# Loaded Participation Rates for Equity-Indexed Annuities

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**Abstract**

In this paper, we will introduce premium principles for equity-indexed annuities (EIAs). We first obtain the fair participation rate based on a fair value of the equity-linked contract. The hedging errors are extracted from the dynamic hedging strategy. Using risk measures, we then obtain a new participation rate based on the tail loading of the hedging error distribution. Risk management strategies reducing risk related to the hedging error are also presented. A detailed numerical analysis is performed for a point-to-point EIA.

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# 1 Introduction

An equity-indexed annuity (EIA) is an insurance product with benefits linked to the performance of an equity market. It provides a limited participation in the performance of an equity index (e.g. S&P 500) while guaranteeing a minimum rate of return. Introduced by Keyport Life Insurance Co in 1995, EIAs have been the most innovative annuity product over the last 10 years. They have become increasingly popular since their debut and the sales of EIAs have broken the $20 billion barrier ($23.1 billion) in 2004 and reached $27.3 billion in 2005. See the 2006 Annuity Fact Book from National Association of Variable Annuities (NAVA).

There are two common approaches to deal with equity-linked products: financial and actuarial approaches. In the latter, it is generally assumed that insurance companies can diversify the mortality risk. Working with this assumption and using the classical Black-Scholes framework, Tiong (2000) and Lee (2002, 2003) use the Esscher transform method developed in Gerber and Shiu (1994) to obtain closed-form formulas for several equity-indexed annuities. Lin and Tan (2003) consider a more general model for equity-indexed annuities, in which the external equity index and the interest rate are general stochastic differential equations. In a discrete time setting, Gaillardetz and Lin (2006) propose loaded participation rates based on implied loaded mortality probabilities using standard life insurance information. The actuarial approach has also been used to evaluate equity-linked products. This method relies heavily on the choice of risk measures (see Artzner et al., 1997, 1999, and Wirch and Hardy, 1999). The financial and actuarial approaches have been compared by Boyle and Hardy (1997). Attempts have been made by Jacques (2003) and Barbarin and Devolder (2005) to combine the two approaches in the management of equity-linked products. Our goal is to integrate these two methodologies in order to protect the equity-linked issuers against the possible losses incurred by a fair valuation. The approach in this paper differs from the above by obtaining a loaded participation rate that is determined using the financial and actuarial approaches.

The traditional insurance and annuity pricing method calculates the net premium of a product as the expected present value of its benefits with respect to a mortality law. In order to protect the insurance company against the mortality risk, the premium is determined as the net premium plus a loading that is based on certain premium principles (see Bowers et al., 1997). However, the traditional actuarial pricing is difficult to extend directly to the valuation of equity-linked products since these products are embedded with various types of financial guarantees. However, in this paper, we propose a premium principle for equity-indexed annuities that protects the issuer against the mortality risk. Our approach derives the fair participation rate based on a fair valuation of the equity-linked contract using the financial approach. The dynamic hedging strategy underlying the fair valuation principle is then extracted. Because of mortality risk, the application of the dynamic hedging leads to some errors in the hedging strategy. Using risk measures, we then obtain a new participation rate based on the tail loading of the hedging error distribution. Risk management strategies reducing risk relating to the hedging errors are also presented.

This paper is organized as follows. The next section presents a binomial model for the equity index and introduces actuarial notation. We then present fair valuation approaches for standard financial contingent claims and equity-linked products. Section 4 focuses on the underlying hedging strategy used in the fair valuation of equity-linked products and the errors caused by this strategy. Section 5 proposes premium principles as well as risk management strategies that reduce the hedging error risk. Each of these sections is followed by numerical examples presenting the implications of the different proposed approaches on EIAs.

## 2 Underlying Binomial Model and Actuarial Notations

Binomial models have widely been used to model stocks, stock indices, interest rates and other financial securities due to their flexibility and tractability. See Panjer et al. (1998) for example. In this section, we employ a modified CRR binomial model (Cox, Ross and Rubinstein, 1979) for a stock index. We then introduce the standard actuarial notation for mortality probabilities.

Let $\delta$ be the force of interest, i.e. $\delta$ is a nominal annual rate of interest compounded continuously. It is assumed that $\delta$ is constant. For each year, assume that there are $N$ trading periods, each with the length of $\Delta = 1/N$. The (stock) index process is denoted as $S(t), t = 0, \Delta, 2\Delta, \cdots$, where $S(0)$ is the initial level of the index. At time $t$, the index process can take exactly $tN+1$ distinct values denoted $S(t,0), S(t,1), \cdots, S(t,tN)$. Indeed, $S(t,i)$ represents the index level at time $t$ that has made "$i$" up moves. For the time period $[t, t+\Delta]$, $t = 0, \Delta, 2\Delta, \cdots$, the index $S(t,i)$ has two possible outcomes : $S(t+\Delta, i) = S(t,i)d$ and $S(t+\Delta, i+1) = S(t,i)u$ with $d < u$. Hence, the index process can move up from $S(t)$ to $S(t)u$, or down to $S(t)d$. Because of the constant assumption of the interest rates, the time-$t$ value $B(t), B(0) = 1$, of the money-market account is given by

$$B(t) = e^{\delta t}, \tag{2.1}$$

for $t = 0, \Delta, 2\Delta, \cdots$.

For the index process $S(t)$, since the interest rates remain constant each year, it is natural to assume that the values of $d$ and $u$ are constant, which will lead to a recombining binomial model. The values of $d$ and $u$ will be obtained using the volatility structure of the index process and they will be specified later. Without loss of generality, let us also assume that the time-0 index value is one unity.

Let $\pi$ and $\widetilde{\pi}$ be the probability that the index value goes up during the period $[t, t+\Delta]$ under the physical probability measure $P$ and the martingale probability measure $Q$

respectively, i.e.

$$\Pr[S(t+\Delta) = S(t,i)u | S(t) = S(t,i)] = \pi, \text{ and } Q[S(t+\Delta) = S(t,i)u | S(t) = S(t,i)] = \widetilde{\pi}, \tag{2.2}$$

for $t = 0, \Delta, 2\Delta, \cdots$.

That the discounted value process $\{S(t)/B(t)\}$ is a martingale implies

$$\pi = \frac{e^{\delta\Delta} - d}{u - d}. \tag{2.3}$$

The no-arbitrage condition thus requires

$$d < e^{\delta\Delta} < u. \tag{2.4}$$

The model assumes the usual frictionless market: no tax, no transaction costs, etc. Gaillardetz and Lakhmiri (2006) determine the price of EIAs under transaction cost. The filtration associated with the index process is the one generated by the process.

We next introduce the standard actuarial notation, as described in Bowers et al. (1997). Let $T(x)$ be the future lifetime of insured $(x)$ at time $t = 0$ and the modified curtate-future-lifetime

$$K(x) = \lfloor N \ T(x) \rfloor \, \Delta, \tag{2.5}$$

the fraction of future $\Delta$ years completed by the insured $(x)$ prior to death. Here, $\lfloor . \rfloor$ is the floor function.

Let ${}_{t|\Delta}q_x$ denote the probability that $(x)$ survives $t$ years and dies within the following $\Delta$ year, i.e.

$${}_{t|\Delta}q_x = \Pr\left[t \le T(x) < t + \Delta\right] = \Pr\left[K(x) = t\right] = Q\left[K(x) = t\right].$$

Here, the martingale and the physical probability measures are assumed to be the same for the mortality. Gaillardetz and Lin (2006) obtained different martingale probability measures for the mortality under certain conditions. Define the probability that $(x)$ survives to $x + t$ by

$${}_tp_x = \Pr\left[T(x) \ge t\right] = Q\left[T(x) \ge t\right],$$

which is also assumed to be the same under both probability measures.

# 3 Fair Valuations

## 3.1 Financial Contingent Claims

In this section, we evaluate financial contingent claims using the arbitrage-free theory. The self-financing hedging strategy is also presented for the binomial model.

Let $\Pi(t, n)$ denote the time-$t$ price of a contingent claim $D(n)$, payable at time $n$ ($n = 0, \Delta, \cdots$ and $t \leq n$). From Harrison and Pliska (1981), the arbitrage-free price of contingent claims using the martingale probability measure is given by

$$\Pi(t, n) = \widetilde{E}\left[\frac{D(n)}{B(n)} B(t) \,|F(t)\right], \tag{3.6}$$

where $F(t)$ is the filtration and $\widetilde{E}[.]$ represents expectation with respect to $Q$.

For notational purposes, let

$$\mathbf{i}_t = \{i_0, i_\Delta, i_{2\Delta}, \cdots, i_t\},$$

which represents the index's realization up to time $t$ and where $i_t \in \{0, 1, \cdots, tN\}$ is the number of up moves up to time $t$ with $i_0 = 0$. Sometimes, we also use $D(t, \mathbf{i}_t)$ or $\Pi(t, n, \mathbf{i}_t)$ to specify the index realization.

Harrison and Pliska (1981) also state that there is a one-to-one relation between a self-financing strategy and a martingale (risk-neutral) probability measure $Q$ defined by (2.3). For $t \leq n$, let $\{a(t, n, \mathbf{i}_t), b(t, n, \mathbf{i}_t)\}$ be a portfolio strategy consisting of $a(t, n, \mathbf{i}_t)$ index shares and an amount $b(t, n, \mathbf{i}_t)$ invested in the money market account. Since the binomial market is complete, the contingent claim $D(n)$ is attainable using a self-financing strategy, i.e.

$$D(n, \mathbf{i}_n) = a(n, n, \mathbf{i}_n) S(n, i_n) + b(n, n, \mathbf{i}_n) = V(n, n, \mathbf{i}_n), \tag{3.7}$$

where $V(t, n, \mathbf{i}_t)$ represents the time-$t$ value process of the replicating portfolio for a contingent claim maturing at $n$, that is

$$V(t, n, \mathbf{i}_t) = a(t, n, \mathbf{i}_t) S(t, i_t) + b(t, n, \mathbf{i}_t). \tag{3.8}$$

Moreover, let $V((t+\Delta)^-, n, \mathbf{i}_{t+\Delta})$ denote the hedge portfolio from $t$ that has accumulated to $t + \Delta$

$$V((t+\Delta)^-, n, \mathbf{i}_{t+\Delta}) = a(t, n, \mathbf{i}_t) S(t+\Delta, i_{t+\Delta}) + b(t, n, \mathbf{i}_t) e^{\delta\Delta}, \tag{3.9}$$

for $t = 0, \Delta, 2\Delta \cdots, n - \Delta$.

The hedging strategy is self-financing, which means that it does not require new investments and no withdrawal from the portfolio during the life of the portfolio, i.e.

$$V((t+\Delta)^-, n, \{\mathbf{i}_t, i_t + j\}) = V((t+\Delta), n, \{\mathbf{i}_t, i_t + j\}), \tag{3.10}$$

for $j = 0, 1$. From (3.8) and (3.9), it follows

$$\begin{aligned} & a(t, n, \mathbf{i}_t) S(t+\Delta, i_t + 1) + b(t, n, \mathbf{i}_t) e^{\delta\Delta} \\ &= a(t+\Delta, n, \{\mathbf{i}_t, i_t + 1\}) S(t+\Delta, i_t + 1) + b(t+\Delta, n, \{\mathbf{i}_t, i_t + 1\}) \\ &= V(t+\Delta, n, \{\mathbf{i}_t, i_t + 1\}), \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} & a(t,n,\mathbf{i}_t)S(t+\Delta,i_t)+b(t,n,\mathbf{i}_t)e^{\delta\Delta} \\ = \;& a(t+\Delta,n,\{\mathbf{i}_t,i_t\})S(t+\Delta,i_t)+b(t+\Delta,n,\{\mathbf{i}_t,i_t\}) \\ = \;& V(t+\Delta,n,\{\mathbf{i}_t,i_t\}). \end{aligned} \tag{3.12}$$

for $t = 0, \Delta, \cdots, n-\Delta$. The solutions of (3.11) and (3.12) are given by

$$a(t,n,\mathbf{i}_t) = \frac{V(t+\Delta,n,\{\mathbf{i}_t,i_t+1\}) - V(t+\Delta,n,\{\mathbf{i}_t,i_t\})}{S(t+\Delta,i_t+1)-S(t+\Delta,i_t)}, \tag{3.13}$$

and

$$b(t,n,\mathbf{i}_t) = \frac{uV(t+\Delta,n,\{\mathbf{i}_t,i_t\}) - dV(t+\Delta,n,\{\mathbf{i}_t,i_t+1\})}{u-d}e^{-\delta\Delta}. \tag{3.14}$$

Under the arbitrage-free condition, the value of the self-financing replicating portfolio must be equal to the price of the contingent claim. Then, for all $t$ and $\mathbf{i}_t$, we must have

$$V(t,n,\mathbf{i}_t) = \Pi(t,n,\mathbf{i}_t), \tag{3.15}$$

which is equivalent to (2.4) for the binomial model. A comprehensive introduction to the binomial model may be found in Shreve (2005) or van der Hoek and Elliott (2005).

## 3.2 Equity-linked Products

Due to their unique designs, equity-linked products involve mortality and financial risk since these contracts provide both death and accumulation/survival benefits. Moreover, the level of their benefits are linked to the financial market performance and an equity index in particular. Consider now an equity-linked product that pays

$$\begin{cases} D(K(x)+\Delta), & \text{if } K(x) = 0, \Delta, \cdots, n-2\Delta, \\ D(n), & \text{if } K(x) = n-\Delta, n, \cdots. \end{cases} \tag{3.16}$$

Note that, in practice, the final payoff $D(.)$ might not be the same function as that of the death benefits, but for simplicity we assume it to be the same.

Let $FV(x,t,n,\mathbf{i}_t)$ denote the fair value at time $t$ ($t = 0, \Delta, \cdots, n$) of the equity-linked contract given that $(x)$ is still alive and the index process has taken the path $\mathbf{i}_t$. Given that $K(x) \geq t$, the fair value may be obtained using the expected discounted payoff of the the equity-linked contract

$$\begin{aligned} FV(x,t,n,\mathbf{i}_t) \;=\; & \widetilde{E}\Bigg[D(K(x)+\Delta)I_{\{K(x)<n-\Delta\}}\frac{B(t)}{B(K(x)+\Delta)} \\ & +D(n)I_{\{K(x)\geq n-\Delta\}}\frac{B(t)}{B(n)}|\mathbf{i}_t, K(x)\geq t\Bigg], \end{aligned} \tag{3.17}$$

for $t = 0, \Delta, \cdots, n$.

It is also natural to assume independence between the policyholder and the financial market under the martingale measure. The fair value given in (3.17) becomes

$$\begin{aligned} FV(x,t,n,\mathbf{i}_t) &= \sum_{l\in A_{n-t-2\Delta}} \widetilde{E}\left[D(t+l+\Delta)\frac{B(t)}{B(t+l+\Delta)}|\mathbf{i}_t\right]\ {}_{l|\Delta}q_{x+t} \\ &\quad +\widetilde{E}\left[D(n)\frac{B(t)}{B(n)}|\mathbf{i}_t\right]\ {}_{n-\Delta}p_{x+t}. \end{aligned} \tag{3.18}$$

where $A_t = \{0, \Delta, \cdots, t\}$. It follows from (3.6)

$$FV(x,t,n,\mathbf{i}_t) = \sum_{l\in A_{n-t-2\Delta}} \Pi(t, t+l+\Delta, \mathbf{i}_t)\ {}_{l|\Delta}q_{x+t} + \Pi(t,n,\mathbf{i}_t)\ {}_{n-\Delta}p_{x+t}. \tag{3.19}$$

The fair value of the equity-linked product can be expressed as a weighted sum of the financial contingent claim prices of diverse maturities. The weight of each contingent claim is determined by the probability that the policyholder dies in the given period or survives to $n - \Delta$.

## 3.3 Equity-indexed Annuities

In this section, we determine the fair value of Equity-Indexed Annuity contracts based on the approach presented previously. EIAs appeal to investors because they offer the same protection as conventional annuities by limiting the financial risk, but also provide participation in the equity market. From Lin and Tan (2003) and Tiong (2000), EIA designs may generally be grouped in two broad classes: Annual Reset and Point-to-Point. The index growth on an EIA with the former is measured and locked in each year. Particularly, the index growth with a term-end point design is calculated using the index value at the beginning and at the end of each year. On the other hand, the index growth with point-to-point indexing is based on the growth between two time points over the entire term of the annuity. Particularly, the index growth with a term-end feature is calculated using the terminal index value. The cost of the EIA contract is reflected through the participation rate. Hence, the participation rate is expected to be lower for expensive designs.

To illustrate the fair valuation, we consider one of the simplest design of EIAs, known as the point-to-point with term-end design. The payoff at time $t$ can be represented by

$$D_\alpha(t) = \max\left[\min\left[1+\alpha R(t), (1+\zeta)^t\right], \beta(1+g)^t\right], \tag{3.20}$$

for $t = 0, \Delta, \cdots, n$, where $\alpha$ represents the participation rate and the "gain" $R(t)$ is defined by

$$R(t) = \frac{S(t)}{S(0)} - 1. \tag{3.21}$$

It also provides a protection against the loss from a down market $\beta(1+g)^t$. The cap rate $(1+\zeta)^t$ reduces the cost of such a contract since it imposes an upper bound on the maximum return. Here, we suppose for simplicity that there is no cap ($\zeta = \infty$).

As explained in Lin and Tan (2003), an EIA is evaluated through its participation rate $\alpha$. Without loss of generality, we suppose that the initial value of EIA contracts is one monetary unit. The present value of the EIA is a function of the participation rate through the payoff function $D_\alpha(t)$, $t = \Delta, 2\Delta, \cdots, n$. By holding all other parameters constant, the fair participation rate $\alpha_1$ is characterized by $FV_{\alpha_1}(x, 0, n, \mathbf{i}_0) = 1$. Here, the subscript $\alpha_1$ is added to point out that the EIA contract is evaluated through the participation rate. Through numerical methods, we may then solve for $\alpha_1$ using (3.19).

## 3.4 Equity-indexed Annuities: Example

Our example involves a five-year EIA issued to a male aged 55 with minimum interest rate guarantee of 3% on 90% of the premium. The mortality of the policyholder is assumed to follow the 1979 − 1981 U.S. Life Table (see Bowers et al., 1997, Table 3.3.1). Moreover, we assume that the insurer $(x)$ may die only at the end of each year. The force interest $\delta$ is set to be constant over time and is equal to 6%. The index is governed by the CRR model introduced previously with $S(0) = 1$ and where the number of trading dates $N$ is 6. The index's volatility is assumed to be constant and is 25%. In other words, $u = e^{\sigma/\sqrt{N}} = 1.107$ ($\sigma = 0.25$) and $d = u^{-1} = 0.903$. Using (2.3) with $N = 6$, the index martingale probability of going up is $\widetilde{\pi} = 0.524$.

The fair participation rate $\alpha_1$ for the point-to-point EIA class with term-end point is equal to 69.31%.

# 4 Dynamic Hedging and Errors

This section presents the dynamic hedging underlying the fair valuation of equity-linked products presented previously and determines the errors caused by the hedging strategy. Indeed, the fair value given by (3.19) does not represent the usual risk-neutral price since the combined insurance and financial markets, even under independence assumption, leads to an incomplete market. This is due to the fact that insurance products cannot be treated as standard financial assets, since they are neither liquid nor accessible to each investor. However, Equation (3.19) is based on the possibility of diversifying mortality risk, which is fundamental in insurance.

From (3.19) and (3.15), the fair value of the equity-linked contract could be written as

$$FV(x, t, n, \mathbf{i}_t) = \sum_{l \in A_{n-t-2\Delta}} V(t, t+l+\Delta, \mathbf{i}_t)\ {}_{l|\Delta}q_{x+t} + V(t, n, \mathbf{i}_t)\ {}_{n-\Delta}p_{x+t}. \quad (4.22)$$

for $t = 0, \Delta, \cdots, n$, where $A_t = \{0, \Delta, \cdots, t\}$. Using (3.7) and (4.22), the dynamic hedging strategy implied by the fair valuation is given by

$$\begin{aligned} FV(x,t,n,\mathbf{i}_t) &= \sum_{l \in A_{n-t-2\Delta}} \left[a(t,t+l+\Delta,\mathbf{i}_t)S(t,i_t) + b(t,t+l+1,\mathbf{i}_t)\right] \ {}_{l|\Delta}q_{x+t} \\ &\quad + \left[a(t,n,\mathbf{i}_t)S(t,i_t) + b(t,n,\mathbf{i}_t)\right] \ {}_{n-\Delta}p_{x+t} \\ &= a^*(x,t,n,\mathbf{i}_t)S(t,i_t) + b^*(x,t,n,\mathbf{i}_t) \\ &= V^*(x,t,n,\mathbf{i}_t), \end{aligned} \tag{4.23}$$

where

$$a^*(x,t,n,\mathbf{i}_t) = \sum_{l \in A_{n-t-2\Delta}} a(t,t+l+\Delta,\mathbf{i}_t) \ {}_{l|\Delta}q_{x+t} + a(t,n,\mathbf{i}_t) \ {}_{n-\Delta}p_{x+t}, \tag{4.24}$$

$$b^*(x,t,n,\mathbf{i}_t) = \sum_{l \in A_{n-t-2\Delta}} b(t,t+l+\Delta,\mathbf{i}_t) \ {}_{l|\Delta}q_{x+t} + b(t,n,\mathbf{i}_t) \ {}_{n-\Delta}p_{x+t}, \tag{4.25}$$

and $V^*(x,t,n,\mathbf{i}_t)$ represents the time-$t$ value process of the replicating portfolio for an equity-linked product maturing at $n$. Hence, the fair value of equity-linked contracts may be expressed as a weighted sum of different replication portfolios. It can be decomposed into $a^*$ index shares and $b^*$ invested in the money market account. Let $V^*(x,(t+\Delta)^-,n,\mathbf{i}_t)$ be the hedge portfolio from $t$ that has accumulated to $t+\Delta$, which is explicitly given by (3.9) with superscript $*$ for $a$, $b$, and $V$.

The hedging strategy underlying the dynamic hedging is not self-financing, which means that the issuer will have to invest or withdraw from the account in order to keep the hedging strategy solvent. The error term is used to refer to the addition or subtraction of money from the replicating portfolio. There are two causes of errors: survival and death. The latter occurs when, in case of death, the accumulated hedging portfolio is not equal to the payoff of the equity-linked contract, while the former occurs when readjustment of the hedging portfolio is needed in case of survival. The error present value is defined by

$$\begin{aligned} &ERROR = \\ &\quad \sum_{l \in A_{K(x)-\Delta}} \left[(FV(x,l+\Delta,n,\mathbf{i}_{l+\Delta}) - V^*(x,(l+\Delta)^-,n,\mathbf{i}_{l+\Delta}))e^{-\delta(l+\Delta)}\right] \\ &\quad + \left[D(K(x)+\Delta,\mathbf{i}_{K(x)+\Delta}) - V^*(x,(K(x)+\Delta)^-,n,\mathbf{i}_{K(x)+\Delta})\right] e^{-\delta(K(x)+\Delta)}, \end{aligned} \tag{4.26}$$

if $K(x) = 0, \Delta, 2\Delta, \cdots, n-2\Delta$, with the physical probability given by

$$\Pr\left[K(x) = t, \mathbf{i}_{t+\Delta}\right] = \ {}_{t|\Delta}q_x \ \pi^{i_{t+\Delta}}(1-\pi)^{(t+\Delta)N - i_{t+\Delta}}. \tag{4.27}$$

If $K(x) = n - \Delta, n, \cdots$, we have

$$ERROR = \sum_{l \in A_{n-2\Delta}} (FV(x, l+\Delta, n, \mathbf{i}_{l+\Delta}) - V^*(x, (l+\Delta)^-, n, \mathbf{i}_{l+\Delta}))e^{-\delta(l+\Delta)}, \quad (4.28)$$

with the physical probability given by

$$\Pr[K(x) \geq n - \Delta, \mathbf{i}_{n-\Delta}] = {}_{n-\Delta}p_x \ \pi^{i_{n-\Delta}}(1-\pi)^{(n-\Delta)N - i_{n-\Delta}}. \quad (4.29)$$

Note that there is no error after time $n - \Delta$, since at this time the product becomes a financial derivative. Given $K(x) \geq n - \Delta$, the policyholder will receive $D(n)$ at time $n$ in either cases, and the underlying contingent claim can then be perfectly hedged using a self-financing strategy. Bear in mind that negative errors represent gains for issuers and, conversely, positive errors represent losses for the insurance company.

## 4.1 Dynamic Hedging and Errors: Example

In this numerical illustration, we consider the same set of parameters as in the previous example. It still involves a five-year point-to-point EIA issued to a male aged 55 with minimum interest rate guarantee of 3% on 90% of the premium. The mortality of the policyholder is assumed to follow the $1979 - 1981$ U.S. Life Table (see Bowers et al., 1997, Table 3.3.1) with death occurring only at the end of each year. The index is governed by the CRR model with $\sigma = 25\%$, $\delta = 6\%$, and $N = 6$. The index physical probability of going up is set such that

$$\pi = \frac{e^{\mu\Delta} - d}{u - d}, \quad (4.30)$$

with $\mu$[1]$= 15\%$, which implies that $\pi = 0.598$.

Figure 1 presents the histogram of the hedging error distribution for the point-to-point with the term-end point design, which is given by (4.26), (4.27), (4.28), and (4.29). We also obtain the expected value, the standard deviation (Std), the 95% conditional tail expectation (CTE), and the 95% value-at-risk (VaR) of the errors.

**Insert Figure 1**

This distribution will serve as a benchmark for alternative hedging strategies. The mean of the hedging errors is zero since the participation rate is designed to allow for exact replication when the mortality of the group is equal to its expectation. Because holdings in portfolios of various maturities do not differ substantially, the inherent dynamic hedging error is small relative to the size of the contract; therefore, most of the risk is transferred to

[1] The parameter $\mu$ may be estimated using the index data.

the hedging strategy. This is observed in the small range of the distribution and its standard deviation of only 1.4%. Although the distribution has zero mean, it is right-skewed with mode close to 1%. Also, the probability that the insurer records a positive loss on the issue of this EIA is 45%; this is consistent with the skew ness of the distribution. The $\mathrm{VaR}_{95\%}$ is 2.8%, which means that 5% of the time, the issuer loses more than 2.8% of the contract value. If the loss exceeds the $\mathrm{VaR}_{95\%}$, the insurer assumes, on average, a ($\mathrm{CTE}_{95\%}$) loss of 3.2% of the contract; thus, catastrophic events typically result in dangerously high losses. Compare this figure to an average loss of 1.25%, given that the issuer experiences a loss. These last measures are good indicators of the inherent risk of the hedging strategy, since they use the entire right tail of the distribution. Gaillardetz and Lakhmiri (2006) present the effect of portfolio diversification on the hedging errors.

# 5 Tail Loading and Risk Management Strategies

This section proposes a new premium principle for equity-linked products based on the dynamic hedging strategy and risk measures. The proposed approach is inspired from actuarial premium principles (see Bowers et al., 1997), which usually loads the premium to protect the insurance company against mortality risk[2]. Similarly, the proposed premium principle loads the equity-linked premium to protects the issuer against the dynamic hedging errors. Different investment strategies are also presented. Those risk management strategies may be grouped in two broad classes: static and dynamic. The former sets an investment strategy for the loading at time 0 and keeps the same positions in the financial market during the contract life. The latter adjusts the investment strategy every $\Delta$ period, based on the underlying dynamic hedging strategy. Each investment strategy is set such its fair value is one unity at time 0.

The pricing principle has three distinct steps. We first obtain the fair value of the equity-linked products. In particular, the participation rate $\alpha_1$ for the equity-indexed annuity is obtained such that the fair value of the contract is one unity. The distribution of the hedging errors is then obtained using the dynamic hedging strategy. Based on this information, a tail loading is determined using a risk measure. For instance, the 95% value-at-risk may be used to obtain a tail loading of 2.8% in the previous example. Let $\epsilon$ ($\epsilon > 0$) be the choice of tail loading based on the risk measure, e.g. $\epsilon = 2.8\%$. A loaded participation rate for the equity-indexed annuity is then obtained by setting the fair value of the contract equal to $1 - \epsilon$. In other words, the participation rate $\alpha_2$ is set such that

$$FV_{\alpha_2}(x, 0, n, i_0) = 1 - \epsilon, \tag{5.31}$$

which is defined using (3.19).

[2]Mortality risk can never be perfectly diversifiable.

Note that the $\alpha_2$ replicating portfolio has a time-$t$ value process $V^*_{\alpha_2}(x, t, n, \mathbf{i}_t)$ and a present value of $1 - \epsilon$ at time 0. It is clear that $\alpha_2 \leq \alpha_1$ if and only if $\epsilon \geq 0$.

## 5.1 Strategy I

In strategy I, the insurer invests $\frac{1}{1-\epsilon}$ in the replicating portfolio $\alpha_2$. Hence, the asset share $a^*_{\alpha_2}$ and the investment in the money market $b^*_{\alpha_2}$ is increased by a factor $\frac{1}{1-\epsilon}$. The fair value of this investment strategy is given by

$$\frac{a^*_{\alpha_2}(x, 0, n, i_0)}{1 - \epsilon} S(0) + \frac{b^*_{\alpha_2}(x, 0, n, i_0)}{1 - \epsilon} = \frac{V^*_{\alpha_2}(x, 0, n, i_0)}{1 - \epsilon} = 1, \tag{5.32}$$

since $V^*_{\alpha_2}(x, 0, n, i_0) = 1 - \epsilon$. In other words, the insurer invests $1 - \epsilon$ in the $\alpha_2$ replicating portfolio and also $\epsilon$ in the $\alpha_2$ replicating portfolio.

The present value of the dynamic hedging errors is now obtained using

$$\begin{aligned} &ERROR = \\ &\sum_{l \in A_{K(x)-\Delta}} \left[ \left( FV_{\alpha_2}(x, l + \Delta, n, \mathbf{i}_{l+\Delta}) - \frac{V^*_{\alpha_2}(x, (l + \Delta)^-, n, \mathbf{i}_{l+\Delta})}{1 - \epsilon} \right) e^{-\delta(l+\Delta)} \right] \\ &+ \left[ D_{\alpha_2}(K(x) + \Delta, \mathbf{i}_{K(x)+\Delta}) - \frac{V^*_{\alpha_2}(x, (K(x) + \Delta)^-, n, \mathbf{i}_{K(x)+\Delta})}{1 - \epsilon} \right] e^{-\delta(K(x)+\Delta)}, \end{aligned} \tag{5.33}$$

if $K(x) = 0, \Delta, 2\Delta, \cdots, n - 2\Delta$, and

$$ERROR = \sum_{l \in A_{n-2\Delta}} \left( FV_{\alpha_2}(x, l + \Delta, n, \mathbf{i}_{l+\Delta}) - \frac{V^*_{\alpha_2}(x, (l + \Delta)^-, n, \mathbf{i}_{l+\Delta})}{1 - \epsilon} \right) e^{-\delta(l+\Delta)}, \tag{5.34}$$

if $K(x) = n - \Delta, n, \cdots$, with physical probabilities given by (4.27) and (4.29), respectively.

## 5.2 Strategy II

Strategy II suggests that the insurer invests 1 in the replicating portfolio $\alpha_1$. It is clear that the fair value of the replicating portfolio is 1 since $\alpha_1$ has been set such that the fair value of the contract is 1.

This investment strategy is inspired from actuarial science. Generally, premiums for standard insurance products are evaluated using premium principles (e.g. standard deviation principle, percentile principle, etc.) and the reserves are then obtained using the equivalence principle (see Bowers et al., 1997).

The present value of dynamic hedging errors is now obtained using

$$
\begin{aligned}
&ERROR = \\
&\quad \sum_{l \in A_{K(x)-\Delta}} \left[ \left( FV_{\alpha_2}(x, l+\Delta, n, \mathbf{i}_{l+\Delta}) - V^*_{\alpha_1}(x, (l+\Delta)^-, n, \mathbf{i}_{l+\Delta}) \right) e^{-\delta(l+\Delta)} \right] \\
&\quad + \left[ D_{\alpha_2}(K(x)+\Delta, \mathbf{i}_{K(x)+\Delta}) - V^*_{\alpha_1}(x, (K(x)+\Delta)^-, n, \mathbf{i}_{K(x)+\Delta}) \right] e^{-\delta(K(x)+\Delta)}, \qquad (5.35)
\end{aligned}
$$

if $K(x) = 0, \Delta, 2\Delta, \cdots, n-2\Delta$, and

$$
ERROR = \sum_{l \in A_{n-2\Delta}} \left( FV_{\alpha_2}(x, l+\Delta, n, \mathbf{i}_{l+\Delta}) - V^*_{\alpha_1}(x, (l+\Delta)^-, n, \mathbf{i}_{l+\Delta}) \right) e^{-\delta(l+\Delta)}, \qquad (5.36)
$$

if $K(x) = n-\Delta, n, \cdots$, with physical probabilities given by (4.27) and (4.29), respectively.

## 5.3 Strategy III

In Strategy III, the issuer invests $1-\epsilon$ in the $\alpha_2$ replicating portfolio and invest $\epsilon$ in the index and/or money market account. This strategy is different than strategies I and II since it is using a static investment strategy for $\epsilon$. Similar to Strategy I, the insurance company invest $1-\epsilon$ in portfolio $\alpha_2$. However, they also invest $\epsilon-\phi$ in the money market account and buy $\phi$ index shares at time 0. Those proportions will remain constant over the contract's life. The fair value of this investment strategy is then given by

$$
\left( a^*_{\alpha_2}(x, 0, n, i_0) + \phi \right) S(0) + b^*_{\alpha_2}(x, 0, n, i_0) + \epsilon - \phi = V^*_{\alpha_2}(x, 0, n, i_0) + \epsilon = 1, \qquad (5.37)
$$

since $S(0) = 1$.

Under Strategy III, The present value of dynamic hedging errors is now obtained using

$$
\begin{aligned}
&ERROR = \\
&\quad \sum_{l \in A_{K(x)-\Delta}} \left[ \left( FV_{\alpha_2}(x, l+\Delta, n, \mathbf{i}_{l+\Delta}) - V^*_{\alpha_2}(x, (l+\Delta)^-, n, \mathbf{i}_{l+\Delta}) \right) e^{-\delta(l+\Delta)} \right] \\
&\quad -(\epsilon-\phi) + \left[ D_{\alpha_2}(K(x)+\Delta, \mathbf{i}_{K(x)+\Delta}) - V^*_{\alpha_2}(x, (K(x)+\Delta)^-, n, \mathbf{i}_{K(x)+\Delta}) \right. \\
&\quad \left. -\phi S(K(x)+\Delta, i_{K(x)+\Delta}) \right] e^{-\delta(K(x)+\Delta)} \qquad (5.38)
\end{aligned}
$$

if $K(x) = 0, \Delta, 2\Delta, \cdots, n-2\Delta$, and

$$
\begin{aligned}
&ERROR = \\
&\quad \sum_{l \in A_{n-2\Delta}} \left[ \left( FV_{\alpha_2}(x, l+\Delta, n, \mathbf{i}_{l+\Delta}) - V^*_{\alpha_2}(x, (l+\Delta)^-, n, \mathbf{i}_{l+\Delta}) \right) e^{-\delta(l+\Delta)} \right] \\
&\quad -(\epsilon-\phi) - \phi S(n-\Delta, i_{n-\Delta}) e^{-\delta(n-\Delta)} \qquad (5.39)
\end{aligned}
$$

if $K(x) = n-\Delta, n, \cdots$, with physical probabilities given by (4.27) and (4.29), respectively.

## 5.4 Risk Management Strategies: Example

In this numerical illustration, we consider the same set of parameters as in the previous examples. It still involves a five-year point-to-point EIA issued to a male aged 55 with minimum interest rate guarantee of 3% on 90% of the premium. The mortality of the policyholder is assumed to follow the 1979 − 1981 U.S. Life Table (see Bowers et al., 1997, Table 3.3.1) with death occurring only at the end of each year. The index is governed by the CRR model with $\sigma = 25\%$, $\delta = 6\%$, $N = 6$, and $\pi$ given by (4.30), where $\mu = 15\%$.

The loaded participation rate is obtained using the premium principle presented previously. The fair participation rate $\alpha_1$ equal 69.31% (see first example). The corresponding hedging error distribution is illustrated in Figure 1. The tail loading $\epsilon$ is obtained using the 95% value-at-risk, which is equal to 2.8%. Hence, the loaded participation rate $\alpha_2 = 59.35\%$, which is evaluated using (4.22); that is $FV_{\alpha_2}(x, 0, n, i_0) = 97.2\%$.

Figures 2 and 3 considers hedging errors for the investment Strategies I and II, respectively. The errors are obtained using (5.33) and (5.34) for Strategy I and (5.35) and (5.36) for Strategy II.

**Insert Figures 2 and 3**

Strategy 1 and 2 require the issuer to invest in a hedging strategy whose value is greater than the EIA contracted on the loaded participation rate $\alpha_2$. Thus, the value of the a portfolio at maturity is always greater than or equal to its corresponding payoff. Since the overall effect of this is to reduce the hedging error in case of death, losses tend to shift to the left. Accordingly, the mean values for these Strategies are −3.3% and −5.7% respectively. In addition, due to the overall leftward shift, there is a reduction in the frequency of positive losses; the probability of experiencing such an event under Strategy 1 and 2 are now only 9% and 7%, a dramatic decrease for the original 45%. Whenever the hedging portfolio is readjusted in the case of survival, errors are also magnified. This magnification causes an increase in the dispersion of the distribution, as reflected in standard deviations of 2.3% for Strategy 1 and 6.67% for Strategy 2. This also gives rise to pathological cases, in with some outcomes produces profit as high as 20% and 80% of the contract value. It is clear that Strategy 2 has distorded the original hedging error distribution more than Strategy 1; however, Strategy 1 outperforms its counterpart in reducing the size of the right tail. The $\text{VaR}_{95\%}$ is now 0.21% and 0.32%, the $\text{CTE}_{95\%}$ measure is 1.16% and 1.77% and the expected loss given a positive loss is 0.72% vs 1.26%, respectively. These are, of course, all below benchmark values.

Figures 4 present hedging errors for the investment Strategy IIIA, where $\phi$ is equal to 0. The errors for this strategy are obtained using (5.38) and (5.39).

**Insert Figure 4**

Under Strategy IIIA with $\phi = 0$, the mean of the hedging errors is $-2.80\%$, because the issuer realizes a net gain of $\epsilon = 2.80\%$ in average. The standard deviation is 1.89%. The $\text{VaR}_{95\%}$ is 0.74%. The $\text{CTE}_{95\%}$ is 1.55%. The probability of recording a positive loss is 13.47%. The expected loss, given the loss is positive is 0.82%.

Figures 5 and 6 still present hedging errors for the investment Strategy III, where $\phi$ is determined according to an optimal investment strategy. We present the hedging errors for $\phi$ that minimizes either the 95% value-at-risk in Figure 5 (Strategy IIIB) or the 95% conditional tail expectation in Figure 6 (Strategy IIIC). There is no short/long and borrowing limit for the index and the money market account, respectively. For both strategies, the errors are still obtained using (5.38) and (5.39).

**Insert Figures 5 and 6**

In Figures 5 and 6 $\phi$ is 5.89% and 3.70%, respectively. For Figure 5, this means that $\epsilon - \phi = -3.09\%$ is held in the money-market account, which constitutes a loan. The same is true for Figure 6, where $\epsilon - \phi = -0.09\%$ is also negative. This seems to suggest that catastrophic losses tend to occur when the index level is high, since the position in stock partially offsets the claim by the insured. The expected value is $-4.83\%$ and $-4.07\%$, respectively. The standard deviation for Figure 5 and 6 is 4.53% and 3.29% respectively. The $\text{VaR}_{95\%}$ is $-0.52\%$ and $-0.18\%$; notice that these static strategies are able to generate profits, at least 95% of the time. The $\text{CTE}_{95\%}$ is 1.32% and 0.99%, respectively. The probability of registering a positive loss is 2.92% and 4.76% respectively. The expected loss, given a positive loss is 2.43% and 1.04% respectively.

# 6 Conclusions

The purpose of this paper is to present a loaded participation rate for equity-linked products that can be implemented in practice. To this end, we introduce a tail loading on the hedging errors. This is an improvement over the standard financial approach since the issuer is protected against the idiosyncratic mortality risk. The errors are extracted from the dynamic hedging strategy, which underlies the fair valuation. We also present risk management strategies that reduces the mortality risk. A detailed numerical analysis is then performed for a point-to-point with term-end design EIA.

Our methodology may be used to evaluate variable annuities (segregated fund contracts in Canada) because of the similarity in payoff structure between EIAs and VAs.


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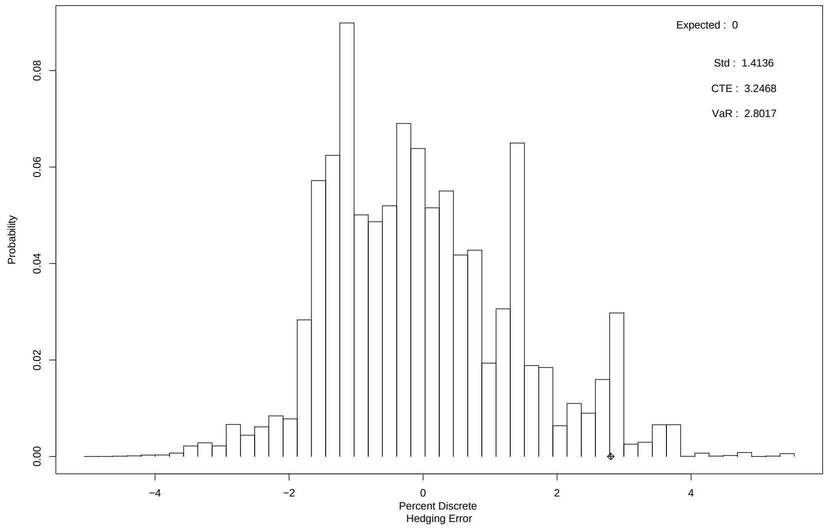


Figure 1: The present value of dynamic hedging errors

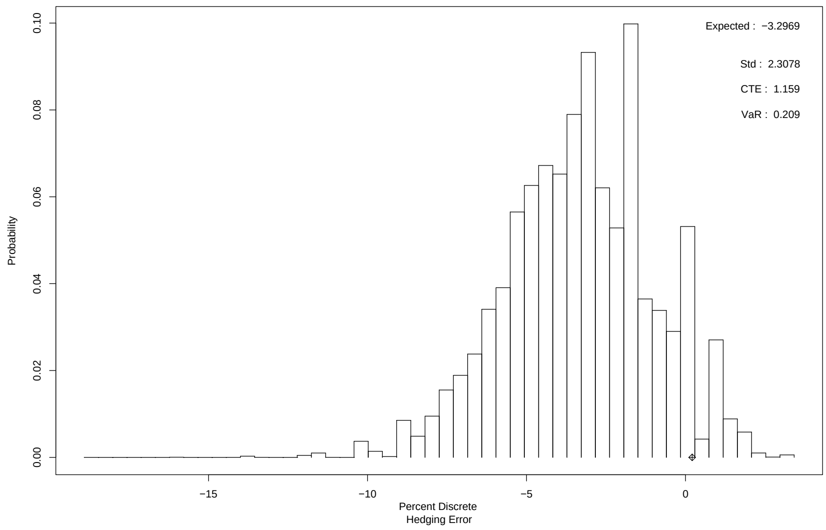


Figure 2: Strategy I: The present value of dynamic hedging errors

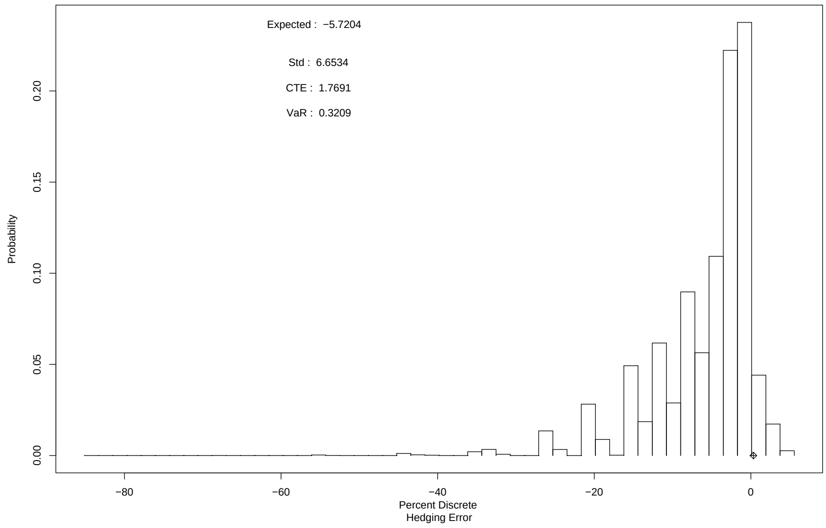


Figure 3: Strategy II: The present value of dynamic hedging errors

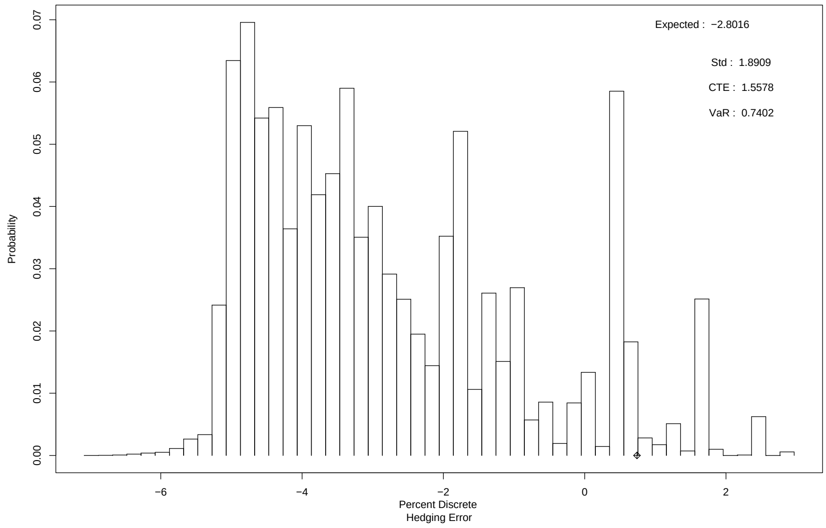


Figure 4: Strategy IIIA: The present value of dynamic hedging errors

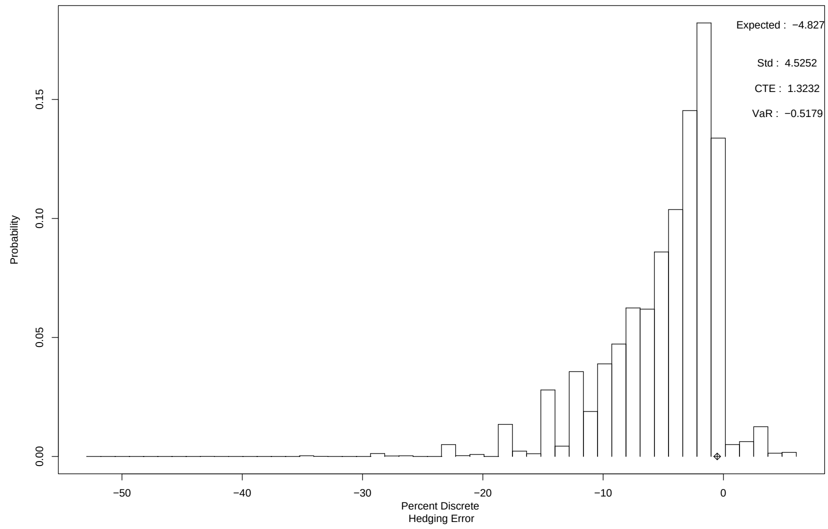


Figure 5: Strategy IIIB: The present value of dynamic hedging errors

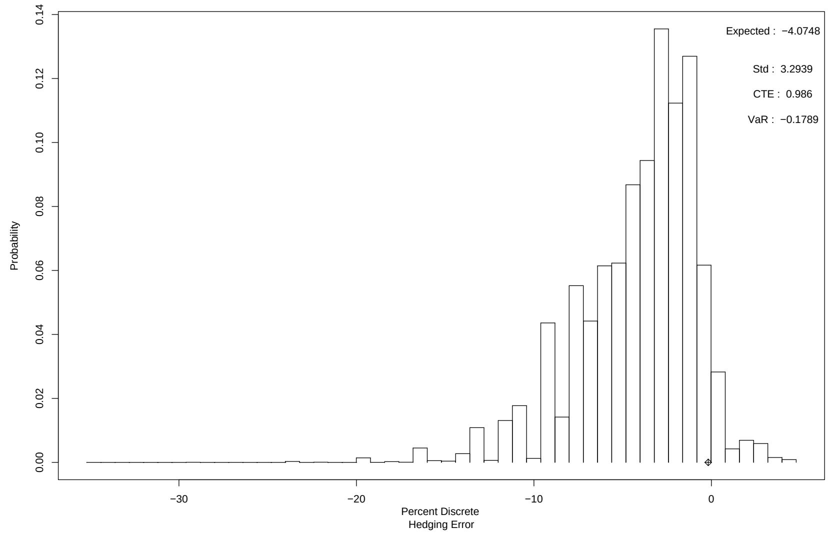


Figure 6: Strategy IIIC: The present value of dynamic hedging errors