

Number of call options to purchase
= RB/C = 198.01/10.87056 = 18.21556

(iii) Break-even participation rate = RB / 10.87056 /100 = 0.1821556

(f)

(i) Calculate the break-even participation rate that can be funded with the risk budget in part (e)(i).

(ii) Recommend whether to switch to an underlying fund with a target volatility.

Commentary on Question:
Most candidates who attempted this part did well.

14. Continued

(i) From part e, we have RB = 198.01
The risk budget should be invested in long ATM call options with maturity of 1 year.
Using the BS framework, the value of the ATM call option is:
$C = N(d1)S_t – N(d2)*K*e^{-rt}$
$d1 = \frac{\ln\left(\frac{S_t}{k}\right)+\left(r+\frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}} = \frac{\ln\left(\frac{100}{100}\right)+\left(0.02+\frac{0.15^2}{2}\right)}{0.15} = 0.20833$
$d2 = d1 - \sigma\sqrt{t} = 0.20833 – 0.15 = .0583$
$C = N(.2083)*100 – N(.0583) * 100 * e^{-.02} = 6.961842$

Break-even participation rate = RB / (6.961842*100) = 0.284427

(ii) Yes, the company should switch to an underlying fund with a target volatility.

The target volatility fund will allow for higher participation rates in the guaranteed structure of the EIA. This is a feature that investors typically look for in a product. This can be seen in the participation rate for the target volatility fund being 28.4% while the equity growth fund is only 18.2%

15. Learning Objectives:

4. The candidate will understand:
   - How to apply the standard models for pricing financial derivatives.
   - The implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory.
   - How to evaluate risk exposures and the issues in hedging them.

5. The candidate will learn how to apply the techniques of quantitative finance to applied business contexts.

Learning Outcomes:

(4a) Demonstrate an understanding of option pricing techniques and theory for equity derivatives.

(4k) Describe and contrast several approaches for modeling smiles, including: stochastic volatility, local-volatility, jump-diffusions, variance-gamma, and mixture models.

(5a) Identify and evaluate embedded options in liabilities, specifically indexed annuity and variable annuity guarantee riders (GMAB, GMDB, GMWB and GMIB).

(5b) Demonstrate an understanding of embedded guarantee risk including: market, insurance, policyholder behavior, and basis risk.

(5e) Demonstrate an understanding of how differences between modeled and actual outcomes for guarantees affect financial results over time.

Sources:

QFIQ-135-22: Structured Product Based Variable Annuities (Sections 2 & 3)

QFIQ-124-20: Variable Annuity Volatility Management: An Era of Risk-Control
The Volatility Smile, Derman, Emanuel and Miller, Michael B., 2016, Ch.18

Commentary on Question:

The question tests the candidates' ability to identify and evaluate embedded options in structured product based variable annuities using standard model and model that incorporate volatility skew. Candidates in general must fully explain all parts of the questions and perform the calculations correctly in order to receive maximum points. Candidates did poorly on this question. Many candidates simply did not attempt this question. Credits were rewarded for answers that were partially correct.

15. **Continued**

Solution:
(a)

(i) Calculate the pro-rated cap value component of the Interim AV.

(ii) Specify and justify a portfolio of bonds and options that provides the maturity payout of the spVA product (for each instrument specify whether it is a long or short position, the strike price, bond principal, bond coupon, and the time-to-maturity).

Commentary on Question:
This question is testing the straight-forward definition of the valuation methodology for structured product based variable annuity product from the reading. Most candidates who know the reading were able to correctly answer part (i). Most candidates were able to provide the instruments for the portfolio in (ii) but did not justify their answer by including the payout structure.

(i) Pro-rated cap value component of the Interim Account Value
$= \frac{1}{3} * 500{,}000 * .25 + 500{,}000 \ = \ 541{,}666.67$

(ii) (1) a 2-year, zero coupon bond with 500,000 principal
(2) a short 2-year European put option with a strike price equal to the buffer level 400,000
(3) a long 2-year European at-the-money call option with a strike price equal to the initial level 500,000
(4) a short 2-year European call option at the cap level 625,000

zero coupon bond maturity payout = 500,000

a short European put option maturity payout
= - max(400,000 – S,0)
a long European at-the-money call option maturity payout = max(S – 500,000,0)
a short European call option maturity payout
= -max(S – 625,000,0)

Portfolio = 500,000 - max(400,000 – S,0) + max(S – 500,000,0) - max(S – 625,000,0)

If S < 400,000 then
S – 500,000 < 0 & S – 625,000 < 0
Portfolio = 500,000 – (400,000 – S)=100,000 + S
=Buffer + S

15. Continued

If 400,000 <= S < 500,000 then
400,000 – S <= 0 & S – 500,000 < 0 & S – 625,000 < 0
Portfolio = 500,000 = Initial
If 500,000 <= S < 625,000 then
400,000 – S < 0 & S – 625,000 < 0
Portfolio = 500,000 + (S – 500,000) = S
If 625,000 <= S then
400,000 – S < 0 & S – 500,000 < 0
Portfolio = 500,000 + (S – 500,000) - (S – 625,000)
= 625,000 = Cap

(b) Your manager believes that it is more accurate to incorporate volatility skew into the model when valuing the underlying options position of the spVA product. Based on the market data, your manager has fitted an implied volatility function $\Sigma(K) = 20\% - .05\%*(K - 254)$, where K is the strike level.

Assess your manager's approach.

Commentary on Question:
Most candidates did well for this question and were able to identify that the manager is applying the sticky strike rule. Candidates need to provide both the benefits and the shortcomings of the sticky strike rule in order to receive full credit.

Your manager is correct in that the model will be more accurate when incorporating volatility skew. However, sticky strike rule is an unsophisticated attempt to preserve the BSM model and permits different volatilities for the same underlier, which is illogical.

The sticky strike rule requires the ATM implied volatility to decrease when the market goes up, this may be a good approximation over short time periods or in extremely calm markets but cannot be true in the long run. Markets can continue to rise indefinitely, but volatility cannot decline forever.

A better way is to use the local volatility model, which unlike sticky strike, is more than a heuristic and provide a consistent model in option pricing.
(Objective 4a, 4k)

15. Continued

(c) Your junior assistant made the following statement:

"Using this modeling approach will also impact our delta hedging strategy for this product"

Critique his statement.

Commentary on Question:
Most Candidates were able to identify that the junior assistant is incorrect. Full credits were rewarded if candidates were able to qualitatively explain their reasoning without using the formula.

Your junior assistant is incorrect.

$$\Delta = \Delta_{BSM} + \frac{\partial C}{\partial \Sigma}\frac{\partial \Sigma}{\partial S}$$

Under sticky strike rule

$$\frac{\partial \Sigma}{\partial S} = 0$$

Therefore, the hedge ratio is the same as the BSM hedge ratio
(Objective 4a, 4k)

(d) Calculate the Interim AV for the policy as of January 3, 2018, using

(i) The fitted implied volatility function provided by your manager.

(ii) The implied volatility function evaluated only at K = 207.5, namely Σ(K=207.5) = 20% − .05%*(207.5 – 254).

Commentary on Question:
Candidates did poorly for this question. Most candidates were only able to correctly identify/calculate some of the options' parameters needed for the final results (r, d, T, sigma and K). Partial credits were rewarded for parts of the calculations that were correct.

15. Continued

d (i)

Initial Premium	500,000	prorated cap value	541,666.67
	Stock Index Level		
2017-01-03	207.5		
2018-01-03	254		
ATM Implied Volatility	20%	Buffer %	20%
risk free rate	3%	Cap %	25%
dividend yield	2%	Buffer level	100,000
Valuation date	2018-01-03	Cap Level	125,000
Maturity date	2020-01-03		
zero coupon bond	470,882.27		
Bond and Options Value	538,998.11		
Interim AV	538,998.11		

	Short Put	Long Call	Short Call
Spot (S) on 2018-01-03	612,048.19	612,048.19	612,048.19
Strike (K)	400,000	500,000	625,000
Option Price			
Risk-Free Rate (r)	3%	3%	3%
Dividend Yield (q)	2%	2%	2%
Strike (for Volatility calculation)	166	207.5	259.38
Volatility (σ)	24.40%	22.33%	19.73%
T (years till maturity)	2	2	2
d1	1.463138624	0.861651944	0.136150102
d2	1.118070515	0.545928766	-0.142891911
Option Value	7,466.13	140,587.23	65,005.25

d (ii)

Initial Premium	500,000.00	prorated cap value	541,666.67
	Stock Index Level		
2017-01-03	207.5		
2018-01-03	254		
ATM Implied Volatility	20%	Buffer %	20%
risk free rate	3%	Cap %	25%
dividend yield	2%	Buffer level	100,000
Valuation date	2018-01-03	Cap Level	125,000
Maturity date	2020-01-03		
zero coupon bond	470,882.27		
Bond and Options Value	532,673.18		
Interim AV	532,673.18		

	Short Put	Long Call	Short Call
Spot (S) on 2018-01-03	612,048.19	612,048.19	612,048.19
Strike (K)	400,000.00	500,000.00	625,000.00
Option Price			
Risk-Free Rate (r)	3%	3%	3%
Dividend Yield (q)	2%	2%	2%
Strike (for Volatility calculation)		207.5	
Volatility (σ)	22.33%	22.33%	22.33%
T (years till maturity)	2	2	2
d1	1.568421567	0.861651944	0.154882321
d2	1.252698389	0.545928766	-0.160840857
Option Value	5,276.51	140,587.23	73,519.81

(e) Evaluate whether including volatility skew will lead to lower or higher interim AV based on the results in part (d) above.

Commentary on Question:
Candidates need to provide explanation in order to receive full credit.

Including volatility skew will lead to a **higher** interim AV. This is because

- The long ATM (as of Jan 2017) call with K=207.5 has the same σ and therefore the same option value in both models.
- The bond value doesn't depend on the volatility, so therefore is same for both models.

15. Continued

- The short put is more OTM due to its lower strike price, therefore it has a lower option value than the short call (i.e., moneyness of the OTM put is ~65% = 400k/612k while the moneyness of the OTM call is ~98% =612k/625k after the increase in the stock index over the 1-year period between Jan 2017 and Jan 2018). The decrease in σ for the short call produce a greater decrease in its option value than the increase in the short put. This results in a lower net value for the short positions (72k vs 79k), thus, a higher interim AV.