Application of Epidemiological Models in Actuarial Mathematics

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Abstract

The emergence of the worldwide SARS epidemic in 2003 led to a revived interest in the study of infectious diseases. Mathematical models have become important tools in analyzing transmission dynamics and measuring the effectiveness of controlling strategies. Research on infectious diseases in the actuarial literature only goes so far as to set up epidemiological models which better reflect the transmission dynamics. In an effort to build a bridge between epidemiological and actuarial modeling, we analyze possible financial arrangements made against expenses resulted from medical treatments given to insured patients.

Based on classical compartment models, the first part of this paper designs insurance policies for susceptible participants facing the risk of infection and formulates the financial obligations of both parties using actuarial methodology. For practical purposes, the second part employs a variety of numerical methods for calculating premiums and reserves. The last part illustrates the methods by designing insurance products for the Great Plague in Eyam and the SARS Epidemic in Hong Kong.

1 Introduction

One beneficial side of the Severe Acute Respiratory Syndrome (SARS) epidemic in 2002 was to draw tremendous attention to the treatment and prevention of infectious diseases and to their implication to general social welfare. The adverse economic impact caused by SARS in East Asia has been compared with that of the 1998 financial market crisis in that area. From a social point of view, an effective protection against diseases depends not only on the development of medical technology to identify viruses and to treat infected patients, but also on a well-designed healthcare system. The latter can reduce the financial impact of a sudden pandemic outbreak such as surging costs of medications, medical

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equipments, prevention measures like vaccination and quarantine. Broader insurance programs can even cover financial losses resulting from the interruption in regular business operations. As a profession with the reputation of applying rigorous techniques to model and quantify financial risk, actuaries are certainly well put to expand their expertise and deal with epidemics within healthcare systems.

Due to their frontline experience with SARS, many health insurers in Asia provided coverage to compensate for medical costs for SARS treatment, by listing the disease as an extended liability on regular health insurance policies. Still many problems arose. Traditional actuarial models for human mortality lack the flexibility required to model infectious diseases, which are significantly different from natural causes of death in many aspects.

One of the remarkable differences is that in a population exposed to an epidemic outbreak there are several mutually dependent groups involved with different levels of vulnerability to the disease. Whereas mortality rates are often assumed to be constant among homogenous age-specific groups. How fast an infectious disease spreads within a population relies on the number of susceptible individuals, the number of infectious individuals and the social structure between these two groups. To be more specific in the context of a health insurance for an initially complete susceptible group, the number of insureds bearing premiums would actually decrease in time, whereas the number of insureds claiming benefits due to infection increases as the epidemic breaks out. Applying traditional life table methods overlooks epidemiological dynamics and dependence between insurance payers and beneficiaries. It consequently violates the fair premium principle assumed in the industry.

The idea of borrowing from epidemiological models to account of several interacting subgroups in a population and modeling the according financial arrangements is suggested in this paper.

To make it self–contained, basic techniques from the mathematics of epidemiology, for instance the simplest three–compartment model, are reviewed in Section 2. The corresponding business model is set to cover potential financial losses from insured clients.

For insurance applications, Section 3 formulates epidemiological models in actuarial notation and analyzes the quantitative relations among some insurance concepts, namely the actuarial present value of continuous payments to hospital and medical services, the actuarial present value of death benefits and the actuarial present value of premium income.

In Section 4, several premium rating methods are presented to price different infectious disease insurance policies. An algorithm is derived to calculate premiums under the fair premium principle.

The solvency aspects of the insurance plans are also studied. Since benefit reserves should reflect a policy's cash value, at least theoretically refundable to the policyholder, these should remain positive. We see that the premiums calculated under a fair value principle yield negative reserves. Therefore Section 5 develops a numerical method to determine safety–loading premium levels that ensure benefit reserve which never fall below a certain tolerable balance level. Based on epidemiological models in the literature, Section 6 analyzes the dynamics of the Great Plague in Eyam and SARS epidemics in Hong Kong. We propose an insurance coverage against the resulting financial losses in a manner that could easily be adapted to enable further analysis of a wide range of scenarios.

2 Epidemiological compartment model

Over the last century, many contributions to the mathematical modeling of communicable diseases have been made by a great number of public health physicians, epidemiological mathematicians and statisticians. Their brilliant work ranges from empirical data analysis to differential equation theory. Many have achieved successes in clinical data analysis and effective predictions. For a complete review of a variety of mathematical and statistical models, the interested readers are referred to Hethcote [14] and Mollison *et al.* [15]. Standing on the shoulders of giants, actuaries could incorporate economical consideration into epidemiological models and make financial and medical arrangements to protect the insured population against infectious diseases. For an account of co-operative opportunities for actuaries and epidemiologists, we refer to a report by Cornall *et al.* [10].

To illustrate the main idea of application, we start off by looking at the simplest deterministic model where a clear actuarial analysis can be conducted. Although most infectious diseases like SARS are more complex, the generalization from the three-compartment model to multi-dimensional models follows similar procedures.

In epidemiological studies, to model an epidemic, a whole population is usually separated into compartments with labels such as S, I and R. These acronyms are used in different patterns according to the transmission dynamics of the studied disease. Generally speaking, class S denotes the group of individuals without immunity, or in other words, susceptible to a certain disease. In an environment exposed to the disease, some individuals come into contact with the virus. Those infected who are able to transmit the disease are considered in class I. Through medical treatment, individuals, removed from the epidemic due to either death or recovery, are all counted in class R. The upper part of Figure 1 gives the transferring dynamics among the three compartments.

Another merit of this partition, from an actuarial perspective, is that the three compartments play significantly different roles in an insurance model. As demonstrated in the lower part of Figure 1, The susceptible individuals facing the risk of being infected in an epidemic each contribute a certain amount of premium to the insurance funds in return for future coverage of medical expenses incurred as a result of infection. During the outbreak, the infected are eligible for claiming benefits for expenditures covered in the policy. Following an individual's death, a death benefit for funeral and burial expenses would be paid to specified beneficiaries. Interest will accrue on the properly managed insurance funds at a certain rate.

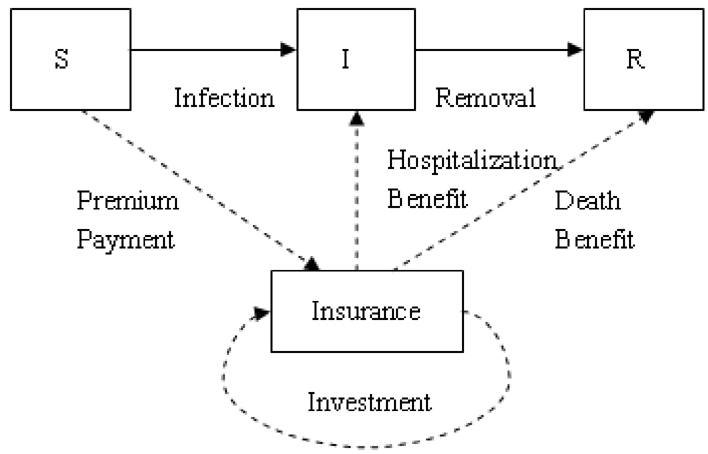

Figure 1: General transfer dynamics and insurance principles among compartments S, I and R.

We start by looking at a typical mathematical model that serves to characterize the interaction among the three compartments. Let $S(t)$ denote the number of susceptible individuals at time t, whereas $I(t)$ be the number of infected individuals, and $R(t)$ be the number of individuals removed from class I. According to the mass action laws commonly used in biological quantitative analysis, compartment sizes are determined in terms of their derivatives. Therefore we assume that the number of members in each compartment is a differentiable function, defined with support on the non-negative side of the real line.

Their qualitative relations are given by the following system of differential equations known as the SIR model.

$$S'(t) = -\beta S(t)I(t)/N \ , \qquad t \geq 0 \ , \tag{1}$$
$$I'(t) = \beta S(t)I(t)/N - \alpha I(t) \ , \qquad t \geq 0 \ , \tag{2}$$
$$R'(t) = \alpha I(t) \ , \qquad t \geq 0 \ , \tag{3}$$

with given initial values $S(0) = S_0$, $I(0) = I_0$ and $S_0 + I_0 = N$.

The model is based on the following assumptions:

1. The total number of individuals keeps constant, $N = S(t) + I(t) + R(t)$, representing the total population size.

2. An average person makes an average number β of adequate contacts (i.e. contacts sufficient to transmit infection) with others per unit time.

3. At any time a fraction α of the infected leave class I instantaneously. α is also considered to be constant.

4. There is no entry into or departure from the population, except possibly through death from the disease. For our purpose of setting up an insurance model, the demographic factors like natural births and deaths are negligible, as the time scale of an epidemic is generally shorter than the demographic time scale.

Since the probability of a random contact by an infected person with a susceptible individual is S/N, then the instantaneous increase of new infected individuals is $\beta(S/N)I = \beta SI/N$. The third assumption implies that the instantaneous number of people flowing out of the infected class I into the removal class R is αI.

3 Actuarial analysis

The idea of setting up an insurance coverage against an infectious disease is akin to that of covering other contingencies like natural death and destruction of property.

Since mortality analysis is based on ratios instead of absolute counts, we now introduce $s(t)$, $i(t)$ and $r(t)$ respectively as fractions of the whole population, in each of class S, I and R. Dividing equations (1)-(3) by the constant total population size N yields

$$s'(t) = -\beta\, i(t)\, s(t)\ , \qquad t \geq 0\ , \tag{4}$$
$$i'(t) = \beta\, i(t)\, s(t) - \alpha\, i(t)\ , \qquad t \geq 0\ , \tag{5}$$
$$r(t) = 1 - s(t) - i(t)\ , \qquad t \geq 0\ , \tag{6}$$

where $s(0) = s_0$, $i(0) = i_0$ given that $s_0 + i_0 = 1$.

One could actually interpret the ratio function $s(t)$, $i(t)$ and $r(t)$ as the probability of an individual being susceptible, infected or removed from infected class respectively at the time spot t.

Since all these ratio functions vary between 0 and 1, we could easily interpret them as the probability of an individual remaining susceptible, infected or removed at the time point t . However, it should be noted that due the laws of mass action, movements among the compartments rely on the sizes of each other. Thus these probabilities represent mutually dependent risks as opposed to the independent hazards one always sees in multiple decrement life insurance models. With these probability functions $s(t)$, $i(t)$ and $r(t)$, we now incorporate actuarial methods to formulate the quantities of interest for an infectious disease insurance.

3.1 Annuity for premium payments and annuity for hospitalization

We assume that the infection disease protection plan works in a simple annuity fashion. Individual premiums are collected continuously as long as the covered person remains susceptible, whereas medical expenses are continuously reimbursed to each infected policyholder during the whole period of treatments. Once the individual recovers from the disease, the protection ends right away.

Following the International Actuarial Notation, the actuarial present value (APV for later use) of premium payments from an insured person for the whole epidemic is denoted by $\overline{a}_0^s$ with the superscript indicating payments from class S , and APV of benefit payments from the insurer is denoted by $\overline{a}_0^i$ with the superscript indicating payments to class I .

On the debit side of the insurance product, the total discounted future claim is given by

$$\overline{a}_0^i = \int_0^\infty e^{-\delta t} i(t) dt \ , \tag{7}$$

while on the revenue side, the total discounted future premiums is

$$\overline{a}_0^s = \int_0^\infty e^{-\delta t} s(t) dt \ , \tag{8}$$

where δ is the force of interest. Our study in this paper is based on the fundamental *Equivalence Principle* in Actuarial Mathematics for the determination of level premiums, which requires

E[present value of benefits]=E[present value of benefit premiums] .

Therefore, the level premium for the unit annuity for hospitalization plan

$$\overline{P}(\overline{a}_0^i) = \frac{\overline{a}_0^i}{\overline{a}_0^s} \ . \tag{9}$$

Just like in life insurance, where the force of mortality is defined as the additive inverse of the ratio of the derivative of the survival function to the survival function itself, we define here the force of infection as

$$\mu_t^s = -\frac{s'(t)}{s(t)} \ , \qquad t \geq 0 \ ,$$

and the force of removal as

$$\mu_t^i = -\frac{i'(t)}{i(t)} \ , \qquad t \geq 0 \ .$$

Specifically from (4)-(5), we see that $\mu_t^s = \beta \, i(t)$ and $\mu_t^i = -\beta \, s(t) + \alpha$.

Note that the above definitions imply that

$$s(t) = \exp\{-\int_0^t \mu_r^s \, dr\} = \exp\{-\beta \int_0^t i(r) dr\} \ , \qquad t \geq 0 \ , \tag{10}$$

and

$$i(t) = \exp\{-\int_0^t \mu_r^i \, dr\} = \exp\{\beta \int_0^t s(r) \, dr + \alpha t\} \ , \qquad t \geq 0 \ . \tag{11}$$

Proposition 3.1. *In the SIR model in (4)-(5),*

$$(1+\frac{\alpha}{\delta})\,\overline{a}_0^i+\overline{a}_0^s=\frac{1}{\delta}\ . \tag{12}$$

Proof. From (4) and (5), we obtain that

$$s'(t)+i'(t)=-\alpha\ i(t)\ , \qquad t\geq 0\ .$$

Integrating from 0 to a fixed t gives

$$s(t)+i(t)-1=-\alpha\int_0^t i(r)\ dr\ , \qquad t\geq 0\ .$$

Multiplying both sides by $e^{-\delta t}$ and integrating with respect to t from 0 to ∞ yields

$$\overline{a}_0^s+\overline{a}_0^i-\frac{1}{\delta}=-\frac{\alpha}{\delta}\,\overline{a}_0^i\ ,$$

where the right hand side comes from exchanging the order of integrals,

$$\begin{aligned}\int_0^\infty \exp\left(-\delta t\right)\int_0^t i(r)\ dr\ dt &= -\frac{1}{\delta}\int_0^\infty\int_0^t i(r)\ dr\ d(\exp\left(-\delta t\right))\\ &= \frac{1}{\delta}\int_0^\infty \exp\left(-\delta r\right)i(r)\ dr=\frac{1}{\delta}\overline{a}_0^i\ .\end{aligned}$$

□

Notice that the right hand side represents the present value of a unit perpetual annuity. The intuitive interpretation of the left hand side is that if every one in the insured group is rewarded with a perpetual annuity, the APV of expenses from class S accounts for $\overline{a}_0^s$, and similarly that of expenses from class I adds $\overline{a}_0^i$ to the cost. Recall that at any time a fraction α of the infected subgroup move forwards to class R . Each of them would receive a perpetual of value $1/\delta$ as well at the time of transition. Therefore, the APV of expenses from this compartment would be $(\alpha/\delta)\overline{a}_0^i$. It is reasonable that it should sum up to the value of a unit perpetual annuity regardless of the policyholder's location among compartments.

With this relation in mind, we could easily find the net level premium for the unit annuity for hospitalization plan,

$$\overline{P}(\overline{a}_0^i)=\frac{\overline{a}_0^i}{\overline{a}_0^s}=\frac{\delta\,\overline{a}_0^i}{1-(\delta+\alpha)\overline{a}_0^i}\ . \tag{13}$$

3.2 Lump sum for hospitalization

The analogy of this plan is with a whole life insurance in actuarial mathematics. When a covered person is diagnosed being infected with the disease and hospitalized, the medical expenses is to be paid immediately in a lump sum and insurance protection ends. Then the APV of benefit payments to the infected denoted by $\overline{A}_0^i$ can be obtained as

$$\overline{A}_0^i \triangleq \beta \int_0^\infty e^{-\delta t} s(t)\, i(t)\, dt\ , \tag{14}$$

since the probability of being newly infected at time t is $\beta\, s(t) i(t)$.

Proposition 3.2.

$$\frac{1}{\delta}\overline{A}_0^i + \overline{a}_0^s = \frac{1}{\delta} s_0\ , \tag{15}$$

and

$$\frac{1}{\delta} i_0 + \frac{1}{\delta}\overline{A}_0^i = \frac{\alpha}{\delta}\overline{a}_0^i + \overline{\alpha}_0^i\ . \tag{16}$$

Proof. Substituting (4) into (14), we have that

$$\begin{aligned} \overline{A}_0^i &= -\int_0^\infty e^{-\delta t}\, s'(t)\, dt \\ &= s(0) - \delta \int_0^\infty e^{-\delta t}\, s(t)\, dt \\ &= s_0 - \delta\, \overline{a}_0^s\ . \end{aligned}$$

Since $\overline{a}_0^s = (1/\delta)[1 - (\delta + \alpha)\, \overline{a}_0^i]$ by (12), it follows that

$$\overline{A}_0^i = (\delta + \alpha)\, \overline{a}_0^i - 1 + s_0\ .$$

□

The above proposition also provides an interesting insight into the breakdown of expenses in each class. To understand (15), we suppose every susceptible individual claims one unit perpetual annuity. The APV of the total cost is s_0/δ . From an another perspective, it is equivalent to give every one a unit annuity as long as the person remains healthy in the group and then grant them each a unit perpetual immediately as he or she becomes infected. The APV of these two payments is exactly $(1/\delta)\overline{A}_0^i + \overline{a}_0^s$. If one thinks of class I as a transit, the left hand side of (16) counts the expenses at the front door. Since expenses for initial members is i_0/δ and other individuals from class S each add $1/\delta$. Hence the total expenses add up to $i_0/\delta + (1/\delta)\overline{A}_0^i$. While at the back door, every one already inside accounts for $\overline{a}_0^i$, and any one leaving the class takes away a perpetual of value $1/\delta$. Thus the right hand side sums up to $(\alpha/\delta)\overline{a}_0^i + \overline{a}_0^i$.

Therefore for the lump sum payment plan with a unit benefit, the equivalence principle gives the net level premium $\overline{P}(\overline{A}_0^i)$:

$$\overline{P}(\overline{A}_0^i) = \frac{\overline{A}_0^i}{\overline{a}_0^s} = \frac{(\alpha + \delta)\overline{a}_0^i - i_0}{1 - (\alpha + \delta)\overline{a}_0^i}\ .$$

3.3 Death benefit

It is necessary to point out that in the epidemiological literature the class R is composed of all individuals removed chronologically from a previous compartment, who either recover with immunity or die due to the disease. A more refined model would have separate compartments for the dead and the recovered. For our purpose of deducing an actuarial analysis based upon epidemiological models, we keep the model as simple as possible by assuming only one R compartment exclusively for deaths caused by the disease, which implies that nobody recovers from the disease. The SIR model remains the same as in (4)-(5), except that α can now be interpreted as the constant rate of fatality from class I .

For most health insurance policies, death benefits differ in amount from healthcare benefits. We assume that in this infectious disease plan, a death benefit of $d_t = 1$ is paid immediately at the moment of death.

Thus, the actuarial present value of a lump sum death benefit payment denoted by $\overline{A}_0^d$ is

$$\begin{aligned} \overline{A}_0^d &\triangleq \alpha \int_0^\infty e^{-\delta t}\, i(t)\, dt \\ &= \alpha\, \overline{a}_0^i \,. \end{aligned}$$

Therefore, the net level premium for the plan with both a unit annuity of hospitalization benefit and a unit death benefit is obtained by

$$\overline{P}(\overline{a}_0^i + \overline{A}_0^d) = \frac{\overline{a}_0^i + \overline{A}_0^d}{\overline{a}_0^s} = \frac{\delta(1+\alpha)\overline{a}_0^i}{1-(\alpha+\delta)\overline{a}_0^i} \,,$$

and the net level premium for the plan with both a lump sum for hospitalization and death benefit is given by

$$\overline{P}(\overline{A}_0^i + \overline{A}_0^d) = \frac{\overline{A}_0^i + \overline{A}_0^d}{\overline{a}_0^s} = \frac{(\delta+\alpha+\delta\alpha)\overline{a}_0^i - i_0}{1-(\alpha+\delta)\overline{a}_0^i} \,.$$

4 Premium rating

So far net premiums have only been expressed in terms of $\overline{a}_0^i$, which is a Laplace transform of $i(t)$. An implicit integral solution to the SIR model in (4)-(5) is as follows,

$$s(t) = \frac{1}{N} \exp\{-\beta \int_0^t \exp\{\beta N \int_0^u s(r)\, dr - \alpha u\} du\} \,,$$

$$i(t) = \frac{1}{N} \exp\{\beta \int_0^t \exp\{\beta N \int_0^u i(r)\, dr\} - \alpha u\, du\} \,.$$

But there is not an explicit method available to solve $s(t)$ and $i(t)$. Therefore we propose numerical formulas and approximations that can provide satisfactory solutions for insurance applications.

4.1 Insurance related quantities and Runge-Kutta method

Among many numerical methods for solving ODE, the Runge-Kutta method is the most popular. It can be adapted for any order of accuracy. For applications in insurance, the fourth order Runge-Kutta method (RK-4), given by the following recursion formulas, represents a good compromise between simplicity and accuracy:

$$\begin{aligned}
y_{i+1} &= y_i + \frac{1}{6}(k_1 + 2k_2 + 3k_3 + k_4)\ , \qquad i = 1, 2, \cdots, n\ , \\
k_1 &= hf(t_i, y_i)\ , \qquad i = 1, 2, \cdots, n\ , \\
k_2 &= hf(t_i + \frac{h}{2}, y_i + \frac{1}{2}k_1)\ , \qquad i = 1, 2, \cdots, n\ , \\
k_3 &= hf(t_i + \frac{h}{2}, y_i + \frac{1}{2}k_2)\ , \qquad i = 1, 2, \cdots, n\ , \\
k_4 &= hf(t_i + h, y_i + k_3)\ , \qquad i = 1, 2, \cdots, n\ ,
\end{aligned}$$

where y_i is given by the ODE:

$$\frac{dy}{dt} = f(t, y)\ ,$$

evaluated at $t = t_i$, and the time step $h = t_i - t_{i-1}$ for $i = 1, 2, \cdots, n$.

Actuaries may particularly be interested in the properties of insurance-related quantities, such as discounted total benefits, discounted total premiums and premium reserves. Based on the RK-4 method, we need to fit these items into a differential equation system. Let $P(t)$ denote the present value of premiums up to time t and $B(t)$ the corresponding present value of benefits at the same time point. We introduce $V(t)$ as the cumulative benefit reserve at time t , which is the difference of accumulated value of premiums and accumulated value of claim expenses.

For practical use, we could as well define insurance plans of finite time duration, say

$$\overline{a}^s_{0:\overline{t}|} \triangleq \int_0^t e^{-\delta z} s(z)\ dz \ \text{ and } \overline{a}^i_{0:\overline{t}|} \triangleq \int_0^t e^{-\delta z} i(z)\ dz\ .$$

Therefore, for the annuity for hospitalization plan, the quantitative relations among these insurance factors could be described by the following ODE system:

$$\begin{aligned}
P'(t) &= P_{AH}\, e^{-\delta t} S(t)\ , \qquad t > 0\ , && (17) \\
B'(t) &= e^{-\delta t} I(t)\ , \qquad t > 0\ , && (18) \\
V'(t) &= P_{AH}\, e^{\delta t} S(t) - e^{\delta t} I(t)\ , \qquad t > 0\ , && (19)
\end{aligned}$$

where $P_{AH} = \overline{P}(\overline{A}^i_0)$ is determined by the equivalence principle. By applying the RK-4 method, we should obtain $\overline{a}^s_{0:\overline{t}|} = P(t)/N/P_{AH}$, $\overline{a}^i_{0:\overline{t}|} = B(t)/N$ and ${}_t\overline{V}(\overline{a}^i_0) = V(t)/N$.

For the lump sum for hospitalization plan, we have the following insurance factor system:

$$\begin{aligned} P'(t) &= P_{SH}\, e^{-\delta t} S(t)\ , \qquad t > 0\ , && (20) \\ B'(t) &= e^{-\delta t}\beta\, S(t)\, I(t)\ , \qquad t > 0\ , && (21) \\ V'(t) &= P_{SH}\, e^{\delta t} S(t) - e^{\delta t}\beta\, S(t)\, I(t)\ , \qquad t > 0\ , && (22) \end{aligned}$$

where $P_{SH} = \overline{P}(\overline{a}^i_0)$ is also from the equivalence principle, $\overline{a}^s_{0:\overline{t}|} = P(t)/N$, $\overline{A}^i_{0:\overline{t}|} = B(t)/N$ and ${}_t\overline{V}(\overline{A}^i_0) = V(t)/N$.

Similarly, the annuity for hospitalization plan with a death benefit is determined by the system

$$\begin{aligned} P'(t) &= P_{AHD}\, e^{-\delta t} S(t)\ , \qquad t > 0\ , && (23) \\ B'(t) &= e^{-\delta t} I(t)(1+\alpha)\ , \qquad t > 0\ , && (24) \\ V'(t) &= P_{AHD}\, e^{\delta t} S(t) - e^{\delta t} I(t)(1+\alpha)\ , \qquad t > 0\ , && (25) \end{aligned}$$

and $P_{AHD} = \overline{P}(\overline{A}^i_0 + \overline{A}^d_0)$ is from the equivalence principle, $\overline{a}^s_{0:\overline{t}|} = P(t)/N/P_{AHD}$, $\overline{A}^i_{0:\overline{t}|} + \overline{A}^d_{0:\overline{t}|} = B(t)/N$ and ${}_t\overline{V}(\overline{A}^i_0 + \overline{A}^d_0) = V(t)/N$.

Finally, for the lump sum for hospitalization plan with a death benefit, the corresponding system is:

$$\begin{aligned} P'(t) &= P_{SHD}\, e^{-\delta t} S(t)\ , \qquad t > 0\ , && (26) \\ B'(t) &= e^{-\delta t} I(t)(\beta S(t) + \alpha)\ , \qquad t > 0\ , && (27) \\ V'(t) &= P_{SHD}\, e^{\delta t} S(t) - e^{\delta t} I(t)(\beta S(t) + \alpha)\ , \qquad t > 0\ . && (28) \end{aligned}$$

Hence, the equivalence principle gives $P_{SHD} = \overline{P}(\overline{a}^i_0 + \overline{A}^d_0)$, $\overline{a}^s_{0:\overline{t}|} = P(t)/N/P_{SHD}$, $\overline{a}^i_{0:\overline{t}|} + \overline{A}^d_{0:\overline{t}|} = B(t)/N$ and ${}_t\overline{V}(\overline{a}^i_{0:\overline{t}|} + \overline{A}^d_{0:\overline{t}|}) = V(t)/N$.

These ODE systems can be readily solved in most mathematical software such as Maple environment. Information about programming with ODE tool kits in Maple can be found in Coombes [9].

4.2 Infection table based approximation

In practice it is difficult to make record of susceptible individuals, partly because of its enormous number in a population and partly due to the fact that an person susceptible to a certain disease appears no different from one with immunity. But we could keep track of infected people using public data from government health agencies and hospitals. Hence we now rely on the function $i(t)$ instead of $s(t)$ for all calculations leading to the premium rating.

A natural analogy here is with the life table in life insurance mathematics, which virtually describes an empirical survival distribution of an average person's longevity. Similarly, an infection table can be generated to keep record of the number of infected cases reported during each sampling period (e.g., every day for SARS). Table 1 in Section 6.1 is a simple example of an infection table dated back to the seventeenth century.

Now from the infection table, we have a piecewise constant empirical approximation of the continuous function $i(t)$ given by

$$\tilde{\imath}(t) = \begin{cases} i_k\ , & k-1 < t \leq k \\ 0\ , & \text{otherwise} \end{cases} \quad .$$

Using this function in place of $i(t)$ in (7) gives an approximation to $\overline{a}^{i}_{0:\overline{t}|}$:

$$\begin{aligned} \overline{a}^{i}_{0:\overline{t}|} = \int_0^t e^{-\delta z}\ i(z)\ dz \quad &\approx \quad \int_0^z e^{-\delta t}\ \tilde{\imath}(z)\ dz \\ &\approx \quad \sum_{k=1}^{n} \frac{e^{-\delta(k-1)} - e^{-\delta k}}{\delta}\ i_k\ , \end{aligned}$$

where $n = [t]$, the integer part of t , for n large enough.

4.3 Power series solutions

The power series method is one of the oldest techniques used to solve linear differential equations. This method can be adapted well to our SIR model.

Since every point in the system is an ordinary point, in particular, $t = 0$, we look for solutions of the form

$$s(t) = \sum_{n=0}^{\infty} a_n\ t^n\ , \qquad t \geq 0\ , \tag{29}$$

$$i(t) = \sum_{n=0}^{\infty} b_n\ t^n\ , \qquad t \geq 0\ . \tag{30}$$

Therefore, differentiating term by term yields

$$s'(t) = \sum_{n=1}^{\infty} n\ a_n\ t^{n-1} = \sum_{n=0}^{\infty} (n+1)\ a_{n+1}\ t^n\ , \qquad t \geq 0\ ,$$

$$i'(t) = \sum_{n=1}^{\infty} n\ b_n\ t^{n-1} = \sum_{n=0}^{\infty} (n+1)\ b_{n+1}\ t^n\ , \qquad t \geq 0\ .$$

Multiplying (29) by (30) gives,

$$s(t)i(t) = \sum_{n=0}^{\infty} c_n\ t^n\ , \qquad t \geq 0\ ,$$

where

$$c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_{n-1} b_1 + a_n b_0\ .$$

From (4), we obtain

$$\sum_{n=0}^{\infty}(n+1)\, a_{n+1}\, t^n + \beta \sum_{n=0}^{\infty} c_n\, t^n = 0\ ,$$

$$\sum_{n=0}^{\infty}(n+1)\, b_{n+1}\, t^n - \beta \sum_{n=0}^{\infty} c_n\, t^n + \alpha \sum_{n=1}^{\infty} n\, b_n\, t^{n-1} = 0\ .$$

To satisfy these equations for all t, it is necessary that the coefficient of each power of t be zero. Hence we obtain the following recurrent relation:

$$a_{n+1} = -\frac{\beta}{n+1}(a_0 b_n + a_1 b_{n-1} + \cdots + a_{n-1} b_1 + a_n b_0)\ ,$$

$$b_{n+1} = -a_{n+1} - \frac{\alpha}{n+1} b_n\ .$$

Therefore,

$$\overline{a}^{i}_{0:\overline{t}|} = \sum_{n=0}^{\infty} \int_0^t a_n\, z^n\, e^{-\delta z}\, dz\ .$$

4.4 Integral equation approach

A considerable difficulty in solving the ODE system (4)-(5) is due to the presence of the nonlinear term $\beta\, S(t)\, I(t)$. However, its multiplicative structure inspires the use of an inverse Fourier transform.

Recall that

$$\int_{\mathbb{R}} e^{itx} f * g(x)\, dx = \int_{\mathbb{R}} e^{itx} f(x)\, dx \int_{\mathbb{R}} e^{itx} g(x)\, dx\ ,$$

where $f * g(x) = \int_{\mathbb{R}} f(x-y)\, g(y)\, dy$. Therefore, if we think of $S(t)$ as a Fourier transform of a certain function $\tilde{S}(x)$ and $I(t)$ as a Fourier transform of a certain function $\tilde{I}(t)$, i.e.

$$\tilde{S}(x) = \frac{1}{2\pi}\int_0^{\infty} S(t) e^{-itx}\, dt, \qquad \tilde{I}(x) = \frac{1}{2\pi}\int_0^{\infty} I(t) e^{-itx}\, dt\ .$$

Now write the nonlinear term as

$$\beta S(t) I(t) = \beta \int_{-\infty}^{\infty} e^{itx} \tilde{S} * \tilde{I}(x)\, dx\ .$$

Therefore, we take the inverse Fourier transform on both sides of (4), giving

$$ix\tilde{S}(x) - \frac{1}{2\pi} S(0) = -\beta \int_{-\infty}^{\infty} \tilde{S}(y) \tilde{I}(x-y) dy\ , \qquad x \in \mathbb{R}\ ,$$

$$ix\tilde{I}(x) - \frac{1}{2\pi} I(0) = \beta \int_{\mathbb{R}} \tilde{S}(y) \tilde{I}(x-y) dy - \alpha \tilde{I}(x)\ , \qquad x \in \mathbb{R}\ . \qquad (31)$$

Adding up the above two equations gives

$$\tilde{S}(x) = \frac{1}{2\pi ix}\big[S(0)+I(0)\big] - \frac{(ix+\alpha)}{ix}\tilde{I}(x)\ , \qquad x\in\mathbb{R} \tag{32}$$

Substitute (32) into (31) to obtain the following integral equation

$$\tilde{I}(x) = \frac{\beta}{(ix+\alpha)}\int_{-\infty}^{\infty}\big[\frac{N}{2\pi iy} - \frac{\alpha+iy}{iy}\tilde{I}(y)\big]\tilde{I}(x-y)\,dy + \frac{1}{2\pi}I(0)\ , \qquad x\in\mathbb{R}\ .$$

By the definition (7), we have then

$$\begin{aligned}\overline{a}^{i}_{0:\overline{t}|} &= \frac{1}{N}\int_0^t e^{-\delta z}\int_{-\infty}^{\infty} e^{izx}\tilde{I}(x)\,dx\,dz \\ &= \frac{1}{N}\int_{-\infty}^{\infty}\frac{e^{(ix-\delta)t}-1}{(ix-\delta)}\tilde{I}(x)\,dx\ .\end{aligned}$$

5 Premium adjustment

In actuarial mathematics, the fact that mortality rises with age leads to the consequence that an insurer's future financial liability is always greater than future revenue from benefit premiums. Therefore the benefit reserve is normally positive in traditional life insurance products. Unlike the "U" shape of mortality curve, a unique feature of epidemics is that the infection rates rapidly increases at the beginning and then drops down after reaching a peak. Figure 2 illustrates a typical path of a benefit reserve function obtained from the insurance quantities system (17) - (19), where the benefit premium is determined by the means employed in (13).

Although the equivalence principle is applied from time 0 to 5, it is dangerous for an insurer to have a long standing negative reserve, which indicates so much more expenses are paid out than premiums collected that the insurer actually becomes an debtor to all policyholders. A negative reserve could severely increase an insurer's risk of insolvency, and in worst scenario might even cause bankruptcy. Another potential hazard is that the insurance policy virtually becomes a certificate of debts. It is likely that policyholder might withdraw from the insurance simply by stopping paying premiums. Therefore a prudent insurer would require additional premium in order to keep reserve above an early warning level, which we choose zero in our analysis.

Before giving an algorithm for determining an added-value premium, we would like to study for a moment the trend of a benefit reserve function $V(t)$ and its dependency on functions $S(t)$ and $I(t)$.

Proposition 5.1. *For the SIR model in (1)-(3), $S(t)$ is a monotonically decreasing function, and $R(t)$ is monotonically increasing. If $S(0)\le \alpha N/\beta$, then $I(t)$ is a monotonically decreasing function; If $S(0)>\alpha N/\beta$, $I(t)$ increases up to the time when $S(t)=\alpha N/\beta$, and then decreases after.*

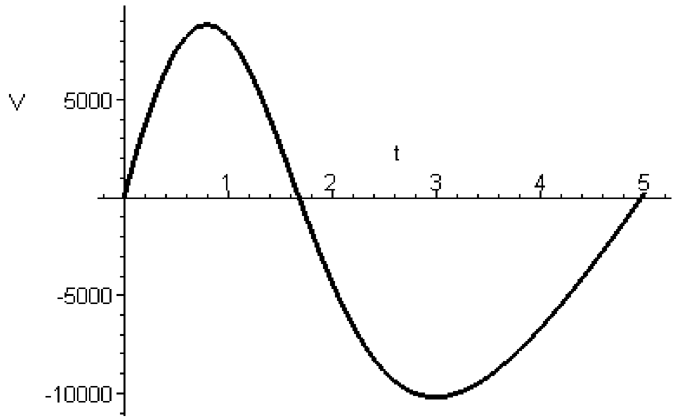

Figure 2: Benefit reserve function $V(t)$ for AH plan for the Great Plague, $P_{AH} = 106.51$. Double arch structure as explained in Proposition 5.3.

Proof. Since $S(t)$ and $I(t)$ are all non-negative, from (1) and (3) we know that $S'(t) = -\beta\ S(t)\ I(t)/N < 0$, for $t > 0$, and $R'(t) = \alpha\ I(t) > 0$. Hence $S(t)$ is a monotonically decreasing function and $R(t)$ is a monotonically increasing function. If $S(0) \leq \alpha/\beta$, then $I'(t) = I(t)[\beta\ S(t)/N - \alpha] < 0$, which means that $I(t)$ is monotonically decreasing. By contrast, if $S(0) > \alpha N/\beta$, because $S(t)$ is monotonically decreasing, then $I'(t) = I(t)[\beta\ S(t)/N - \alpha] > 0$, as long as $S(t) > \alpha N/\beta$. Thus $I(t)$ reaches its local maximum at the point where $S(t) = \alpha N/\beta$. When $S(t)$ continues to decrease below $\alpha N/\beta$, $I'(t) < 0$ and $I(t)$ is monotonically decreasing thereafter. □

From now on in this section, we study the benefit reserve by limiting our focus on an AH plan where the force of interest is considered zero. The generalization to other plans follows the same idea. From (19), we know that $V'(t) = P_{AH}\ S(t) - I(t)$, for $t \geq 0$. The sign of an instantaneous change in $V(t)$ depends on the two competing forces, monotonically decreasing $P_{AH} \cdot S(t)$ and increasing-then-decreasing $I(t)$. There are only two possible geometrical structures in the trend of $V(t)$: Single arch structures as shown in Figure 3 and double arch structures typically illustrated in Figure 2. The following propositions indicate conditions under which the two structures may appear, respectively.

Proposition 5.2. *(Single Arch Structure) In the insurance quantities system (17) - (19), the benefit reverse $V(t)$ is concave, if the premium*

$$P_{AH} > \frac{\alpha N}{\beta S_\infty} - 1 \ , \tag{33}$$

where the constant $c = I_0 + S_0 - \alpha N/\beta\ \log(S_0)$ and $S_\infty = \lim_{t\to\infty} S(t)$.

Proof. To check the concavity of $V(t)$, we look at $V''(t)$,

$$\begin{aligned} V''(t) &= P_{AH}S'(t) - I'(t) \\ &= -\frac{\beta}{N}\, P_{AH}\; S(t)\; I(t) - \frac{\beta}{N}\; S(t)\; I(t) + \alpha\; I(t) \\ &= I(t)\; [\alpha - \frac{\beta}{N}\; (P_{AH} + 1)S(t)]\;. \end{aligned}$$

It follows that when

$$P_{AH} > \frac{\alpha N}{\beta S(t)} - 1\;, \qquad \text{for all } t > 0\;,$$

$V(t)$ is concave downward. Since $S(t)$ is monotonically decreasing, thus condition (33) is required. □

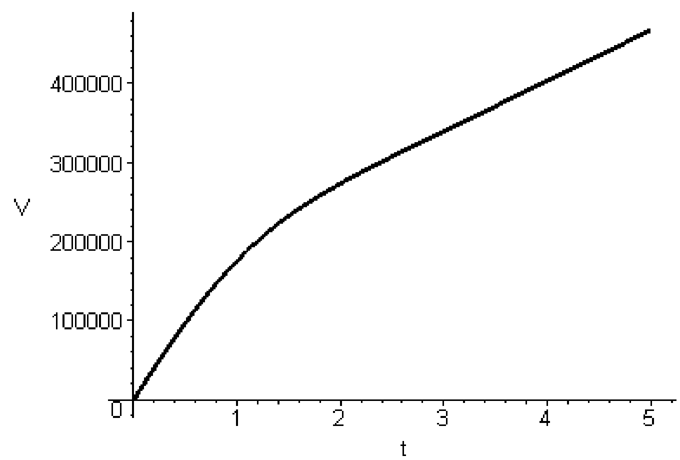

Figure 3: Benefit reserve function $V(t)$ for AH plan for the Great Plague, $P_{AH} =$ 843.38, Single arch structure as explained in Proposition 5.2.

Proposition 5.3. *(Double Arch Structure) If*

$$\frac{\alpha N}{\beta S_0} - 1 < P_{AH} < \frac{\alpha N}{\beta S_\infty} - 1\;, \tag{34}$$

then the benefit reserve $V(t)$ changes from concave to convex, with a point of inflection t_f such that

$$S(t_f) = \frac{\alpha N}{(1 + P_{AH})\beta}\;. \tag{35}$$

Proof. In view of (34), $V''(t)$ changes from negative to positive at time t_f, when

$$S(t_f) = \frac{\alpha N}{(1+P_{AH})\beta} \ .$$

Therefore, no matter whether $V'(t)$ starts from a negative or positive value, $V(t)$ goes through two phases, from concave to convex. □

The next question that comes to mind is how to control the extent of the "deficit" in the reserve by premium adjustment, while preserving the equivalence principle. From the double arch structure, we know that the biggest "deficit" in reserves comes at the local minimum in the second arch. Therefore we are interested in locating the minimum point on the time scale and looking into the connection between the premium rates and the local minimum point .

We observe from (35) that t_f is an increasing function of the premium rate P_{AH} , i.e. as we increase the premium, the point of inflection between two arches moves forward on the time scale. Another time point of interest denoted by t_m is the one when the reserve function reaches its local minimum in the second arch,

$$\frac{I(t_m)}{S(t_m)} = P_{AH} \ .$$

We would show in the next proposition that t_m is an increasing-then-decreasing function with respect to P_{AH} . Therefore, as the premium rates increase, the local minimum would eventually move backward on the time scale. As one can imagine, when the point of inflection gets closer to the next local minimum, the curve in between becomes flatter. It is a natural conjecture that as the premium P_{AH} rises to a critical value $P^{\star}_{AH}$, there must be a corresponding time point when t_f overlaps with t_m, as shown in Figure 4. Thus,

$$\frac{\alpha N}{\beta[1 + I(t_m)/S(t_m)]} = S(t_m) \ ,$$

which implies that $S(t_m) + I(t_m) = \alpha N/\beta$. It is not surprising to obtain the same critical value in the following proposition.

Proposition 5.4. *For the insurance quantities system in (17) - (19), the reserve $V(t)$ is concave and strictly increasing, if*

$$P_{AH} > P^{\star}_{AH} = \frac{\alpha N}{\beta} \ \exp\left(\frac{\beta c}{\alpha N} - 1\right) - 1 \ , \tag{36}$$

where the constant $c = I_0 + S_0 - \alpha N/\beta \ \log(S_0)$.

Proof. To ensure that $V'(t) > 0$, we need

$$P_{AH} > \frac{I(t)}{S(t)} \ , \qquad \text{for all } t \ ,$$

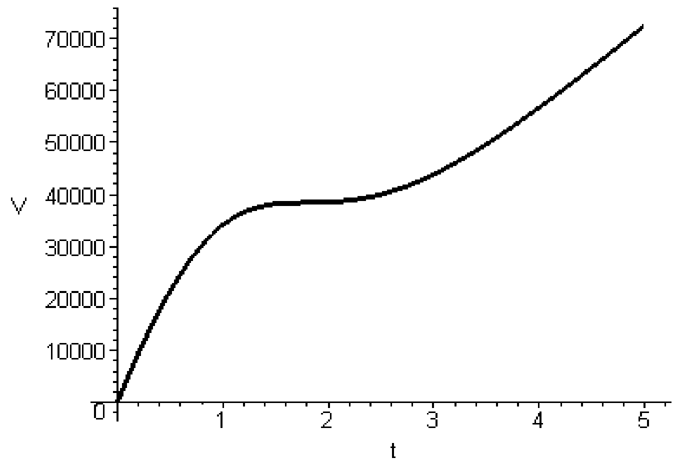

Figure 4: Benefit reserve function $V(t)$ for AH plan for the Great Plague, $P_{AH} = 202.17$. Double arch structure and strictly increasing as explained in Proposition 5.4.

or equivalently,

$$\log P_{AH} > \log I(t) - \log S(t) \;, \qquad \text{for all } t \;.$$

Let $f(t) = \log I(t) - \log S(t)$, then

$$\begin{aligned} f'(t) &= \frac{I'(t)}{I(t)} - \frac{S'(t)}{S(t)} \\ &= \frac{\beta}{N}[S(t) + I(t)] - \alpha \;, \qquad \text{by (1) and (2).} \end{aligned}$$

Since $S(t) + I(t) = N - R(t)$ is monotonically decreasing, at the time t_m when

$$S(t_m) + I(t_m) = \alpha N/\beta \;, \tag{37}$$

$f'(t)$ changes from positive to negative and $f(t)$ reaches its maximum at time t_m. Thus P_{AH} is required to be greater than $I(t_m)/S(t_m)$.

Now since

$$\frac{I'(t)}{S'(t)} = \frac{dI(t)}{dS(t)} = \frac{(\beta S(t)/N - \alpha)I(t)}{-\beta S(t)I(t)/N} = -1 + \frac{\alpha N}{\beta S(t)} \;,$$

integrating to find the orbits of the (S, I)-plane gives:

$$I(t) + S(t) - \frac{\alpha N}{\beta} \log S(t) = c \;, \tag{38}$$

where c is a constant of integration for each specific orbit, say $c = I_0 + S_0 -$

$\alpha N/\beta\ \log(S_0)$. Combining (37) and (38), we can solve for $S(t)$ and $I(t)$, as

$$S(t_m) = \exp\left(1 - \frac{\beta c}{\alpha N}\right) , \quad (39)$$

$$I(t_m) = \frac{\alpha N}{\beta} - \exp\left(1 - \frac{\beta c}{\alpha N}\right) . \quad (40)$$

Hence

$$\log P_{AH} > f(t_m) . \quad (41)$$

Substituting (39) and (40) into (41) gives the condition (36). □

From the above analysis, we realize that as the premium tends to $P^{\star}_{AH}$, the local minimum in the second arch and the point of inflection move towards each other, which implies that the local maximum in the first arch approaches the local minimum in the second arch as well. They all converge at the time point t_m . Therefore $V(t_m)$ should shift upwards as the premium rates increase. We can infer that a proper premium rate between $\overline{P}(\overline{A}^{i}_{0:\overline{t}|})$ and $P^{\star}_{AH}$ exists in order to fulfill certain requirements on the reserves. However, it can not be found in a closed algebraic expression. Instead, an easy algorithm can determine the value.

1. Specify an early warning level which the reserve function should never go below. For example, $V(t) \geq 0$, for all t .
2. Start by setting premium rate at $P^{(0)} = \overline{P}(\overline{A}^{i}_{0:\overline{t}|})$.
3. Increment the premium each time by a monetary unit, say, $P^{(n)} = P^{(n-1)} + 0.01$.
4. Calculate the resulting $V(t_m)$, and see if it is greater than zero. If yes, $P^{(n)}$ gives a reasonable adjusted premium. Otherwise, repeat the last step.
5. By the fair premium principle, a survival benefit should be awarded to the remaining susceptible policy holders when the policy duration t ends. The benefit amount is determined by $V(t)/S(t)$.

6 Numerical examples

The epidemiological model in our first numerical example of Great Plague in Eyam was originally studied by Raggett [16], and has been considered as a classical case study in many textbooks because predictions from the model are remarkably close to actual data. The second example of six compartment model came from Chowell *et al.* [8], in which parameters were primarily used for measuring basic reproduction number.

6.1 Great plague in Eyam

The village of Eyam near Sheffield, England, suffered an horrific outbreak of bubonic plague in 1665-1666. The plague was survived by only 83 of an initial population of 350 villagers, and detailed records were preserved as shown in Table 1. In Raggett [16], the disease in Eyam was fitted by the SIR model, over the period from mid-May to mid-October 1666, measured in months with an initial population of 7 infectives and 254 susceptibles, and a final population of 83. Since the disease was fatal at that time, infected individuals eventually died due to the disease.

Date	**Susceptibles**	**Infectives**
Initial	254	7
July 3/4	235	14.5
July 19	201	22
August 3/4	153.5	29
August 19	121	21
September 3/4	108	8
September 19	97	8
October 4/5	Unknown	Unknown
October 20	83	0

Table 1: Eyam plague observation of susceptible and infective populations in 1666. Data source: Raggett [16], Table II.

According to (38),

$$I_0 + S_0 - \frac{\alpha N}{\beta} log S_0 = I_\infty + S_\infty - \frac{\alpha N}{\beta} log S_\infty \ ,$$

from which we obtain an expression for $\beta/(\alpha N)$ in terms of the measurable quantities S_0, I_0, S_∞ and I_∞, namely

$$\frac{\beta}{\alpha N} \approx \frac{log \frac{S_0}{S_\infty}}{S_0 - S_\infty} \ .$$

The relation with $S_0 = 254$, $I_0 = 7$ and $S_\infty = 83$ gives $\alpha N/\beta = 153$. The parameter α is determined by its reciprocal, which has the clinical meaning of an average infectious period. From clinical observations, an infected person stays infectious for an average of 11 days or 0.3667 months before death, so that $\alpha = 2.73$ and $\beta/N = 0.0178$. The resulting graphs of S and I, as functions of time t, are given in Figures 5.

Insurance coverage would not directly reduce the transmission of the disease, but a well-designed insurance system could have provided financial incentives for prevention measures and compensate for hospitalization and other medical costs and services. To develop this insurance model, we assume that everyone in the village foresees the coming of the Great Plague and willingly chips in the mutual insurance group at the beginning of the epidemic. The insurance funding

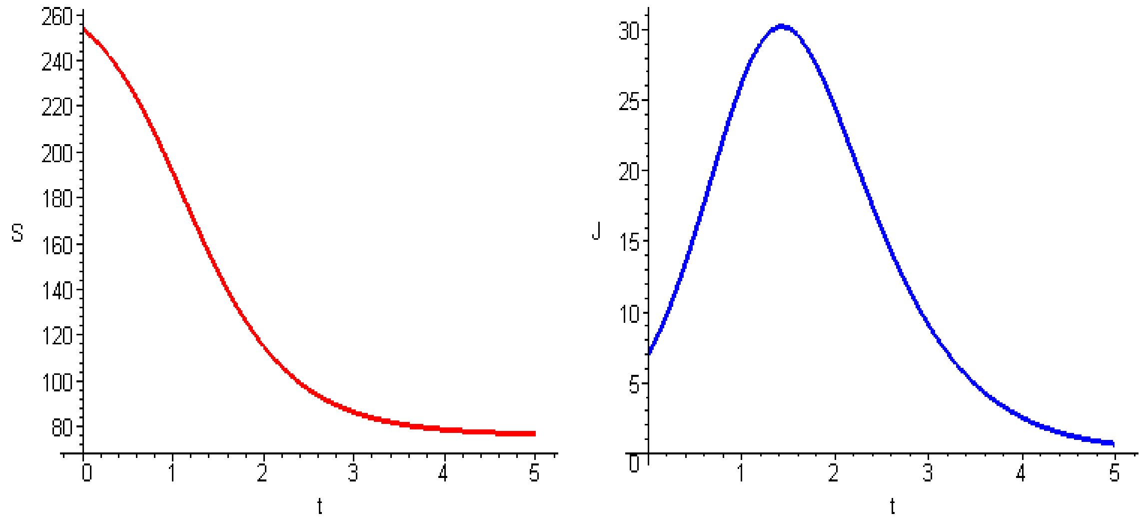

Figure 5: Function of susceptibles S(t) and infectives I(t)

is secured with a monthly force of interest of 0.2%. The insurance period lasts 5 months which matches the duration of the epidemic.

Plan	P.V. Benefits	P.V. Premiums	Level Premium
1	65015.62	610.41	106.51
2	242508.27	610.41	397.29
3	172385.38	610.41	282.41
4	349878.02	610.41	573.18

Table 2: Eyam plague premium rating (dollar)

1. **Annuity for Hospitalization (AH)**:

   This plan provides infection benefits continuously at the rate of $1000 per month until death for every infected individual regardless of how long he or she has entered the class. The insurance liability is terminated after death. It is purchased continuously by susceptible individuals.

2. **Annuity for Hospitalization with Death Benefit (AHD)**:

   This plan contains all of the same benefits as in the previous one plus an additional death benefit of $1000 payable at the moment of death. The insurance liability is terminated after death. It is also purchased in the same pattern.

3. **Lump Sum for Hospitalization (SH)**:

This plan provides a lump sum infection benefit of $1000 at the moment of the individual being diagnosed infected. The insurance liability is terminated after death. It is purchased continuously by susceptible individuals.

4. **Lump Sum for Hospitalization with Death Benefit (SHD)**:

   This plan contains all of the same benefits as in the previous one. In addition, a death benefit of $1000 is payable to specified beneficiaries at the moment of the insured's death. The insurance liability is terminated after death. It is purchased continuously by susceptible individuals.

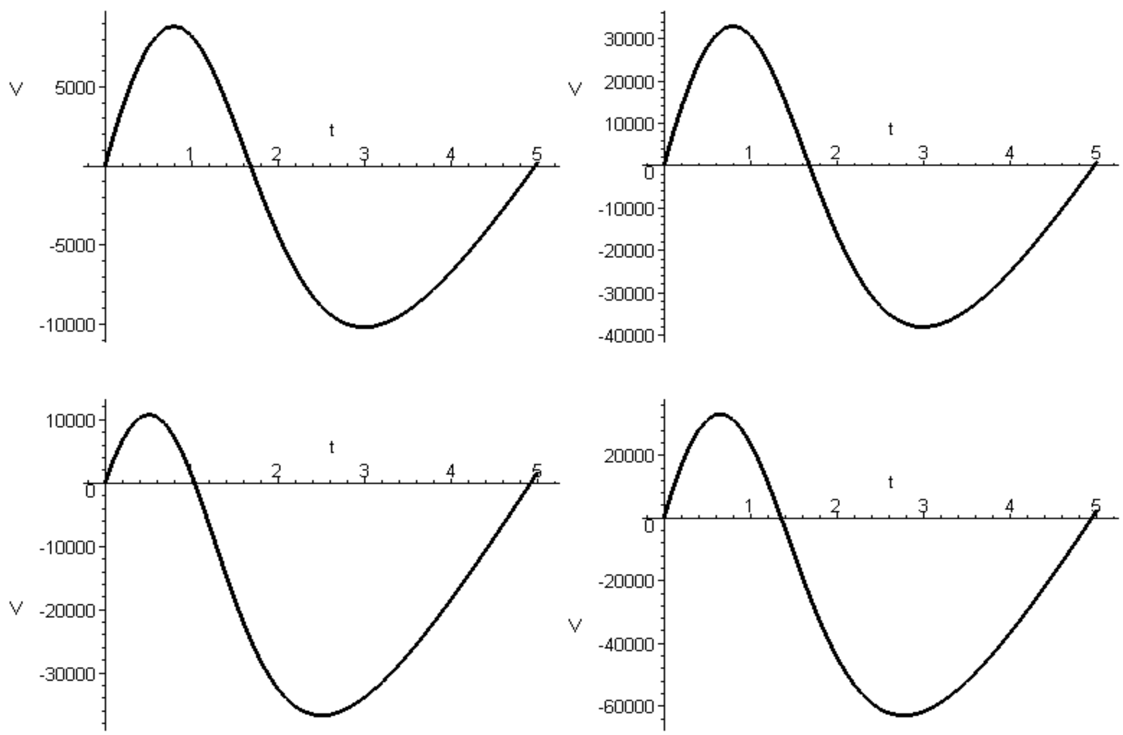

Figure 6: Benefit reserve $V(t)$ with premiums determined by the equivalence principle. Clockwise from the top left corner Plan AH, AHD, SHD and SH.

Table 2 gives net level premiums for each plan determined by the original equivalence principle. It is probably against what one might expect that the premium of the annuity for hospitalization plan is not even half the premium of the lump sum for hospitalization plan. Note that although it seems to cost more for the first plan to cover not only the newly infected, but also all existing patients. The fact is that few of them ever survived longer than a month once the disease spread out. Therefore, providing the infected with immediate reimbursement costs an insurer much more than a monthly annuity in the case of an acute fatal disease.

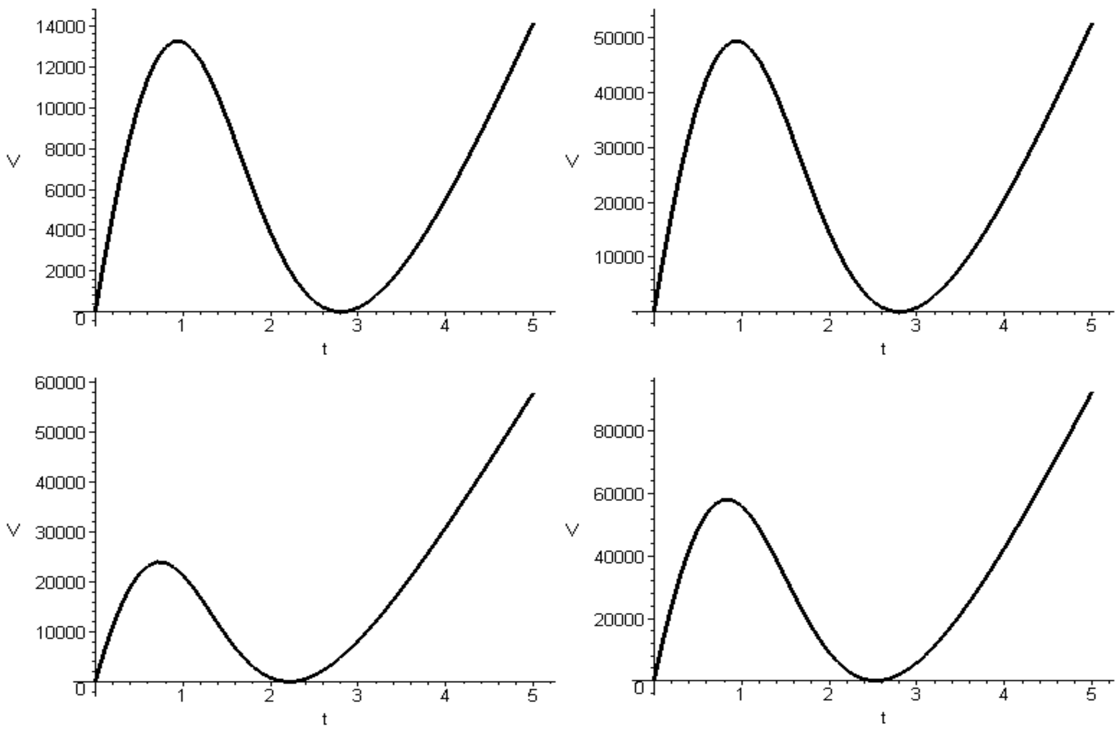

Figure 7: Benefit reserve $V(t)$ with adjusted premiums and survival benefits. Clockwise from the top left corner Plan AH, AHD, SHD and SH.

Like benefit reserves in life insurance, the reserve functions, shown in Figure 6 for the infectious disease coverage, reaches zero at the end of policy duration, when the protection ends. However, as discussed in Section 5, every reserve function in our example appears to go through a negative phase, which dangerously reduces the insurer's financial solvency. Thus there is a need for our algorithm that adjusts premiums to meet the financial requirement such as that benefit reserve must not be allowed to go under zero.

Since the death and hospitalization benefits remain the same and only premiums are raised to ensure insurers' nonnegative reserve, the equivalence cannot be established unless the insurer clears off the reserve balance in the form of another benefit for the remaining survivors. Therefore, in each plan we add a new insurance liability that every survivor of the epidemic is entitled to an equal payment of dividends. Table 3 illustrates the adjusted premiums for each plan and the final dividend payment to survivors.

Plan	Adjusted Premium	Survival Dividend	Terminal Reserve
1	128.38	184.74	14170.81
2	478.86	689.11	52858.76
3	370.76	756.95	58062.73
4	715.02	1209.73	92793.84

Table 3: Eyam plague adjusted premiums (dollar)

6.2 SARS epidemic in Hong Kong

In the classical SIR model, the implicit assumption that the mixing of members from different compartments is geographically homogeneous is probably unrealistic. The susceptible people in geographical neighborhoods of an infectious virus-carrier are more likely to be infected than those who are remote from the carrier. For instances, health care workers are at higher risk of infection than most other groups in a population.

To distinguish different levels of vulnerability or infectiousness within different social groups, spatial structures were introduced and developed in epidemiological studies. A typical example of a spatial structure applied to the SARS epidemic in Hong Kong is defined by Chowell *et al.* [8] in the following ODE system,

$$S_1'(t) = -\beta S_1(t)\frac{I(t)+qE(t)+lJ(t)}{N}\,, \qquad t \geq 0\,, \tag{42}$$

$$S_2'(t) = -\beta p S_2(t)\frac{I(t)+qE(t)+lJ(t)}{N}\,, \qquad t \geq 0\,, \tag{43}$$

$$E'(t) = \beta(S_1(t)+pS_2(t))\frac{I(t)+qE(t)+lJ(t)}{N} - kE(t)\,, \qquad t \geq 0\,, \tag{44}$$

$$I'(t) = kE(t) - (\alpha+\gamma_1+\delta)I(t)\,, \qquad t \geq 0\,, \tag{45}$$

$$J'(t) = \alpha I(t) - (\gamma_2+\delta)J(t)\,, \qquad t \geq 0\,, \tag{46}$$

$$R'(t) = \gamma_1 I(t) + \gamma_2 J(t)\,, \qquad t \geq 0\,. \tag{47}$$

In this model, there are two distinct susceptible compartments with different levels of exposure to the SARS, namely S_1 for the most susceptible urban community and S_2 for the less susceptible rural population. Initially, $S_1(0) = \rho N$ and $S_2(0) = (1-\rho)N$, where ρ is the proportion of urban inhabitants in total population. An average highly susceptible person (in the Class S_1) makes an average number of β adequate contacts (i.e. contacts sufficient to transmit infection) with others per unit time. Because of less frequent visits to public areas where viruses concentrate, an average lower susceptible person (in the Class S_2) would only be exposed to an average number of $p\beta$ adequate contacts with others per unit time.

Because an individual infected with SARS may experience an incubation period of 2-7 days before the onset of any visible symptom. An infectious class is set up for those infected but not yet symptomatic. The parameter q is used to measure the lower level of infectivity during the incubation. As the

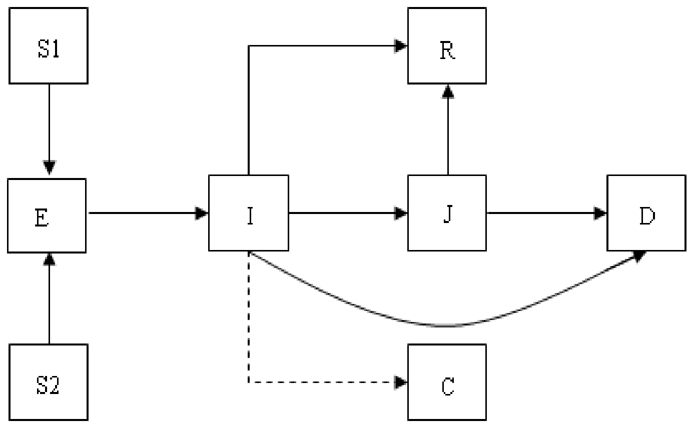

Figure 8: Transfer diagram of the SARS epidemic dynamics, Reprinted from the Figure 1 in Chowell [8].

time elapses, the infected individual would develop observable symptoms and become fully infectious in Class I with $q = 1$. In order to distinguish their potential disease transmission to general public, the Class I is separated for those infectious individuals that are undiagnosed. Since almost all diagnosed cases are quarantined in hospitals, the Class J has a lower infectivity level reflected by a reduction factor l.

The rates of population transferring from E, I and J to their chronologically adjacent compartments I, J and the recovered class R are respectively k, α and γ_2. Considering that even before being diagnosed SARS patients may either recover naturally at the rate of γ_1 or die at the force of fatality δ, we also have the class D keeping track of deaths as a result of the SARS from two sources I and J. The patients under medical treatments in Class J suffer death at the rate assumed to be the same as the mortality in Class I.

Notice that both E and I are undiagnosed phases, there is literally no statistical data for estimating their parameters. Therefore, another compartment C for reported probable cases is set aside to trace back the original time of incidences by a time series. Figure 8 gives transfer directions among the different compartments.

To avoid getting into details of parameter inference, we make use of parameter values estimated in the original paper as summarized in Table 4. These parameter values were used to compute the basic reproductive number R_0 in the original article. A defect of this model is that there appears to be some negative numbers in Classes I and J.

From an insurer's point of view, this model presents many business opportunities. On the one hand, individuals in Classes S_1 and S_2 are potential buyers facing the risk of infection with SARS. On the other hand, there is an evident need for insurance covering vaccination costs in both S_1 and S_2, medical exam-

Parameter	Moving from/to	Value
β	$S_1, S_2/E$	0.75
q	reduced infectiousness	0.1
l	reduced infectiousness	0.38
p	reduced susceptibility	0.1
k	E/I	1/3
α	I/J	1/3
γ_1	I/R	1/8
γ_2	J/R	1/5
δ	$I, J/R$	0.006
ρ	reduced contacts	0.4

Table 4: Parameter values that fit the SARS model for Hong Kong, adapted from the Table 1 in Chowell *et al.* [8].

ination expenses for probable cases in Class I, hospitalization and quarantine expenses for Class J and death benefit for Class D. Since a number of parties are involved in the health care system, such as insurance companies, policyholders, government health agencies, and hospitals. Numerous business models could be designed to bring them together in order to reduce the financial impact to the lowest level. To illustrate an easy example of such an infectious disease insurance, we design the following two plans.

1. Annuity for Hospitalization Plan

   Every participant of the mutual insurance funding purchases the coverage by means of an annuity. Rural inhabitants are charged lower premiums proportional to their reduced susceptibility. From the time of policy issue to the end of the epidemic, every insured is eligible for claiming a medical examination fee of $100,000 once observed with suspicious symptoms, and hospitalization expenses of $100,000 per day, in the form of a life annuity for the period under medical treatment in hospital. Specified beneficiaries are entitled to a death benefit of $100,000 after a covered person's death due to the infectious disease. The protection ends at the ealiest time of either the end of the epidemic or the time of the policyholder's death.

2. Lump Sum for Hospitalization Plan

   This plan contains all of the same benefits as in the previous one with the exception of a lump sum payment, instead of an annuity, of $100,000 after the policyholder is diagnosed positive with the disease. The protection also ends at the ealiest time of either the end of the epidemic or the time of the policyholder's death.

The discounted total benefits and premiums in Table 5are calculated under the assumption that all Hong Kong residents during the pandemic are enrolled in the policy. We obtained surprisingly low net level premiums for both plans, which are determined by the equivalence principle. This reinforces our assertion that an insurance plan of fairly low cost could cushion the sharp blow from a pandemic to our healthcare system.

Plan	P.V. Benefits	P.V. Premiums	Level Premium
1	3.0571×10^8	1.71604×10^8	1.78
2	1.3231×10^8	1.71604×10^8	0.77

Table 5: SARS insurance premium rating (dollar)

7 Future work

Since research in this emerging type of insurance is just at the infancy stage, much more work needs to be done to generalize the models in order to fit other aspects and features of different diseases. There have been abundant and extensive studies in epidemiological stochastic modeling. We are looking forward to having some stochastic models incorporated in our actuarial applications.

Acknowledgement

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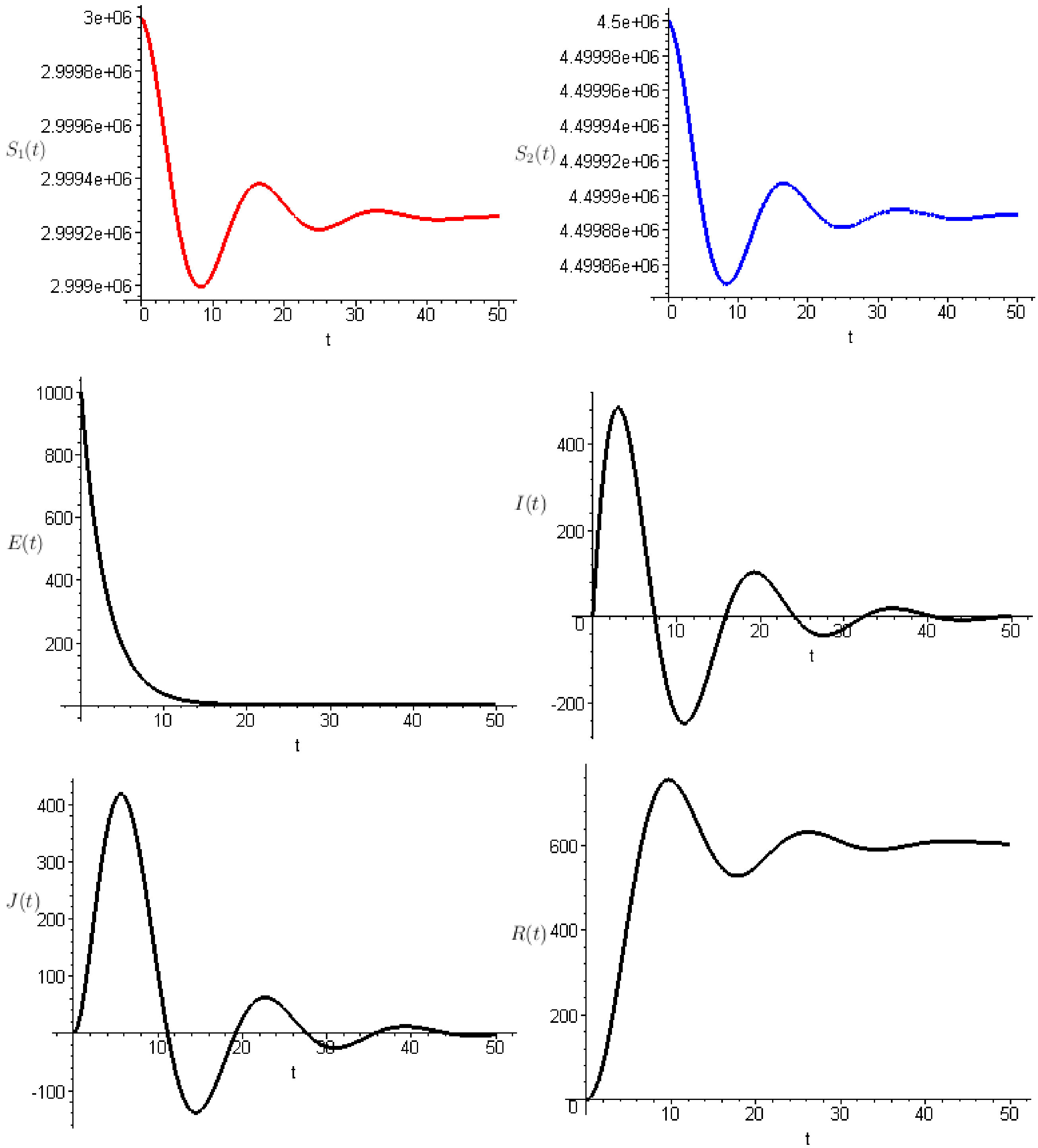

Figure 9: Functions of individual numbers in each compartment $S_1(t)$, $S_2(t)$, $E(t)$, $I(t)$, $J(t)$ and $R(t)$.