Classical Risk Model with Multi-layer Premium Rate

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Abstract

A classical risk model with a multi-layer premium rate is considered in this paper. In the two-layer case, an explicit expression is obtained for the joint distribution of the maximal surplus up to ruin, the surplus immediately before ruin and the deficit at ruin. Such an expression involves some known results on the joint distribution at ruin for a risk model with a constant premium rate. In the multi-layer case, a scheme is proposed to compute its ruin probability.

Keywords: risk model, multi-layer premium rate, joint distribution at ruin.

1. Introduction and preliminaries

The following risk model with a varying premium rate is considered in this paper. Let N_t be a Poisson process with intensity λ. Let $U_j, j = 1, 2, \ldots$, be i.i.d. positive random variables with a common density function f and a finite mean μ. c is a function taking nonnegative values. We write the surplus process R as

$$R_t = u - \sum_{j=1}^{N_t} U_j + \int_0^t c(R_s)ds,$$

where $u \geq 0$ represents the initial surplus, N_t represents the number of claims up to time t, U_j represents the size of the j-th claim, and $c(r)$ represents the rate at which the premium is collected when the current surplus is r. R stands for a model in which the insurance company would adjust its premium rate according to the current surplus level. A remarkable feature of this model is that it is not spatially homogenous, which leads to additional difficulties in its study. In this paper we only concern an n-layer model in which c is a positive step function; i.e.

$$c(r) := c_i \text{ for } v_{i-1} \leq r < v_i,$$

where $0 \equiv v_0 < v_1 < v_2 < \ldots < v_{n-1} < v_n \equiv \infty$. Such a model has been proposed and discussed in Chapter VII of Asmussen (2000) for the case $n = 2$; also see Zhou (2004).

Write

$$R_t^i := u + c_i t - \sum_{j=1}^{N_t} U_j, i = 1, 2, \dots, n.$$

Then R^i is a model with a single premium rate c_i. It is clear that R_t follows the probability law of R_t^i when its value is between v_{i-1} and v_i.

Put

$$\tau := \inf\{t \geq 0 : R_t < 0\}$$

with the convention that $\inf \emptyset := \infty$. Put

$$\psi(u) := \mathbb{P}\{\tau < \infty | R_0 = u\} \quad \text{and} \quad \bar{R} := \sup_{0 \leq t \leq \tau} R_t.$$

τ is the so-called ruin time. $\psi(u)$ is the probability of ever ruin given that the initial surplus is u. $\bar{R}$ is the maximal value of the surplus before ruin ever occurs. Notice that $\psi(u)$ is not necessary one even if the positive safety loading condition

$$c_i > \lambda \mu, \; i = 1, 2, \dots, n,$$

is violated.

Similarly, we can define $\bar{R}^i$, τ_i and ψ_i for model $R^i, i = 1, 2, \dots, n$, accordingly in the obvious way. The Laplace transform for $\psi_i(u)$ is well known. $\psi_i(u)$ can also be expressed using the Pollaczeck-Khinchine formula. See Chapter III of Asmussen (2000) for detailed accounts.

In this paper we are interested in the joint distribution of $(\bar{R}, R_{\tau-}, -R_\tau)$, i.e. the joint distribution of the maximal surplus before ruin, the surplus immediately before ruin and the deficit caused by ruin. The distribution of $(R_{\tau-}, -R_\tau)$ is well understood for the classical risk model with a single premium rate. Earlier work in this respect can be found in Dufresne and Gerber (1988) and in Dickson (1992). Expressions for such a joint distribution are also obtained in Schmidli (1999) for models with either positive safety loading or negative safety loading, and in those models the claim size distribution does not have to be absolutely continuous. We refer to Schmidli (1999) for a summary of related work and references therein. Recent work in this direction can be found in Wu et al. (2003).

For $u, x, y, z \geq 0$, let

$$G(u; x, y, z) := \mathbb{P}\{\tau < \infty, \bar{R} \leq x, R_{\tau-} > y, -R_\tau > z | R_0 = u\}.$$

It will be clear from the proof for Proposition 2.1 that $G(u;x,y,z)$ is differentiable in both y and z. Write

$$g(u;x,y,z) := \frac{\partial^2}{\partial y \partial z} G(u;x,y,z),$$

i.e.

$$\mathbb{P}\{\tau < \infty, \bar{R} \le x, R_{\tau-} \in dy, -R_\tau \in dz | R_0 = u\} = g(u;x,y,z)dydz.$$

In the classical risk model with a single premium rate, the expression for $G(u;\infty,y,z)$ is known. For $i = 1,2,\ldots,n$, write $G_i(u;x,y,z)$ for the corresponding joint distributions for R_t^i; write

$$g_i(u;x,y,z) := \frac{\partial^2}{\partial y \partial z} G_i(u;x,y,z)$$

for the defective density function of $G_i(u;x,y,z)$.

With positive safety loading $c_i > \lambda\mu$, it is obtained in Gerber and Shiu (1997) and Schmidli (1999) that

$$g_i(u;\infty,y,z) = \frac{\lambda}{c_i - \lambda\mu}\left[(1-\psi_i(u))f(y+z) - 1_{\{u>y\}}(1-\psi_i(u-y))f(y+z)\right].$$

An expression for $g_i(u;\infty,y,z)$ under negative safety loading condition can be found in Schmidli (1999).

When there is no restriction on the safety loading, such a result is also available in the fluctuation theory for Lévy processes. Write ρ_i for the unique nonnegative solution to equation

$$c_i t + \lambda(\hat{f}(t) - 1) = 0,$$

where $\hat{f}$ denotes the Laplace transform for f. By Theorem 1 and Corollary 2 in Bertoin (1997), and Theorem VII.1 in Bertoin (1996), we can show that

$$g_i(u;\infty,y,z) = \left[W_i(u)e^{-\rho_i y} - 1_{\{u\ge y\}}W_i(u-y)\right]\lambda f(y+z), \tag{1.1}$$

where W is determined by the following Laplace transform

$$\int_0^\infty e^{-tx}W_i(x)dx = \frac{1}{c_i t + \lambda(\hat{f}(t)-1)}, \quad t > \rho_i. \tag{1.2}$$

Notice that under positive safety loading, we have

$$\rho_i = 0 \quad \text{and} \quad W_i = \frac{1-\psi_i}{c_i - \lambda\mu}.$$

Let

$$T_i(x) := \inf\{t \ge 0 : R_t^i \notin [0,x]\}$$

and

$$p_i(u;x) := \mathbb{P}\{R^i_{T_i(x)} = x | R^i_0 = u\}, 0 \le u \le x.$$

$p_i(u;x)$ is just the probability that, starting from u, the surplus process R^i ever reaches level x before ruin. It is known (eg. Theorem VII. 8 in Bertoin (1996)) that

$$p_i(u;x) = \frac{W_i(u)}{W_i(x)},\ x \ge u. \tag{1.3}$$

Under positive safety loading, (1.3) becomes

$$p_i(u;x) = \frac{1-\psi_i(u)}{1-\psi_i(x)},\ x \ge u.$$

$p_i(u;x)$ will be needed in our study on the risk model with a multi-layer premium rate.

We further point out that for $x > u$ and $x > y$,

$$g_i(u;x,y,z) = g_i(u;\infty,y,z) - p_i(u;x)g_i(x;\infty,y,z). \tag{1.4}$$

To see this, observe that for the event $\bar{R} > x > u$ to occur, the surplus process R has to first move above level x before a possible ruin occurs. Applying the strong Markov property at the time when it first reaches level x we have that

$$\begin{aligned} G_i(u;\infty,y,z) - G_i(u;x,y,z) &= \mathbb{P}\{\bar{R}_1 > x, R_{\tau_i -} > y, -R_{\tau_i} > z\} \\ &= p_i(u;x)G_i(x;\infty,y,z). \end{aligned} \tag{1.5}$$

Then (1.4) follows by taking derivatives on both sides of (1.5). $g_i(u;x,y,z)$ will also be needed to reach an expression for $g(u;x,y,z)$.

One of the main results in this paper concerns a model with a two-layer premium rate. In Section 2 we recover an explicit expression for g in terms of ψ_i and $g_i, i = 1,2$. It generalizes Proposition 7.1.10 in Asmussen (2000). To our best knowledge such a result is new. Observe that R is not even a Lévy process. One would not expect such an expression to be as neat as the one for the classical model.

Since the surplus process R only allows negative jumps, it has to reach level v before it ever upcrosses v. This fact plays a key role in our proofs. By applying the strong Markov property at the time when R first downcrosses or upcrosses level v, and taking use of some known results for models with a constant premium rate.

Our approach can be adapted to study a model with an n-layer premium rate. We discuss the ruin probability for such a model in Section 3. We also obtain some numerical results for a model with exponential claims. If the premium rate changes continuously according to the current surplus, the problem remains open except for certain special examples; see e.g. Paulsen and Gjessing (1997).

Positive safety loading is not assumed in Section 2 and Section 3.

2. Joint distribution for a model with a two-layer premium rate

In this section we only consider a model with a two-layer premium rate. In this relatively simple case we could work out an explicit expression for $g(u;x,y,z)$. Expression for G follows by taking integrations.

In the next proposition we first find a formula for $g(v_1;x,y,z)$. Then using $g(v_1;x,y,z)$ we can obtain a general expression for $g(u;x,y,z)$. Eventually, g can be written in terms of W_1, W_2 and f.

Proposition 2.1. *For the risk model R with a two-layer premium rate, given $u,x,y,z>0$, we have that*

$$g(v_1;x,y,z) = \frac{g_2(0;x-v_1,y-v_1,v_1+z)}{1-I_2(x,0)} 1_{\{v_1 \le y < x\}} + \frac{I_1(x,y,0)}{1-I_2(x,0)} 1_{\{y<v_1<x\}} \tag{2.1}$$

and

$$\begin{aligned} &g(u;x,y,z) \\ &= g_1(u;x,y,z)1_{\{u<v_1, y<x\le v_1\}} + p_1(u;v_1)g(v_1;x,y,z)1_{\{u<v_1\le y<x\}} \\ &\quad + (g_1(u;v_1,y,z) + p_1(u;v_1)g(v_1;x,y,z))1_{\{u<v_1\le x, y<v_1\}} \\ &\quad + (g_2(u-v_1;x-v_1,y-v_1,v_1+z) + g(v_1;x,y,z)I_2(x,u-v_1))\, 1_{\{v_1<u\le x, v_1\le y<x\}} \\ &\quad + (I_1(x,y,u-v_1) + g(v_1;x,y,z)I_2(x,u-v_1))\, 1_{\{y\le v_1<u\le x\}}, \end{aligned} \tag{2.2}$$

where

$$I_1(x,y,r) := \int_0^{v_1} g_1(v_1-w;v_1,y,z)dw \int_0^{x-v_1} g_2(r;x-v_1,y',w)dy'$$

and

$$I_2(x,r) := \int_0^{v_1} p_1(v_1-w;v_1)dw \int_0^{x-v_1} g_2(r;x-v_1,y',w)dy'.$$

Proof. We are going to show (2.1) first. Given $R_0 = v_1$ and $y \ge v_1$, in order for the event $\{\tau<\infty, \bar{R} \le x, R_{\tau-} \in dy, -R_\tau \in dz\}$ to occur, the process R has to follow the law of R^2 until it jumps downwards across level v_1. If this jump is large enough it causes ruin. Otherwise, R first jumps to somewhere between 0 and v_1, and then starting from there it follows the law of R^1 and comes back to level v_1 before ruin, and then starts all over again from v_1. During this period the maximum of R keeps below x. As a result of the strong Markov property, it yields that

$$
\begin{aligned}
&g(v_1; x, y, z) \\
&= g_2(0; x - v_1, y - v_1, v_1 + z) \\
&\quad + g(v_1; x, y, z) \int_0^{v_1} p_1(v_1 - w; v_1) dw \int_0^{x - v_1} g_2(0; x - v_1, y', w) dy'.
\end{aligned}
\tag{2.3}
$$

Solve it for g we will reach the first part of (2.1).

If $y < v_1$, given that $R_0 = v_1$, to make the event $\{\tau < \infty, \bar{R} \le x, R_{\tau-} \in dy, -R_\tau \in dz\}$ happen process R has to first jump downwards across lever v_1 to somewhere between 0 and v_1. Starting from between 0 and v_1, either a ruin occurs before R goes back to level v_1 or R reaches level v_1 before ruin. Then we obtain another equation

$$
\begin{aligned}
&g(v_1; x, y, z) \\
&= \int_0^{v_1} g_1(v_1 - w; v_1, y, z) dw \int_0^{x - v_1} g_2(0; x - v_1, y', w) dy' \\
&\quad + g(v_1; x, y, z) \int_0^{v_1} p_1(v_1 - w; v_1) dw \int_0^{x - v_1} g_2(0; x - v_1, y', w) dy'.
\end{aligned}
\tag{2.4}
$$

We thus obtain the second part of (2.1) by solving the above equation.

As to (2.2), if $v_1 < u$ and $v_1 \le y$, for the ruin to occur and the event $\{R_{\tau-} \in dy\}$ to happen as well the process R has to first jump downwards across level v_1. If it overshoots then ruin occurs. Otherwise, R first jumps to between 0 and v_1 and then from there it comes back to level v_1 without ruin; i.e.

$$
\begin{aligned}
&g(u; x, y, z) \\
&= g_2(u - v_1; x - v_1, y - v_1, v_1 + z) \\
&\quad + g(v_1; x, y, z) \int_0^{v_1} p_1(v_1 - w; v_1) dw \int_0^{x - v_1} g_2(u - v_1; x - v_1, y', w) dy'.
\end{aligned}
\tag{2.5}
$$

Similarly, for $y \le v_1 < u$, R has to downcross level v_1 to between 0 and v_1 before ruin. Then either ruin occurs before R comes back to level v_1 or R comes back to level v_1 before ruin, i.e.,

$$
\begin{aligned}
&g(u;x,y,z) \\
&= \int_0^{v_1} g_1(v_1-w;v_1,y,z)dw \int_0^{x-v_1} g_2(u-v_1;x-v_1,y',w)dy' \\
&\quad + g(v_1;x,y,z)\int_0^{v_1} p_1(v_1-w;v_1)dw \int_0^{x-v_1} g_2(u-v_1;x-v_1,y',w)dy'.
\end{aligned}
\tag{2.6}
$$

Finally, consider the case for $u < v_1$. If $y < x \leq v_1$, then R can never reach level v_1, and it has to follow the law of R^1 until ruin. If $v_1 \leq y < x$, then R first reaches level v_1 before ruin. If $y \leq v_1 < x$, then either ruin occurs before R ever reaches level v_1, or R reaches level v_1 before ruin.

Combining all the cases, (2.2) follows. □

3. The n-layer case

For $u, z > 0$ and $i = 1, 2, \ldots, n$, put

$$g_i(u;z) := \int_0^\infty g_i(u;\infty,y,z)dy,$$

i.e.

$$g_i(u;z)dz = \mathbb{P}\{\tau_i < \infty, -R_{\tau_i} \in dz\}.$$

Now we consider the risk model with an n-layer premium rate. By an argument similar to the proof of Proposition 2.1, we can show that,

- for $0 \leq u < v_1$, we have

$$\psi(u) = \frac{W_1(u)}{W_1(v_1)}\psi(v_1) + \int_0^\infty g_1(u;z)dz - \frac{W_1(u)}{W_1(v_1)}\int_0^\infty g_1(v_1;z)dz; \tag{3.1}$$

- for $v_k \leq u < v_{k+1}$ and $1 \leq k \leq n-2$, we have

$$
\begin{aligned}
\psi(u) =& \frac{W_{k+1}(u-v_k)}{W_{k+1}(v_{k+1}-v_k)}\psi(v_{k+1}) + \int_0^{v_k} g_{k+1}(u-v_k;v_k-z)\psi(z)dz \\
&- \frac{W_{k+1}(u-v_k)}{W_{k+1}(v_{k+1}-v_k)}\int_0^{v_k} g_{k+1}(v_{k+1}-v_k;v_k-z)\psi(z)dz \\
&+ \int_{v_k}^\infty g_{k+1}(u-v_k;z)dz - \frac{W_{k+1}(u-v_k)}{W_{k+1}(v_{k+1}-v_k)}\int_{v_k}^\infty g_{k+1}(v_{k+1}-v_k;z)dz;
\end{aligned}
\tag{3.2}
$$

- for $u \geq v_{n-1}$, we have

$$\psi(u) = \int_0^{v_{n-1}} g_n(u-v_{n-1};v_{n-1}-z)\psi(z)dz + \int_{v_{n-1}}^\infty g_n(u-v_{n-1};z)dz. \tag{3.3}$$

Given $\{g_i\}$ and $\{W_i\}$, we can solve this system of equations explicitly by the following scheme. First, by letting $u = 0$ in (3.1), we can write $\psi(v_1)$ as a linear transformation of $\psi(v_0)$. Then by (3.1) again we can write $\psi(u), 0 \le u < v_1$, linearly in terms of $\psi(v_0)$. Now let $u = v_1$ in (3.2). We see that $\psi(v_2)$ can also be written as a linear transform for $\psi(v_0)$. Repeating the previous procedure we can eventually express $\psi(u)$ linearly in terms of $\psi(v_0)$ for all $0 \le u \le v_{n-1}$. Then applying (3.3) for $u = v_{n-1}$, we can obtain a linear equation on $\psi(v_0)$ and solve it. At the end, we are able to find an expression of $\psi(u)$ for all $0 \le u < \infty$.

An analytical expression for $\psi(u)$ in terms of $\{g_i\}$ and $\{W_i\}$ appears to be quite complex even for the case that $n = 3$. But a numerical solution is certainly possible. At the end of this paper we consider a risk model with an exponential claim size and with $n = 4$.

Suppose that U_i follows an exponential distribution with mean $1/\beta$. Then

$$f(x) = \beta \exp\{-\beta x\} \quad \text{and} \quad \hat{f}(t) = \beta/(\beta + t).$$

By inverting the Laplace transform (1.2) we obtain

$$W_i(x) = \frac{1}{c_i\beta - \lambda}\left(\beta - \frac{\lambda}{c_i}\exp\left\{\frac{(\lambda - c_i\beta)x}{c_i}\right\}\right).$$

It follows from (1.1) that, for $c_i\beta > \lambda$,

$$\rho_i = 0 \quad \text{and} \quad g_i(u; z) = \frac{\lambda}{c_i}\exp\left\{\frac{(\lambda - c_i\beta)u}{c_i} - \beta z\right\};$$

for $c_i\beta < \lambda$,

$$\rho_i = \frac{\lambda - c\beta}{c} \quad \text{and} \quad g_i(u; z) = \beta \exp\{-\beta z\}.$$

To implement the above-mentioned numerical procedure, we consider two cases. In the first case, let $\beta = 1,\ \lambda = 1,\ v_1 = 5, v_2 = 10, v_3 = 15,\ c_1 = 1.4, c_2 = 1.3, c_3 = 1.2, c_4 = 1.1$ and $u = 2$. We can obtain numerical values of the ruin probability $\psi(u)$ for different u values. The curve in the bottom in Figure 1 is a plot of $\psi(u)$ in this case.

To see what happens when the positive loading condition does not hold, we further consider the second case in which $\beta = 1,\ \lambda = 1,\ v_1 = 5, v_2 = 10, v_3 = 15,\ c_1 = 1.4, c_2 = 0.9, c_3 = 1.2, c_4 = 1.1$ and $u = 2$. Then the plot of $\psi(u)$ is given by the curve at the top in Figure 1. The abnormality of the part of the curve corresponding to $5 < u < 10$ is clearly due the lack of positive loading.

Some numerical values for $\psi(u)$ in the two cases are recorded in the following table.

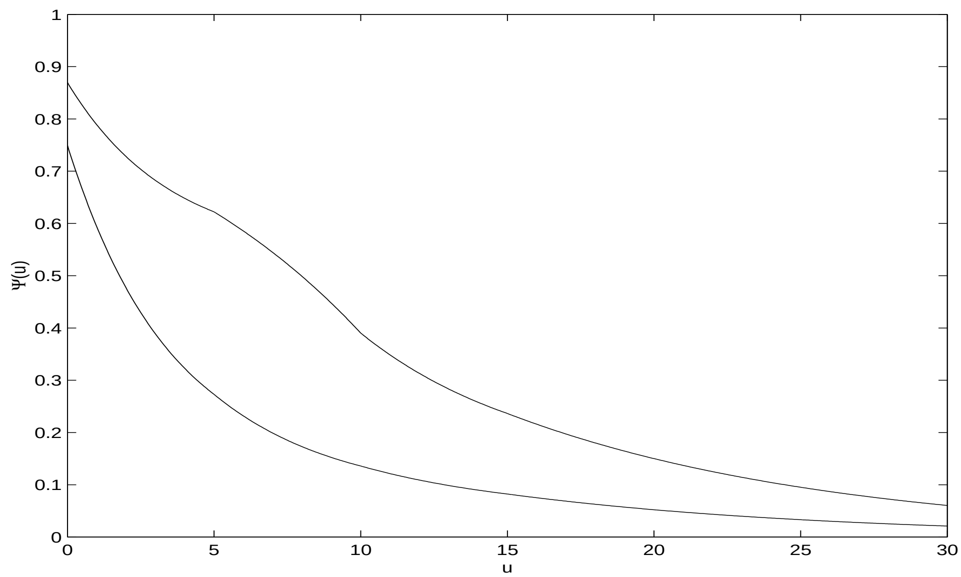

Figure 1. Ruin probabilities

u	0	5	10	15	20	30
First Case	0.7494	0.2730	0.1359	0.0823	0.0523	0.0211
Second Case	0.8697	0.6222	0.3903	0.2364	0.1501	0.0605

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