Bayesian Inference Resistant to Outliers, using Super Heavy-tailed Distributions, for the Calculation of Premiums

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Introduction.

- Context: A sample of n claims is collected for a specified product of an insurance company.
- Objective: Determine a distribution for the next claim which is robust to outliers.
- Method: Robust combination of the n claims with the prior information, using the Bayesian model and super heavy-tailed densities.

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Bayesian context.

- Let $X_1, ..., X_n$ be n random variables conditionally independent given the scale parameter σ, corresponding to the amount of claims.
- Let the conditional densities of $X_i|\sigma$ be given by $\frac{1}{\sigma} f_i(\frac{x_i}{\sigma})$, where $X_i \in \mathbb{R}^+, \sigma \in \mathbb{R}^+$, $i = 1, ..., n$.
- The prior density of σ is $\frac{1}{x_0}\pi_\sigma(\frac{\sigma}{x_0})$, where $x_0 \in \mathbb{R}^+$ is a known scale parameter.
- The posterior density of the scale parameter σ is given by

$$\pi(\sigma|x_1, \ldots, x_n) = \frac{\frac{1}{x_0}\pi(\frac{\sigma}{x_0}) \prod_{i=1}^{n} \frac{1}{\sigma} f_i(\frac{x_i}{\sigma})}{\int_0^\infty \frac{1}{x_0}\pi(\frac{\sigma}{x_0}) \prod_{i=1}^{n} \frac{1}{\sigma} f_i(\frac{x_i}{\sigma}) d\sigma}.$$

- The predictive distribution of a next claim X_{n+1} is given by

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Conditions of robustness.

- Robustness to outliers depends on the choice of the prior and the likelihood.
- For example, log-normal distributions produce sensitive inference to outliers.
- Outliers in this context is conflicting information, which can be an extreme observation as well as a misspecification of the scale parameter of the prior density.
- The tails of the prior and the likelihood determine if the posterior density of σ and the predictive distribution of a next claim are robust to outliers.

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Conditions of robustness.

- Our paper established the conditions of robustness. Simply stated, the theoretical results say that:
- 1) if the tails of the prior and the likelihood are sufficiently heavy,
- 2) if the number of conflicting information is less or equal to half of the observations,

  then
- $\sigma|\underset{\sim}{x}_n \xrightarrow{\mathcal{L}} \sigma|\underset{\sim}{x}_k$ as the outliers tend to 0 or infinity, where $\underset{\sim}{x}_k$ is the vector of non-outliers, and the density of the random variables $\sigma|\underset{\sim}{x}_n$ and $\sigma|\underset{\sim}{x}_k$ evaluated at the point y are given by $\pi(y|\underset{\sim}{x}_n)$ and $\pi(y|\underset{\sim}{x}_k)$.

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Conditions of robustness.

- What “sufficiently heavy” means?
- Densities with exponential tails such as Normal and gamma densities are not sufficiently enough to produce robust inference.
- Heavy-tailed densities such as Student and Pareto are not sufficiently enough to produce complete robust inference.
- However, they will produce “partial” robustness, in the sense that an outlier will have an impact on the inference, but this impact will be limited.
- Super heavy-tailed densities, such as log-Student or log-Pareto densities are sufficiently heavy and satisfy the condition of complete robustness.
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Example.

- We observe 5 claims $X_1, \ldots, X_5 = 380, 420, 600, 650, 760$.
- We choose a non-informative distribution for the prior:
- $$\frac{1}{x_0}\pi_\sigma\left(\frac{\sigma}{x_0}\right) \propto \frac{1}{\sigma}$$
- We compare two models for the likelihood: the log Normal and the log Student.
- Log Normal:
$$\frac{1}{\sigma}f_i\left(\frac{x_i}{\sigma}\right) = \frac{1}{sx_i}N\left(\frac{\log x_i - \log\sigma}{s}\right),$$
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$$\frac{1}{\sigma} f_i(\frac{x_i}{\sigma}) = \frac{1}{s x_i} T \left(\frac{\log x_i - \log \sigma}{s} \right),$$

where $T(\cdot)$ is the density of a Student with 5 degrees of freedom.

- The scale parameter σ behave more like a location parameter while the parameter s behave like a scale parameter.
- When the densities are expressed in term of σ as it is the case in the posterior density, x_i becomes the scale parameter (and behave as a location parameter).

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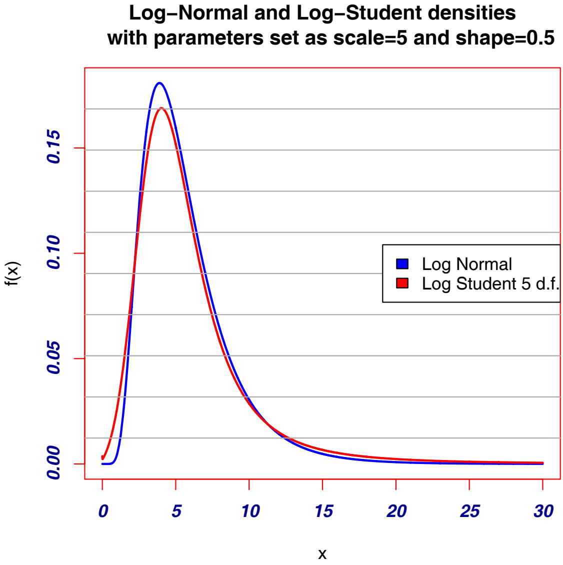
Log−Normal and Log−Student densities
with parameters set as scale=5 and shape=0.5
f(x)
0.00
0.05
0.10
0.15
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5
10
15
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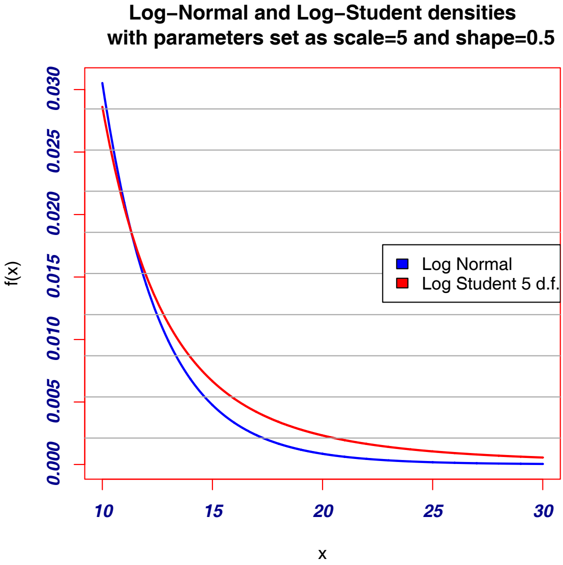
25
30
x
Log Normal
Log Student 5 d.f.


Log–Normal and Log–Student densities
with parameters set as scale=5 and shape=0.5
f(x)
0.000
0.005
0.010
0.015
0.020
0.025
0.030
10
15
20
25
30
x
Log Normal
Log Student 5 d.f.

- Since the tails are too heavy for the posterior mean to exists, we estimate σ with the posterior median.
- We look at the posterior median of σ for different values of x_5 for both models.
- If the observation x_5 is removed from the analysis, we find that:
- the posterior median of σ for the log Normal model is 4.8 (all numbers are expressed in hundreds)
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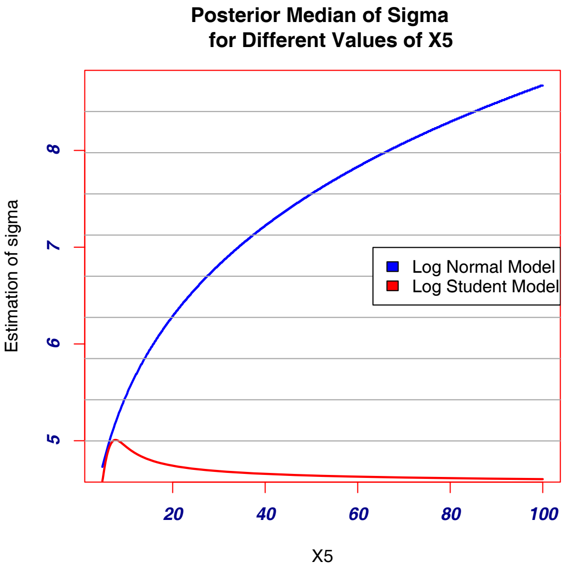
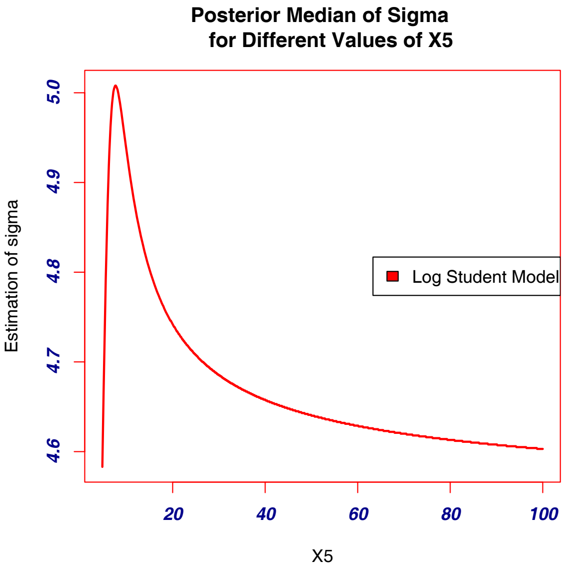
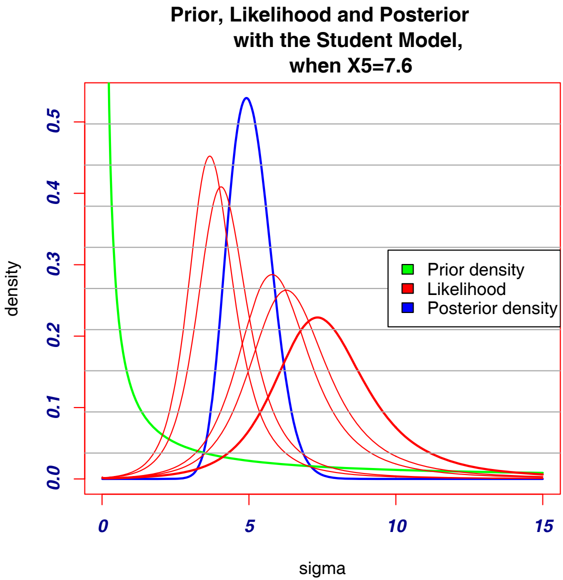
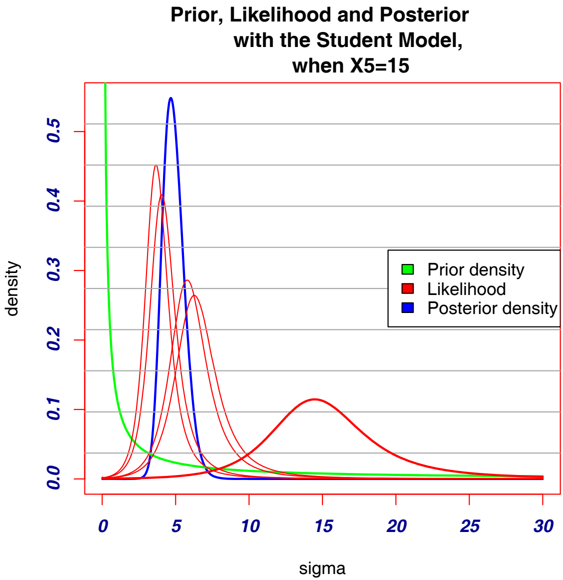

Posterior Median of Sigma
for Different Values of X5
Estimation of sigma
5
6
7
8
20
40
60
80
100
X5
Log Normal Model
Log Student Model


Posterior Median of Sigma
for Different Values of X5
Estimation of sigma
5.0
4.9
4.8
4.7
4.6
Log Student Model
20
40
60
80
100
X5


Prior, Likelihood and Posterior
with the Student Model,
when X5=7.6
density
0.0
0.1
0.2
0.3
0.4
0.5
0
5
10
15
sigma
Prior density
Likelihood
Posterior density


Prior, Likelihood and Posterior
with the Student Model,
when X5=15
density
0.0
0.1
0.2
0.3
0.4
0.5
0
5
10
15
20
25
30
sigma
Prior density
Likelihood
Posterior density

Conclusion.

- Robust Bayesian Inference for scale parameter is possible.
- Modelling the prior and the likelihood using super heavy-tailed distributions satisfy the conditions of robustness.
- Calculation of premiums resistant to outliers is then possible using the predictive distribution.

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