Threshold Life Tables and Their Applications

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Motivations

- The emergence of centenarians / supercentenarians often leads to low frequency high severity losses.
- Examples include:
  1. Life Annuities.
  2. Pensions.
  3. Reverse Mortgages.

The Problems

- Exposures-to-risk at very advanced ages are very small.
- Age at death may not be reported correctly.
- Life tables are often closed at an arbitrarily chosen age.
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Objectives and Outline of the Presentation

- The main objective of this study is to model mortality at the advanced ages using modern extreme value theory.
- Outline of the presentation:
  1. Current solutions to modeling old-age mortality.
  2. Our proposed solution: The Threshold Lifetable.
  3. Application of the Threshold Lifetable to the prediction of the highest attained age.

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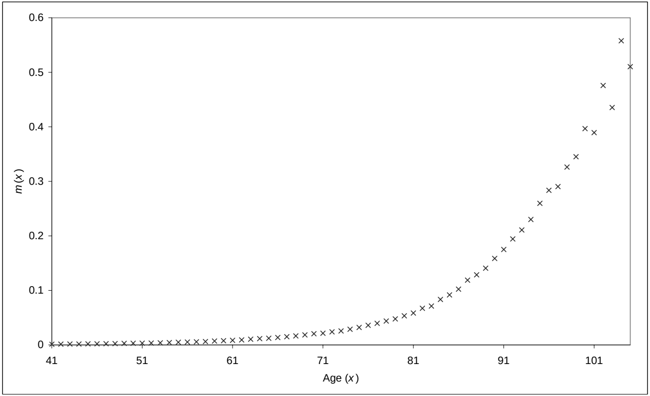
Empirical Central Rate of Death



Current Methods of Old-Age Mortality Modeling

- Cubic polynomial extrapolation (Panjer and Russo, 1991; Panjer and Tan, 1995).
- The Heligman-Pollard Model (Heligman and Pollard, 1980).
- The Coale-Kisker Method (Coale and Guo, 1989; Coale and Kisker, 1990).
- The Old-age Mortality Standard (Himes et al., 1994; United Nations, 1997).

Notations

- X : age at death random variable.
- $F(x)$: d.f. of X.
- $s(x) = 1 - F(x)$: survival function.
- d_x: number of deaths between ages x and $x + 1$.
- E_x: number of exposures-to-risk between ages x and $x + 1$.
- l_x: number of survivors to age x.
- $m_x = d_x / E_x$: central rate of death.
- $q_x = d_x / l_x$: probability of death between ages x and $x + 1$, conditioning on the survival to age x.

The Balkema-de Haan-Pickands Theorem

- Let $Y = X - d | X > d$ be the excess over the threshold d.
- The Balkema-de Haan-Pickands Theorem states that under certain regulatory conditions, the limiting distribution of Y is a Generalized Pareto Distribution as d tends to infinity.
- The Generalized Pareto Distribution:

$$F_Y(y) = 1 - \left(1 + \gamma \frac{y}{\theta}\right)^{-1/\gamma}.$$

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The Proposed Model

Specification of the Threshold Life Table:

For $x \leq N$,

$$F(x) = 1 - \exp\left(-\frac{B}{\ln C}(C^x - 1)\right);$$

For $x > N$,

$$F(x) = 1 - p\left(1 + \gamma\left(\frac{x-N}{\theta}\right)\right)^{-1/\gamma},$$

where $p = F(N)$, and N is the *threshold age*.

Estimation of Parameters

- Suppose that we are estimating a threshold life table for age 65 and above, and that we are given $l_{65}, \ldots, l_{100}$.
- Then the likelihood function is given by

$$L(B, C, \gamma, \theta; N) = \prod_{x=65}^{99} \left(\frac{s(x) - s(x+1)}{s(65)} \right)^{d_x} \left(\frac{s(100)}{s(65)} \right)^{l_{100}}.$$

- It can be shown that the log-likelihood function can be decomposed into two parts, i.e.,

$$l(B, C, \gamma, \theta; N) = l_1(B, C; N) + l_2(\gamma, \theta; N)$$

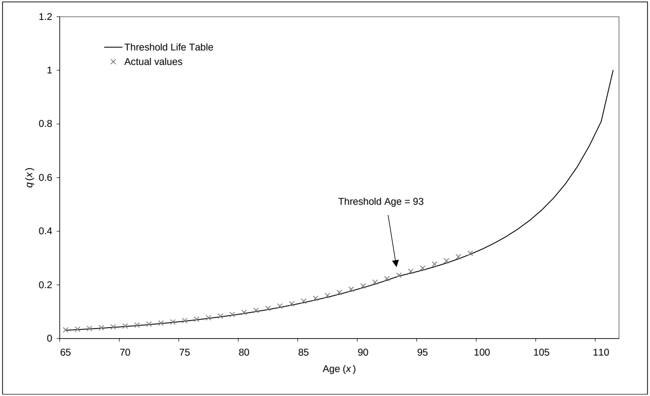
Estimation of Parameters

- The choice of N relies on the maximization of the profile likelihood:

$$l_p(N) = l(\hat{B}(N), \hat{C}(N), \hat{\gamma}(N), \hat{\theta}(N); N),$$

  where $l = \ln(L)$; $\hat{B}(N)$, $\hat{C}(N)$, $\hat{\gamma}(N)$ and $\hat{\theta}(N)$ are the maximum likelihood estimates of B, C, γ and θ for fixed N, respectively.
- Find the value of N which maximizes the profile likelihood $l_p(N)$.

A Fitted Threshold Life Table



Distribution of the Highest Attained Age

- Suppose that there are n members of a birth cohort survive to the threshold age N.
- Let Y_i, $i = 1, 2, ..., n$, be the exceedances over the threshold age. They follow a common Generalized Pareto Distribution independently.
- Let $M_n = \max\{Y_i, i = 1, 2, ..., n\}$. The distribution of M_n is given by

$$F_{M_n}(y) = [F_Y(y)]^n,$$

where $F_Y(y)$ is the distribution function of a Generalized Pareto Distribution.

Distribution of the Highest Attained Age

- Recall that when $\gamma < 0$, the distribution of Y has a finite right-hand-end support $-\theta/\gamma$.
- Therefore, the limiting distriubtion of M_n degenerates at $-\theta/\gamma$, i.e.,

$$\lim_{n\to\infty} F_{M_n}(y) = \begin{cases} 0 & y < -\theta/\gamma \\ 1 & y \geq -\theta/\gamma. \end{cases}$$

- In other words, the highest attained age of the cohort converges to $\omega = N - \frac{\theta}{\gamma}$.

Distribution of the Highest Attained Age

- Variability in the estimate of ω is inherited from that in the estimates of θ and γ.
- Hence, the asymptotic variance of $\hat{\omega}$ can be computed by the delta method, i.e.,

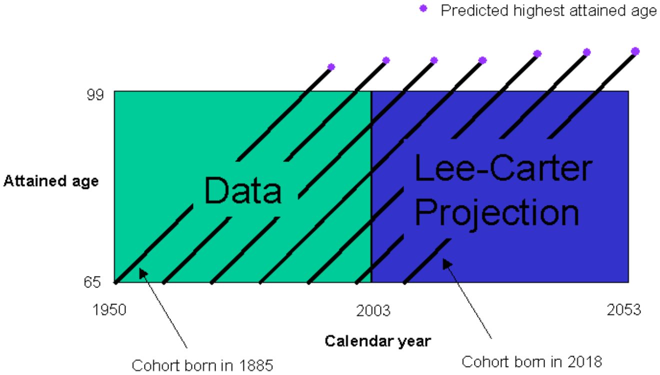
$$\mathrm{Var}(\hat{\omega}) = \left[\begin{array}{cc} \frac{\partial \omega}{\partial \gamma} & \frac{\partial \omega}{\partial \theta} \end{array} \right] [I(\gamma, \theta)]^{-1} \left[\begin{array}{c} \frac{\partial \omega}{\partial \gamma} \\ \frac{\partial \omega}{\partial \theta} \end{array} \right],$$

  where $I(\gamma, \theta)$ is the information matrix in estimating γ and θ.
- The confidence interval of ω can be obtained by assuming normality holds.

A Numerical Illustration

- We use the mortality data of the Japanese population (1950-2003). Central death rates and exposures-to-risk are available up to age 99.
- We consider age 65 and above.
- For older cohorts, death rates are available from the historical data, e.g., for cohort born in 1885 (aged 65 in 1950), we have $m_{65,1950}, m_{66,1951}, \ldots, m_{99,1984}$.
- For younger cohorts, death rates at the higher ages have to be estimated, e.g., for cohort born in 1925 (aged 65 in 1990), we have $m_{65,1990}, \ldots, m_{78,2003}, \hat{m}_{79,2004}, \ldots, \hat{m}_{99,2034}$.

A Graphical Illustration of the Methodology



Empirical Results

Year of birth	Age in 2006	Predicted omega
1893	113	114.0719 (108.0013, 120.1424)
1911	95	115.0247 (107.8065, 122.2430)
1921	85	114.6435 (109.0398, 120.1731)
1931	75	119.9964 (110.0398, 132.1508)
1941	65	121.0953 (107.0769, 132.9158)

Supercentenarians in Japan

Yone Minagawa (1893 – , 113) in Fukuoka, is the oldest *living* person.

Ura Koyama (1890 – 2005, 115) in Fukuoka, is the oldest person.

Mitoyo Kawate (1889 – 2003, 114) in Hiroshima, is the second oldest person.

Concluding Remarks

- Features of the Threshold Life Table:
  1. It does not require a subjective decision on the age at which extrapolation begins
  2. The form of the extrapolation is justified by the results of the extreme value theory.
  3. The end point of life table can be determined statistically.

- Uncertainty about future death probabilities would affect the prediction of the highest attained age.