TRANSACTIONS OF SOCIETY OF ACTUARIES
1964 VOL. 16 PT. 1 NO. 45

DISCUSSION OF PAPERS PRESENTED AT THE REGIONAL MEETINGS

A STATISTICAL APPROACH TO PREMIUMS AND RESERVES IN MULTIPLE DECREMENT THEORY

JAMES C. HICKMAN

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CECIL J. NESBITT:

Very often actuarial models, particularly if rates of increment are involved, are considered from a deterministic viewpoint only. It is assumed that certain given rates will apply exactly, and no random variation is taken into account. However, actuarial models based on rates of decrement may be considered from a probabilistic viewpoint, and estimates may be made of variance arising from finite random sampling. In this paper, the author has systematically developed multiple decrement theory from the probabilistic viewpoint and, in doing so, has clarified connections between actuarial and statistical theory. It should be illuminating to students and others to see the time t until benefit payment and the cause i of decrement introduced as basic random variables, from which premium and reserve theory follow by consideration of the distribution functions of these variables and their application to loss functions dependent on the basic variables. Of particular interest is the expression for the reserve as the expected value of a conditional loss function.

Besides developing multiple decrement theory from a statistical approach, the paper lays the groundwork for an exposition of individual risk theory. In fact, the formulas for the total premium and total reserve, specialized to the case $m = 1$, provide an outline of the continuous model risk theory with mortality as cause of decrement. To my mind, there has been a need for some while for a comprehensive exposition of individual risk theory that would be readily available to our students and others interested, and the paper should be of service in that regard. There is a classical treatment by Harold Cramér in the *Skandia* Jubilee Volume (1930), *On the Mathematical Theory of Risk*, but many will not have ready access to that volume.

The author has neatly obtained Hattendorf's theorem for the continuous case discussed in his paper. For such case, there are a number of calculus devices which permit one to prove the theorem but leave one somewhat in the dark as to why it holds. In the *Skandia* Jubilee Volume, Cramér gives an elegant proof, adapted from one by F. P. Cantelli, for the discrete case with $m = 1$, and from this proof one gains some insight into the theorem. R. Butcher has developed a related but somewhat more explicit proof, and I shall outline it here for the discrete multiple decrement case. Now t will take on integral values $1, 2, \ldots, n$ denoting the n years of insurance. We consider a person insured from age x to age $x + n$ with benefit of $B_{x+t}^{(i)}$ payable at end of year t in case of decrement due to cause i during that year ($t = 1, 2, \ldots, n; i = 1, 2, \ldots, m$) and of B_{x+n} in case of survival to the end of n years (that is, $t = n, i = m + 1$), with level premiums of P payable at the beginning of each year during the insurance term and with reserve ${}_tV$ at the end of year t.

Following the author, we define a loss function

$$L(t, i:x) = v^t B_{x+t}^{(i)} - P\ddot{a}_{\overline{t|}} \; (1 \le t \le n, i = 1, 2, \ldots, m)$$
$$= v^n B_{x+n} - P\ddot{a}_{\overline{n|}} \; (t = n, i = m + 1).$$

We also define component functions ${}_sL\ (t, i:x)$, $1 \le s \le n$, where

$${}_sL(t, i:x) = 0 \; (1 \le t < s)$$
$$= v^s B_{x+s}^{(i)} - v^{s-1}({}_{s-1}V + P) \; (t = s, i = 1, 2, \ldots, m)$$
$$= v^s {}_sV - v^{s-1}({}_{s-1}V + P) \; (t = s + 1, \ldots, n, i = 1, 2, \ldots, m; \; t = n, i = m + 1).$$

The probability density for each of $L(t, i:x)$ and ${}_sL(t, i:x)$ is ${}_{t-1|}q_x^{(i)}$ for $1 \le t \le n$, $i = 1, 2, \ldots, m$ and is ${}_np_x$ for $t = n$, $i = m + 1$.

The following results may then be obtained:

$$L(t, i:x) = \sum_{s=1}^{n} {}_sL(t, i:x). \tag{1}$$

$$\mathcal{E}\,[{}_sL(t, i:x)] = 0 \; (s = 1, 2, \ldots, n). \tag{2}$$

$$\mathcal{E}\,[L(t, i:x)] = 0. \tag{3}$$

$$\text{Var}\,[{}_sL(t, i:x)] = v^{2(s-1)}\, {}_{s-1}p_x \,\text{Var}\,[Y(i:s)], \tag{4}$$

where Var $[Y(i:s)]$ denotes the variance for a one-year term multiple decrement insurance for a person aged $x + s - 1$, providing a net risk benefit of $B_{x+s}^{(i)} - {}_sV$ $(i = 1, 2, \ldots, m)$ in case of decrement due to cause i. Here $Y(i:s)$ may be regarded as a random variable representing the discounted benefit outgo, namely,

$$Y(i:s) = v(B_{x+s}^{(i)} - {}_sV) \; (i = 1, 2, \ldots, m)$$
$$= 0$$

in event of survival over the year, or, alternatively, as a loss function

$$Y(i:s) = v(B^{(i)}_{x+s} - {}_sV) - [P - (v_sV - {}_{s-1}V)] = vB^{(i)}_{x+s} - ({}_{s-1}V + P) \quad (i = 1, 2, \ldots, m)$$

$$= -[P - (v_sV - {}_{s-1}V)]$$

in event of survival. Here the expression in square brackets is the one-year term premium for the multiple decrement insurance, and, as such premium is a constant, it does not affect the variance.

$$\text{Covar}\,[{}_rL(t, i:x), {}_sL(t, i:x)] = 0, r \neq s. \quad (5)$$

This key result may be obtained by observing that, if $r < s$, then

$$\text{Covar}\,[{}_rL(t, i:x), {}_sL(t, i:x)] = [v^r{}_rV - v^{r-1}({}_{r-1}V + P)]\,\mathcal{E}[{}_sL(t, i:x)],$$

which is zero by formula (2). Relating as they do to the single insured individual, the random variables ${}_rL(t, i:x)$, ${}_sL(t, i:x)$ are not stochastically independent; nevertheless their covariance is zero, and this paves the way for the Hattendorf theorem, which from the foregoing now follows as

$$\text{Var}[L(t, i:x)] = \sum_{s=1}^{n} \text{Var}[{}_sL(t, i:x)]$$
$$= \sum_{s=1}^{n} v^{2(s-1)}\,{}_{s-1}p_x\,\text{Var}[Y(i:s)]. \quad (6)$$

One quirk of the discrete theory is that Var $[Y(i:s)]$ reduces to

$$\sum_{i=1}^{m} v^2 p_{x+s-1} q^{(i)}_{x+s-1} (B^{(i)}_{x+s} - {}_sV)^2,$$

if the sums insured $B^{(i)}_{x+s}$ $(i = 1, 2, \ldots, m)$ are all equal, but may not do so if the sums insured are distinct. This restriction does not appear in the author's continuous case. By the way, it is possible that the theorem for the continuous case could be obtained as a limiting case of the discrete theorem with the insurance term subdivided into intervals much smaller than years, but the details of such a proof would be somewhat formidable.

The aforementioned quirk is not the only instance where the discrete theory is somewhat more complex then the continuous theory. As I recall, D. R. Schuette in his thesis research found that in the discrete case Covar $[{}^bL^{(k)}(t, i:x), {}^bL^{(j)}(t, i:x)]$, $j \neq k$, where b denotes functions based on the dependent decomposition, is not zero exactly, unlike the continuous case. Also, in the discrete case, Schuette has shown that if the maturity benefit in respect to cause j is zero, then the Loewy premium ${}^{L}P^{(j)}$ satisfies the equation

$$ {}^{L}P^{(j)}\ddot{a}^{\,j}_{x:\overline{n}|} = \sum_{t=1}^{n} v^{t}\,{}_{t-1}p^{\,j}_{x}\;{}_{j}q^{(j)}_{x+t-1}\left(B^{(j)}_{x+t} - {}_{t}V^{\underline{j-1}}\right) $$

$$ + \sum_{t=1}^{n}\sum_{i=1}^{j-1} v^{t}\,{}_{t-1}p^{\,j}_{x}\left({}_{j}q^{(i)}_{x+t-1} - {}_{\underline{j-1}}q^{(i)}_{x+t-1}\right)\left(B^{(i)}_{x+t} - {}_{t}V^{\underline{j-1}}\right), $$

where ${}_{j}q^{(i)}_{x+t-1}$ is the probability that an individual aged $x + t - 1$ and subject to causes of decrement $1, 2, \ldots, j$ will terminate within one year due to cause i. As ${}_{j}q^{(i)}_{x+t-1}$ is not necessarily equal to ${}_{j-1}q^{(i)}_{x+t-1}$, it may result that the addition of a benefit $B^{(j)}_{x+t} = {}_{t}V^{\underline{j-1}}$ produces a Loewy component ${}^{L}P^{(j)} \neq 0$, so that $P^{\underline{j}} \neq P^{\underline{j-1}}$, contrary to one's intuition regarding the situation.

T. N. E. Greville has pointed a way out of this difficulty for Loewy components in the discrete case by considering an insurance paying benefits $B^{(j)}_{x+t}$ with probabilities

$$ \tilde{q}^{(j)}_{x+t-1} = q^{\underline{j}}_{x+t-1} - q^{\underline{j-1}}_{x+t-1},\; j = 1, 2, \ldots, m $$

in respect to insurance year t. It follows readily that

$$ q^{\underline{j}}_{x+t-1} = \sum_{i=1}^{j} \tilde{q}^{(i)}_{x+t-1} $$

and fairly easily that

$$ \tilde{q}^{(j)}_{x+t-1} = p^{\underline{j-1}}_{x+t-1}\; q'^{(j)}_{x+t-1}, $$

where $q'^{(j)}_{x+t-1}$ is the independent probability of decrement due to cause j in insurance year t. Thus $\tilde{q}^{(j)}_{x+t-1}$ is a probability that an individual aged $x + t - 1$ will survive the first $(j - 1)$ causes of decrement over the year but be affected by cause j, with no condition whatsoever in regard to causes $j + 1, \ldots, m$. The mathematical formulation of the Loewy components for this modified discrete model avoids the difficulty encountered by Schuette, but there remains some question as to the actuarial significance of the model.

In regard to the author's statement in the discussion of Independent Premiums and Reserves that a change in $B^{(i)}_{x+t}$, $i \neq j$ will leave ${}^{a}\bar{P}^{(j)}$ and ${}^{a}_{s}\bar{V}^{(j)}$ unchanged, one qualification is necessary, namely, that the change does not affect $B^{(j)}_{x+t}$. If, for instance, $B^{(j)}_{x+t}$ is defined in terms of the total reserve, then $B^{(j)}_{x+t}$ depends on $B^{(i)}_{x+t}$, and ${}^{a}P^{(j)}$, ${}^{a}_{s}\bar{V}^{(j)}$ may vary with change in $B^{(i)}_{x+t}$.

The author has succeeded in developing multiple decrement theory from a statistical approach and in doing so has illuminated it for us and has also contributed to our understanding of individual risk theory. There are still some loose ends in multiple decrement theory, and the paper should be a stimulus to further investigations.

(AUTHOR'S REVIEW OF DISCUSSION)

JAMES C. HICKMAN:

The Society's practice of publishing discussions of formal papers has done a great deal to promote the development of actuarial science. Frequently rather modest papers have stimulated discussions containing interesting and original ideas. Professor Nesbitt's discussion illustrates this point. The ideas, developed by Butcher, Greville, Schuette, and Nesbitt and presented in this discussion, point out that many interesting and surprising results may be derived by using a discrete model to tackle the problems considered in the paper.

The decision to use the continuous model in this paper was influenced by a desire (1) to limit the paper to an acceptable length, (2) to complement the earlier paper by Bicknell and Nesbitt, and (3) to avoid the rather tedious details that seem inevitably to arise in using a discrete multiple decrement model. The consequences that followed from this decision were not entirely happy ones. First, as was indicated in the paper, at several places interesting and important points involving mathematical rigor were rather hurriedly passed over. Second, the continuous model, despite its elegance, often does not provide the same insight into practical actuarial problems that the discrete model does. The development of the Hattendorf theorem is a good example of the second disadvantage of the continuous model.

The individual risk theory portion of the Society's study notes on Risk and Reinsurance is largely devoted to a study of the mortality risk involved in a one-year term insurance for the net amount at risk. Rosenthal's paper (*RAIA*, XXXVI, 6–22), which also is on the syllabus for the risk topic, studies the determination of retention limits by using this year-by-year approach. There seem to be several good reasons for this emphasis on an annual study of risk rather than on an examination of the over-all mortality risk involved in an individual policy (measured rather imperfectly by Var $[L(t, i{:}x)]$ as is done in the paper under discussion. The year-by-year approach does not compel the long-term estimation of the interest rate and of the distribution of time until death. It seems to isolate the mortality risk problem rather than to obscure it behind long-range estimation problems. The year-by-year approach is also consistent with traditional life insurance practice of keeping predetermined reserves which are usually related to guaranteed cash values, and of determining and analyzing surplus annually.

The important point in all this, however, is that the Hattendorf theorem provides actuaries with a convenient bridge between these two aspects of individual risk theory. Thanks to equation (5) in the discussion, it can be seen that there is a simple way to combine the variances associated with the risk on one-year term insurance for the amount at risk to form the variance associated with a longer period.

Thanks are also due to Professor Nesbitt for his correction of the rather imprecise statement in the paper concerning the impact of a change in $B_{x+t}^{(i)}$, $i \neq j$, on ${}^{a}\bar{P}^{(j)}$ and ${}_{s}^{a}\bar{V}^{(j)}$.

HEALTH INSURANCE CLAIM RESERVES AND LIABILITIES

JOHN M. BRAGG

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FRED H. HOLSTEN:

Mr. Bragg's paper covers a subject that has grown in importance and complexity with the vigorous growth and expansion of Health insurance. This expansion to new benefits and benefit provisions may have outmoded some concepts that were suitable for the older benefits. At the least, judging from the differences in practices and opinions on some of the newer coverages, generalization of the older concepts to cover the newer situations has apparently been far from uniform. It therefore is appropriate to look into this matter of basic concepts.

To this end it might be helpful to first seek to establish as sharp a distinction as possible between claim obligations and other obligations. One approach that seems reasonable would be to postulate that claim obligations outstanding at a given time, such as year-end, should not be affected by the basis on which premiums are determined, their frequency of payment, the premium periods, or rights of renewal. The claim incurral concepts arrived at by considering only the "pure" situation where the given time is at the expiry of a term-premium period, with no unearned premium or level premium reserves, should not change if there happened to be a different situation solely as to premium or renewal aspects at that time. Any differences from the "pure" situation might well require appropriate recognition, but this matter would appear to belong in the area of traditional premium reserves or contingency reserves.

It would follow, therefore, that the concept of claim incurral, and therefore of outstanding incurred claim obligations, should proceed only from consideration of the situation where earned premium development terminates. This will be recognized as the "minimum requirement" described by Mr. L. S. Wagenseller in his discussion of Mr. Bertram N. Pike's paper on incurred claims for group health dividend purposes appearing in *TSA*, Volume X.

By way of clarification, the expression "earned premium development" is used here in the sense that level premium reserves, like unearned premium reserves, can be looked upon as reflecting premiums as yet not earned; these reserves therefore develop future earned premiums unless erased by contract provisions. Thus, an individual policy that becomes paid-up at a given age would continue to develop earned premiums after that age is attained.

This concept of claim incurral may be stated more specifically as follows:

The incurral date of the claim payment for any given day of benefit accrual is the earliest date on which earned premium development for the claimant could have terminated without extinguishing the obligation to make that payment.

While, in group insurance, termination of earned premium development could result if the entire group terminated, this would not be a proper basis if termination with respect to an individual (without termination of the group) could leave the company with a larger obligation to that individual. This principle, which makes provision for a reduction in in-force on a continuing group, is reflected in the majority of the responses to item (*b*) in the section of the paper on maternity reserves. If, however, termination of the entire group could extinguish or reduce any remaining obligation with respect to previous individual terminations, it would seem proper to incorporate the proper probability of this occurrence in the claim reserve valuation for all groups combined.

In some cases, such as loss-of-time and hospital benefits, the termination of earned premium development provisions is such that a common incurral date for different days of benefit accrual results—for example, where these days of benefit accrual are linked together in a single period of disability or a single period of hospital confinement as defined in the policy. This association of different days of benefit accrual into a single claim combination having a common incurral date is clearly explained in the paper, and its advantages are demonstrated by the author in his subsequent presentation of techniques.

The incurral-date concept that I have given would not change this grouping into a claim combination except that it would add that the claim combination would be one determined by the provisions applicable at termination of earned premium development. This might have considerable effect in the case of certain major medical plans. For example, although the major medical claim accruals occurring while earned premiums are being developed may be linked together for the purpose of applying a deductible and a maximum, this would not establish the claim combination should the basis for tying together claim accruals after termination of earned premium development be more restrictive as to conditions covered, maximum period of benefit accrual, and so forth. However, proceeding on the basis described would generally lead to the same claim combination for loss-of-time and hospital expense benefits where there usually are no such cutbacks at termination of earned premium development.

A good test of the reasonableness of an incurral-date basis, and of a claim-combination basis, is the benefit provided under the "Comprehensive Plan" of doctor's-visits coverage, as described in Mr. Morton D. Miller's paper on "Group Medical Expense Insurance," appearing in *TASA*, XLIX, 2. Under such a plan, non-disabling as well as disabling conditions are covered, the plan can be written on a "first-visit" basis, and coverage (and therefore accrual of benefits) can continue indefinitely on an individual so long as earned premiums continue to develop. It is difficult to see how any reasonable incurral date or claim combination can be established except on a "termination-of-premium-development" basis, when certain restrictions, such as limitation to a disabling condition and a time limit, generally become applicable. Any attempt to bring more than this into the claim reserve and liability picture would appear to be reflecting deterioration of risk in relation to earned premiums —a matter that, turning to Life insurance for precedent, falls into the area of premium reserves or contingency reserves.

In application, it would seem advisable to consider modifications in the interests of practicality, particularly as to basic claim data recording. In Group insurance, the claimant is usually still insured at claim accrual, so that the precise incurral date is generally of no consequence to the claim administrator. Considering the volume of individual claim transactions, the multiplicity of benefit types, and variety in paypoints (including group policyholders whose outlook on the matter might be different from that of a company claim administrator or an actuary), appreciable expense savings can be realized by not calling for determination of this precise incurral date. It would be more economical and lead to more uniform application to record some other definitive date that is more readily obtainable and, on the average, is either not too far removed from the incurral date or has a natural relationship to the incurral date. Then this difference or relationship would be reflected in appropriate over-all techniques to come up with the proper total claim reserves and liability.

If such a reference date is used as if it were the incurral date in the method adopted, then the appropriate adjustment is simply a generalization of Version III in the paper. In the case where coverage of maternity benefits continues on outstanding pregnancies after termination of earned premium development, a natural relationship of nine-twelfths of a year exists between the incurral and reference dates, so that, as indicated in the paper, allowance is made for the expected level of maternity costs incurred during the last nine-twelfths of the year. In the case of non-maternity hospital benefits with the common three-month extension at termination of premium development, the incurral date would be the date of disablement but not earlier than three months before the entry into the hospital. Here the recorded reference date could again simply be the date of entry into the hospital; the recognition of the difference between this date and an incurral date averaging perhaps several days sooner would translate into the use of a fraction (corresponding to the nine-twelfths in the case of maternity) that would be of the order of one one-hundredth. In the case of other benefits, this fraction could be greater, particularly in the case of major medical, depending upon the basis adopted for the reference date.

As the paper points out, the determination of claim reserves and liabilities is often an art rather than a science. One cannot simply set up a mechanistic basis and accept its results. Some further consideration is required as to whether circumstances might call for some adjustment in which they might affect the rate of claim incurral or claim lags. For example, an unusual increase or decrease in insurance in force toward the end of the year, such as could occur in the case of group insurance, might require special attention if the basic formulas did not incorporate this possibility. Knowledge of an unusual health situation, such as an epidemic prevalent at or before the year-end, also requires appropriate consideration. Still another might be some disruption in the mail channels just before the year-end. Nevertheless, to start with, one must have a reliable basis for the normal situation together with an understanding of the factors that are at work. In these and other respects, Mr. Bragg has given us a welcome and valuable paper.

STUART F. CONROD:

Mr. Bragg is to be congratulated for his pioneering spirit in presenting such an excellent and comprehensive paper on a subject which has heretofore been largely neglected in the *Transactions*.

In Section I, Mr. Bragg discusses the two opposing viewpoints with respect to reserves for deferred maternity benefits under individual policies. It would appear to me that there should be a similarity of treatment in the calculations of the reserve for deferred maternity benefits and the reserve for incurred but unreported claims. If a company uses the dates of disablement as the incurred dates for determining the reserve for incurred but unreported claims, then it would appear both logical and consistent to assume that maternity claims are incurred from the dates of conception.

Mr. Bragg's paper deals mainly with the theoretical aspects of the subject in which he treats the calculation of claim reserves and claim liabilities as separate entities. This may be so, but, from a small company viewpoint, it is sometimes more practical to combine the two in the calculations and then separate the reserve for future unaccrued payments from the accrued liability by means of ratios determined from past company experience.

As the methods used by our company for computing its claim reserves on pending claims under its noncancellable loss-of-time policies might be of interest to actuaries of smaller companies or of companies new to the noncancellable field, I am setting out the methods we use in detail.

We break down our pending claims into six separate groups, as follows: (1) specific estimates; (2) accident lifetime indemnity; (3) claims more than one year old (other than lifetime accident); (4) approved claims less than one year old (other than lifetime accident); (5) claims in course of settlement; and (6) special partial accident.

The specific estimates are largely for accidental deaths and dismemberment benefits attached to loss-of-time policies in process of settlement and hence are included in the claim liability for the full amount.

Tabular reserves (except where modified by the $3\frac{1}{2}$ times rule) are set up for Groups 2 and 3. In addition, the accrued liability to December 31 is calculated separately for each claim and included in the claim liability.

The claims in Group 4 are divided into three sub-groups, namely, (*a*) accident claims, (*b*) short-term sickness claims, and (*c*) long-term sickness claims; average claim factors per $1 monthly indemnity in accordance with the accompanying table are applied to each claim on a seriatim basis.

Month Incurred	Accident Claims	Short-Term Sickness	Long-Term Sickness
December	$2.60	$3.00	$ 3.50
November	2.80	3.50	4.50
October	3.00	3.75	6.00
September	3.20	4.00	8.00
August	3.40	4.25	10.00
July	3.60	4.50	12.50
June	3.80	4.75	15.00
May	4.00	5.00	17.50
April	4.40	5.25	20.00
March	4.80	5.50	23.00
February	5.40	5.50	26.00
January	6.00	5.50	30.00

The aggregate reserve and liability for Group 4 are then divided between reserve for future payments and accrued liability in the ratios of 0.75 and 0.25, respectively.

The factors used for the Group 4 claims are intended in the aggregate not only to cover any future payments for the basic loss-of-time benefits but also any extraneous benefits (other than principal sum) included in the policies. We are presently conducting studies for the purpose of converting our factors for Group 4 claims to a tabular basis which will require a separation of the liability under extraneous benefits into a separate category and certain changes in our claim records procedure.

The claims in Group 5 are also divided into the same three sub-groups as Group 4, and the following average claim factors per $1 monthly indemnity are applied: (1) accident claims, $1.30; (2) short-term sickness claims, $2.10; (3) long-term sickness claims, $4.00.

The aggregate reserve and liability for Group 5 claims are divided between reserve for future payments and accrued liability in the ratios of 0.40 and 0.60, respectively.

This leaves only the special partial accident claims in Group 6. These are accident claims more than one year old which are running on partial disability following a prolonged period of total disability. These claims are very few in number. Our practice is to set up a reserve equal to 150 per cent of the basic monthly indemnity under the policy.

GERALD A. LEVY:

Mr. Bragg has presented a useful survey of a subject which has received little coverage in actuarial publications. I would like to comment on the use of the incurral date to calculate the claim reserve.

The incurral date is of importance for the following reasons: (1) it is a starting point to determine when a claim reserve will be required, since only claims incurred prior to the year-end generate a reserve; and (2) it is used to determine the claim duration and thus the size of the reserve for future payments.

If the elimination period is short, the difference between setting the incurral date at the beginning of the elimination period and setting it at the end is minor. However, where the elimination period is longer, say from one to three months or more, the definition of the incurral date can have a significant effect on the claim reserve, especially the incurred but unreported portion of this reserve.

I believe it is more consistent with the contractual obligation of the company to set the incurral date at the end of the elimination period.

A company could choose the incurral date to be at the beginning of the elimination period, thus producing a larger claim reserve. But the Internal Revenue Service might object to this result, especially as it increases the value of the incurred but unreported reserves. Apparently the IRS has shown a keen interest in this liability, allowing only a small margin of conservatism between actual and expected payments.

Mr. Bragg refers to pages 196 and 197 of the text by Bartleson giving the NAIC recommendations, reported in 1941, of a special committee on Reserves for Non-Cancellable A&H Insurance. They recommended that for policies with a *waiting period*, the duration of disablement should be considered as dating from the time that benefits would have begun to accrue had there been no waiting period. Current use of the term "waiting period" equates it to the more definite term "elimination period." However, earlier use of the term "waiting period" appears also to have referred to a benefit which had retroactive payments. An example of this is in *TASA*, XXXIII, "Monetary Values for Disability Benefits, Based on 150% and 165%, Modifications of Class (3)," by James T. Phillips. That is, it is used as a qualification period as distinguished from an elimination period and allows retroactive payment of benefits. This may account for the NAIC recommendation, since it is consistent with the company's obligation to accrue a reserve during the waiting period. Herb Feay, *RAIA*, XXXIII, "Valuation of Disability Claims (Life Insurance)," discusses the date of disability or incurral date. He concludes that an adequate method is to use the end of the waiting period.

The question of the incurral date also has significance in the calculation of the reserve offset in certain reinsurance agreements. I bring to your attention the type of reinsurance agreement covering a Loss of Time Benefit for excess of loss payments which, in the jargon of accident and health reinsurers, is commonly called "Extended Elimination Period coverage." Many companies, especially smaller ones, are concerned about their potential liabilities under a long duration disability claim. They have sought reinsurance arrangements whereby the reinsurer would take over a share of the claim after an extended elimination period of, say, two to five years or more. In this circumstance the interpretation of the incurral date is obviously significant to the reinsurance reserve offset. The reinsurer, if given a choice, would probably interpret the incurral date to assist the ceding company. But, possibly because of the long duration of this elimination period and the contingent nature of the reinsurers' liability, the State Insurance Department might not favor any reinsurance reserve offset which significantly reduces the direct writing company's liability.

E. PAUL BARNHART:

I wish to express my profound thanks to Jack Bragg for the tremendous undertaking which he has brought to completion in the presentation of this paper. We who have been associated with him on the Actuarial and Statistical Committee of HIAA are well aware of the enormous difficulty of the task he faced in preparing a comprehensive survey of the wide and often confusing variety of claim-reserve techniques in use in the business.

I feel compelled to make a number of critical comments concerning various topics in the paper, and these are here presented with due respect and regard for the broad and difficult task which the paper represents.

1. "*Reserve for deferred maternity and similar benefits.*"—The author discusses the two viewpoints concerning the need for this reserve in a way that suggests that the question is a matter of opinion quite independent of the contract provisions involved. It seems to me that the matter hinges directly on the underlying contractual obligations. If the contract must be in force at the time of termination of pregnancy, then claims may fairly be regarded as incurred at termination and no deferred reserve should be necessary. Provision of a maternity extension, however, has the effect of contractually defining the claims to be incurred at conception, and surely a reserve for deferred benefits is proper.

The idea that reserves for deferred maternity are needed only when a contract has actually terminated I consider actuarially unsound. In a group case, particularly, if no reserve is established unless and until the group policy is terminated, it is quite likely that the group's account will not be sufficiently solvent to fund the reserve. The proper approach is to fund the reserve in a manner consistent with establishment of reserves for all other incurred claims, treating maternity claims as "incurred" at conception.

2. "*The problems of incurral date and period of disability.*"—Similar considerations apply here. If the contract establishes liability for disability commencing while the policy is in force, then it seems obvious that the claim must be regarded as incurred at the outset of the elimination period. If the policy must be in force at the onset of *compensable* disability, then the claim may fairly be regarded as incurred at the end of any elimination period. I do not see how such questions can be answered without reference to the specific contractual terms.

Toward the end of the paper, the author discusses rules for assignment of incurred date under calendar year deductible major medical. Here again it seems to me that the contractual terms covering the establishment of liability, especially in respect to continuation of liability upon contract termination, clearly define the correct basis for assigning incurred date. The liability is incurred within the period during which it is contractually created and should be charged against premiums earned during that period.

3. "*The tabular method.*"—I must confess that I am very much in doubt about the practical usefulness of this method and even the statistical validity of several of the steps involved. Even with high-powered computing equipment I do not regard the technique as being easily applied. The need to establish separate reserve and liability tables for every significant plan variation and the further necessity of subdividing incurred claims into weekly and quarterly intervals render this approach, to my mind, downright formidable.

However, the question of practicality is not a sufficient criterion for judging the basic value of a theoretical development. The boundaries of actuarial science and technique are not likely to be extended if we refuse to consider or accept a particular approach merely because we doubt its practical utility, and I think too many of us have tended to adopt this stifling attitude. The tabular method presented by the author is of considerable theoretical interest and deserves careful consideration.

I have the following comments and criticisms:

a) The implicit assumption is that Rj, the "reporting lag," applies equally to claims of all sizes and durations. I think that some testing of this assumption is desirable.

b) It seems to me that the most easily applied version of the method in actual use would be to back into the table from known paid claims rather than to attempt to generate incurred claims from some tabular expected claim basis, a procedure which appears to me to be fraught with many serious pitfalls. Thus, using Table 1, if, as of December 31, claims paid and incurred in the sixth week of the fourth quarter amount to $10,000, division by 0.5077 readily gives an estimate of total incurred to be approximately $19,800, and the various statement reserve and liability items are readily derived. It should be recognized that this technique is essentially the same as the development method.

The author mentions that this back in method "may be somewhat open to question where paid claims are subject to wide random fluctuation." However, when this situation prevails, the entire tabular method is of questionable value, unless some direct account is taken of paid claims and the year-end pending claim inventory. Regardless of the stability of the claim pattern, I think it is quite dangerous to derive the various reserve and liability items directly from purely tabular sources without some checking or adjustment against known paid and pending claims. At the very least, the tabular basis in use should be carefully tested year by year against prior developing claim results to prove its essential reliability and adequacy.

c) When the attempt is made to generate incurred claims on a quarterly and weekly basis and to account for shifting in-force business, the matter of "easily" prorating the claims over the year, a matter to which the author gives merely a passing reference, becomes a whole subject of research in itself, in my opinion. Any prorating is not advisably accomplished by some merely arbitrary rule, if its importance in relation to the business pattern is great enough to bother with it at all. One might prorate on the basis of a linear increase each week or a constant percentage increase. Suppose, however, that production is up heavily only for the last few months because of a sales effort or introduction of new plans. If this is not accounted for, results may be badly in error. Then, too, what about application of factors on a select and ultimate basis? Such coverage as major medical is known to exhibit a highly pronounced wearing-off of selection even into its second year. Uniform use of ultimate expected costs would, of course, give conservative results, although very highly conservative in some situations.

In any case, application of tabular costs to the in-force data to generate estimated incurred claims bears far more cautious investigation, I believe, than the author appears to suggest.

d) Specific and careful attention ought to be given to the type of benefit in which claims occur infrequently but have a large average value, such as a high deductible major medical benefit or long-term disability with a long elimination period. These benefits are troublesome to reserve in any case, especially with respect to unreported, but I believe use of the tabular method is prone to especially serious pitfalls.

If the method is applied on an expected tabular basis without making adjustments for reported pending claims, the estimate of incurred claims may be equivalent to no more than one or two average claims, unless an enormous in-force exposure is involved. If it so happened that, say, half a dozen claims were actually pending at December 31, the tabular method would produce a grossly insufficient reserve, when an easy inventory of actual pending claims would have shown an evident need for thousands of dollars more. While the unreported will still be a widely fluctuating unknown, surely maximum account should be taken of what can be known about the actual liability at December 31.

e) In discussing Version II, the author regards each day's claim accrual as separately reported and says that this accords with "practical facts" and represents approximately "actual conditions." I am unable to see that this can be true at all. The typical hospital claim is reported after hospital discharge, the entire hospital claim being usually reported at one time. In view of this, it appears to me that small claims of brief duration will be reported much sooner, on the average, than larger claims involving more lengthy confinement. If this is so, than it is fallacious to treat Rj, the reporting lag, as applying independently of the size of claim. This assumption could result in significant error in the whole division between reported and unreported. The question becomes particularly acute in considering the validity of formulas (16) and (17), where I suspect the tabular result will be a serious overstatement in the proportion of CU_t which is allocated to RCU_t, and corresponding understatement of $NRCU_t$.

For similar reasons I am inclined to challenge the author's statement that the reporting lag is "probably independent of the particular plan of benefit involved." A longer reporting lag would reasonably seem to be characteristic of a ninety-day plan than would be the case with a thirty-day plan, at least for the long confinement claims. It does not help the problem to suggest that under modern verification of coverage procedures an early reporting of potential liability will occur in either case, since the approach here is to define a day's accrual as reportable only on or after its accrual date.

This entire definition of reporting will be inconsistent with the practice of those companies that identify reporting period by means of the claim number, since identification by a claim number assigned before December 31 would no longer demonstrate reported status of any item. Thus, in setting up statistical history records, it would be necessary to resort to a somewhat involved system of auxiliary reporting dates for every item in the total claim.

f) It should be recognized that formulas (16) and (17) revert to the conventional definition of "reported." This inconsistency with the other formulas in Version II does no harm, so far as I can see, except that it is somewhat confusing and should be carefully understood in following the theory. I have already commented on the possible distortion created by separating $NRCU_t$ from RCU_t by these formulas.

g) In Version III, the author's description seems incorrect. Applying a factor of nine-twelfths to the year-end in-force seems incorrect unless the in-force were stationary throughout the last nine months of the year. Ignoring miscarriages and premature births, it would appear to me to be more correct to measure the mean in-force over the period April 1–December 31 and apply a nine-twelfths factor to the expected claims of this period. If the in-force were increasing rapidly, application of the factor only to the year-end in-force would seriously overstate the reserve.

4. As the author implies, one object of using the tabular method is to establish a clear-cut technique using a "recognized standard" as a morbidity basis, in order to establish a more certain qualification of claim reserves for federal income tax purposes.

While there is perhaps a desirable objectivity in the tabular method, I think it would be extremely unfortunate if companies felt compelled to resort to this basis in order to qualify their reserves as life insurance reserves for federal tax purposes. Other methods discussed by the author, such as average claim factor and development methods, certainly contain the company's own "experience table" in implicit form, and if these produce reserves judged to be adequate and proper under insurance department examination, I think they should surely be acceptable to IRS as "recognized standards." The tabular method is, in my opinion, sufficiently ponderous and sufficiently full of potential difficulties that it surely should not end up as a *required* basis for IRS approval, and I would hope that the industry would resist any such outcome with determination. The more direct methods in use have long standing acceptability, and their practicality and convenience should commend them as qualified methods for federal tax purposes.

In this connection, a reasonable confidence level in claim reserves is also surely in order. The author suggests a confidence likelihood of 3 to 1, or 75 per cent. I have usually aimed at an 80 per cent, or 4 to 1, confidence level. While the exact level is a matter of opinion, there should be no objection from IRS concerning reasonable conservatism in establishing qualified health claim reserves.

Aside from these several criticisms, the paper comprises a badly needed survey of claim-reserve methods and will be of much value to actuaries and statisticians dealing with this important area.

WILLIAM T. TOZER:

I would like to congratulate Mr. Bragg on his very excellent paper. He has rendered the health insurance actuary a great service.

Mr. Bragg provides a very excellent explanation of the tabular method. However, I have one objection: Mr. Bragg's method starts with an assumed amount of incurred claims and dissects this into its various components. There are many figures which make up incurred claims which are known at the end of the year. One very good example is the total amount of claims paid. I personally feel that a company would have a much more accurate answer if it develops the various parts of incurred claims and then sums these various components to arrive at the total incurred claims. In other words, I suggest that a company build incurred claims from its various components rather than by dissecting an assumed incurred claims into its various parts.

If a company desires to do this, one of the problems that it has is determining the incurred and not reported claim liability. The following is a method which may be used to determine this liability, if all other items are known. The example used below is for the purpose of illustration and does not propose to be representative.

TABLE 1

BREAKDOWN OF ONE MONTH'S PAID CLAIMS BY SPREAD BETWEEN INCURRED AND REPORTED DATES

t*	Amount Paid	Per Cent
0	$171,760	21.47%
1	384,080	48.01
2	152,000	19.00
3	42,640	5.33
4	17,120	2.14
5	15,360	1.92
6	5,200	.65
7	4,320	.54
8	3,200	.40
9	2,080	.26
10	1,120	.14
11	400	.05
Over 11	720	.09
Total	$800,000	100.00%

* t = The calendar month report less the calendar month incurred.

Reported lag table.—The first step is to develop a reported lag table. A type of reported lag table has been developed in Table 1.

Table 1 was developed from a typical month's paid claims. Each claim carried an incurred date and a reported date. From these dates a t was calculated for each claim. This was accomplished by subtracting the incurred calendar month from the reported calendar month. For example, a claim reported July 13 and incurred on July 4 would have a t of 0; however, a claim reported on July 3 and incurred on June 30 would have a t of 1.

Next, these paid claims were grouped by their respective t values. Table 1 shows that $171,760 of paid claims had a t value of 0, $384,080 of paid claims had a t value of 1, and so forth.

The final step is to determine the per cent of the total paid claims that falls in each of the t classes. This is shown in the last column of Table 1.

Per cent of incurred claims unreported.—The next step in this method of determining the incurred and not reported claim liability is to determine the per cent of incurred claims that is unreported. This has been done in Table 2.

According to Table 1, 21.47 per cent of the incurred claims are reported in the same month they are incurred. Consequently, on December 31, the percentage of December incurred claims reported is 21.47

TABLE 2

PER CENT OF INCURRED CLAIMS UNREPORTED ON DECEMBER 31

Incurred Month	Per Cent
December	78.53%
November	30.52
October	11.52
September	6.19
August	4.05
July	2.13
June	1.48
May	0.94
April	0.54
March	0.28
February	0.14
January and prior	0.09
Total	136.41%
Average	11.37%

per cent. This leaves 78.53 per cent unreported. The November claims have 30.52 per cent unreported, because 21.47 per cent were reported in November and 48.01 per cent were reported in December. The remaining months were determined by the same method of subtracting from 100 per cent appropriate percentages from Table 1.

By summing these percentages and dividing this sum by twelve, the per cent of the annual incurred claims that are incurred and not reported is found.

Amount of incurred claims unreported.—The next step is to determine the amount of incurred claims that have been reported. This has been done in Table 3.

The per cent of incurred claims not reported must be converted from an incurred claim base to a reported incurred claim base. This is done by dividing the percentage determined in Table 2 (11.37 per cent) by the difference of 100 per cent and this percentage (100.00 − 11.37 = 88.63

per cent). This gives us a new percentage of 12.83 per cent (11.37 ÷ 88.63 per cent).

The amount of incurred claims unreported can now be determined by multiplying the amount of reported incurred claims ($8,500,000) by our new percentage (12.83 per cent). This gives us an incurred and not reported claim liability of $1,100,000.

Conclusion.—This method is appropriate when the other components of incurred claims are (1) known, (2) easily estimated, or (3) the estimated items are small in relation to this liability. This condition is often met in hospitalization insurance. Also, the incurred and reported dates are needed to determine the reported lag table.

TABLE 3

AMOUNT OF REPORTED INCURRED CLAIMS

Claims paid this year	$9,500,000
Less amount paid on prior year's claim reserves and liabilities	1,600,000
Claims incurred and paid this year	$7,900,000
Plus this year's claim reserves and liabilities except incurred and not reported	600,000
Amount of reported incurred claims	$8,500,000

Again I would like to thank Mr. Bragg for his worthwhile contribution. This discussion is not meant to detract from Mr. Bragg's work. In fact, he made a passing reference to this type of approach in his tabular method. I hope that we will have the benefit of additional excellent papers on health actuarial science.

ANTHONY T. SPANO:

Mr. Bragg's paper is certainly a welcome addition to the Society's literature. An especially commendable feature is that his summary of reserve methods is very comprehensive and thus presents the actuary with enough choice to permit him to select those procedures which best fit his company's type of operation. Due to a relative scarcity of published morbidity data and the consequently frequent need to use methods which are different from a routine application of stable unit reserve factors, Mr. Bragg's paper should be especially valuable to companies which are just entering the health insurance field. On the other hand, the tabular method contains enough refinement and sophistication to serve adequately the needs of companies with health insurance portfolios of considerable volume and variety.

At the Equitable we use the procedure that Mr. Bragg classifies as the "development method" for all our individual hospital and medical expense coverages. We have been quite satisfied with the use of this method and feel that it would probably also yield favorable results if applied to noncancellable disability income claims where, as of valuation date, the disablement has not extended beyond a relatively short period. Due to the requirements contained in Section 219 of the New York Insurance Law, however, we have been unable to consider the extension of this method to disability income policies. We are pleased that the industry advisory committee which has been exploring the matter of reserves for these policies has recommended that companies be permitted greater latitude in determining disabled life reserves during the first two years of disablement. We currently use the "three-and-one-half-times rule" for claims which are within the first year of disablement, and our experience to date indicates that this method has resulted in a significant overstatement of claim reserves. We would therefore welcome a procedure that would produce a more realistic valuation.

We apply the development method in a manner which is fundamentally the same as that discussed in the paper. On the basis of subsequent experience, we redetermine the claim reserves and liabilities actually required at previous year-ends and relate these to the amount of premiums in force in the years during which the claims were incurred. This procedure is followed separately for each policy form, and we then determine graphically a set of percentage factors that appears to be appropriate for the current valuation. Due to the inflationary trend in medical-care costs and the increasing average duration of our business, these factors have been rising from year to year, in some cases rather markedly.

The use of the development method as described by Mr. Bragg assumes that the volume of claims still in course of processing bears a fairly constant relationship, from one year-end to the next, to the amount of premiums in force. This may not be generally true in some companies, particularly if the amount of health insurance business is changing rapidly, and at the Equitable we have noticed significant fluctuations through the years. It is clear that the use of the method without any modification for variations in the relative volume of unprocessed claims at the valuation date can lead to substantial misstatements in the amount of claim reserves and liabilities. Accordingly, the final step in the development of our reserve factors is a modification to reflect any such significant changes.

We have recently been particularly pleased that our procedures have continued to function satisfactorily despite a number of unstabilizing influences on our premium volume, such as sizable premium increases for certain classes of policies, uneven amounts of new business, and substantial conversions from one policy form to another. We feel that the method involves unusual simplicity when measured against the effectiveness of the results.

EDWARD A. GREEN:

In his comprehensive paper on health insurance claim reserves and liabilities, Mr. Bragg makes only passing references to interest. This may be on the premise that the interest involved is of relatively little importance. Such a premise would be based on the concept that obligations existing at the end of an accounting period, such as a calendar or policy year, are discharged in large part in the early days of the ensuing accounting period.

As a practical matter, obligations arise initially at the time of issue of a group policy or a block of personal health business and exist continuously as long as the policies are in force. By way of illustration, premiums and liability for claim payments under a group policy begin to accrue the moment it is put in force. However, the various types of lag between the incurral of claim liability and the payment of benefits described in Mr. Bragg's paper delay for a length of time, depending on the type of coverage involved, the emergence of claim disbursements at their ultimately expected level. Once this ultimate level is reached, income and outgo can be expected to be roughly in balance until premium receipts cease and the lag in benefit payments catches up. The same general principle would apply to level premium personal health business provided premium income is adjusted for changes in the level premium reserve. It is from the excess of premium receipts over benefit payments in the early days of the policy or block of business that funds are accumulated to meet the excess of benefit payments over premium receipts following termination. These funds are held continuously over the life of the policy or block of business, with modification in amount to reflect changes in the size or nature of the risk, and can be kept invested, earning a rate of return.

It seems to me that the short-term concept of funds held to meet health insurance claim obligations ignores the practical fact that as fast as one obligation is met another arises to take its place. The total value of outstanding obligations for a constant block of business reaches a relatively stable level early in its existence, and therefore the funds held to meet them may be viewed as not being short-term at all.

A long-term concept does not affect the requirements of sound accounting for holding the funds or the validity of the techniques described by Mr. Bragg for determining their level. It may produce different answers, however, to such questions as whether to and how to recognize the interest earnings on such funds in rating and valuation practices, whether to use short-term or long-term interest rates in any such recognition, and what is the tax status of the earnings on such funds.

(AUTHOR'S REVIEW OF DISCUSSION)

JOHN M. BRAGG:

I certainly wish to thank the seven gentlemen who discussed this paper. The subject is difficult and to some extent controversial. I was pleased to see the widespread nature of the discussion. The commendatory remarks are much appreciated.

Mr. Holsten, Mr. Conrod, Mr. Levy, and Mr. Barnhart have all discussed the question of "incurral date," and their remarks are a valuable contribution to the paper. The heart of this controversial question can be illustrated by a loss-of-time policy with a two-month elimination period, which provides that no payments are made unless the policy is in force at the end of the elimination period. Should a claim be considered incurred at the beginning or end of the elimination period? Mr. Holsten, Mr. Levy, and Mr. Barnhart would all seem to take the strictly contractual attitude and feel that the claim is incurred at the end of the period. However, the opposing attitude holds that if an insured is disabled and in the elimination period, he is almost certainly not going to allow his insurance to terminate, and there is therefore almost no chance that the claim can be avoided if it goes beyond the elimination period. As a matter of practical reality, therefore, an appropriate reserve should be established for benefits expected, and the claim should be considered incurred at the beginning of the elimination period. The 1941 NAIC recommendations seem to bear out this attitude. On this and other questions the paper does not take sides but merely points out the different viewpoints.

Mr. Conrod has pointed out the parallel between this and the maternity claim situation. (In the latter the "waiting period" is always nine months!) I also wish to thank Mr. Conrod for his description of the methods used by his company and the factors he outlines.

Mr. Barnhart's thorough and searching discussion of the tabular method is much appreciated. The method is primarily an attempt to discover the nature of the beast by analytical means. Its practical application is undoubtedly difficult and it is to this area that most of Mr. Barnhart's comments are directed. The method will be of most practical value only for truly "unknown" items such as those arising from brand new blocks of business or the item "Present Value of Amounts Not Yet Due on Unreported Claims." Certainly, other methods are preferable where concrete information is available, such as pending claim inventories.

Mr. Barnhart makes several comments about Version II of the tabular method, particularly concerning reporting lag. This version regards each day's claim accrual as separately reported. This means, of course, that if one could work out a composite average reporting date for a twenty-day claim, it would be later than that for a two-day claim, simply because there are eighteen additional, and later, days to be reported. It is in this way that the method attempts to comply with Mr. Barnhart's requirement "small claims will be reported much sooner, on the average, than larger claims involving more lengthy confinement." Version I would not accomplish this, incidentally, since that method measures all reporting lag from the incurral date by the same probability factors, regardless of the length of the claim.

The Version II method takes the attitude that any particular day of confinement (say the eighth day) will be reported with about the same average lag, regardless of whether the insured has purchased thirty- or ninety-day coverage. This seems fairly reasonable to the author and results in the statement that the reporting lag is "probably independent of the particular plan of benefit involved." Mr. Barnhart makes the statement, "A longer reporting lag would reasonably seem to be characteristic of a ninety-day plan than would be the case with a thirty-day plan, at least for the long confinement claims." Version II does indeed seek to bring this about, in the same manner pointed out in the previous paragraph.

Mr. Tozer's description of a modified tabular method designed to give the incurred but unreported liability is indeed a valuable contribution, for which I want to thank him.

Mr. Spano's comments about his company's use of the development method are most interesting. The development method seems to be the most widely used and one of the most reliable methods, particularly if handled with the caution Mr. Spano suggests.

I also wish to thank Mr. Green for his valuable discussion of interest rates and his comments concerning the validity of a long-term concept relating to health insurance claim reserves and liabilities.

APPROXIMATE PROBABILITY STATEMENTS ABOUT LIFE ANNUITY COSTS

ROBERT L. FRETWELL AND JAMES C. HICKMAN

SEE PAGE 55 OF THIS VOLUME

JOHN M. BOERMEESTER:

The authors of this paper stated they did not attempt to solve the fundamental problem concerning attaching probability statements to pension-system liabilities. They have, nevertheless, performed a valuable service to actuaries by exploring the possible use of probability-inequality theory for a situation involving a distribution of annuity values.

A reader of this paper is immediately confronted with the fact that the three probability-inequality analyses which they performed showed such high bounds in comparison to those derived by the other methods. This situation leads one to ask two questions.

First, I wonder if the authors would now venture making a conclusion that the three specific probability-inequality tests which they examined would generally produce bounds which would be too high for practical use in connection with annuity probability statements? It would seem to me that they might so conclude simply because of the conservative theoretical considerations which are involved in each of the particular probability-inequality tests.

Second, under what conditions would the bounds produced by the other three methods be considered not conservative enough for use in connection with annuity probability statements? In particular, the authors noted that the bounds produced by the Monte Carlo method are subject to a random error which may be reduced by the repetition of the number of trials.

I would like to make the observation that the highly increased speed of more modern computational equipment has removed many of the time limitations which existed a decade ago with respect to reducing random error under the Monte Carlo method. Where formerly one hundred trials could be considered for practical reasons, perhaps as many as ten thousand could be made today, since the speeds of machines in many instances have increased a hundred fold.

DONALD A. JONES:

Many thanks to Mr. Fretwell and Dr. Hickman for their fresh approach to a long-standing problem. Their paper provided stimulating material for our actuarial mathematics seminar at Michigan, and it should stimulate other research on the problem now that computer hardware is within reach of most of us.

There are three attributes of the authors' probability bounds which I believe were undersold. First, I think the form of the authors' answer is an improvement over previous answers. It may be my conservative nature, but I prefer to know an upper bound for the probability that total costs (C) will exceed a given limit (L). The Monte Carlo, Normal, and Pearson Type III answers are not bounds but approximations to the probability that $C \geq L$ and thus may be less than or greater than the true probability. Moreover, since the Monte Carlo answer is in fact a statistical estimate of $Pr\{C \geq L\}$, there is no way to determine the sign, much less the magnitude, of the error involved.

Second, the authors' remark in paragraph 3, "The advantage of such bounds would be their independence of any specific assumption as to the structure of the distribution . . . ," should be underscored. The probability inequalities are valid for *all* distributions with the given moments. In fact, the results given here do not exploit either the nature of the probability distribution of the cost per life, that is, the a1949 Male Mortality Table or the fact that the total cost is the sum of the independent costs per life. The authors' report that Bernstein's inequality, which does use the independent sum property, was no better than the Uspensky inequality is disappointing. Nevertheless improvement in the probability bounds may come by introducing more information into the inequalities.

Third, in comparing these approximations of $Pr\{C \geq L\}$ one must keep in mind that Monte Carlo is not practical without a digital computer. Thus, a reasonable question would seem to be, "Given n units of time on a computer, should it be invested in a Monte Carlo statistical estimate of $Pr\{C \geq L\}$ or should it be used to find $d \geq Pr\{C \geq L\}$ by a more refined probability inequality?"

The answer to this last question is not obvious to me. I used the University of Michigan IBM 7090 to apply the eighth moment inequality,

$$Pr\{X \geq b\} \leq \frac{1}{1+\frac{(b^4-E[X^4])^2}{E[X^8]-(E[X^4])^2}},$$

as derived in the appendix of this discussion. In less than fifty seconds, the program (written in FORTRAN) was compiled and executed for the case $I_{10} = 10$. The output of the execution was invalidated by an error; however, this probably did not affect the time.

The results in the accompanying table were computed by our IBM

7090 using the eighth moment inequality above, with b as defined by the authors below Table 1 and X as defined by the authors at the end

EIGHTH MOMENT

I_{10}	0.01	0.05	0.10	0.20
1.........	19.27	16.36	15.35	14.45
2.........	19.43	16.48	15.46	14.54
5.........	20.60	17.37	16.23	15.21
10.........	22.21	18.63	17.35	16.19
25.........	24.45	20.43	18.97	17.63
50.........	25.73	21.46	19.90	18.47

of their paper. Thus the columns headed 0.05 and 0.10 correspond to those in Table 1. The time required for the entire computation was 48.1 seconds; 37.3 seconds of this were used for FORTRAN compilation, 2.4 seconds for execution, and the balance for loading and processing.

APPENDIX

Cantelli inequality.—If $E[Y] = 0$, Var $(Y) = E[Y^2] - (E[Y])^2 = 1$, and $\lambda > 0$, then $Pr\ \{Y \geq \lambda\} \leq 1/(1 + \lambda^2)$. In other words, every probability distribution with mean zero and standard deviation one has no more than $1/(1 + \lambda^2)$ probability in the right tail commencing at λ. This bound on the right tail probability is best in the sense that, for each λ, there exists a probability distribution such that $Pr\ \{Y \geq \lambda\} = 1/(1 + \lambda^2)$.

Figure 1 shows the geometry of the authors' method of proof as it

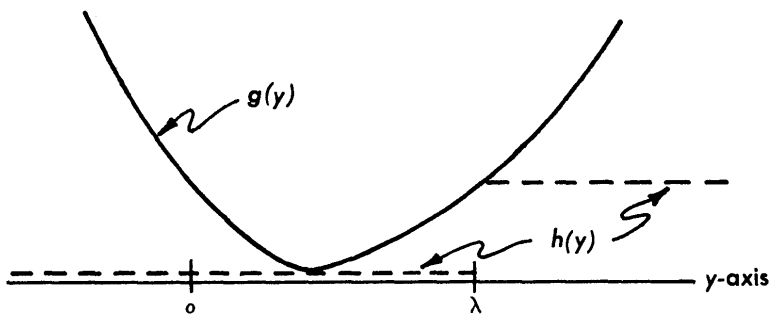

FIG. 1.—Graph of $h(y)$ and $g(y)$

applies here and gives some insight into the "quality" of the inequalities. If the step at λ in $h(y)$ is unity, then $E[h(Y)] = Pr\ \{Y \geq \lambda\}$. For every second degree polynomial which is everywhere greater than $h(y)$, like

$g(y)$ in the figure, we have $E[g(Y)] \geq E[h(Y)] = Pr\{Y \geq \lambda\}$. The best such bound is given by the polynomial with smallest expected value, that is, the one which passes through $(\lambda, 1)$ and is tangent to the y-axis at $-1/\lambda$. Thus we see the discrepancy between the bound and the desired probability for a given distribution is the weighted (by the given distribution) average of the differences between the parabola and the step function.

As corollaries to the Cantelli inequality we have the Uspensky, the fourth moment, and, in fact, the whole family referred to by the authors. As an example, consider the fourth moment inequality (the notation is the authors').

$$Pr\{X \geq \sqrt{kv}\} \leq Pr\{X^2 \geq kv\}.$$

In order to satisfy the hypothesis of the Cantelli inequality, we "translate" the latter event to one in terms of a standardized variable.

$$Pr\{X^2 \geq kv\} = Pr\left\{\frac{X^2 - v}{\sqrt{t - v^2}} \geq \frac{kv - v}{\sqrt{t - v^2}}\right\}$$

$$\leq \frac{1}{1 + \left(\frac{kv - v}{\sqrt{t - v^2}}\right)^2}.$$

More generally,

$$Pr\{X \geq \sqrt[N]{KE[X^N]}\} \leq Pr\{X^N \geq KE[X^N]\}$$

$$= Pr\left\{\frac{X^N - E[X^N]}{\sqrt{E[X^{2N}] - E[X^N]^2}} \geq \frac{(K-1)E[X^N]}{\sqrt{E[X^{2N}] - E[X^N]^2}}\right\}$$

$$\leq \frac{1}{1 + \left[\frac{(K-1)E[X^N]}{\sqrt{E[X^{2N}] - E[X^N]^2}}\right]^2}.$$

With $N = 1$, 2, and 4 we obtain the Uspensky, fourth moment, and eighth moment inequalities, respectively.

THOMAS M. YOUNG:

Frequently I have been asked by my business associates to comment on the relationship between the size of a group of employees covered by a pension plan and the probability that they will experience mortality that is unfavorable from the standpoint of cost of the plan. I say "frequently" because my comments have usually been lacking in mathematical preciseness and leave the questioner with a feeling that perhaps next time the question will be answered in more specific terms.

The article written by Dr. Hickman and Mr. Fretwell does not, of course, answer this fundamental question of how large a group should be before they can safely have an uninsured pension plan with very little risk of unfavorable mortality. There will probably always be differences of opinion among actuaries on this question. The paper does, however, present some interesting figures and statements regarding the probability that life annuity costs will vary from the assumed costs.

Not being enough of a statistician to discuss the significance of the relationship between the "Bounds on Life Annuity Costs" due to assuming different distributions of the random variable as presented in Table 1, I would like to make some comments regarding the probable size of I_{10} as defined by the authors and some observations regarding the effect of varying the interest rate. My comments shall be related to the Monte Carlo approach, the Normal distribution, and the Pearson Type III distribution only.

As shown in Table 1 of the paper, the larger the size of the income payable to any one individual, or a few individuals, in relation to the average income payable to the other members of the covered group, the greater the probability that the actual life annuity costs will vary from the assumed. Generally the size of the benefits under a pension plan is a function of the compensation of the individuals. It would seem then that the most common value of I_{10} for small pension cases would be between 5 and 10. From the standpoint of variation in actual pension plan costs from the assumed costs, the larger probability of excess cost due to mortality variation that is associated with plans in which one employee has a benefit more than ten times the size of the average benefits for the other employees may be offset by the fact that in these situations the employee with the large benefit is far more inclined to postpone retirement, thus producing actuarial gains.

The authors leave the development of probability statements concerning life annuity costs where both the factors of time until death and interest are variable for later development. Such later development would prove very interesting. As the authors point out, however, actuaries have a greater wealth of knowledge about the distribution of the random variable time until death than they do about the distribution of the random variable interest rates.

Table 1 of the paper suggests that for an I_{10} of 5 there is a 90 per cent probability that the actual life annuity costs will not exceed 126 per cent of the assumed costs. The value of a life annuity at age 65 on the a1949 Male Mortality Table and 4 per cent interest is 87.5 per cent of the value using $2\frac{1}{2}$ per cent. What then is the probability that 4 per cent will be earned instead of $2\frac{1}{2}$ per cent? In today's competitive pension market, there seems to be more emphasis placed on investment performance than on mortality guarantees. Does this represent a feeling that there is apt to be more variation in the random variable interest rates than in the random variable time until death and that such variation will produce wider variations in life annuity costs?

Note also that pension costs are based on estimates made far in advance of age 65, and fixed changes in the interest rate and mortality assumptions will produce greater variation in an annual cost calculated at age 35, for example, than in life annuity costs at age 65.

The authors present in Table 1 the upper bounds on life annuity costs for specified distributions and given probabilities. There is, of course, the probability that the actual life annuity costs will be less than assumed. Can we assume that the lower bounds for similar distributions and probabilities are approximately as far below the assumed costs as the upper bounds are above the assumed costs?

The authors have presented a very readable and interesting paper and are thus to be commended. I particularly appreciate their efforts, since I can now state that if the random variable time until death follows the Pearson Type III distribution and one of ten members has a benefit five times as large as the other nine, then there is a 90 per cent probability that actual life annuity costs will not exceed 126 per cent of the assumed costs. I will not have done much more in the way of answering the fundamental question that was stated above, but I will have failed to do so with mathematical preciseness.

(AUTHORS' REVIEW OF DISCUSSION)

ROBERT L. FRETWELL AND JAMES C. HICKMAN:

The members who have discussed this paper have asked some very penetrating questions and, what is even more, they have also indicated the direction that further research should take to find the answers. Mr. Boermeester was quite correct when he ventured the conjecture that we might consider the bounds, calculated by using probability inequalities, in our example too high for practical use. We will admit that we were initially disappointed with the results. We had hoped for tighter bounds. However, the somewhat more satisfactory bounds found by using an inequality that depends on eighth moments, and reported by Professor Jones in his discussion, give us some encouragement concerning the usefulness of probability inequalities in this problem.

Professor Jones's question concerning the relative economy of using a high speed computer and a Monte Carlo method or alternatively a probability-inequality method now becomes the central question. Mr. Boermeester has already supplied a flow chart for building a program for using the Monte Carlo method on a pension system with a variety of annuity benefits. Now it is necessary to insert a random number subroutine in the program and experiment with modern equipment, as was suggested in Mr. Boermeester's discussion. More work remains to be done in expanding Jones's program to provide for the computation of probability inequalities based on rather high order moments for annuity costs arising from a general pension system. Since we have not produced such programs, a final judgment by us on this question would be based almost entirely on intuition rather than facts. At the moment, however, based on our subjective feelings, we are not optimistic about the practical superiority of probability-inequality methods.

Mr. Boermeester's second question is an excellent statement of the difficulty that has always inhibited the application of individual risk theory to practical problems. Actuaries do not have a sufficiently detailed list of rules and error formulas to guide them to a "best" method of determining the probability distribution of total costs for a collection of life annuity contracts. We have no new results that can help solve this problem. Since we have not determined the exact distribution of annuity costs under the assumptions of this simple example, we cannot even make absolute comparisons among the methods illustrated in the paper. No theoretical difficulties are involved in finding the distribution of the sum of ten independent random variables, yet the computational details are rather considerable.

Mr. Young's remarks bring some very important practical problems to our attention. He reports, we believe quite correctly, that random variation in mortality costs in pension systems is usually of less concern and of smaller magnitude than are variations in costs due to interest rate changes. Although the question is far from closed, we are not certain what modern statistics can contribute to statements about interest rates. Past experience is certainly not a perfect guide to future interest yields. It appears that the best that we can hope for is a statistical theory which will require actuaries to formulate their estimates about future interest rates in a form that will lead to a probability type weighting of interest rates on a restricted range of possible rates, say, from 2 to 6 per cent. Such a technique might serve to make more explicit differences of opinions among actuaries concerned with estimating future interest rates.

We want to thank the members who have discussed this paper. They have proposed a stimulating set of questions, and have made several constructive additions to the paper. We especially appreciate Professor Jones's development of the general family of one-sided probability inequalities depending on an even order moment. His geometric representation helped us in understanding the nature of these inequalities.

ACTUARIAL STUDENTS, EXAMINATIONS, AND THE PROFESSION

CARL H. FISCHER

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EDMUND C. BERKELEY:

It seems to me that this paper is both important and interesting. In general, the degree of importance of some factor relative to some result is directly related to how much difference variation in that factor can produce in that result. In this paper we have a large increment of knowledge about actuarial students, examinations, and the profession. I think this increment of knowledge, if acted upon by the Society, will produce a great increase in the supply of actuaries in years to come, and this would be highly beneficial.

Second, I think this paper is important because it provides a fine illustration of the principle "The work of science is to substitute facts for appearances, and demonstrations for impressions." Instead of the assumptions and armchair reasoning of the past in regard to actuarial students and examinations, colored by one's own feelings as he passed the examinations, we have an impressive array of statistical evidence. Of course, in any study of observed statistics a deductive demonstration is hardly possible; but, nevertheless, a study using questionnaires in which more than 80 per cent are returned is as close to a demonstration as one might reasonably expect.

Finally, this paper is interesting to me because its subject departs from the ordinary actuarial topics and applies the actuarial point of view to a subject not in the syllabus. It seems to me that there are many more fields outside the syllabus in which the peculiarly practical —and mathematical—point of view of the actuary can be usefully applied. For example, one such field is the mathematics of estimating and inspecting—which all actuaries learn from their work and which should be taught everywhere. Another such field is, I think, programming languages for electronic computers, which I believe will have a profound effect on actuarial work.

I would like to see actuaries engaged more in plowing in new fields and engaged less in plowing again and again in old fields.

JOSEPH B. MACLEAN:

This paper has interested me greatly. As there will probably be a full discussion, I shall make only a brief comment.

When we seek recruits, as we do, on the basis that the work of an actuary is essentially and primarily mathematical in character and with the implication that mathematical skill is the main road to advancement, we are, I think, laying the ground for disillusion and disappointment.

Most of the successful actuaries, both of the present and the past, are, or were not, "mathematicians" in the academic sense. They have all, of course, possessed the basic knowledge of mathematics which all actuaries must have and which, actually, is relatively elementary. First-rate mathematicians have been comparatively few in the profession and, with some notable exceptions, have not in general attained high company rank.

Dr. Fischer shows that this situation is now being realized in the colleges and that it is deterring some of those students who are primarily interested in mathematics from taking up an actuarial career.

The actuarial profession has much to offer to capable and ambitious young men—not only, or chiefly, to mathematicians. There should, I think, be no shortage of recruits if more information as to the real character of actuarial work and as to its wide scope and variety were made more generally available to college students. In doing this, less stress should be laid on the importance of mathematics and more on the value of a well-rounded education and of such qualities as initiative and personality.

RALPH GARFIELD:

Because many members of the Society are probably not familiar with the English system of training actuaries, I thought it might be worthwhile to describe that system.

Tutoring for the examinations of the Institute and Faculty of Actuaries is provided by the Actuarial Tuition Service (A.T.S.). This is an official organization set up and controlled by the Institute and Faculty and is responsible (under direction of the councils of those two bodies) for all arrangements for tutoring for their examinations.

With the exception of the preliminary examination in mathematics, the A.T.S. provides tutoring for *all* parts of the British actuarial syllabus. Tutoring for each subject is in the form of a correspondence course which is designed as a supplement to, *not* as a substitute for, the official reading matter. The course notes are divided into lessons and, at the end of each lesson, there is a test paper based on the material covered in that lesson. At the end of all the lessons there is a test paper which covers all the material studied. This paper can, therefore, be considered as a trial examination. There is no compulsion to purchase this correspondence course, although in practice everyone does. The student is recommended to complete all the test papers, which, as far as possible, include old examination questions. There is a high correlation between students who do the test papers and those who pass the examination. (Unlike the Society, the A.T.S. does not provide illustrative solutions to actual examination papers.)

Each test paper is corrected by a tutor assigned to the student by the A.T.S. The tutor gives a mark to each question on the paper, the total is recorded by the A.T.S., and the paper is returned to the student together with a model set of answers. (In general, students do not pay enough attention to these answers.) Tutors' comments can be most helpful, particularly in the later parts. The tutor is always available, through the official channels, to discuss special problems the student may have. For the Intermediate examination, which includes probability, elementary statistics, finite differences and compound interest, the A.T.S. also provides oral tutoring. For the remaining portion of the syllabus, discussion classes are arranged. These discussion classes are led by the tutor and are designed to discuss and resolve difficulties students experience. Many of these classes are very valuable, particularly in those subjects in which there is no single answer.

Once the student has passed that particular portion of the syllabus, he must return the lesson notes, the test papers, and the model answers. A student who fails the examination the first time may retain the notes and so forth for an additional fee and, in addition, may purchase what is called the "revision course." This consists of a set of test papers, each covering the whole subject.

Tutors are Fellows of the Institute or Faculty who have had practical experience in the subjects they tutor. For example, a tutor on pension funds would come from a consulting office, a tutor on Life Assurance would come from a Life office, one on finance and investment from a stockbroker or the investment department of an insurance company, and so on. Currently there are about eighty tutors. In a typical year over 5,000 test papers are marked.

Tutors are appointed by a committee under the direction of the Councils of the Institute and the Faculty and are usually the more recently qualified Fellows. The principal tutors help to co-ordinate any revision of courses that current developments may require. There is a natural line of succession in that many tutors eventually become examiners. Furthermore, most, if not all, of the members of the Council of the Institute were at one time tutors and examiners.

I do not recall the actual statistics, but I do not think there is a significant difference between the time students of the Institute and students of the Society take to attain their fellowships. However, I would think that, on the average, students of the Institute attain their fellowships at younger ages than students of the Society. Ages like 23 and 24 are by no means uncommon. An important point to bear in mind here is that a significant proportion of students *without* university degrees go directly from the school system into Life Assurance companies or consulting offices and thereby commence their actuarial studies at an earlier age. Furthermore, conscription into the armed forces ceased about five years ago.

Unless things have changed in the last few years, with one exception, no university in Great Britain provides courses in actuarial mathematics. The exception is the London School of Economics and Political Science, which, as a part of its special degree in Statistics, gives courses in finite differences and actuarial statistics.

Like the Society, the Institute has a public relations problem. To the general public, actuaries are virtually unknown. The Council of the Institute has appointed a committee which pursues an active policy of recruitment. This committee has prepared booklets on the actuarial profession and arranges talks at schools and universities where contact is maintained with career counselors and appointments boards. Judging by the increased awareness of the profession in these institutions, this committee is doing an excellent job.

JOHN C. MAYNARD:

In one section of this illuminating paper there is recorded some students' comments on the examinations. One-half of the recorded comments are critical of the later examinations on the old course of reading, because they are too dependent on memorization. This criticism has been made before, and, in reply, it has been pointed out that it is difficult to set searching questions when related topics are dealt with in separate examinations and when the examination time which can be given to one topic is limited.

The architects of the new examination syllabus intended that the questions on the advanced examinations would be deeper and more intensive, mainly for two reasons: (1) the background of the earlier basic examinations could be presumed and (2) there would be more examination time per topic.

The first examinations under the new syllabus were held in November, 1963. With the thought that students' comments on the first advanced examination 9I might be of interest, a questionnaire was sent to the 32 students who enrolled for the Canadian Association of Actuaries' study group for this examination. Part 9I is offered in two fractional parts, 9IA and 9IB. Part 9IA covers gross premiums, reserves, non-forfeiture values and changes; 9IB covers dividends, investments, asset valuation. Twenty-two replies to the questionnaire were received, of which four did not write, three wrote the full examination 9I, and fifteen wrote part 9IB only. The students were asked to classify the examination as having one or more of five characteristics. The characteristics and the number of responses are shown in the accompanying table. This sample of opinion is encouraging.

Characteristics	Full 9I	9IB
A good test of the understanding of advanced actuarial principles..............	3	9
Not a good test of the understanding of advanced actuarial principles	....	....
A test with too much emphasis on memorization..............................	....	2
Not a test with too much emphasis on memorization.......................	....	6
A test with other characteristics*.........	....	5

* Those who chose "other characteristics" commented widely.

RUSSELL E. MUNRO:

Dr. Fischer is to be commended on his very extensive analysis of the problem posed by the shortage of qualified actuaries.

Since 1947 the Canadian Life Insurance Officers Association has granted cash awards to successful undergraduate students in Canada achieving the highest marks on Part 2, General Mathematics, or G.M.E. and Part 3, Probability and Statistics, or P.&S. Usually these awards have been $100 each, but for several years a cash award of $200 was given the top student in each examination.

In the early years as many as eighteen or nineteen persons won the awards each year. The Part 2 award was conditional on the student's also passing the language aptitude Part 1 test not later than a year following success in Part 2.

In recent years from ten to twelve individuals shared in these awards. It is the Association's plan to continue eighteen prizes of $100 each, twelve based upon the spring examinations and six based upon the fall examinations. The awards are divided equally between the two examinations (new Parts 1 and 2). The fall examination awards have been effective since November, 1962.

Table 1, developed from the Association's news bulletins, can be compared with Dr. Fischer's Table 8, which shows the examination progress beyond the G.M.E. results to and including May, 1962, have been analyzed. This table adjusts for those who succeeded in winning awards in both examinations. It will be noted that 70 per cent of all winners and 73 per cent from 1956–58 winners progressed beyond Part 3, which is to be compared with 28 and 26 per cent, respectively, in Table 8.

Table 2 can be compared with Dr. Fischer's Table 18, which shows progress of prize winners toward fellowship according to a year grouping of winners. While the G.M.E. winners have been credited with approxi-

TABLE 1

EXAMINATION PROGRESS BEYOND THE G.M.E.

Examination Part	All Winners			1956–58 Winners		
	G.M.E. (104 Winners)	P.&S. (98 Winners)	Net* (168 Winners)	G.M.E. (13 Winners)	P.&S. (15 Winners)	Net* (22 Winners)
2 (G.M.E.)	26%	....	16%	15%	....	9%
3 (P.&S.)	15	12%	14	23	13%	18
4A	5	5	6	8	0	4
4B	7	8	7	8	7	5
5	11	17	13	31	27	27
6	7	8	7	15	20	14
7	3	5	4	0	20	14
8	26	45	33	0	13	9

* Net is G.M.E. and P.&S. winners less double winners G.M.E.

TABLE 2

PROGRESS OF PRIZE WINNERS TOWARD FELLOWSHIP

Years	Average Number of Examinations Credited through May, 1962		Proportion of Fellows	
	G.M.E. Winners	P.&S. Winners	G.M.E. Winners	P.&S. Winners
1947–49	5.3	6.6	50%	69%
1950–52	5.1	6.8	45	65
1953–55	3.9	7.0	11	57
1956–58	3.6	5.5	....	13
1959–62	2.6	3.9	....	5

mately one more examination and show just slightly better proportions of Fellows, the P.&S. winners had much better results in both categories.

The CLIOA has also made grants to "organizations engaged in promoting an increased student interest in actuarial and mathematical careers," that is, to support the mathematical contest for high school students. In addition, the Association has undertaken to finance the printing in English and French of a brochure on the actuarial profession in co-operation with the Canadian Association of Actuaries for distribution to high schools and universities.

The education committee of the Canadian Association of Actuaries has been instrumental in encouraging the students and in directing their preparation for all associate and fellowship examinations. Doubtless this program has been most effective in achieving the progress indicated in the tables, particularly for those who are engaged in actuarial office duties following graduation.

The evidence seems to support Dr. Fischer's second suggestion, which was to award prizes for success in one of the later examinations. Perhaps the prizes should be awarded on the new Part 2 P.&S., with the condition that Part 1 G.M.E. is passed not later than the year following the Part 2 success.

WILLIAM H. SCHMIDT:

I am sorry that I do not have a written discussion, but in reading this paper I felt there were three short points that I would like to add for the record. I am a member of the Public Relations Committee and, as such, the Society's representative on the MAA National High School Mathematics Contest. The statement is made in the paper that the Society "supports" the contest. This could be a little misleading. The Society gives $5,000 to the contest committee, but the committee operates on a budget of $25,000 or $30,000. They wanted support from a professional body such as ours. As a rule we have tried to comport ourselves in such a way that we have been assisting the professionals, in this case the mathematicians, in the pursuit of their aims.

The second point I would like to make is that as far as the aims of the contest go they seem to be pretty well achieved. In 1958 the contest was given in 2,900 schools, and 80,000 candidates wrote the examination. In 1964, there were 6,300 schools and 225,000 candidates wrote it. At least 200,000 got the career booklet which gives a brief description of the actuarial career, as well as other careers. To help measure the impact of the contest, the number of mathematical B.A.'s granted in 1958 was 6,900, and in 1962 the number of mathematical B.A.'s granted was

14,600, which is more than double the 1958 number. In the number of successful General Mathematics Examination candidates, the Society has a threefold increase between 1958 and 1962.

The third point I would like to make is on another matter entirely. In Table 7 Dr. Fischer discusses the trials required to pass. In 1958, 57 per cent passed it the first time; in 1962, 74 per cent passed. He surmises any one of three reasons, the last being that it could be due to the lower passing standard. I personally think that it is entirely due to that. The Education and Examination Committee analyzed the grades of those who took the first mathematical examination. They found that, while there was a significantly better chance for future success in the later parts for those who got 7 or above and significantly less for those who got 4 or lower, there was little difference between those who got 5 and 6. Accordingly, with the Board of Governors' approval, the passing standard was dropped from 6 to 5. I think this is largely the reason for the result shown in Table 7.

KENNETH P. VEIT:

There is probably no subject of more universal interest to all actuaries than the examinations of the Society. From the oldest to the youngest, we have all participated in the struggle and are prone to examine, with much reflection, any statistics pertaining to them.

Table 20 is of particular interest, and I am certain that every actuary who noted it paused to compare his own experience with the averages presented by Dr. Fischer. The mean (and median) length of time of roughly ten years between the date of the first examination and attainment of fellowship is slightly deceptive, as there may be a tendency to conclude that the *current* rate of speed through the examinations is about ten years. As Dr. Fischer was quick to point out, the results would be quite different if all those who started (rather than finished) the examinations at a certain time were studied.

Out of curiosity, I decided to test, for various periods, the hypothesis that the median length of time to complete the examinations was ten years. Since the only data I had available were some old *Year Books* and *Transactions*, I was forced to keep my tests fairly simple. For various years, I followed the progress of all those who passed the first mathematics examination (the language aptitude exam was ignored, as it was only given briefly) to determine how many ultimately attained fellowship. This was done by matching lists of successful candidates against the fellowship roster ten years later and at various other points. The results, shown in Table 1, may be only approximate, as certain similarities between names and the possibility of early death could have caused me to add or delete a person here and there. Also, the figures shown do not measure the period from first exam *taken* to fellowship but rather the duration between passing the General Mathematics Examination and attainment of fellowship. However, within these relatively minor limitations, they do provide useful and somewhat startling information.

The upper section of the table shows that, for ten sample years be-

TABLE 1

PERFORMANCE OF SUCCESSFUL CANDIDATES ON GENERAL MATHEMATICS EXAMINATION

Year in Which Passed G.M.E.	Number Passing (1)	Per Cent of (1) Attaining F.S.A. within 10 Years of Passing G.M.E. (2)	Per Cent of (1) Ultimately Attaining F.S.A. (3)	Per Cent of F.S.A.'s Attaining Degree within 10 Years of G.M.E. [(2) divided by (3)×100%] (4)
1925	65	19%	32%	59%
1929–30	158	10	32	31
1934–35	186	13	26	50
1939–40	99	10	44	23
1944–45	47	21	34	62
1950	271	20	35*	57
Total	826	16%†	33%	50%†
1953	169	19	....	....
1954	167	25‡	....	....
1955	177	29§	....	....
1956	168	29‖	....	....

* Estimate, based on thirteen years' experience (25 per cent through January 1, 1964).
† Excluding 1939–40 group as not typical, due to World War II.
‡ Estimate, based on nine years' experience (20 per cent through January 1, 1964).
§ Estimate, based on eight years' experience (20 per cent through January 1, 1964).
‖ Estimate, based on seven years' experience (15 per cent through January 1, 1964).

tween 1925 and 1950, some 16 per cent of the successful candidates on the G.M.E. became Fellows within ten years, and 33 per cent *ultimately* attained fellowship. This last figure was most surprising to me, and, checking with various other actuaries, I found that most of them guessed the proportion to be less than one in ten. These percentages should prove useful in recruiting. Another interesting figure (not shown in my table) is the percentage of successful students on the life contingencies examination (ranked as the hardest in Dr. Fischer's paper) who eventually become Fellows. Here the proportion is somewhat less than might be expected, being roughly two out of three, based on the results of those that passed this examination ten, fifteen, and twenty years ago. However, this proportion appears to be increasing rapidly.

Finally, the last column of the table shows (in a roundabout fashion) that, while a median figure of ten years between passing the G.M.E. and the last fellowship examination is correct in the aggregate, there are great variations between decades. The lower half of the table, based on experience from 1953 to date, shows a significant change in the percentage of successful G.M.E. candidates who finished their exams within ten years. Although the figures are partially estimated, it is clear that this percentage is rapidly increasing. This means that either the median duration to fellowship is decreasing, or that a larger percentage of those that pass the G.M.E. now go on to complete the fellowship requirements, or both. If the proportion of those passing Part 1 who become F.S.A.'s is indeed increasing, then with 700 to 800 students a year currently passing Part 1, we will not have to worry about the shortage of actuaries much longer!

Dr. Fischer has presented us with a most interesting study. With automation making such things easier and easier, I would like to see this and similar studies presented regularly, and updated continuously, perhaps by biennial, simple-to-code questionnaires being completed by all members of the Society. Studies such as Dr. Fischer's are a relatively rare source of statistical information about the Society; they provide an excellent means for evaluation of the over-all examination system and the course of the Society's growth.

BERT A. WINTER:

There is no subject of greater continuing interest to the members of a profession than the recruitment and training of their successors. We actuaries of the United States and Canada are thus fortunate to have available the substantial body of facts on this subject represented by the 754 replies to Professor Fischer's questionnaire and particularly fortunate to have these facts developed and analyzed by someone with Professor Fischer's experience in this field.

My own association with the education and examination work of the Society, like Professor Fischer's, began before 1947, when our predecessor organizations first granted prizes to those undergraduates scoring best in our General Mathematics Examination. I have not formed the impression, during this association, that a primary purpose of the prizes has ever been to attract to the profession the particular individuals who won the prizes. Indeed, it is quite clear from Charles Spoerl's paper in

TSA, Volume I, that the original purpose of the G.M.E. and the related prizes was to improve familiarity with and the standing of the profession in the undergraduate mathematics departments of the colleges—not just "pure mathematicians" but any student with sufficient facility in the mathematics he was being taught as an undergraduate so that mathematical illiteracy, at least, would be no bar to his passing our later examinations, and, of course, including the teachers and advisers of these students. Over the past seventeen years, the Society has strengthened its means to accomplish this same general purpose with the co-operative program at the high-school level with the Mathematical Association of America, the Actuarial Aptitude Test, the Graduate Record Examination alternative, and the continued effort to keep G.M.E. content and standards in line with current undergraduate mathematics teaching, and the minimum prerequisite to success in our later examinations. Over these years, possible candidates for the profession have been subject to many influences from outside it—the vagaries of the draft and the changing opportunities elsewhere for those with mathematical talents, as Korea, computers, the Cold War and the Space Age all had their effect. The net result of all this on the familiarity of the colleges with our profession may be summarized by the statistics shown in the accompanying tabulation from

Year of Examination	No. of Candidates	No. of History Cards Found	No. of Colleges Shown on Cards
1947	628	598	174
1955	756	713	228
1963	1,907	1,896	416

the 3,207 history cards readily available in the files of the Education and Examination Committee for the 3,291 candidates who took the G.M.E. in 1947, 1955, or 1963.

Even with this steady progress in numbers, both of candidates and of colleges aware of our profession, the Society's expenditures must be used as effectively as possible to carry out its aims. To this end, a subcommittee of the Public Relations Committee is studying the G.M.E. prizes, with particular reference to their geographical distribution. I am sure that, in this study, the subcommittee will want to consider Professor Fischer's interesting suggestions for improving the administration of these prizes.

CARL J. STRUNK:

Dr. Fischer has given a great deal of vital information about the examination system which will be useful to members. He suggests a further study to obtain information from those students who fail the General Mathematics Examination. I submit that such information would be incomplete without a simultaneous census of the present membership of the Society.

The three prerequisites of becoming a Fellow of the Society would seem to be high mathematical aptitude, high native intelligence, and drive or determination. The first of these can be measured to some extent by mathematics grades in high school or college, the second may be estimated by I.Q. tests, while the third probably cannot be measured.

A census of members would reveal characteristics associated with success in the examinations. If these were contrasted with those of the unsuccessful students, it would more clearly delineate the better candidates.

LAWRENCE MITCHELL:

Dr. Fischer mentions that 50 per cent of the students fail each year. And yet, this fact is one which should provide the student with an edge over his competition. No matter how poorly we, the students, write an exam (and there are many examiners who will testify to the "how poorly") 50 per cent of us will pass!

If the student will fortify this with the belief that the purpose of the exams is to find a way to pass him on to the next higher level, he will have taken a giant step toward hurrying through the syllabus. He will no longer panic when he cannot supply all—or even most—of the facts about a particular question; he will not waste time looking for the hidden meanings in questions; he will write legibly in the hope that the examiner will be able to find some points pertinent to the question; and he will remember that his competition is probably in worse shape than he is.

HARRY M. SARASON:

Dr. Fischer's paper should be put in perspective for students. My remarks are addressed entirely to students.

1. The critical comments reported in the paper are commonplace for all educational systems up to and including training on the job. My educational grapevines stated that the complaints were commonplace for any course in which any students had any unusual difficulties. One teacher interrupted my explanation to tell me practically verbatim what one of Dr. Fischer's correspondents had said in his comment. Correspondingly, actuaries will debate such comments at the drop of a hat. The students can hardly be objective about their personal experiences, and the ones who comment are obviously likely to be the critical ones.

2. Do not start our examinations, except as a sideline while in college,

unless you are a very good mathematician and are willing to drill yourself in solving problems. Our prize winners are a few in millions. Are you one math student in ten? Do you work problems other math students do not work? Are you interested enough to check your answers for reasonableness? Dr. Fischer's 9.87 years and 50 per cent failing have applied in the past to competent serious students. Do not automatically expect any college or other organized course to help you on an actuarial examination unless it is directed to that purpose by someone who knows how to help you on our examinations.

3. Our examinations require the mastering of facts, including considerable memorizing as part of mastering. The ability to memorize facts can be developed, but gradually, like the ability to run five miles. Just five minutes of continuous effort in memorizing is a definite accomplishment for a mind which is not accustomed to such exercise. When we understand the significance of the facts to be mastered and memorized, we make the memorizing easier (a little easier). A good approach to an actuarial understanding and memorizing job is to consider the purposes, the causes of the underlying trends or the history, and all the people involved (with *their* proper purposes, *their* particular methods, and *their* personal motives—their profit motive, their expansion motive, their fairness motive, their emotional motives). Make a point of memorizing the facts so fundamental as not to be mentioned ordinarily and also the facts ordinarily delegated to others for consideration.

4. The examination preparation and writing procedures are good exercises in model business procedure, which always involves the mastery of facts. The most influential and respected person in any conference often has his facts and other people's "facts" and motives so completely at his command that he can bring up and discuss the facts important to him at the psychological moment. One way to improve your writing is to write an answer to one broad essay-type examination question on each topic (book open). Repeat the process in two weeks; compare, outline, and improve in at least one respect. Repeat twice more on each question. Analyze each examination question by underlining or taking notes; then organize your facts by writing key words and numbering them tentatively before you start writing.

5. Teachers, books, examination systems, supervisors, rules, and experience do not provide all the "answers," or even the underlying facts, by any means. Look to your own resources: your own knowledge of facts, your own books and men from whom to obtain more knowledge, your own enterprise. I am reminded of a young lady who spent ten years on a well-organized actuarial clerical job; when she went to work as general

office assistant in a small manufacturing firm, the owner had to tell her continually, "You can do it"—and she could. I am reminded of the college student (my daughter) who complained to me that there were no rules to guide her for her part-time work in the actuarial department of a rapidly expanding life insurance company. I told her to write her own rules, just for herself. She did, and those rules were used for years until some actuary revised them to fit changing conditions. I am reminded of the small chemical manufacturing firm which was faced with a court order in a patent suit giving it just sixty days to stop making its sole product. The founder, an engineer, called his force of eleven, including nine other engineers, together and told them they were out of business unless they could develop a new product. They actually developed eight new products in those sixty days, every one of which was more valuable than the one that they had thought was indispensable.

6. The safety and ultimate utility of tens of billions of dollars belonging to tens of millions of policyholders, beneficiaries, taxpayers, and others are influenced by the business integrity, business foresight, and business acumen of actuaries. Our examinations are a strong buttress to our business and professional integrity. Actuarial education provides the mathematical models and the background for long-time foresight. Being a well-rounded individual with emphasis on responsibility (such as doing today's work today and doing it well) is more important than formal education. Education should help you to prepare yourself for harder and more responsible work and for lifelong learning of your own choice—from books of your own choice, from people of your own choice, from experiences of your own choice, and by thought processes of your own choice.

(AUTHOR'S REVIEW OF DISCUSSION)

CARL H. FISCHER:

I should like to thank all who took part in the discussion of this paper. It is very gratifying to find persons like Mr. Berkeley and Mr. Maclean in substantial agreement with the author. Mr. Garfield has given an interesting account of the English system of educating actuaries. The differences between their system and ours can doubtless be ascribed chiefly to the differences in the proportions of the populations attending college and also to the centralization of the English insurance industry as contrasted with the wide dispersion of our own.

Mr. Maynard offered data that tended to show that the character of one of the later examinations now meets with student approval. It is to be hoped that this trend will continue. I was pleased to note the statistical evidence produced by Mr. Munro supporting the theory that, at least for Canadian students, a prize offer at a later stage than Part 1 is more effective.

Mr. Schmidt, Mr. Veit, and Mr. Winter all helped clarify various aspects of the paper. Mr. Strunk believes that a comprehensive survey should be made of the entire Society membership to try to determine the characteristics of the successful actuary so that these can be compared with the characteristics of the unsuccessful G.M.E. candidates. It would not be easy to design and carry out such a study to produce meaningful results, but here is a real challenge.

Mr. Mitchell and Mr. Sarason directed their remarks primarily at students. Mr. Sarason has had long experience at tutoring and is eminently well qualified to advise the student. His paper, "A Technique for Facing the Actuarial Examinations," *RAIA*, Volume XXX (1941), attracted a great deal of attention at the time. Incidentally, he found then that the average length of time to become a Fellow was ten years, just as it is today! Mr. Sarason stated that student complaints about examinations are commonplace, and so they are, but most come from mediocre or unsuccessful students. Teachers generally pay some attention to criticisms from their "A" students, even though they may have to be discounted a bit. The comments in this paper were from successful students, mostly from Fellows, and I do not believe they should be completely ignored.

KING'S DATING METHOD IN A HEALTH INSURANCE VALUATION SYSTEM

E. PAUL BARNHART

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JOHN H. MILLER:

There is little one can say in discussion of such a complete and well-written paper on a technical subject other than to commend the author and to thank him for bringing to our attention a contribution by one of the great leaders of the profession and demonstrating its current utility. I might suggest that Mr. Barnhart would also find it of interest to investigate the method of valuation by "ages passed through" developed many years ago by another of the British actuaries in connection with the sickness benefits of friendly societies.

(AUTHOR'S REVIEW OF DISCUSSION)

E. PAUL BARNHART:

I wish to thank Mr. Miller for his comments concerning the paper and for calling my attention to another method for determination of ages in a valuation system. A merely cursory consideration of such seemingly minor items as the manner of determining age may well suggest that such items are not deserving of extensive study, but further investigation certainly shows that they have importance and are of considerable practical consequence.

In connection with my own investigation, I found it most interesting to be reminded of the fact that half-forgotten methods proposed so many years ago by imaginative actuaries of that day can still prove to be quite timely and useful.