Fuzzy Volatility Forecasts and Fuzzy Option Values

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Content

Introduction

Volatility

- Important factor in Risk Management.
  Volatility modeling provides a simple approach to calculating Value at Risk (VaR) of financial position.
- Important factor in Option Trading.
  Black-Scholes formula for European options - Conditional variance of log return of the underlying stock (volatility) plays an important role.
- Modeling the volatility of a time series can improve the efficiency in parameter estimation and the accuracy in forecat intervals.

Characteristics of Volatility

Stock volatility is not directly observable. The characteristics that are commonly seen in asset returns:

- Volatility clusters exist.
- Volatility jumps are rare.
- Volatility varies within some fixed range.
- Volatility react differently to a big price increase/decrease.

Preliminaries

Some definitions

Let A be a fuzzy subset of X. Then the support of A, S(A), is

$$S(A) = \{x \in X : \mu_A(x) > 0\}$$

The height $h(A)$ of A is

$$h(A) = \underset{x \in X}{Sup} \mu_A(x)$$

If $h(A) = 1$, then A is called a *normal* fuzzy set.
The α-cut (interval of confidence at level-α) of the fuzzy set A

$$A[\alpha] = \{x \in X : \mu_A(x) \geq \alpha\}, \ \alpha \in [0, 1]$$

Some definitions

A fuzzy number $\bar{N}$ is a fuzzy subset of $\Re$: and

(i) the core of $\bar{N}$ is non-empty;

(ii) α-cuts of $\bar{N}$ are all closed, bounded intervals, $\alpha \in (0, 1]$; and

(iii) the support of $\bar{N}$ is bounded.

Properties

Extension Principle: Any $h : [a, b] \rightarrow \Re$ may be extended to $H(\bar{X}) = \bar{Z}$ as follows

$$\bar{Z}(z) = \sup_{x} \left\{ \bar{X}(z) | \quad h(x) = z, \ a \leq x \leq b \right\}$$

$$\begin{aligned} z_1(\alpha) &= \min \left\{ h(x) \mid x \in \bar{X}(\alpha) \right\} \\ z_2(\alpha) &= \max \left\{ h(x) \mid x \in \bar{X}(\alpha) \right\} \end{aligned}$$

for $0 \leq \alpha \leq 1$.

Properties

Properties: Let $\bar{A}$ and $\bar{B}$ be two fuzzy numbers such that

$$\begin{aligned} \bar{A}[\alpha] &= [a_1(\alpha), a_2(\alpha)] \\ \bar{B}[\alpha] &= [b_1(\alpha), b_2(\alpha)]. \end{aligned}$$

Then, for all $\alpha \in (0,1]$,

(a) $\bar{C}[\alpha] = \bar{A}[\alpha] + \bar{B}[\alpha]$

(b) $\bar{C}[\alpha] = \bar{A}[\alpha] - \bar{B}[\alpha]$

(c) $\bar{C}[\alpha] = \bar{A}[\alpha].\bar{B}[\alpha]$

(d) $\bar{C}[\alpha] = \dfrac{\bar{A}[\alpha]}{\bar{B}[\alpha]}$,

provided that zero does not belong to $\bar{B}[\alpha]$ for all $\alpha \in (0,1]$.

Mean, Variance and nth Moment

$$\bar{X} \sim N(\bar{\mu}, \bar{\sigma}^2)$$

$$\bar{M}[\alpha] = \left\{ \int_{-\infty}^{\infty} x f(x; \mu, \sigma^2) dx \mid \mu \in \bar{\mu}[\alpha], \sigma^2 \in \bar{\sigma}^2[\alpha] \right\}$$

$$\bar{V}[\alpha] = \left\{ \int_{-\infty}^{\infty} (x - \mu)^2 f(x; \mu, \sigma^2) dx \mid \mu \in \bar{\mu}[\alpha], \sigma^2 \in \bar{\sigma}^2[\alpha] \right\}$$

$$\bar{E}(X - \mu)^n [\alpha] = \left\{ \int_{-\infty}^{\infty} (x - \mu)^n f(x; \mu, \sigma^2) dx \mid \mu \in \bar{\mu}[\alpha], \sigma^2 \in \bar{\sigma}^2[\alpha] \right\}$$

Kurtosis and Fuzzy Paramer

$$
\begin{aligned}
\bar{K}[\alpha] &= \frac{\bar{E}\,(X-\mu)^4\,[\alpha]}{\left(\bar{V}[\alpha]\right)^2} \\
&= \frac{\left\{\int_{-\infty}^{\infty}(x-\mu)^4\,f(x;\mu,\sigma^2)dx \mid \mu\in\bar{\mu}[\alpha],\sigma^2\in\bar{\sigma}^2[\alpha]\right\}}{\left\{\left(\int_{-\infty}^{\infty}(x-\mu)^2\,f(x;\mu,\sigma^2)dx \mid \mu\in\bar{\mu}[\alpha],\sigma^2\in\bar{\sigma}^2[\alpha]\right)^2\right\}}.
\end{aligned}
$$

Fuzzy Parameter: Place the confidence intervals, one on top of the other, to produce a triangular shaped fuzzy number $\overline{\theta}$ whose α-cuts are the confidence intervals.

$$
\overline{\theta}(\alpha) = [\theta_1(\alpha),\theta_2(\alpha)], \qquad 0\le\alpha\le 1.
$$

Class of Volatility Models

ARMA(l,m) with GARCH(p,q) Errors

$$y_t - \bar{\mu} = \sum_{j=0}^{\infty} \bar{\psi}_j a_{t-j},$$

where $\overline{\psi}_j$'s are such that $\sum_{j=0}^{\infty} \int_0^1 (\psi_{j1}^2(\gamma) + \psi_{j2}^2(\gamma))\gamma \, d\gamma < \infty$.

The series $\{a_t\}$ is a GARCH(p,q) process given by

$$\begin{aligned} a_t &= \sqrt{h_t}\epsilon_t, \\ h_t &= \omega + \sum_{i=1}^{p} \bar{\varphi}_i a_{t-i}^2 + \sum_{j=1}^{q} \bar{\beta}_j h_{t-j} \end{aligned}$$

with mean zero, variance $\bar{\sigma}_a^2$ and kurtosis $K^{(a)}$.

ARMA(l,m) with GARCH(p,q) Errors

Let $u_t = a_t^2 - h_t$ and $\sigma_u^2 = \mathrm{Var}(u_t)$, then

$$\begin{aligned} a_t^2 - u_t &= \bar{\omega} + \sum_{i=1}^{p} \bar{\varphi}_i a_{t-i}^2 + \sum_{j=1}^{q} \bar{\beta}_j h_{t-j}, \\ \left[1 - \sum_{i=1}^{p} \bar{\varphi}_i B^i - \sum_{j=1}^{q} \bar{\beta}_j B^j\right] a_t^2 &= \bar{\omega} - \sum_{j=1}^{q} \bar{\beta}_j B^j u_i, \\ \Phi(B) a_t^2 &= \bar{\omega} + \beta(B) u_t, \\ a_t^2 &= \bar{\omega} + \sum_{i=1}^{r} \overline{\Psi}_i u_{t-i}, \end{aligned}$$

where, $\Phi(B) = 1 - \sum_{i=1}^{r} \overline{\Phi}_i B^i$, $\overline{\Phi}_i = (\bar{\varphi}_i + \bar{\beta}_i)$, $\beta(B) = 1 - \sum_{j=1}^{q} \bar{\beta}_j B^j$ and $r = \max(p, q)$.

Stationary Assumptions

(1) all the zeroes of the polynomial $\Phi(B)$ lie outside of the unit circle. (This assumption ensures that $u_t's$ are uncorrelated with zero mean and finite variance).

(2) $\sum_{i=0}^{\infty}\int_0^1(\overline{\Psi}_{i1}^2(\gamma)+\overline{\Psi}_{i2}^2(\gamma))\gamma\ d\gamma<\infty$, where $\Psi(B)\ \Phi(B)=\beta(B)$ with $\Psi(B)=\sum_{i=0}^{\infty}\overline{\Psi}_i B^i$.

We can show that

$$Var(y_t)=\bar{\sigma}_a^2\sum_{j=0}^{\infty}\overline{\psi}_j^2,\ \text{where}\ \bar{\sigma}_a^2=\frac{\bar{\omega}}{1-\sum_{i=1}^{r}\bar{\Phi}_i}.$$

Kurtosis of $\{y_t\}$ and Forecast Error

$$K^{(y)} = \frac{K^{(a)}\left[\sum_{j=0}^{\infty} \overline{\psi}_j^4\right] + 6\sum_{i<j}^{\infty} \overline{\psi}_i^2 \overline{\psi}_j^2}{\left(\sum_{j=0}^{\infty} \overline{\psi_j^2}\right)^2},$$ provided $\sum_{j=0}^{\infty} \overline{\psi_j^4} < \infty$, where

$$K^{(a)} = \frac{E(\epsilon_t^4)}{E(\epsilon_t^4) - [E(\epsilon_t^4) - 1]\sum_{j=0}^{\infty} \overline{\Psi}_j^2}.$$

Let $y_n(l)$ be the l-steps ahead forecast of y_{n+l}. Then

$$E[y_{n+l} - y_n(l)]^2 = \frac{\bar{\omega}}{1 - \overline{\Phi}_1 - \overline{\Phi}_2 - \quad \ldots \overline{\Phi}_r} \sum_{j=0}^{l-1} \overline{\psi}_j^2.$$

FC-GARCH(1,1) Model

The classical GARCH(1,1) model takes the form

$$\begin{aligned} y_t &= \mu + a_t, \qquad a_t = \sqrt{h_t}\epsilon_t, \\ h_t &= \omega + \varphi a_{t-1}^2 + \beta h_{t-1}, \end{aligned}$$

where $E(\epsilon_t) = 0$, $Var(\epsilon_t) = 1$, $E(\epsilon_t^4) = K^{(\epsilon)} + 3$.
$K^{(\epsilon)}$ is the excess kurtosis of ϵ_t.

$$Var(a_t) = E(h_t) = \frac{\omega}{1-\varphi-\beta}; \qquad E(a_t^4) = (K^{(\epsilon)}+3)E(h_t^2)$$

$$E(h_t^2) = \frac{\omega^2(1+\varphi+\beta)}{(1-\varphi-\beta)[1-\varphi^2(K^{(\epsilon)}+2)-(\varphi+\beta)^2]}$$

Kurtosis of GARCH(1,1) Model

Kurtosis of $\{a_t\}$:

$$K^{(a)} = \frac{E(a_t^4)}{[E(a_t^2)]^2} = \frac{(K^{(\epsilon)}+3)[1-(\varphi+\beta)^2]}{1-2\varphi^2-(\varphi+\beta)^2-K^{(\epsilon)}\varphi^2}.$$

When ϵ_t is normally distributed $K^{(\epsilon)}=0$.

$$K^{(a)} = \frac{3[1-(\varphi+\beta)^2]}{1-2\varphi^2-(\varphi+\beta)^2}.$$

With fuzzy Coefficients,

$$\bar{K}^{(a)} = \frac{3[1-(\bar{\varphi}+\bar{\beta})^2]}{1-2\bar{\varphi}^2-(\bar{\varphi}+\bar{\beta})^2}.$$

UNCV and Forecast

With fuzzy Coefficients, the UNC variance of $\{a_t\}$ is
$E(\bar{h}_t) = E(\bar{\sigma}_t^2) = \dfrac{\bar{\omega}}{1 - \bar{\varphi} - \bar{\beta}}$.
At the forecast origin n, the one-step ahead volatility forecast is given by

$$\bar{\sigma}_n^2(1)[\alpha] \quad = \quad \bar{\omega} + \bar{\varphi}\bar{a}_n^2 + \bar{\beta}\bar{\sigma}_n^2$$

The ℓ-step ahead volatility forecast is

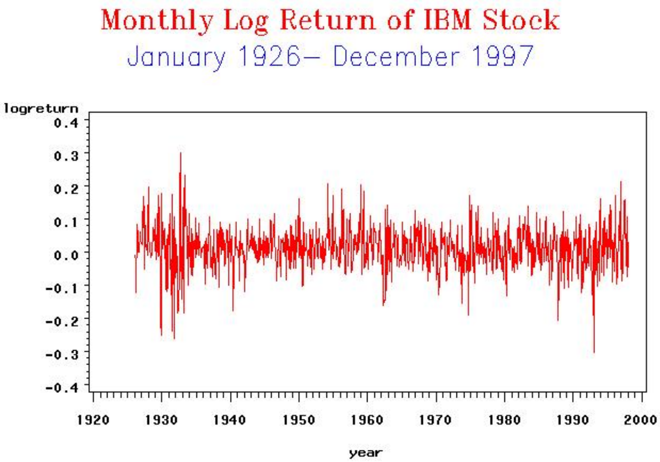
$$\bar{\sigma}_n^2(\ell)[\alpha] \quad = \quad \bar{\omega} + (\bar{\varphi} + \bar{\beta})\bar{\sigma}_n^2(\ell - 1), \quad \ell > 1$$

The starting value of σ_0^2 is set to zero or $E(\sigma_t^2)$.

Numerical Example

Monthly Log Return of IBM Stock

Figure: 1



Monthly Log Return of IBM Stock

Monthly Log Returns of IBM Stock - Fitted GARCH(1,1) model. Assume ε_t are i.i.d standard normal.

- Fuzzy Parameters.
- Fuzzy UC variance.
- Fuzzy kurtosis
- Fuzzy Forecast.
- Fuzzy Call Option Value.

Parameter Estimators

Monthly Log Returns of IBM Stock - Fitted GARCH(1,1) model. Assume ε_t are i.i.d standard normal.

$$
\begin{aligned}
y_t &= \mu + a_t, \qquad a_t = \sigma_t \varepsilon_t \\
h_t &= \sigma_t^2 = \omega + \varphi a_{t-1}^2 + \beta \sigma_{t-1}^2 \\
\widehat{\mu} &= 0.0130(0.0023) \\
\widehat{\omega} &= 0.00035(0.00011) \\
\widehat{\varphi} &= 0.0999(0.0214) \\
\widehat{\beta} &= 0.8187(0.0422)
\end{aligned}
$$

UNC Variance and Kurtosis

$$E(h_t) = E(\sigma_t^2) = \frac{\hat{\omega}}{1-\hat{\varphi}-\hat{\beta}} = \frac{0.00035}{1-0.0999-0.8187} = 0.0043$$

$$\begin{aligned} K^{(a)} &= \frac{3[1-(\hat{\varphi}+\hat{\beta})^2}{1-2\hat{\varphi}^2-(\hat{\varphi}+\hat{\beta})^2} \\ &= \frac{3[1-(0.0999+0.8187)^2}{1-2(0.0999)^2-(0.0999+0.8187)^2} = 3.4396 \end{aligned}$$

Fuzzy Estimation

Fuzzy Coefficients with their α-cuts:

$$\begin{aligned}
\overline{\omega} &= [\omega_1(\alpha), \omega_2(\alpha)] = \left[0.00035 - Z^*_{\frac{\alpha}{2}}(0.00011),\ 0.00035 + Z^*_{\frac{\alpha}{2}}(0.00011)\right] \\
\overline{\varphi} &= [\varphi_1(\alpha), \varphi_2(\alpha)] = \left[0.0999 - Z^*_{\frac{\alpha}{2}}(0.0214),\ 0.0999 + Z^*_{\frac{\alpha}{2}}(0.0214)\right] \\
\overline{\beta} &= [\beta_1(\alpha), \beta_2(\alpha)] = \left[0.8187 - Z^*_{\frac{\alpha}{2}}(0.0422),\ 0.8187 + Z^*_{\frac{\alpha}{2}}(0.0422)\right]
\end{aligned}$$

Fuzzy UNC Variance and Kurtosis

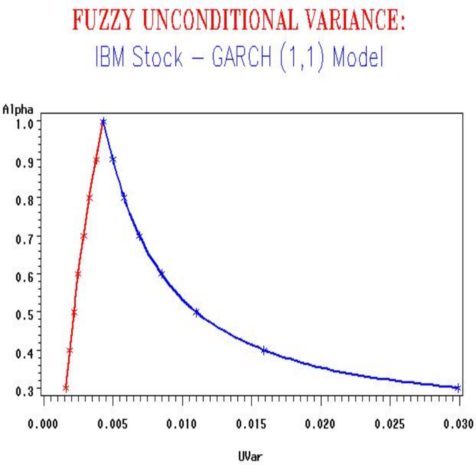
$$E[\bar{h}_t] = [E(h_{t,1}), E(h_{t,2})] = \left[\frac{\omega_1(\alpha)}{1-\varphi_1(\alpha)-\beta_1(\alpha)}, \quad \frac{\omega_2(\alpha)}{1-\varphi_2(\alpha)-\beta_2(\alpha)}\right].$$

$$\overline{K}^{(a)} = [K_1^{(a)}(\alpha), K_2^{(a)}(\alpha)]$$

$$= \left[\frac{3[1-(\varphi_1(\alpha)+\beta_1(\alpha))^2]}{1-2\varphi_1^2(\alpha)-(\varphi_1(\alpha)+\beta_1(\alpha))^2}, \quad \frac{3[1-(\varphi_2(\alpha)+\beta_2(\alpha))^2]}{1-2\varphi_2^2(\alpha)-(\varphi_2(\alpha)+\beta_2(\alpha))^2}\right].$$

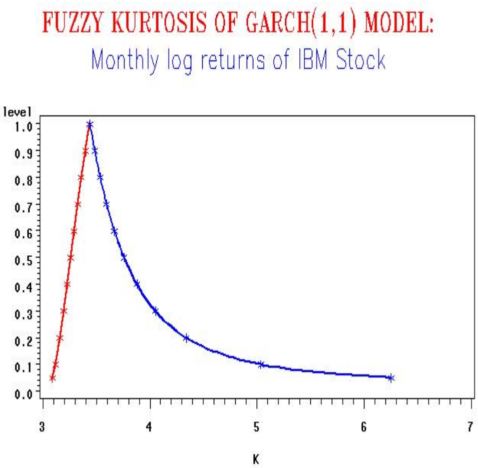
Fuzzy UNC Variance

Figure: 2



Fuzzy Kurtosis

Figure: 1



Fuzzy Forecast

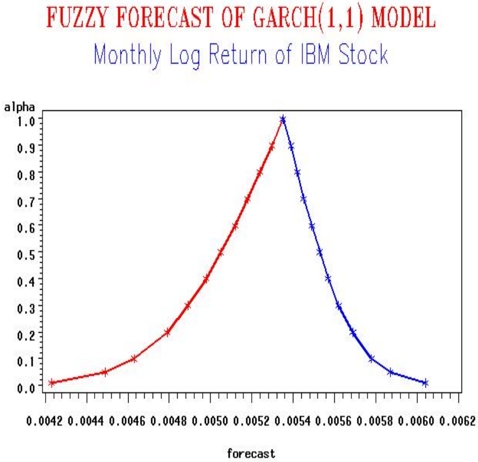
At the forecast origin n, the one-step ahead volatility forecast is given by

$$\begin{aligned}\sigma_n^2(1) &= \omega + \varphi a_n^2 + \beta\sigma_n^2 \\ &= 0.00035 + 0.0999(0.00342) + 0.8187(0.00569) \\ &= 0.00535\end{aligned}$$

With the fuzzy coefficients,

$$\begin{aligned}\bar{\sigma}_n^2(1)[\alpha] &= \bar{\omega} + \bar{\varphi}\bar{a}_n^2 + \bar{\beta}\bar{\sigma}_n^2 \\ &= [\bar{\omega}_1(\alpha) + \bar{\varphi}_1(\alpha)\bar{a}_n^2 + \bar{\beta}_1\bar{\sigma}_n^2, \quad \bar{\omega}_2(\alpha) + \bar{\varphi}_2(\alpha)\bar{a}_n^2 + \bar{\beta}_2\bar{\sigma}_n^2].\end{aligned}$$

Figure: 3


FUZZY FORECAST OF GARCH(1,1) MODEL
Monthly Log Return of IBM Stock
alpha
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
0.0042 0.0044 0.0046 0.0048 0.0050 0.0052 0.0054 0.0056 0.0058 0.0060 0.0062
forecast

Black-Scholes Formula

European Call Option Value:

$$C = S_0 N\left(\frac{\log(\frac{S_0}{X}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right) - Xe^{-rT} N\left(\frac{\log(\frac{S_0}{X}) + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right)$$

where

$$N(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-\frac{z^2}{2}} dz.$$

Fuzzy Call Option Values

The α - cuts of call option value:

$$\overline{C}(\alpha) = [C_1(\alpha),\ C_2(\alpha)]$$

$$C_1(\alpha) = Min\left\{S_0 N(d1)) - Xe^{-rT} N(d2) \,|\, \sigma \in \bar{\sigma}[\alpha]\right\}$$

and

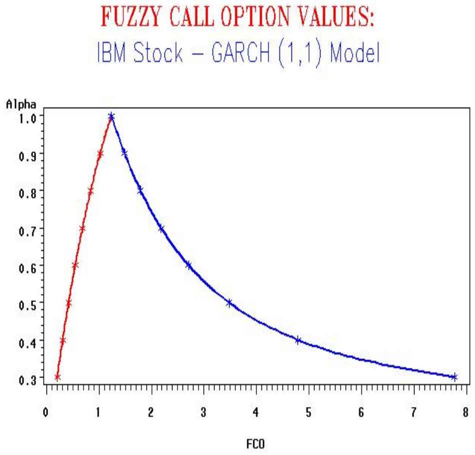
$$C_2(\alpha) = Max\left\{S_0 N(d1) - Xe^{-rT} N(d2) \,|\, \sigma \in \bar{\sigma}[\alpha]\right\}$$

where

$$d1 = \left(\frac{\log(\frac{S_0}{X}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right),\ \sigma \in \bar{\sigma}[\alpha]$$

$$d2 = \left(\frac{\log(\frac{S_0}{X}) + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right),\ \sigma \in \bar{\sigma}[\alpha].$$

Figure: 4



$(S = 80, X = 90, r = 0.08, and T = 3 months)$

Thank you for your attention!