Student's Note: King's Z-Method and Compound Statuses: An Algebraic Approach

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Abstract. The intent of this note is two-fold. First, a derivation for the Z-Method is presented. This derivation is algebraic (in the abstract sense) rather than combinatorial in nature. Second, a clarification of the use of compound statuses results in simpler calculations and largely eliminates the need to return to first principles when calculating probabilities.

0. King's Z-Method for representing the probabilities that exactly r or at least r of m events will occur was presented originally as a mnemonic for Waring's formulas, also called inclusion-exclusion formulas. No underlying mathematical meaning was assumed; the formula reflects a fortuitous agreement among coefficients and significantly eased the memorization of the formulas.

In this paper an appropriate algebraic structure is constructed for the inclusion-exclusion formulas and King's Z is defined as formal operator on that algebraic structure. The structure is square-free symmetric polynomials and Z is old-fashioned polynomial integration. (Note: Probabilists and Statisticians tend to read "expectation" whenever "integration" appears. In this application, "integral" in the sense of a linear operator is more appropriate, I expect.)

In addition, a simplification of current exposition may be possible with an explicit interpretation of compound statuses. This is presented in section 2 with examples in section 3.

To keep the abstract algebra understandable, every attempt to keep the proofs and definitions concrete has been made. For those readers who feel more comfortable with polynomial rings, quotient rings, lifts and projections, section 1 may provide an entertaining exercise.

The references cited in the bibliography reflect the results of a literature search conducted on inclusion-exclusion formulas. The closest results to any given in section 1 (following Lemma 2) appeared in [5, pp. 261-262] . While formally identical with our Theorem 2, Loeve's result is really a modification of King's Z-Method.

1. In the general multi-life status,

we are primarily concerned with the probabilities
${}_np_{\overline{x_1\cdots x_m}}^{[r]} = F^{[r]}({}_np_{x_1},\cdots,{}_np_{x_m})$ and ${}_np_{\overline{x_1\cdots x_m}}^{r} = F^{r}({}_np_{x_1},\cdots,{}_np_{x_m})$
denoting the probability that exactly r and at least r of the m lives will survive n years. Three cases have special notation:

$${}_np_{x_1\cdots x_m} = {}_np_{\overline{x_1\cdots x_m}}^{m} \qquad \text{(joint survivorship)}$$

$${}_np_{\overline{x_1\cdots x_m}} = {}_np_{\overline{x_1\cdots x_m}}^{1} \qquad \text{(last survivorship)}$$

$${}_nq_{x_1\cdots x_m} = {}_np_{\overline{x_1\cdots x_m}}^{[0]} = {}_nq_{x_1}\cdots{}_nq_{x_m} = (1 - {}_np_{x_1})(\cdots)(1 - {}_np_{x_m})$$

The rest of this section will express F^r and $F^{[r]}$ as functions of $X_1,\cdots,X_m$.

Let X_i be the probability that event i will occur.

Let $F^{[r]}(X_1,\cdots,X_m)$ be the probability that exactly r of the m independent events listed will occur.

Let $F^r(X_1,\cdots,X_m)$ be the probability that at least r of the m independent events will occur.

Where confusion will not arise, the "event list" $(X_1,\cdots,X_m)$ will be omitted.

The following are clear.

Lemma 1. $F^r = F^{[r]} + F^{r+1}$ for $r \geq 0$.

Lemma 2. $F^{[r]}(X_1,\cdots,X_m) = X_1 \cdot F^{[r-1]}(X_2,\cdots,X_m) + (1 - X_1)\cdot F^{[r]}(X_2,\cdots,X_m)$

Corollary: $F^{[r]}$ is linear in each X_i.

Theorem 1. $F^{[r]}(X_1,\cdots,X_m) = \frac{1}{r} \sum_{i=1}^{m} X_i \cdot F^{[r-1]}(X_1,\cdots,\hat{X}_i,\cdots,X_m)$
where $\hat{X}_i$ means that X_i has been dropped from the event list.

Proof: Since $F^{[r]}$ is linear in each variable, it suffices to check the 2^m possibilities where $X_i = 0$ or 1. Since the right and left-hand sides (RHS and LHS) are symmetric, this reduces to the $m + 1$ cases where j of the X_i are 1 and $m - j$ are 0 as j runs from 0 to m.

Let $X_1 = X_2 = \cdots = X_j = 1$ and $X_{j+1} = X_{j+2} = \cdots = X_m = 0$.

If $j \neq r$, LHS $= 0$ and if $j = r$, LHS $= 1$ by definition. For the RHS, if $j > r$, then $j - 1 > r - 1$ so that RHS $= 0$.
Similarly, if $j < r - 1$, $j - 1 < r - 1$ so that RHS $= 0$.

If $j = r - 1$, $X_i \cdot F^{[r-1]}(X_1,\ldots,\hat{X}_i,\ldots,X_m) = \begin{cases} 1\cdot 0 = 0 \text{ if } i \leq j \\ 0\cdot 1 = 0 \text{ if } i > j \end{cases}$

so that RHS $= 0$.
If $j = r$, $X_i \cdot F^{[r-1]}(X_1,\ldots,\hat{X}_i,\ldots,X_m) = \begin{cases} 1\cdot 1 = 1 \text{ if } i \leq j \\ 0\cdot 0 = 0 \text{ if } i > j \end{cases}$

so that RHS $= \frac{1}{r}\left[r\cdot 1 + (m-r)\cdot 0\right] = 1 =$ LHS.

Note that $F^{[r]}$ and F^r are symmetric square-free polynomials. In the following remarks we will derive certain properties of such polynomials. To this end, we introduce the following algebraic relation: For all i, $X_i^2 = 0$. Since $F^{[r]}$ is linear in each X_i, $F^{[r]}$ and F^r are unchanged by this. The calculations in the rest of this section are carried out modulo these relations.

Let $S = S(X_1,\ldots,X_m) = X_1 + X_2 + \cdots + X_m$. We shall see that powers of S generate the homogeneous square-free symmetric polynomials in m variables of each degree. Let $s_j = \frac{1}{j!} S^j$. Then s_j is a homogeneous symmetric square-free polynomial of degree j. Moreover, the term $X_1X_2\cdots X_j$ has co-efficient 1 in s_j.

Let P be homogeneous, symmetric and square-free of degree j. Then for some permutation σ of the numbers 1 through m, the term $X_{\sigma(1)} X_{\sigma(2)} \cdots X_{\sigma(j)} \cdot A_{\sigma(1)\ldots\sigma(j)}$ appears in the expansion of $P = P(X_1,\ldots,X_m)$. Since P is symmetric $P = P(X_{\sigma^{-1}(1)},\ldots,X_{\sigma^{-1}(m)})$ but $A_{\sigma(1)\ldots\sigma(j)}$ is now the co-efficient of $X_1X_2\cdots X_j$. Hence, $A_{1\ldots j} = A_{\sigma(1)\ldots\sigma(j)}$ and it follows that $P = A_{1\ldots j}\, s_j$ (since the full symmetric group acts transitively on cycles.).

If T is a formal power series which is symmetric and square-free, then $T = \sum_{j=0}^{\infty} a_j S^j$. Since $S^{m+1} = 0$, T reduces to a polynomial of finite degree. By $T(X_1, \cdots, \hat{X}_i, \cdots, X_m)$ we mean the polynomial obtained by setting $X_i = 0$. Equivalently $T(X_1, \cdots, \hat{X}_i, \cdots, X_m) = \sum_{j=0}^{\infty} a_j (S - X_i)^j$. This follows from $S(X_1, \cdots, \hat{X}_i, \cdots, X_m) = S - X_i$.

Lemma 3. $\sum_{j=1}^{m} X_i S^j(X_1, \cdots, \hat{X}_i, \cdots X_m) = S^{j+1}$ for $j \geq 0$.

Proof for $j > 0$: $\sum_{i=1}^{m} X_i S^j(X_1, \cdots, \hat{X}_i, \cdots, X_m) = \sum_{i=1}^{m} X_i (S - X_i)^j$

$$= \sum_{i=1}^{m} X_i \left[S^j - \binom{j}{1} S^{j-1} X_i + \cdots \right]$$

$$= \sum_{i=1}^{m} X_i S^j = S^j \sum_{i=1}^{m} X_i = S^j \cdot S = S^{j+1}$$

If $j = 0$, $S^j = 1$ and $\sum_{i=1}^{m} X_i \cdot S^j (X_1, \cdots, \hat{X}_i, \cdots, X_m) = \sum_{i=1}^{m} X_i = S = S^{1+0}$.

Corollary: If T is a symmetric square-free formal power series, then $\sum_{i=1}^{m} X_i \cdot T(X_1, \cdots, \hat{X}_i, \cdots, X_m) = S \cdot T(X_1, \cdots, X_m)$

Lemma 4: $F^{[0]} = e^{-S}$

Proof: $F^{[0]}(X_1, \cdots, X_m) = (1 - X_1)(1 - X_2)(\cdots)(1 - X_m)$

$$= e^{-X_1} \cdot e^{-X_2} \cdots e^{-X_m}$$

$$= e^{-S}$$

Since $X_i^2 = 0$, $e^{-X_i} = 1 - X_i$.

Theorem 2: $F^{[r]} = \frac{1}{r!} S^r e^{-S}$

Proof: Since $F^{[r]}$ satisfies the conditions for the Corollary to Lemma 3, $F^{[r]} = \frac{1}{r} \sum_{i=1}^{m} X_i F^{[r-1]}(X_1, \cdots, \hat{X}_i, \cdots, X_m)$

$$= \frac{1}{r} S \cdot F^{[r-1]}$$

$$= \frac{1}{r!} S^r F^{[0]}$$

$$= \frac{1}{r!} S^r e^{-S} \quad \text{, by Lemma 4.}$$

Define a linear map Z on the symmetric square-free power series as follows: Let $T = \sum_{j=0}^{\infty} A_j S^j = T(S)$

Then $Z(T) = \int_0^S T(S')\,dS'$

In particular, we have

Lemma 5: $Z(\mathcal{S}_j) = \mathcal{S}_{j+1}$

Proof: Since $\mathcal{S}_j = \frac{1}{j!} S^j$,

$$Z(\mathcal{S}_j) = \frac{1}{j!} \int_0^S (S')^j dS' = \frac{1}{(j+1)!} (S')^{j+1} \Big|_0^S = \frac{S^{j+1}}{(j+1)!} = \mathcal{S}_{j+1}$$

Lemma 6: $F^1 = Z\,(F^{[0]})$

Proof: From Lemma 1, $F^1 + F^{[0]} = F^0 = 1$

or $F^1 = 1 - F^{[0]} = 1 - e^{-S}$.

$$Z\,(F^{[0]}) = \int_0^S e^{-S'} dS' = -\,e^{-S'} \Big|_0^S = 1 - e^{-S} = F^1.$$

Lemma 6 suggests a relationship between F^r and $F^{[r]}$ which is indeed true.

Theorem 3. $F^r = Z(F^{[r-1]})$

Proof: This follows by induction from Lemma 1. Lemma 6 provides the result for r = 1. Assume $r \geqslant 1$ and $F^r = Z(F^{[r-1]})$. We shall calcuate F^{r+1}. From Lemma 1,

$$F^{r+1} = F^r - F^{[r]} = Z(F^{[r-1]}) - F^{[r]}$$

$$= \left[\int_0^S \frac{1}{(r-1)!} S'^{\,r-1} e^{-S'} dS' \right] - \frac{1}{r!} S^r e^{-S}$$

$$= \left[\frac{(S')^r}{r!} e^{-S'} \Big|_0^S + \int_0^S \frac{1}{r!} S'^{\,r} e^{-S'} dS' \right] - \frac{1}{r!} S^r e^{-S}$$

$$= \frac{S^r}{r!} e^{-S} + \int_0^S \frac{1}{r!} S'^{\,r} e^{-S'} dS' - \frac{1}{r!} S^r e^{-S} = Z(F^{[r]})$$

It is important to note that $r \geqslant 1$ allows us to use integration by parts, since $Se^S = S \frac{de^S}{dS}$ but $e^S \neq \frac{de^S}{dS}$. The difficulty could be avoided by lifting the formulas out of the quotient ring, but the added machinery is more cumbersome than helpful.

Theorem 4: $F^{[r]} = \dfrac{Z^r}{(1+Z)^{r+1}}$ and $F^r = \dfrac{Z^r}{(1+Z)^r}$

Proof: Since $F^r = F^{[r]} + F^{r+1} = F^{[r]} + Z\,(F^{[r]}) = (1+Z)(F^{[r]})$
it follows that $(1+Z)\,F^{[r]} = Z\,(F^{[r-1]})$
or $F^{[r]} = (1+Z)^{-1}\,Z\,(F^{[r-1]})$

Since Z is a lower triangular matrix when expressed with respect to the basis $\left\{S_j\right\}_{j=0}^{m}$, $(1+Z)^{-1}$ is well-defined.

Moreover, since $(1+Z)^{-1}\cdot(1+Z) = (1+Z)\cdot(1+Z)^{-1}$

or $(1+Z)^{-1}\cdot(1) + (1+Z)^{-1}\cdot(Z) = (1)\,(1+Z)^{-1} + (Z)\cdot(1+Z)^{-1}$

it follows that $(1+Z)^{-1}\,Z = Z\,(1+Z)^{-1}$
and $\left[(1+Z)^{-1}\,Z\right]^r = (1+Z)^{-r}\,Z^r$.

$$\text{Thus } F^{[r]} = \frac{Z^r}{(1+Z)^r}\,F^{[0]}$$

$$= \frac{Z^r}{(1+Z)^r}\cdot\frac{1}{(1+Z)}\,F^0\,.$$

$$= \frac{Z^r}{(1+Z)^{r+1}}\,(1).$$

$F^r = \dfrac{Z^r}{(1+Z)^r}$ follows from Theorem 3.

2. Compound Statuses

In Section 1, formulas were developed to calculate the probabilities of exactly r and at least r of m independent events occurring. Compound statuses can be handled with the formulas already developed by observing that "r out of m events occurring" is itself an event.

To show how this may be applied to statuses we will set up
$_nP_{\overline{x_1x_2:\overline{x_3x_4}}}$ and $_nP^{[2]}_{\overline{x_1:\overset{2}{\overline{x_2x_3x_4}}:x_5x_6}}$.

The notation is simplified by setting $X_i = {}_nP_{x_i}$.

Then $_nP_{\overline{x_1x_2:\overline{x_3x_4}}} = {}_nP^{1}_{\overset{2}{\overline{x_1x_2}}:\overset{1}{\overline{x_3x_4}}} = F^1\left(F^2(X_1,X_2),\ F^1(X_3,X_4)\right)$

and $_nP^{[2]}_{\overline{x_1:\overset{2}{\overline{x_2x_3x_4}}:x_5x_6}} = F^{[2]}\left(X_1,\ F^2(X_2,X_3,X_4),\ F^2(X_5,X_6)\right)$

The full calculation of these expressions is relegated to Appendix 1. Appendix 2 presents a table of $F^{[r]}$ and F^r for $m = 6$. This table can be used for $m < 6$ by recognizing that $S^{m+1}(X_1,\dots,X_m) = 0$ and that $S(X_1,\dots,X_m) = S(X_1,\dots,X_{m-1})$ with $X_m = 0$.

For the sake of completeness, we should mention the status $\overline{n|}$. Let ${}_tp_{\overline{n|}} = 1$ if $0 \leq t \leq n$ and 0 otherwise. Then ${}_tp_{\overline{n|}}$ represents the probability of "survivorship" of the term certain, and the symbol $\overline{n|}$ can be used in the status calculation. The rest of the formalism is clearly applicable. (NOTE: This formula for ${}_tp_n$ is appropriate for $A_{x:\overline{n|}}$ and $a_{x:\overline{n|}}$. However, $\ddot{a}_{x:\overline{n|}}$ requires a shift function: ${}_t\ddot{p}_{\overline{n|}} = {}_{t+1}p_{\overline{n|}}$.)

3. Annuities and insurances with compound statuses.

The basic goal is to express A_α and a_α as a sum of joint insurances or annuities, α denoting an arbitrary status.

Since $a_\alpha = \sum_{t=1}^{\infty} v^t \, {}_tp_\alpha$ and $A_\alpha = \sum_{t=0}^{\infty} v^{t+1} \, {}_{t|}q_\alpha$

$= \sum_{j=0}^{\infty} v^{t+1} ({}_tp_\alpha - {}_{t+1}p_\alpha)$ the approach of Section 2 can be used to expand ${}_tp_\alpha$ as a sum of joint probabilities, and the observation to be made is that: if $\alpha = \alpha_1 + \alpha_2$ then ${}_np_\alpha = {}_np_{\alpha_1} + {}_np_{\alpha_2}$ so that $A_\alpha = A_{\alpha_1} + A_{\alpha_2}$ and $a_\alpha = a_{\alpha_1} + a_{\alpha_2}$.

The formalism here may be confusing in that it is likely that α_1 and α_2 are algebraic expressions which may not in themselves be meaningful statuses.

As an example, refer to Appendix 1, part 1, which can be compared with the insurance calculated in Jordan, page 214, where

$$\alpha = \overline{\overline{wx:yz}}$$

Since $\alpha = F^1(F^2(w,x), F^1(y,z))$, we have from the comments made above and Appendix 1:

$$A_{\overline{\overline{wx:yz}}} = A_y + A_z + A_{wx} - A_{yz} - A_{wxy} - A_{wxz} + A_{wxyz}.$$

as expected.

Consider $a_{\overline{(x:\overline{n|})(y:\overline{m|})}}$ (Jordan, pg. 217).

$$\begin{aligned}
\text{Here } \alpha &= F^1(F^2(x,\overline{n|}), F^2(y,\overline{m|})) \\
&= F^2(x,\overline{n|}) + F^2(Y,\overline{m|}) - F^2(x,\overline{n|})\,F^2(y,\overline{m|}) \\
&= x(\overline{n|}) + y(\overline{m|}) - (x)(\overline{n|})(y)(\overline{m|}) \\
&= (x:\overline{n|}) + (y:\overline{m|}) - (xy:\overline{n|}:\overline{m|})
\end{aligned}$$

so that $a_\alpha = a_{x:\overline{n|}} + a_{y:\overline{m|}} - a_{xy:\overline{m|}}$ if $m \leq n$, since ${}_tp_{\overline{n|}} \cdot {}_tp_{\overline{m|}} = {}_tp_{\overline{m|}}$.

4. A loose end.

In ${}_np_{\underset{x_1 \cdots x_m}{\underline{[r]}}}$ if the ages of $x_1, \cdots, x_m$ are all x, we know that

$${}_np_{\underset{x \cdots x}{\underline{[r]}}} = \binom{m}{r} ({}_np_x)^r (1 - {}_np_x)^{m-r}.$$

Using Section 1, we can give a alternate proof:

$$F^{[r]} = \frac{1}{r!} S^r e^{-S}$$

$$S^r = (X_1 + \cdots X_m)^r = r!\, s_r.$$

If all X_i have the same numeric value X,

$$S^r (X_1, \cdots, X_m) = r! \binom{m}{r} X^r$$

Then

$$F^{[r]}(X, \cdots, X) = \frac{S^r}{r!} \sum_{j=0}^{\infty} (-1)^j \frac{S^j}{j!}$$

$$= \sum_{j=0}^{m-r} (-1)^j \frac{S^{r+j}}{r!j!}$$

$$= \sum_{j=0}^{m-r} \frac{(r+j)!}{r!j!} \binom{m}{r+j} X^{r+j} (-1)^j$$

$$= X^r \sum_{j=0}^{m-r} (-1)^j \frac{(r+j)!}{r!j!} \cdot \frac{m!}{(r+j)!(m-r-j)!} X^j$$

$$= \frac{m!}{r!(m-r)!} X^r \sum_{j=0}^{m-r} (-1)^j \frac{(m-r)!}{j!(m-r-j)!} x^j$$

$$= \binom{m}{r} X^r (1-X)^{m-r}$$

So that ${}_nP_{\underline{X \cdots X}}^{[r]} = \binom{m}{r} {}_nP_X{}^r (1 - {}_nP_X)^{m-r}$, as expected.

Appendix 1. Calculation of Two Statuses

$$ {}_nP_{\overline{wx:\overline{yz}}} = {}_nP_{\overline{\overline{wx}:\overline{yz}}^{\,1}} = F^1\left(F^2({}_nP_w, {}_nP_x),\ F^1({}_nP_y, {}_nP_z)\right) $$

$$ \begin{aligned} F^1\left(F^2(w,x),\ F^1(y,z)\right) &= F^2(w,x) + F^1(y,z) - F^2(w,x)\ F^1(y,z) \\ &= wx + y + z - yz - wx(y + z - yz) \\ &= wx + y + z - yz - wxy - wxz + wxyz \end{aligned} $$

$$ {}_nP_{\overline{wx:\overline{yz}}} = {}_nP_{wx} + {}_nP_y + {}_nP_z - {}_nP_{yz} - {}_nP_{wxy} - {}_nP_{wxz} + {}_nP_{wxyz} $$

(Compare Jordan, page 214.)

$$ {}_nP_{\overline{x_1:\overset{2}{\overline{x_2x_3x_4}}:x_5x_6}^{\,[2]}} = F^{[2]}\left({}_nP_{x_1},\ F^2({}_nP_{x_2},\ {}_nP_{x_3},\ {}_nP_{x_4}),\ F^2({}_nP_{x_5},\ {}_nP_{x_6})\right) $$

$$ \begin{aligned} &F^{[2]}\left(X_1,\ F^2(X_2,\ X_3,\ X_4),\ F^2(X_5,\ X_6)\right) \\ &= X_1F^2(X_2,\ X_3,\ X_4) + X_1\ F^2(X_5,\ X_6) + F^2(X_2,\ X_3,\ X_4)\ F^2(X_5,\ X_6) \\ &\qquad - 3X_1\ F^2(X_2,\ X_3,\ X_4)\ F^2(X_5,\ X_6) \end{aligned} $$

$$ \begin{aligned} &= X_1\ (X_2\ X_3 + X_2\ X_4 + X_3\ X_4 - 2X_2\ X_3\ X_4) + X_1\ X_5\ X_6 + (X_2\ X_3 + X_2\ X_4 \\ &\qquad + X_3\ X_4 - 2X_2\ X_3\ X_4)\ (X_5\ X_6) - 3X_1\ (X_2\ X_3 + \\ &\qquad X_2\ X_4 + X_3\ X_4 - 2X_2\ X_3\ X_4)\ X_5\ X_6 \end{aligned} $$

$$ \begin{aligned} &= X_1\ X_2\ X_3 + X_1\ X_2\ X_4 + X_1\ X_3\ X_4 - 2X_1\ X_2\ X_3\ X_4 + X_1\ X_5\ X_6 \\ &\qquad + X_2\ X_3\ X_5\ X_6 + X_2\ X_4\ X_5\ X_6 + X_3\ X_4\ X_5\ X_6 \\ &\qquad - 2X_2\ X_3\ X_4\ X_5\ X_6 - 3X_1\ X_2\ X_3\ X_5\ X_6 \\ &\qquad - 3X_1\ X_2\ X_4\ X_5\ X_6 - 3X_1\ X_3\ X_4\ X_5\ X_6 \\ &\qquad + 6X_1\ X_2\ X_3\ X_4\ X_5\ X_6 \end{aligned} $$

$$ \begin{aligned} {}_nP_{\overline{x_1:\overset{2}{\overline{x_2x_3x_4}}:x_5x_6}^{\,[2]}} &= {}_nP_{x_1x_2x_3} + {}_nP_{x_1x_2x_4} + {}_nP_{x_1x_3x_4} - 2\,{}_nP_{x_1x_2x_3x_4} + {}_nP_{x_1x_5x_6} \\ &\qquad + {}_nP_{x_2x_3x_5x_6} + {}_nP_{x_2x_4x_5x_6} + {}_nP_{x_3x_4x_5x_6} \\ &\qquad - 2\,{}_nP_{x_2x_3x_4x_5x_6} - 3\,{}_nP_{x_1x_2x_3x_5x_6} \\ &\qquad - 3\,{}_nP_{x_1x_2x_4x_5x_6} - 3\,{}_nP_{x_1x_3x_4x_5x_6} \\ &\qquad + 6\,{}_nP_{x_1x_2x_3x_4x_5x_6} \end{aligned} $$

Appendix 2. $F^{[r]}$ and F^{r} for $m = 6$

r	$F^{[r]}$: $F^{[r]} = \frac{S^r}{r!} e^{-S}$	F^{r}: $F^{r} = \int_0^S F^{[r-1]}\, dS'$
0	$e^{-S} = 1 - S + \frac{S^2}{2} + \frac{S^3}{6} + \frac{S^4}{24} - \frac{S^5}{120} + \frac{S^6}{720}$ $= 1 - \not{S}_1 + \not{S}_2 - \not{S}_3 + \not{S}_4 - \not{S}_5 + \not{S}_6$	$F^0 = 1$
1	$Se^{-S} = S - S^2 + \frac{S^3}{2} - \frac{S^4}{6} + \frac{S^5}{24} - \frac{S^6}{120}$ $= \not{S}_1 - 2\not{S}_2 + 3\not{S}_3 - 4\not{S}_4 + 5\not{S}_5 - 6\not{S}_6$	$F^1 = S^2 - \frac{S^2}{2} + \frac{S^3}{6} - \frac{S^4}{24} + \frac{S^5}{120} - \frac{S^6}{720}$ $= \not{S}_1 - \not{S}_2 + \not{S}_3 - \not{S}_4 + \not{S}_5 - \not{S}_6$
2	$\frac{S^2}{2} e^{-S} = \frac{S^2}{2} - \frac{S^3}{2} + \frac{S^4}{4} - \frac{S^5}{12} + \frac{S^6}{48}$ $= \not{S}_2 - 3\not{S}_3 + 6\not{S}_4 - 10\not{S}_5 + 15\not{S}_6$	$F^2 = \frac{S^2}{2} - \frac{S^3}{3} + \frac{S^4}{8} - \frac{S^5}{30} + \frac{S^6}{144}$ $= \not{S}_2 - 2\not{S}_3 + 3\not{S}_4 - 4\not{S}_5 + 5\not{S}_6$
3	$\frac{S^3}{6} e^{-S} = \frac{S^3}{6} - \frac{S^4}{6} + \frac{S^5}{12} - \frac{S^6}{36}$ $= \not{S}_3 - 4\not{S}_4 + 10\not{S}_5 - 20\not{S}_6$	$F^3 = \frac{S^3}{6} - \frac{S^4}{8} + \frac{S^5}{20} - \frac{S^6}{72}$ $= \not{S}_3 - 3\not{S}_4 + 6\not{S}_5 - 10\not{S}_6$
4	$\frac{S^4}{24} e^{-S} = \frac{S^4}{24} - \frac{S^5}{24} + \frac{S^6}{48}$ $= \not{S}_4 - 5\not{S}_5 + 15\not{S}_6$	$F^4 = \frac{S^4}{24} - \frac{S^5}{30} + \frac{S^6}{72}$ $= \not{S}_4 - 4\not{S}_5 + 10\not{S}_6$
5	$\frac{S^5}{120} e^{-S} = \frac{S^5}{120} - \frac{S^6}{120}$ $= \not{S}_5 - 6\not{S}_6$	$F^5 = \frac{S^5}{120} - \frac{S^6}{144}$ $= \not{S}_5 - 5\not{S}_6$
6	$\frac{S^6}{720} e^{-S} = \frac{S^6}{720}$ $= \not{S}_6$	$F^6 = \frac{S^6}{720}$ $= \not{S}_6$

Note: $S = X_1 + X_2 + X_3 + X_4 + X_5 + X_6$

Bibliography

1. M. Frechet, Probabilités associés à un systéme d'évènements dèpendants et compatibles, Hermann et Cie, 1940, pp. 1-35, 85-87.

2. B. R. Johnson, An Elementary Proof of Inclusion-Exclusion Formulas, American Mathematical Monthly, Vol. 87, No. 9 (November 1980), pp. 750-751.

3. C. W. Jordan, Jr., Life Contingencies, Society of Actuaries, Chicago, 1975, pp. 210-221.

4. G. King, Institute of Actuaries' Text Book, Part II, Life Contingencies, 902. pp. 14-19.

5. M. Loève, Sur les systémes d'évènements; applications à deux théorémes classiques. Comptes Rendus de l'Academie Scientifique; Vol. 212, 1941, pp. 261-264.

6. M. Neuts, Probability, Allyn and Bacon, Boston, 1973, pp. 144-148.

7. R. P. White and T. N. E. Greville, On Computing the Probability that Exactly k of n Independent Events Will Occur, TSA XI, pp. 88-99 (including discussions).

ADDENDUM "King's Z-Method: An Algebraic Approach"

The following argument eliminated the inductive proof in the main exposition.

Theorem 3. $F^r = Z(F^{[r-1]})$ for $r \geq 1$.

The proof involves two lemmas about polynomial derivatives and integrals. Let $\mathcal{S}_2$ denote the ring of square-free symmetric polynomials.

Lemma 1. If $T(S) \in \mathcal{S}_2$ has no constant term, then $Z(T') = T$.

Proof: Clear, since $Z((S^k)') = Z(kS^{k-1}) = S^k$ if $0 < k \leq m$.

Lemma 2. If $u,v \in \mathcal{S}_2$, then $Z((uv)') = Z(u'v + uv')$.

Proof: Since both the RHS and LHS are bilinear in u,v, it suffices to check $u = S^i$, $v = S^j$. This reduces to three cases: (1) $i+j \leq m$, (2) $i+j = m+1$ and (3) $i+j > m+1$.

1. If $i+j \leq m$, then $u'v + uv' = (iS^{i-1})S^j + S^i(jS^{j-1}) = (i+j)S^{i+j-1} = (uv)'$, and applying Z gives the result.

2. If $i+j = m+1$, then $uv = 0$ and $Z((uv)') = Z(0) = 0$.

If $i = 0$, $u' = v = 0$ and there is nothing to show. If $i \neq 0$, then $i,j \leq m$ and $u'v + uv' = (m+1)S^m$ so that $Z(u'v + uv') = Z((m+1)S^m) = 0$. For case (3), similar reasoning leads to $u'v = uv' = uv = (uv)' = 0$.