Quantitative Finance and Investment Portfolio Management Formula Sheet

Fall 2022

The exam computer will provide access to a PDF of this formula package. The exam committee believes that by providing many key formulas, candidates will be able to focus more of their exam preparation time on the application of the formulas and concepts to demonstrate their understanding of the syllabus material and less time on the memorization of the formulas.

The formula sheet was developed sequentially by reviewing the syllabus material for each major syllabus topic. Candidates should be able to follow the flow of the formula package easily. We recommend that candidates use the formula package concurrently with the syllabus material. Not every formula in the syllabus is in the formula package. **Candidates are responsible for all formulas on the syllabus, including those not on the formula sheet.**

Candidates should carefully observe the sometimes subtle differences in formulas and their application to slightly different situations. Candidates will be expected to recognize the correct formula to apply in a specific situation of an exam question.

Candidates will note that the formula package does not generally provide names or definitions of the formula or symbols used in the formula. With the wide variety of references and authors of the syllabus, candidates should recognize that the letter conventions and use of symbols may vary from one part of the syllabus to another and thus from one formula to another.

We trust that you will find the inclusion of the formula package to be a valuable study aide that will allow for more of your preparation time to be spent on mastering the learning objectives and learning outcomes.

Credit-Risk Modelling, Bolder

Chapter 1

(1.3) $\mathrm{VaR}_\alpha(L) = \inf\Big(x : \mathbb{P}(L \leq x) \geq 1-\alpha\Big)$

(1.6) $\mathrm{VaR}_\alpha(L) = F_L^{-1}(1-\alpha)$

Chapter 2

(2.26) $\widehat{\mathbb{E}(L_N)} = \frac{1}{M}\sum_{m=1}^{M} L_N^{(m)}$

(2.27) $\widehat{\sigma(L_N)} = \sqrt{\frac{1}{M-1}\sum_{m=1}^{M}\Big(L_N^{(m)} - \widehat{\mathbb{E}(L_N)}\Big)^2}$

(2.28) $\widehat{\mathrm{VaR}_\alpha(L_N)} = \tilde{L}_N(\lceil \alpha \cdot M \rceil)$

(2.29) $\widehat{\mathcal{E}_\alpha(L_N)} = \frac{1}{M-\lceil \alpha\cdot M\rceil+1}\sum_{m=\lceil \alpha\cdot M\rceil}^{M} L_N^{(m)}$ This is a correction to an error in the text.

(2.45) $\mathbb{E}\Big((\mathbb{D}_N - Np)^2\Big) = Np(1-p)$

(2.60) $f_{\mathbb{D}_N}(k) = \mathbb{P}(\mathbb{D}_N = k) = \frac{e^{-N\lambda}(N\lambda)^k}{k!}$

(2.61) $F_{\mathbb{D}_N}(m) = \mathbb{P}(\mathbb{D}_N \leq m) = \sum_{k=0}^{m}\frac{e^{-N\lambda}(N\lambda)^k}{k!}$

Chapter 3

(3.10) $\mathbb{E}(\mathbb{D}_N) = N\bar{p}$

(3.11) $\mathrm{var}(\mathbb{D}_N) = N\bar{p}(1-\bar{p}) + N(N-1)\mathrm{var}(p(Z))$

(3.17) $f_Z(z) = \frac{1}{B(\alpha,\beta)}z^{\alpha-1}(1-z)^{\beta-1}$

(3.18) $B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt$

(3.19) $B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$

(3.20) $\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}dx$

(3.21) $\mathbb{E}(Z) = \frac{\alpha}{\alpha+\beta}$

(3.22) $\mathrm{var}(Z) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

(3.26) $\rho_{\mathcal{D}} = \rho\left(\mathbb{I}_{\mathcal{D}_n}, \mathbb{I}_{\mathcal{D}_m}\right) = \frac{1}{\alpha+\beta+1}$

(3.27) $\mathbb{E}(Z) = \bar{p} = \frac{\alpha}{\alpha+\beta}$

(3.35) $p_1^{-1}(z) \equiv y = \frac{1}{\sigma_1}\left(\ln\left(\frac{z}{1-z}\right) - \mu_1\right)$

(3.36) $p_2^{-1}(z) \equiv y = \frac{\Phi^{-1}(z)-\mu_2}{\sigma_2}$

(3.39) $f_{p_1(Z)}(z) = \frac{1}{z(1-z)\sqrt{2\pi\sigma_1^2}}exp\left[-\frac{\left(\ln\left(\frac{z}{1-z}\right)-\mu_1\right)^2}{2\sigma_1^2}\right]$

(3.52) $\mathbb{P}(\mathbb{D}_N = k) = \int_{\mathbb{R}_+}\frac{e^{-\lambda(s)}\lambda(s)^k}{k!}f_S(s)ds$

(3.53) $f_S(s) = \frac{b^a e^{-bs}s^{a-1}}{\mathit{\Gamma}(\alpha)}$

(3.54) $\mathbb{P}(\mathbb{D}_N = k) = \frac{\mathit{\Gamma}(a+k)}{\mathit{\Gamma}(k+1)\mathit{\Gamma}(a)}q_1^a(1-q_1)^k, \quad q_1 = \frac{b}{b+1}$

(3.81) $\sigma(p_n(S)) = p_n\sqrt{\frac{\omega_1^2}{a}}$

(3.83) $\mathbb{E}(\mathbb{I}_{\{X_n\geq 1\}}\mathbb{I}_{\{X_m\geq 1\}}) = p_n p_m\left(\omega_0^2 + 2\omega_0\omega_1 + \omega_1^2\mathbb{E}(S^2)\right)$

(3.84) $\mathbb{E}(S^2) = 1 + \frac{1}{a}$

(3.85) $\mathbb{E}(\mathbb{I}_{\{X_n \geq 1\}} \mathbb{I}_{\{X_m \geq 1\}}) = p_n p_m \left(1 + \frac{\omega_1^2}{a}\right)$

(3.86) $\rho(\mathcal{D}_n, \mathcal{D}_m) = \left(\frac{\omega_1^2}{a}\right)\left(\frac{p_n p_m}{\sqrt{p_n(1-p_n)}\sqrt{p_m(1-p_m)}}\right)$

(3.95) $\sigma(p_n(\mathbf{S})) = p_n \sqrt{\sum_{k=1}^{K} \frac{\omega_{n,k}^2}{a_k}}$ This is a correction to an error in the text.

(3.99) $\rho(\mathbb{I}_{\mathcal{D}_n}, \mathbb{I}_{\mathcal{D}_m}) = \left(\sum_{k=1}^{K} \frac{\omega_{n,k}\omega_{m,k}}{a_k}\right)\left(\frac{p_n p_m}{\sqrt{p_n(1-p_n)}\sqrt{p_m(1-p_m)}}\right)$

Chapter 4

(4.1) $y_n = aG + b\epsilon_n$

(4.5) $Y_n = a \cdot G + \underbrace{\sqrt{1-a^2}}_{b}\epsilon_n$

(4.7) $y_n = \sqrt{a} \cdot G + \sqrt{1-a}\epsilon_n$

(4.18) $p_n(G) = \Phi\left(\frac{\Phi^{-1}(p_n) - \sqrt{\rho}G}{\sqrt{1-\rho}}\right)$

(4.22) $\rho(\mathbb{I}_{\mathcal{D}_n}, \mathbb{I}_{\mathcal{D}_m}) = \frac{\mathbb{P}(\mathcal{D}_n \cap \mathcal{D}_m) - p_n p_m}{\sqrt{p_n p_m (1-p_n)(1-p_m)}}$

(4.23) $\mathbb{P}(\mathcal{D}_n \cap \mathcal{D}_m) = \Phi\left(\Phi^{-1}(p_n), \Phi^{-1}(p_m); \rho\right)$

(4.30) $\rho(\mathbb{I}_{\mathcal{D}_n}, \mathbb{I}_{\mathcal{D}_m}) = \frac{\overbrace{\Phi\left(\Phi^{-1}(p_n), \Phi^{-1}(p_m); \rho\right)}^{\mathbb{P}(\mathcal{D}_n \cap \mathcal{D}_m} - p_n p_m}{\sqrt{p_n p_m (1-p_n)(1-p_m)}}$

(4.31) $\rho(\mathbb{I}_{\mathcal{D}_n}, \mathbb{I}_{\mathcal{D}_m}) = \frac{\Phi\left(\Phi^{-1}(\bar{p}), \Phi^{-1}(\bar{p}); \rho\right) - \bar{p}^2}{\bar{p}(1-\bar{p})}$

(4.44) $\text{var}(\breve{\mathbb{D}}_N | G) = \frac{p(G)(1-p(G))}{N}$

(4.46) $\text{F(x)} = \mathbb{P}(G \leq -p^{-1}(x))$

(4.49) $F(x) = h(b(x))$

(4.50) $h(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{\mu^2}{2}} du$ This is a correction to an error in the text

(4.51) $b(x) = \frac{\sqrt{1-\rho}\Phi^{-1}(x) - \Phi^{-1}(p)}{\sqrt{\rho}}$

(4.55) $F'(x) = \sqrt{\frac{1-\rho}{\rho}} e^{\frac{(\Phi^{-1}(x))^2}{2} - \frac{b(x)^2}{2}}$

(4.69) $y_n = \sqrt{\frac{\nu}{W}}\left(\sqrt{\rho}G + \sqrt{1-\rho}\epsilon_n\right)$

(4.72) $\mathbb{E}\left(\frac{1}{W}\right) = \frac{1}{\nu - 2}$

(4.73) $\text{cov}(y_n, y_m) = \rho\left(\frac{\nu}{\nu-2}\right)$

(4.74) $\rho(y_n, y_m) = \rho$

(4.81) $y_n = \sqrt{V}\left(\sqrt{\rho}G + \sqrt{1-\rho}\epsilon_n\right)$

(4.87) $p_n = F_{\mathcal{NV}}(K_n)$

(4.103) $y_n = \underbrace{a_{n,K+1}\epsilon_n}_{\text{Idiosyncratic}} + \underbrace{\sum_{k=1}^{K} a_{n,k} Z_k}_{\text{Systematic}}$

(4.107) $\text{cov}(y_n, y_m) = a_n^T a_m$

(4.108) $p_n(Z) = \Phi\left(\frac{\Phi^{-1}(p_n) - a_n^T Z}{a_{n,K+1}}\right)$

(4.110) $y_n = \sqrt{h(V)}\left(a_{n,K+1}\epsilon_n + \sum_{k=1}^{K} a_{n,k} Z_k\right)$

(4.113) $\rho(y_n, y_m) = a_n^T a_m$

(4.115) $p_n(Z, V) = \Phi\left(\frac{\sqrt{\frac{1}{h(V)}} F_{\mathcal{NV}}^{-1}(p_n) - a_n^T Z}{a_{n,K+1}}\right)$

(4.116) $y_n(k) = \sqrt{h(V)}\left(\underbrace{\sqrt{a}G + \sqrt{(1-a)b_k}R_k}_{\text{Systematic element}} + \underbrace{\sqrt{(1-a)(1-b_k)}\epsilon_n}_{\text{Idiosyncratic element}}\right)$

(4.120) $\rho\left(y_n(k), y_m(j)\right) = a + \mathbb{I}_{k=j}(1-a)\sqrt{b_k b_j}$

(4.123) $p_n(G, R_k, V) = \Phi\left(\frac{\sqrt{\frac{1}{h(V)}} F_{\mathcal{NV}}^{-1}(p_n) - \sqrt{a}G - \sqrt{(1-a)b_k}R_k}{\sqrt{(1-a)(1-b_k)}}\right)$

Modern Investment Management: An Equilibrium Approach, B. Litterman (QFIP 140-19)

Chapter 7

(7.3) $\mu^* = [(\tau\Sigma)^{-1} + P'\Omega^{-1}P]^{-1}[(\tau\Sigma)^{-1}\Pi + P'\Omega^{-1}Q]$

Modern Investment Management: An Equilibrium Approach, B. Litterman (QFIP 142-19)

Chapter 10

(10.5) $RACS_t \equiv \frac{E_t[S_{t+1} - S_t(1+R_f)]}{\sigma_t[S_{t+1} - S_t(1+R_f)]} = \frac{E_t[S_{t+1} - S_t(1+R_f)]}{\sigma_t[S_{t+1}]}$

(10.A.1) $R_{L,t} - R_{f,t} = \beta\left(R_{B,t} - R_{f,t}\right) + \varepsilon_t$

(10.A.2) $V = Ce^{-r(T-t)}$

(10.A.3) $dV = \frac{\partial V}{\partial C}dC + \frac{\partial V}{\partial r}dr + \frac{\partial V}{\partial t}dt = V\frac{dC}{C} - (T-t)V dr + rV dt$

(10.A.4) $\frac{dV}{V} = \frac{dC}{C} - (T-t)dr + rdt$

(10.A.5) $V = \int_t^\infty C_T e^{-r_T(T-t)}dT$

(10.A.6) $dV = -C_t dt + \int_t^\infty \left[dC_T - (T-t)C_T + r_T C_T\right] e^{-r_T(T-t)}dT$

(10.A.7) $d\varepsilon_t = \frac{\int_t^\infty dC_T e^{-r_T(T-t)}dT}{\int_t^\infty C_T e^{-r_T(T-t)}dT}$

(10.A.8) $E_t[d\varepsilon_t] = 0$ (There is an error in the paper.)

(10.A.9) $E_t[d\varepsilon_t^2] = \sigma_\varepsilon^2 dt$

(10.A.10) $S_{t+1} = A_t[\alpha(1 + R_{E,t+1}) + (1-\alpha)(1 + R_{B,t+1})] - L_t[1 + R_f + \beta(R_{B,t+1} - R_f) + \varepsilon_{t+1}]$

(10.A.11) $\frac{S_{t+1}}{A_t} = \alpha(1 + R_{E,t+1}) + R_{B,t+1}\left(1 - \alpha - \frac{L_t}{A_t}\beta\right) - \frac{L_t}{A_t}\varepsilon_{t+1} + 1 - \alpha - \frac{L_t}{A_t}[1 + (1-\beta)R_f]$

(10.A.12) $\min_\alpha Var_t\left(\frac{S_{t+1}}{A_t}\right) = \alpha^2\sigma_E^2 + \left(1 - \alpha - \beta\frac{L_t}{A_t}\right)^2\sigma_B^2 + \left(\frac{L_t}{A_t}\right)^2\sigma_\varepsilon^2 + 2\alpha\left(1 - \alpha - \beta\frac{L_t}{A_t}\right)\rho\sigma_E\sigma_B$

(10.A.13) $\alpha\sigma_E^2 + \left(\alpha - 1 + \beta\frac{L_t}{A_t}\right)\sigma_B^2 + \left(1 - 2\alpha - \beta\frac{L_t}{A_t}\right)\rho\sigma_E\sigma_B = 0$

$$\alpha = \frac{\left(1-\beta\frac{L_t}{A_t}\right)\left(\sigma_B^2-\rho\sigma_E\sigma_B\right)}{\sigma_E^2+\sigma_B^2-2\rho\sigma_E\sigma_B} \tag{10.A.14}$$

$$E_t(S_{t+1}) = E_t\{A_t[\alpha R_{E,t+1}+(1-\alpha)R_{B,t+1}] - L_t[R_f+\beta(R_{B,t+1}-R_f)+\varepsilon_{t+1}]\} \tag{10.A.15}$$

$$\alpha = \frac{\mu_B\left(\beta\frac{L_t}{A_t}-1\right)+\frac{L_t}{A_t}\left[R_f(1-\beta)+\eta\right]}{\mu_E-\mu_B} \tag{10.A.16}$$

$$F_1 = \frac{1}{1-p}F_0(1+R_{x,1}) - \frac{p}{1-p} = aF_0(1+R_{x,1})+b \quad a=\frac{1}{1-p}, b=-\frac{p}{1-p} \tag{10.A.17}$$

$$\begin{aligned} F_2 &= aF_1(1+R_{x,2})+b = a[aF_0(1+R_{x,1})+b](1+R_{x,2})+b \\ &= a^2F_0(1+R_{x,1})(1+R_{x,2})+ab(1+R_{x,2})+b \end{aligned} \tag{10.A.18}$$

$$F_t = a^tF_0\prod_{\substack{1\le s\le t\\ s\in N}}(1+R_{x,s}) + b\sum_{i=0}^{t-1}a^i\prod_{\substack{1\le j\le i\\ j\in N}}(1+R_{x,t-(j-1)}) \tag{10.A.19}$$

$$E_0(F_t) = a^tF_0E_0\left[\prod_{\substack{1\le s\le t\\ s\in N}}(1+R_{x,s})\right] + b\sum_{i=0}^{t-1}a^iE_0\left[\prod_{\substack{1\le j\le i\\ j\in N}}(1+R_{x,t-(j-1)})\right] \tag{10.A.20}$$

$$E_0\left[\prod_{\substack{1\le s\le t\\ s\in N}}(1+R_{x,s})\right] = \prod_{\substack{1\le s\le t\\ s\in N}}E_0[1+R_{x,s}] = (1+\mu_x)^t \tag{10.A.21}$$

$$E_0\left[F_t\right] = \left(\frac{1+\mu_x}{1-p}\right)^tF_0 + p\frac{1-\left(\frac{1+\mu_x}{1-p}\right)^t}{\mu_x+p} \tag{10.A.22}$$