A Bias Reduction Technique for Monte Carlo Pricing of Early Exercise Options

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Overview

- Biased Monte Carlo Estimators.
- Bias Evaluation and Removal.

American-style Derivatives

- An American-style option that pays the holder P_w upon exercise at time w has time-t value $(t \leq T)$ given by

$$B_t = \sup_{\tau} \mathbb{E}\left[\mathrm{e}^{-r(\tau - t)} P_\tau \middle| \mathcal{F}_t \right].$$

- Working backwards in time from the option maturity, the option can be priced using a recursive scheme.
- Drawbacks
  - Methods break down in high dimensions.
  - No information on path properties.

Monte Carlo

- Compute the continuation value of the claim.
  Carriere 1996, Longstaff and Schwartz 2001,
  Broadie and Glasserman 1995, 1997, 1998.
- MC methods typically generate estimators that are biased (but consistent).
- It is common to use both a high- and low-biased estimator.
  Here we discuss only high-biased estimators.
- In a stochastic tree, replace the exact values

$$H_w = \mathbb{E}\left[\mathrm{e}^{-r\Delta T} B_{w+1} \middle| \mathcal{F}_w\right] \text{ and } B_w = \max(P_w, H_w).$$

with the estimators

$$\tilde{H}_w^{\mathbf{i}} = \frac{1}{M} \sum_{j=1}^{M} \mathrm{e}^{-r\Delta T} \tilde{B}_{w+1}^{\mathbf{i},j} \text{ and } \tilde{B}_w^{\mathbf{i}} = \max(P_w^{\mathbf{i}}, \tilde{H}_w^{\mathbf{i}}).$$

Estimator Error

	Held: $\tilde{H}_{w+1} > P_{w+1}$	Exercised: $\tilde{H}_{w+1} < P_{w+1}$
Should Hold: $H_{w+1} > P_{w+1}$	$\tilde{H}_{w+1} - H_{w+1}$	$P_{w+1} - H_{w+1}$
Should Exercise: $H_{w+1} < P_{w+1}$	$\tilde{H}_{w+1} - P_{w+1}$	0

- The error is

$$X_{w+1} = \mathbb{1}_{H_{w+1} > P_{w+1}} \left(\tilde{H}_{w+1} - H_{w+1} \right) + \mathbb{1}_{H_{w+1} < P_{w+1}} \mathbb{1}_{\tilde{H}_{w+1} > P_{w+1}} \left(\tilde{H}_{w+1} - P_{w+1} \right) + \mathbb{1}_{H_{w+1} > P_{w+1}} \mathbb{1}_{\tilde{H}_{w+1} < P_{w+1}} (P_{w+1} - H_{w+1})$$

Bias

- Let $\tilde{Y}_{w+1} = \tilde{H}_{w+1} - P_{w+1}$ and $Y_{w+1} = H_{w+1} - P_{w+1}$.
- The bias is

$$\mathbb{E}\left[\mathrm{e}^{-r\Delta T} X_{w+1} \left|\mathcal{F}_w\right.\right] = \mathrm{e}^{-r\Delta T}\mathbb{E}\left[\mathbb{1}_{Y_{w+1}>0}\left(\tilde{Y}_{w+1} - Y_{w+1}\right) + \mathbb{1}_{Y_{w+1}<0}\mathbb{1}_{\tilde{Y}_{w+1}>0}\tilde{Y}_{w+1} - \mathbb{1}_{Y_{w+1}>0}\mathbb{1}_{\tilde{Y}_{w+1}<0}Y_{w+1}\middle|\mathcal{F}_w\right]$$

Note:

1. $\left(\tilde{H}_{w+1}, H_{w+1}, P_{w+1}\right)$ are functions of $\mathbf{S}_{w+1}$.
2. The bias is an integral over the joint density of $\mathbf{S}_{w+1}$.
3. Can express this as an integral over the joint density of $\left(\tilde{Y}_{w+1}, Y_{w+1}\right)$.
4. Evaluation?

Bias

Reminder:

$$\tilde{H}^{\mathbf{i}}_{w+1} = \frac{1}{M}\sum_{j=1}^{M} \mathrm{e}^{-r\Delta T}\tilde{B}^{\mathbf{i},j}_{w+2} \quad \text{and} \quad H_{w+1} = \mathbb{E}\left[\mathrm{e}^{-r\Delta T} B_{w+2} \middle| \mathcal{F}_{w+1}\right]$$

Assumptions:

1. $\mathbb{E}\left[\tilde{H}_{w+1} \middle| \mathcal{F}_{w+1}\right] = H_{w+1}$.
2. By the CLT $\tilde{H}_{w+1} - P_{w+1} \sim \mathrm{N}\left(H_{w+1} - P_{w+1}, \frac{\Sigma_{w+1}}{M}\right)$

Implication

- Bias can be approximated by an integral over the joint density of $\left(\tilde{Y}_{w+1}, \Sigma_{w+1}\right)$.

Bias and Correction

- Bias can be expressed as

$$\mathbb{E}\left[\mathrm{e}^{-r\Delta T}X\right] \approx \mathrm{e}^{-r\Delta T}\int_{\infty}^{\infty}\int_{0}^{\infty}|\tilde{y}|\,\Phi\left(\frac{-|\tilde{y}|}{\sigma/\sqrt{M}}\right)f_{\tilde{Y},\Sigma}(\tilde{y},\sigma)d\sigma d\tilde{y}$$

- Thus the bias-corrected estimator for the hold value is

$$\tilde{H}_w^{\mathbf{i}} = \frac{1}{M}\sum_{j=1}^{M}\mathrm{e}^{-r\Delta T}\left(\tilde{B}_{w+1}^{\mathbf{i},j} - \left|\tilde{H}_{w+1}^{\mathbf{i},j} - P_{w+1}^{\mathbf{i},j}\right|\Phi\left(\frac{-\left|\tilde{H}_{w+1}^{\mathbf{i},j} - P_{w+1}^{\mathbf{i},j}\right|}{\sqrt{\Sigma_{w+1}^{\mathbf{i},j}/M}}\right)\right)$$

Example: Setup

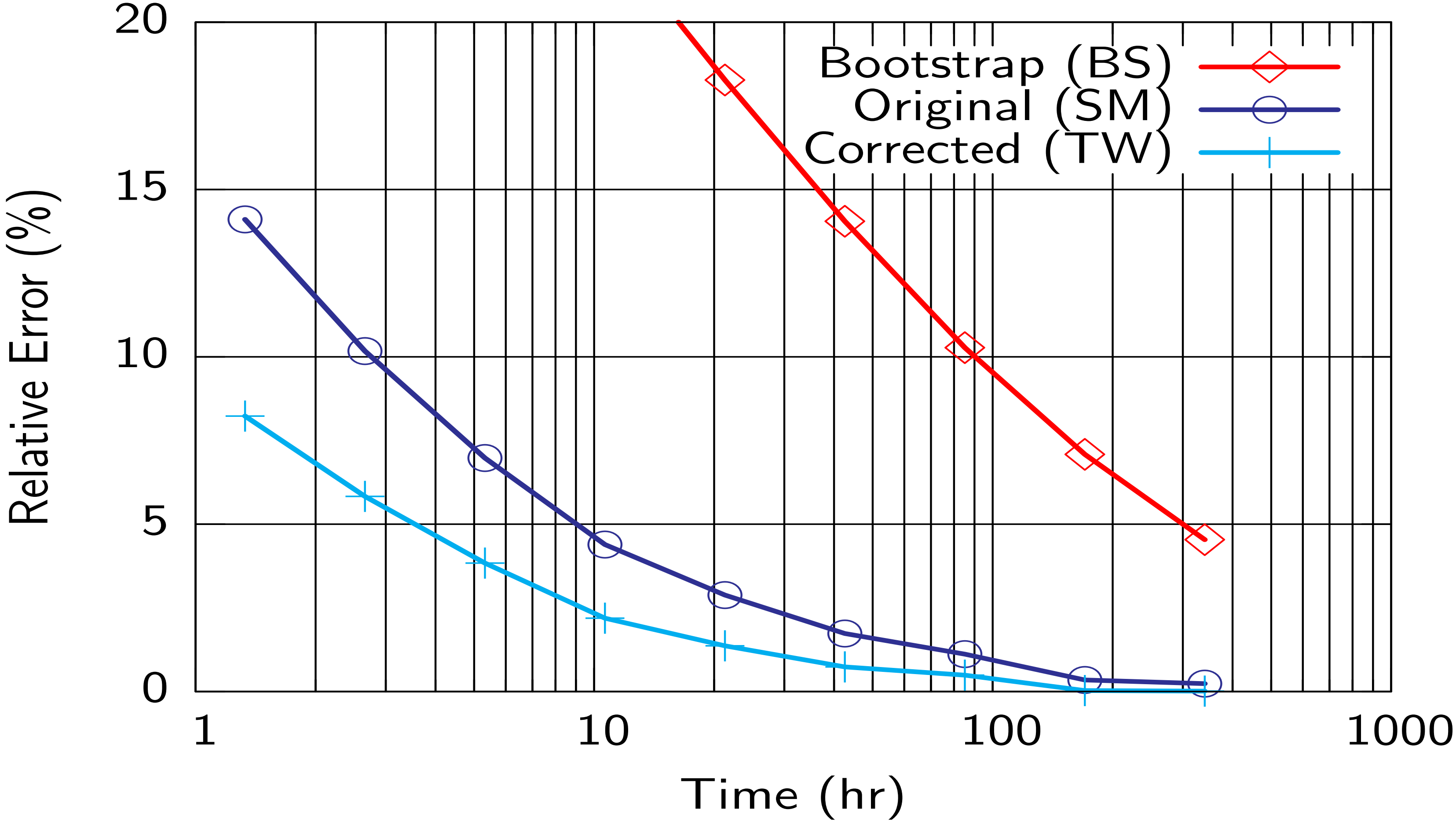
- 5 underlying stocks $\mathbf{S} = (S^1, S^2, S^3, S^4, S^5)'$
- Price a 3-year American-style max-max call struck at \$100 that, upon exercise at τ, pays the holder

$$\max([S^1_\tau - 100]^+, [S^2_\tau - 100]^+, [S^3_\tau - 100]^+, [S^4_\tau - 100]^+, [S^5_\tau - 100]^+).$$

- In the discretized version, $\tau \in \{1, 2, 3\}$.
- 100 bootstrap samples drawn to estimate the bias using the bootstrap (Broadie and Glasserman 1995).
- Stochastic mesh technique used.

Broadie and Glasserman

Example: Results



- Comparisons for a fixed standard deviation (≈ 0.01) and computational time.

Summary

- Like Guinness, reducing bias is good for you.
  and
- It is very cheap.

Example: Computational Notes

- Stochastic mesh technique used.
- Runtime is $O(M^2)$ where M is the number of mesh points (simulations).
- Standard deviation of the estimator is $O(1/\sqrt{M})$.
- Can reduce the standard deviation of the estimator by repeated, independent valuations.
  This does NOT reduce the bias.
- Computations were done using many resources, including SHARCNet, involving four different architectures and many Linux versions and execution environments.
- Total computational time of 282 CPU days, performed over a two to four week period.
- 641,350 independent runs.
- Produced approximately 500 MB of output.