Optimal Retention Levels in Dynamic (Re)insurance Markets

Enrico Biffis

Faculty of Actuarial Science and Insurance
Cass Business School, London

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1. Introduction
2. Setup
3. Dynamic constrained MV problem
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Intro & Motivation

General problem statement:

optimal dynamic insurance strategy for given risk exposure,

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- setting: sources of randomness, financial market...

  my setup:

  - Brownian filtration
  - financial market
  - randomness in claims/market parameters

The model

$(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, with $\mathbb{F} \doteq \overline{\mathbb{F}}^{B}$ (B Brownian motion)

Wealth and shocks:

- Wealth/exposure process: $X(t)$ \$
- Proportional wealth shocks: $\mathrm{d}X(t) = -X(t)\,[\delta(t)\mathrm{d}t + \sigma(t) \cdot \mathrm{d}B(t)]$

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Insurance market:

- Exposure: $w(t)$ \$ covered by insurance, $v(t) = X(t) - w(t)$ \$ retained.
- Premium rate per unit of insured capital/wealth: $\pi(t)$

$$w(t)\pi(t)\mathrm{d}t = \text{premium paid to instantaneously insure exposure } w(t)$$

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Financial market:

- money market account: $r(t)$ risk-free rate
- risky stocks: $\mathrm{d}S_i(t) = S_i(t)\,[\mu_i(t)\mathrm{d}t + \overline{\sigma}_i(t) \cdot \mathrm{d}B(t)] \qquad (i = 1, \ldots, m)$
- $\delta, \sigma, \mu, \overline{\sigma}$ are $\mathbb{F}$-adapted

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- no shorting: $0 \leq \overline{v}_{k+1}(t), \ldots, \overline{v}_m(t)$ $(k = 0, \ldots, m+1)$;
- 'safer' assets $S_1, \ldots, S_k$, 'riskier' assets $S_{k+1}, \ldots, S_m$: $\sum_{i=1}^{k} \overline{v}_i(t) \geq v(t)$;
- combinations of the above.

The Optimization Problem

- Constrained dynamic MV problem:

$$\begin{cases} \min & V\left[X(T)\right] \\ \text{sub} & E\left[X(T)\right] = z^* \\ & (X, u) \quad \text{admissible} \\ & u \in \mathcal{C} \doteq \text{constraints set} \end{cases}$$

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- Stochastic LQ control and BSDE theory: Hu-Zhou (2005).
- Efficient strategy:

$$u^*(t) = \begin{cases} \left[X(t) - A(t)\right] \xi_1^*(t) & X(t) > A(t) \\ \\ \left[A(t) - X(t)\right] \xi_2^*(t) & X(t) \leq A(t) \end{cases}$$

...and mean-variance frontier:

$$E[X(T)] = f\left(\sqrt{V[X(T)]}; A(0), X(0)\right)$$

Key BSDEs

$A(\cdot)$ depends on the solutions to the following BSDEs:

$$\begin{cases} dP_{1,2}(t) = f(t, P_{1,2}, \Lambda_{1,2}, H^*_{1,2})dt + \Lambda_{1,2}(t) \cdot dB(t) \\ P_{1,2}(T) = 1 \\ P_{1,2}(t) > 0 \text{ a.s.} t \in [0, T] \end{cases}$$

$$H^*_{1,2}(t, \omega) \doteq \min_{u \in \mathcal{C}} H_{1,2}(t, \omega, u, P_{1,2}, \Lambda_{1,2})$$

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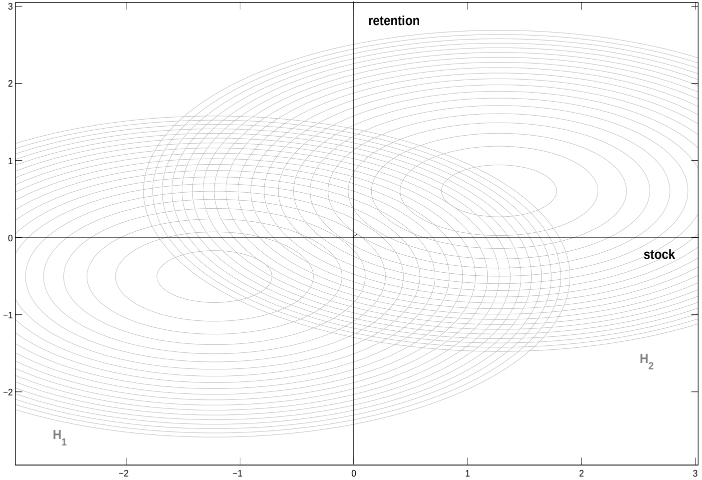
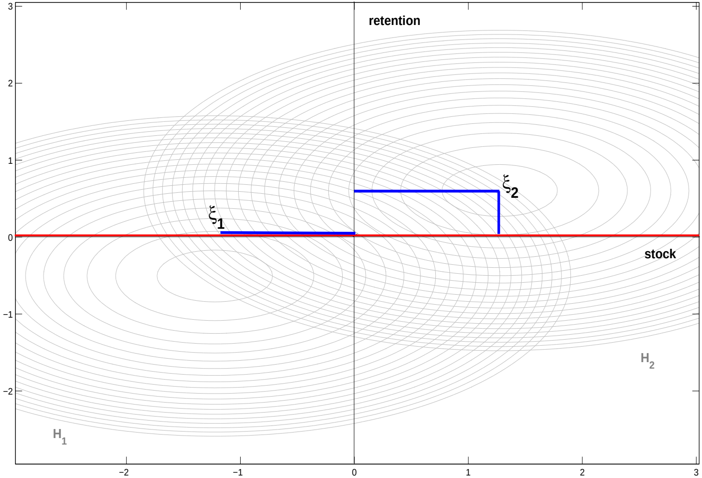
$\xi_{1,2}(\cdot)$ are given by:

$$\xi_{1,2}(t, \omega, P_{1,2}, \Lambda_{1,2}) \doteq \arg\min_{u \in \mathcal{C}} H_{1,2}(t, \omega, u, P_{1,2}, \Lambda_{1,2})$$

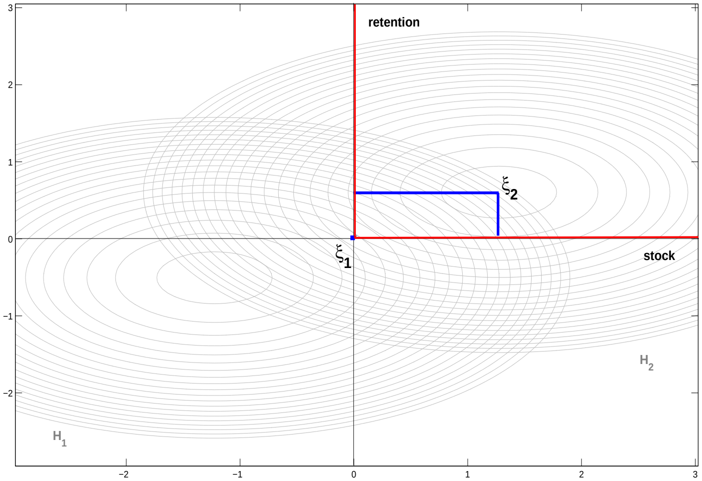
Example: $u(t) = (v(t), \overline{v}(t))^{\mathsf{T}}$



Example: $v(t) \geq 0$, $\overline{v}(t)$ unrestricted



Example: $v(t), \overline{v}(t) \geq 0$



Conclusion

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  - Insurance market only:

$$v^*(t) = \frac{\pi(t) - \delta(t)}{\sigma^2(t)} \left(A(t) - X^*(t)\right)$$

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- De Finetti & Markowitz:
  - efficient frontier
    $\longrightarrow$ for fixed $\epsilon \in (0,1), x^* \geq 0$ choose z^* such that $\mathbb{P}\left(X(T) < x^*\right) \leq \epsilon$
  - constant retention levels are inefficient
    $\longrightarrow$ market portfolio inefficient

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