QFI Quantitative Finance Exam

Exam QFIQF

AFTERNOON SESSION

Date: Tuesday, April 28, 2020
Time: 1:30 p.m. – 3:45 p.m.

INSTRUCTIONS TO CANDIDATES

General Instructions

1. This afternoon session consists of 6 questions numbered 10 through 15 for a total of 40 points. The points for each question are indicated at the beginning of the question.

2. Failure to stop writing after time is called will result in the disqualification of your answers or further disciplinary action.

3. While every attempt is made to avoid defective questions, sometimes they do occur. If you believe a question is defective, the supervisor or proctor cannot give you any guidance beyond the instructions on the exam booklet.

Written-Answer Instructions

1. Write your candidate number at the top of each sheet. Your name must not appear.

2. Write on only one side of a sheet. Start each question on a fresh sheet. On each sheet, write the number of the question that you are answering. Do not answer more than one question on a single sheet.

3. The answer should be confined to the question as set.

4. When you are asked to calculate, show all your work including any applicable formulas. When you are asked to recommend, provide proper justification supporting your recommendation.

5. When you finish, insert all your written-answer sheets into the Essay Answer Envelope. Be sure to hand in all your answer sheets because they cannot be accepted later. Seal the envelope and write your candidate number in the space provided on the outside of the envelope. Check the appropriate box to indicate morning or afternoon session for Exam QFIQF.

6. Be sure your written-answer envelope is signed because if it is not, your examination will not be graded.

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Printed in the U.S.A.

****BEGINNING OF EXAMINATION****
Afternoon Session
Beginning with Question 10

10. (*7 points*) Given the following data as of Dec 31, 2018:

- A six-month Treasury bill with $100 face value was priced at $97
- A one-year Treasury note with $100 face value and paying 3% coupons semi-annually was priced at $95.85

(a) (*0.5 points*) Calculate the discount factors Z(0,0.5) and Z(0,1) as of Dec 31, 2018.

(b) (*1.5 points*) Describe two disadvantages of yield curve bootstrapping and how alternative approaches can be used to overcome each of the disadvantages.

(c) (*1 point*)

(i) Calculate the Macaulay duration of the one-year Treasury note.

(ii) Explain why the Macaulay duration of the one-year Treasury note is shorter than one year.

(d) (*1 point*) Explain the differences between cash flow matching and immunization in terms of hedging a stream of liability annuity cash flows.

Assume that you have purchased a one-year Treasury note at the market price and are worried about the interest rate risk.

(e) (*1 point*) Recommend a duration hedge strategy that uses the six-month Treasury bill to mitigate the interest rate risk.

10. Continued

Consider now a portfolio P with factor durations D_1 and D_2 with respect to level and slope factors. To implement factor neutrality, a short-dated zero-coupon bond, denoted by P^S, and a long-dated zero-coupon bond, denoted by P^L, are selected.

Notations:

- D_1^S and D_2^S are the factor durations of the short-dated zero-coupon bond.
- D_1^L and D_2^L are the factor durations of the long-dated zero-coupon bond.
- k_S is the number of short-dated zero-coupon bond that are selected.
- k_L is the number of long-dated zero-coupon bond that are selected.

(f) (*2 points*) Determine k_S and k_L in terms of $P, P^S, P^L, D_1, D_2, D_1^S, D_2^S, D_1^L,$ and D_2^L such that the portfolio P plus the short-dated and the long-dated bonds is immunized against changes in the level and slope factors.

11. (*9 points*)

(a) (*1.5 points*) Compare and contrast the LIBOR market model, the Vasicek model, and the Black, Derman, and Toy (BDT) model in terms of:

(i) The interest rates being modeled;

(ii) Diffusion processes for interest rates.

(b) (*0.5 points*) Describe briefly the common approach of calibrating the BDT results to observed prices of caplets.

Consider the BDT model for the short-term interest rate r_t.

Let $y_t = \ln(r_t)$, $y_0 = \ln(r_0)$

$$dy_t = \left(\theta_t + \frac{\frac{\partial \sigma_t}{\partial t}}{\sigma_t} y_t \right) dt + \sigma_t dX_t$$

where θ_t, σ_t are functions of t only and X_t is a standard Brownian motion.

(c) (*1.5 points*) Prove that $r_t = r_0^{\frac{\sigma_t}{\sigma_0}} e^{\sigma_t \int_0^t \frac{\theta_s}{\sigma_s} ds} e^{\sigma_t X_t}$.

Hint: First find the SDE for $\frac{y_t}{\sigma_t}$.

(d) (*1.5 points*) Evaluate $E(r_t)$ and $Var(r_t)$, assuming $\sigma_t = \sigma$ for all $t \geq 0$ where σ is a constant.

11. **Continued**

As the BDT model does not allow for a closed-form formula for bond prices, you are asked to use the following Vasicek model, which is calibrated to the current market bond prices, to value at time 0 a 2.1-year European call option on a semi-annual coupon bond that will mature in 3 years:

$$dr_t = \left(0.002 - 0.05r_t\right)dt + 0.015dX_t$$

where X_t is a standard Brownian motion under the risk-neutral measure.

The price of a zero-coupon bond with \$1 principal at time t with maturity date T is given by $Z\left(r,t;T\right) = e^{A(t,T)-B(t,T)r_t}$.

You are given the following:

- Annual coupon rate = 2%
- Principal = \$100
- Strike price = \$99
- Initial short-term interest rate r_0 = 3%
- $B(2.1,3)$ = 0.88005036, $A(2.1,3)$ = –0.00077155

Denote by r_K^* the interest rate that makes the price at time 2.1 years of the coupon bond equal to the strike price of \$99.

(e) (*1.5 points*) Demonstrate that r_K^* = 3.32325%.

In addition, you are given the following:

- $Z\left(r_0,0;2.1\right)$ = 0.938245
- $Z\left(r_0,0;2.5\right)$ = 0.926848
- The price at $t = 0$ of the 2.1-year European call with strike price of \$0.97042824, written on a zero-coupon bond with \$1 principal and maturity of 3 years, is 0.007823.

(f) (*2.5 points*) Compute the value at time $t = 0$ of the above European call option on the coupon bond.

12. (*4 points*) The short-term interest rate r_t follows an Ito process:

$$dr_t = a(r)\,dt + \sigma(r)\,dX_t$$

where $a(r)$ and $\sigma(r)$ are functions of r_t, and X_t is a standard Wiener process.

Let $Z(r,t;T)$ be the price at time t of a zero-coupon bond with \$1 principal maturing at time T.

(a) (*1 point*) Show that $Z(r,t;T)$ follows the process:

$$\frac{dZ(r,t;T)}{Z(r,t;T)} = a(r,t;T)\,dt - q(r,t;T)\,dX_t$$

where

$$a(r,t;T) = \frac{1}{Z(r,t;T)}\left[a(r)\frac{\partial Z}{\partial r} + \frac{1}{2}\sigma^2(r)\frac{\partial^2 Z}{\partial r^2} + \frac{\partial Z}{\partial t}\right]$$
$$q(r,t;T) = -\frac{1}{Z(r,t;T)}\frac{\partial Z}{\partial r}\sigma(r)$$

At time t, a portfolio with two zero-coupon bonds is created by longing one unit of zero-coupon bond with maturity date T_1 and shorting N units of zero-coupon bonds with maturity date T_2.

Let $\Pi(r,t)$ be the value of this portfolio at time t when the current interest rate is r:

$$\Pi(r,t) = Z(r,t;T_1) - N\ Z(r,t;T_2)$$

(b) (*1 point*) Show that for a delta-hedged portfolio

$$N = \frac{Z(r,t;T_1)\ q(r,t;T_1)}{Z(r,t;T_2)\ q(r,t;T_2)}$$

(c) (*1 point*) Demonstrate that $\Pi(r,t)$ with a delta-hedged position has no volatility.

(d) (*1 point*) Show that:

$$\frac{a(r,t;T_1) - r}{q(r,t;T_1)} = \frac{a(r,t;T_2) - r}{q(r,t;T_2)}$$

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13. (*7 points*) Your company uses Black's model with caplets' forward volatilities to price caps and is interested in offering more general options. You have chosen the LIBOR market model for pricing these options.

Given:

- $f_n(t,\tau,T)$ is the n times compounded annual forward rate as seen at time t for the period $[\tau,T]$.
- $r_n(\tau,T)$ is the n times compounded annual LIBOR spot rate at time τ for the period $[\tau,T]$.

(a) (*1 point*) Describe the distribution of $r_n(\tau,T)$ in the LIBOR market model.

(b) (*1 point*) Define caplet forward volatilities, $\sigma_f^{Fwd}(T_{i+1}),\ i=0,\ 1,\ ...,$ and identify their advantages in pricing caps.

Consider a call option on the "square root" of the 6-month LIBOR rate with payoff $P(\tau,T)=N\ \max\left(\sqrt{r_2(\tau,T)}-K,0\right)$, given that:

- N is the notional amount and it is set at $1 million.
- τ = 0.5.
- $f_n(0,\ 0.5,\ 1)$ = 3%.
- The option strike price K is set at 0.2.
- The one-year forward caplet volatility $\sigma_f^{Fwd}(1)$ is 0.2.
- The one-year discount factor $Z(0,1)$ is 0.95.

(c) (*2.5 points*) Calculate the value of the call option.

Suppose that the payoff function in the above is changed to $P(\tau,T)=Ne^{-|r_n(\tau,T)-K|}$.

(d) (*1 point*) Outline an algorithm to calculate the option value using the Monte Carlo method.

13. **Continued**

You plan to set the forward rate volatility $\sigma_f^{i+1}(t)$ of $f_n(t, T_i, T_{i+1})$ as given below:

$$\sigma_f^{i+1}(t) = \begin{cases} S_1, \ t < T_1, \ i = 0 \\ S_2, \ T_1 \le t < T_2, \ i = 1 \\ S_3, \ T_2 \le t < T_3, \ i = 2 \\ \quad\quad \cdots \\ S_M, \ T_{M-1} \le t < T_M, \ i = M-1 \end{cases}$$

where $S_1, S_2, \ldots$ are positive value constants.

Your colleague suggests to set $\sigma_f^{i+1}(t)$ as the same constant for all i.

(e) (*0.5 points*) Critique your colleague's suggestion in light of the relationship between $\sigma_f^{i+1}(t)$ and $\sigma_f^{Fwd}(T_{i+1})$ for $i = 0, 1, 2, \ldots, M-1$.

Given the following information:

$\boldsymbol{i}$	$\boldsymbol{T_i}$	$\boldsymbol{\sigma_f^{Fwd}(T_{i+1})}$
1	0.5	0.03
2	1	0.045
3	1.5	0.05
4	2	0.045

(f) (*1 point*) Calculate the corresponding forward rate volatilities S_i in your plan.

14. (*6 points*) As a pricing actuary of an insurer, you are asked to provide comments on the new variable annuity (VA) business with GMAB, GMMB, GMIB, and GMWB riders.

The following features are offered in the products:

- No floor value in the first 5 policy years
- After the 5^{th} policy year, an annual ratchet feature if no withdrawals were made

(a) (*1 point*) Explain how these features impact the value of the embedded options in the riders.

When pricing the products, your mortality assumption is based on the industry average.

(b) (*1 point*)

(i) Explain how this assumption could be adjusted in order to make the products more competitive.

(ii) Suggest one way to manage the longevity risk after making this change.

The Chief Risk Officer of the company is mostly concerned about the risks inherent in GMWBs.

(c) (*1 point*)

(i) Identify the financial instruments that can be used to hedge against the following financial risks inherent in the GMWB:

- stock market volatility
- increase in stock market volatility.

(ii) State two reasons why the risks in the GMWB cannot be perfectly hedged.

14. Continued

Suppose that the VA benefits are tied to a reference portfolio, whose value follows either a geometric Brownian motion (Black-Scholes) or a Regime-Switching-GARCH model (RS-GARCH). The actuarial risk management team utilizes four assumption sets to analyze the effectiveness of hedging GMMBs, assuming delta-hedging is used. The hedging is performed based on the hedging assumptions below, while the test assumptions are used to simulate outcomes for the analysis of hedge effectiveness.

Assumption Set	Hedging Assumption	Test Assumption
I	Assume that the policyholder will not surrender his contract	The policyholder conforms to the hedging assumption
II	Assume that the policyholder will lapse his contract if the moneyness ratio hits 150%	The policyholder conforms to the hedging assumption
III	Assume that the policyholder will not surrender his contract	The policyholder does not conform to the hedging assumption and lapses when the moneyness ratio hits 150%
IV	Assume that the policyholder will lapse his contract if the moneyness ratio hits 175%	The policyholder actually lapses his contract once the moneyness ratio hits 150%

Your assistant provides the table below showing key statistics for the net hedging error for each combination of assumption set and model:

	Mean		StDev		95% CTE		99% VaR	
Assumption Set	Black-Scholes	RS-GARCH	Black-Scholes	RS-GARCH	Black-Scholes	RS-GARCH	Black-Scholes	RS-GARCH
I	0.0	-0.7	0.7	1.8	1.6	3.6	1.9	4.4
II	-0.1	-1.1	0.8	2.0	1.9	3.6	2.2	4.4
III	0.5	-0.6	1.7	2.4	2.7	3.7	2.9	4.3
IV	1.2	0.0	3.8	4.0	8.2	7.7	8.6	8.3

You have reviewed the results and tell your assistant that the results for Assumption Set III and IV may have been switched.

Question 14 continued on the next page.

14. **Continued**

(d) (*3 points*)

(i) Justify your assertion.

(ii) Identify and explain the conclusions that can be drawn from comparing the above results after switching the results for Assumption Set III and IV.

(iii) Compare the above results with respect to the following aspects between the Black-Scholes and RS-GARCH models (assuming that insurer uses delta-hedging under the Black-Scholes model to manage the risk of the GMMBs):

- Risk measures (including standard deviation)
- Model risk
- Hedging error

(iv) Explain whether dynamic lapsation should be hedged, based on the comparison in part (iii).

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15. (*7 points*) You work in the hedging department of LMN, a company selling variable annuities with a GMWB rider. Your manager is interested in understanding more about volatility management strategies and asks you to prepare a memo.

(a) (*1.5 points*) Describe principal objectives of volatility management strategies of equity-based guarantee products from the perspectives of manufacturer and client respectively.

LMN currently uses a capped volatility strategy to manage volatility risks. LMN targets a 60% allocation to the S&P500 index. The trigger level for the company is 30%. The sum of squared daily returns of the portfolio from the last 21 business days is now 0.0081 (i.e., $\sum_{j=1}^{21} r_j^2 = 0.0081$, where r_j = daily return of the portfolio from j business days ago).

Assume that there are 252 business days in a year and 100% is the maximum equity allocation.

(b) (*1.5 points*) Determine the equity allocation of the portfolio after any changes driven by the capped volatility strategy.

(c) (*0.5 points*) Describe actions, if any, to take to achieve the changes in equity allocation in part (b).

Your co-worker mentions that he had just read about VIX-indexed volatility management strategies. He stated that the VIX-indexed fee rider enables the company to charge clients for all of the hedging costs of the company as they occur and that this in turn makes it a good strategy for dealing with spikes in volatility.

(d) (*1 point*) Critique your coworker's thoughts on VIX-indexed volatility management strategies.

15. Continued

LMN is considering adopting a different volatility management strategy that balances both LMN's and the clients' perspectives. The company has been provided the following information on various volatility management strategies.

	Capped Volatility	**Target Volatility**	**Capital Preservation**	**VIX-indexed fees**	**Joint VIX fee & cap volatility**
Reduction in volatility cost	15%	61%	94%	26%	40%
Vega	0.40%	0.12%	0.03%	0.36%	0.24%
Cumulative fees	100	100	100	101	101
Returns 2000-09	-0.25%	-0.55%	-0.06%	-0.73%	-0.61%
Returns 2010-17	6.05%	5.40%	2.82%	6.20%	6.06%
Volatility 2000-09	11.05%	8.19%	5.26%	12.92%	11.05%
Volatility 2010-17	8.52%	7.60%	4.55%	8.65%	8.51%

A consulting firm recommended a joint VIX-indexed and capped volatility strategy as the volatility management strategy.

(e) (*2.5 points*) Evaluate the recommended strategy.

****END OF EXAMINATION****
Afternoon Session

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