A NOTE ON MULTIPLE DECREMENTS

by

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In the 1985.1 issue of ARCH, Jim Conner derived an interesting corollary in his paper: "A Multiple Decrement Theorem". He later raised the question of whether the converse of his corollary was true. The purpose of this note is twofold: (a) to develop a counterexample, thus demonstrating that the converse is not true in general, and (b) to develop a "partial" converse by adding an additional condition.

Due to the unfortunate limitations of my printer and word processor, I find it necessary to depart from our beloved actuarial notation. Hopefully, the following conventions will not be too confusing to the reader.

Let $u_j(t)$ denote the force of decrement due to cause (j) at age x+t for $0<t<1$. Assume the decrements are continuous.

Let $u_T(t)$ denote the total force of decrement due to all causes combined.

Let $U_j(t) = \int_0^t u_j(s)ds$ and let $U_T(t) = \int_0^t u_T(s)ds$.

Note that $dU_j(t)/dt = u_j(t)$ and $dU_T(t)/dt = u_T(t)$.

Let $p'_j(t) = \exp[-U_j(t)]$ and $p_T(t) = \exp[-U_T(t)]$.

Finally, let $q_j(t) = \int_0^t p_T(s)u_j(s)ds$ and let

$q_T(t) = \int_0^t p_T(s)u_T(s)ds$.

Consider the following two conditions:

(1) $q_j(1)/q_T(1) = U_j(1)/U_T(1)$

(2) $u_j(t)/u_T(t)$ is constant for $0<t<1$

In his paper, Jim Conner proved that (2) implies (1). That the converse is false can be demonstrated by the following simple example.

Let $u_T(t) = 1$ and $u_J(t) = a + bt + ct^2$, $0<t<1$, where $a = b = 1/2$ and $c = (9-e)/(20e-56)$. It can be verified that $0<u_J(t)<u_T(t)$ for $0<t<1$. Let $r = (14e-39)/(20e-56)$. Direct calculation will confirm the following:

$U_T(1) = 1 \quad U_J(1) = r \quad q_T(1) = 1-e^{-1} \quad q_J(1) = r(1-e^{-1})$

Thus (1) is satisfied but (2) clearly does not hold. Therefore, (1) does not in general imply (2).

Now define $H(t) = U_T(t)/U_J(t)$ for $0<t<1$. Note that

$p_T(t) = [p_J'(t)]^{H(t)}$. We can restate (1) as follows:

$$(3) \qquad \int_0^1 [p_J'(t)]^{H(t)} u_J(t)dt = [1-p_T(1)]/H(1)$$

However, the following is true by direct calculation:

$$(4) \qquad \int_0^1 [p_J'(t)]^{H(1)} u_J(t)dt = \{1-[p_J'(1)]^{H(1)}\}/H(1)$$

Combining (3) and (4) yields:

$$(5) \qquad \int_0^1 u_J(t)\{ [p_J'(t)]^{H(t)} - [p_J'(t)]^{H(1)} \}dt = 0$$

Now if H is monotone (either non-increasing or non-decreasing) for $0<t<1$, then the sign of the integrand in (5) does not change. Therefore, it must be identically zero. That is, $H(t) = H(1)$ for all t. But then $U_T(t)=H(1)U_J(t)$ for all t, and by differentiation, $u_T(t)=H(1)u_J(t)$ for all $0<t<1$. This, however, is just condition (2).

In summary, we have proven the following partial converse to Jim Conner s result: if (1) holds <u>and</u> H (as defined above) is monotone on (0,1), then (2) holds.

Under either the constant force or uniform distribution of decrements assumption, H is constant and therefore trivially monotone. In the counterexample given above, $H(0)=2$, $H(0.73)=1.69$, and $H(1)=1.73$. Thus, in this case H is not monotone, and the converse fails.