Macroeconomics-Based Economic Scenario Generation

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Caveat and Disclaimer

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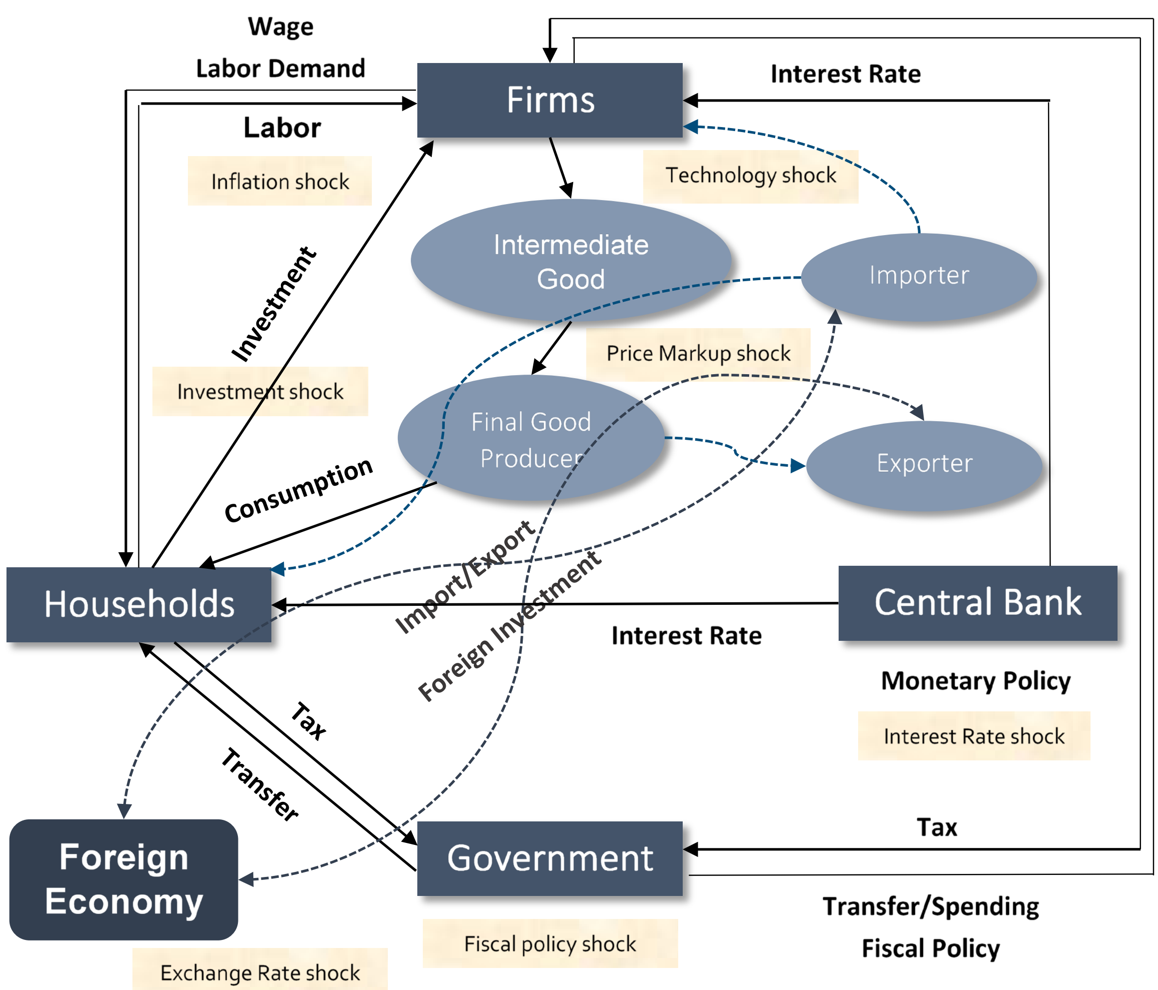
Executive Summary

Economic scenario generators (ESGs) are used in the insurance industry to assess the uncertainty of economic conditions. Many real-world ESGs model asset returns and yield curves directly based on historical data. Others may model systemic risk separately using a macroeconomic model. However, some macroeconomic models can be data-driven as well. They may reflect the historical realization of an economic system but not necessarily how the economic system really works and other possible outcomes.

As a unique part of this new type of ESG, dynamic stochastic general equilibrium (DSGE) models are complex macroeconomic models and have been increasingly popular in central banks for analyzing monetary policies. Behaviors of economic agents such as households, firms, central banks and governments are usually defined explicitly, as shown in Figure E.1. Based on that, observable economic variables such as real GDP growth rate, inflation, imports/exports and employment are estimated while acknowledging economic shocks to the causes, such as labor supply and technology development, but not to the results, such as unemployment rate and GDP growth rate. With the given nonlinear cause-and-effect relationships defined, historical data are used to calibrate the models. As a potential candidate for modeling macroeconomic factors in an ESG, DSGE models are well equipped for maintaining economic patterns in individual scenarios and explaining the causes behind individual scenarios.

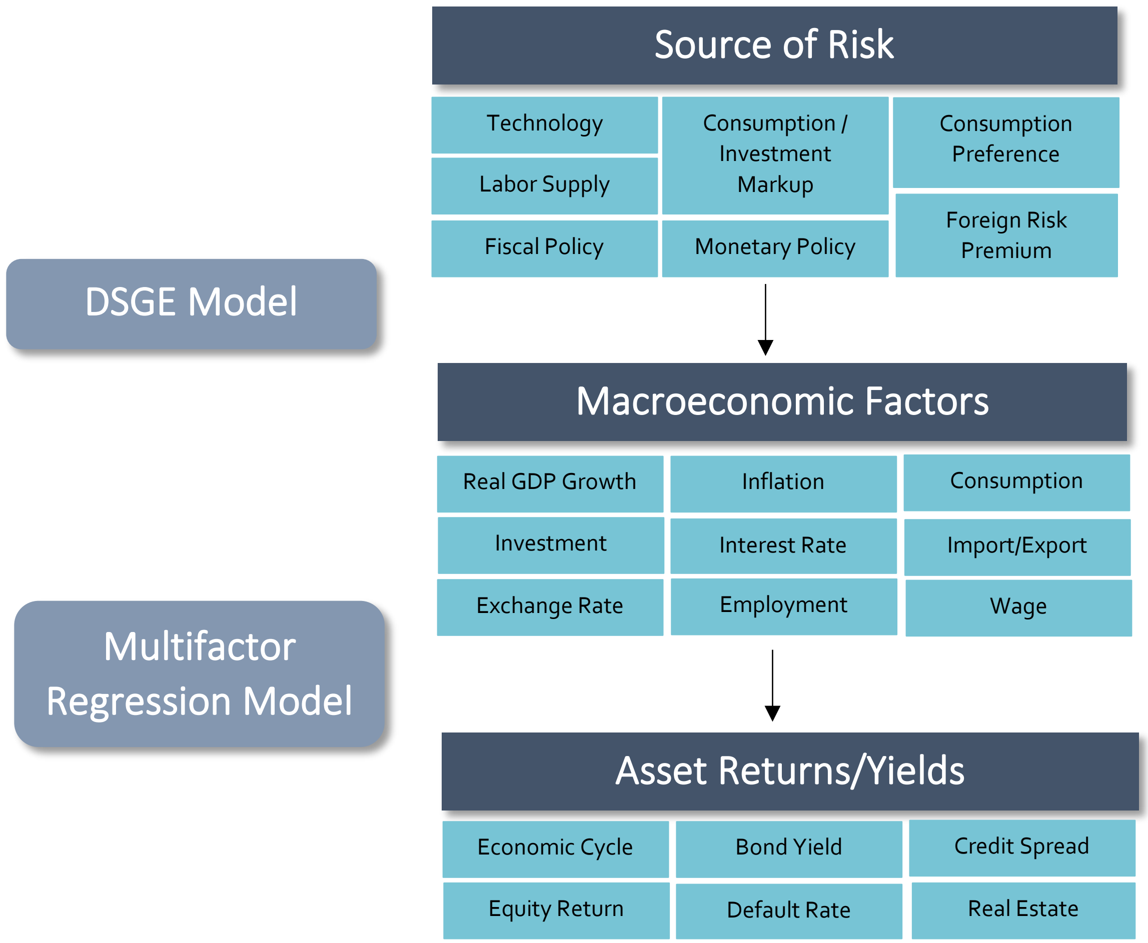
Figure E.1

Sample DSGE Model Structure



This report builds an ESG that uses a complex macroeconomic model type to generate macroeconomic factors. The ESG consists of two parts: a DSGE model that generates macroeconomic factors (see Figure E.2) and multifactor regression models that generate asset returns and bond yields.

Figure E.2
Sample Structure of DSGE Model-Based ESG



With this structure, sources of risk are used to generate scenarios for modeled variables, instead of using the features of the modeled variables directly. The ESG is a real-world scenario generator that tries to model systemic risk based on macroeconomic models. Therefore, it is not mark to daily market conditions but reflects economic patterns that persist in a longer time horizon. That means it is not suitable for hedging, financial option pricing, market consistent valuation and tactic asset allocation. It is more suitable for real-world strategy analysis such as strategic asset allocation, business planning, economic forecasting and long-term capital management. It may be used for reserving and capital management with an ongoing view of the business. As a rule of thumb, the ESG is not preferred for any decision-making that considers patterns that persist only less than a quarter to be important. While this may seem to be a disadvantage of this ESG, it allows users to incorporate their own forward-looking views by adjusting the input data, the models or the scenarios.

The U.S. economy is used as an example to demonstrate the DSGE model. The Bayesian Monte Carlo Markov Chain (MCMC) method is used to calibrate the model based on historical macroeconomic data, while some model parameters are determined based on additional economic analysis to overcome the difficulties in model calibration. Economic factors generated by the DSGE model govern the systemic risk in the remaining ESG process. Asset returns and bond yields are simulated using multifactor regression models, reflecting both systemic risk and idiosyncratic risk.

Several regression models are tested, including linear regression, Lasso, ridge regression, Elastic Net, K-nearest neighbors (KNNs), classification and regression trees (CARTs), artificial neural networks (ANNs) and gradient-boosting machines (GBMs). Given the data volume, overfitting is a critical issue in the example. ANN models showed very good estimation on the training data but the worst estimation on the validation data. With only 103 data records and many more model parameters, the superficial performance of ANN models is caused by overfitting with too many parameters compared with other models. CART models face the issue of overfitting as well but less severe. In addition, the possible values predicted by CART models are limited given that the estimation is based on the average of a subset of historical data. KNN models had similar issues as CART models with limited possible values and unsatisfactory prediction accuracy based on validation data. Linear regression models, especially Elastic Net models, did a good job predicting government bond yields, credit spreads and dividend yields. For public equity, REITs and commodities, idiosyncratic factors contribute much to the volatility. GBM models have better prediction accuracy than linear regression models for REIT equity returns. However, the improvement of accuracy may not justify a much more complicated model type. For succinctness, Elastic Net models are used for multifactor regression in this example. These models show relatively high prediction accuracy in most cases. Therefore, they are chosen to describe the relationships between asset returns and economic factors. Additional adjustments to the correlation among idiosyncratic risks are made to incorporate nonlinearity in the capital market during economic recessions.

Generated sample real-world economic scenarios can be compared to historical data as in Table E.1 or market predictions to assess their reasonableness and identify areas of further improvement.

Table E.1
Moments of Asset Returns and Bond Yields

Asset Class		Mean (%)		Standard Deviation (%)	
		Historical	Simulated	Historical	Simulated
Treasury bond, 0 rate (for given term)	1 year	2.62	2.12	2.18	1.21
	2 years	2.93	2.39	2.24	1.27
	3 years	3.25	2.52	2.25	1.28
	5 years	3.49	2.98	2.18	1.20
	7 years	3.93	3.34	2.02	1.16
	10 years	4.26	3.60	1.90	1.15
	20 years	4.51	3.92	1.75	1.18
	30 years	4.82	4.11	1.56	1.12
AAA-rated corporate bonds	Credit spread	0.47	0.60	0.58	0.64
	Default rate	0.00	0.00	0.00	0.00
AA-rated corporate bonds	Credit spread	0.58	0.71	0.78	0.68
	Default rate	0.02	0.03	0.08	0.34
A-rated corporate bonds	Credit spread	1.04	1.08	0.93	0.65
	Default rate	0.06	0.05	0.11	0.35

BBB-rated corporate bonds	Credit spread	1.83	1.80	1.08	0.67
	Default rate	0.18	0.15	0.25	0.30
Public equity	Dividend yield	2.00	1.93	0.54	0.60
	Capital return	2.04	2.28	9.33	5.49
REITs	Cap rate	1.56	1.21	0.44	1.51
	Capital return	1.43	1.49	9.09	5.58
Crude oil	Total return	2.37	2.48	16.18	11.99
Gold	Total return	1.37	1.45	6.75	5.92

Because of the complexity of the DSGE model, efforts need to be made to mitigate model risk. Although we can rely on traditional measures such as loglikelihood and the Akaike information criterion to compare the model against others, it assesses only the goodness of fit to data inputs. It does not guarantee that generated models are reasonable. Sensitivities to the sources of risk can be examined to make sure that expected economic patterns are reflected in simulated scenarios. The range of forecasts can be compared with those provided by the Fed and professional forecasters. Individual scenarios can be checked to see if they preserve economic cycles and the coexistence of a low GDP growth rate, low interest rates, higher credit spreads and a bear equity market during economic recessions.

If the ESG is used as a tool for regular tasks such as reserving and asset allocation, the calibrated model can be updated with new data or recalibration if deemed necessary. The existing calibration may have already covered a long history, and so an additional quarter's data may not have a material impact on the calibration, and, therefore, recalibration frequency does not need to be quarterly but could be semiannual or yearly. However, with new data coming in each quarter, the starting point of the projection will be changed. The data input for the simulation part includes the latest two quarters' economic data and capital market data. They need to be updated before the scenario generation.

Overall, this research contributes to existing literature in three ways. First, it contains a step-by-step derivation of the DSGE model and details on model calibration. The purpose is to provide enough information for people to understand and be able to customize the DSGE model to reflect their own economic views. Second, it embeds DSGE models into economic scenario generation. DSGE models have been used for analyzing monetary policies but are seldom used in other areas. With a DSGE model-based ESG, users have the flexibility to incorporate prior knowledge of the economic system and the potential to analyze the causes of individual scenarios. It may be attractive for users who need to make decisions based on individual scenarios. The author is not aware of any previous efforts made in this area. Third, sample codes are made available for educational purposes; they are hosted at: https://github.com/Society-of-actuaries-research-institute/LP152-Macroeconomics-based-ESG.

The ESG presented in this report serves as an example of DSGE based on scenario generators. By no means is it perfect and can be used directly without adjustment. More efforts are needed to improve the ESG and make it attractive for practical applications.

Section 1: Introduction

Economic scenarios are used by insurers for many purposes, including pricing, valuation, hedging, business planning and risk assessment. Deterministic scenarios, either expected or stressed, can be used to assess their financial impacts on insurance portfolios. In other situations where options are embedded in the portfolio, or when the range of possible outcome is of interest, stochastic scenarios are used to generate distributions. Most existing economic scenario generators (ESGs) used in the insurance industry model asset returns and yield curves directly. The underlying assumption is that the historical data of modeled variables are enough to explain their changes by time. It is mainly a data-driven process. This approach is reasonable when the generated scenarios are used for short-term exercises such as hedging or fair valuation. In both hedging and fair valuation, current market conditions are the driving factor. Historical development and future evolving do not matter much, and models are calibrated to perfectly match current market conditions.

For other usages with a longer time horizon, modeling an economic variable by assuming it evolves based on recent history, and randomness can be problematic. This approach does not investigate the causes of an individual scenario. Is a bear market scenario caused by an unsatisfactory labor market condition, high financial leverage or international trade? Is this a one-time shock or a persistent long-term structural change? Being able to answer these questions give modelers advantages to capture cause-and-effect relationships, explain individual scenarios, easily adjust for new forward-looking economic views and, most importantly, help inform decision-makers under what scenarios decisions are being made. However, answering these questions can be difficult by looking at yield curves or asset returns because they are the outcomes, not the causes. For example, it is difficult to explain the movement from a high-interest-rate regime to a low-interest-rate regime using interest rate data. Is the movement a trend or a cyclical phenomenon? More details of the economic system need to be considered. A reasonable explanation is the increase in money supply by central banks through stimulating policies such as quantitative easing (QE). More money supply lowers marginal utility of money and therefore the borrowing cost.

The cyclical pattern of the economy is not always considered in existing scenario generators as well. The economic cycle is an important part when designing investment strategies. The market shock such as in the 2008 financial crisis may be deemed significant. But with a diversified equity portfolio and a long time horizon, the losses can be recovered, as evidenced by the new highs of U.S. stock indices. It is better to reflect the cyclical pattern in real-world scenarios.

Interest rate, inflation rate, credit spread, credit default and equity return are linked in the economic system. Reflecting their dependency is important for maintaining the logic of economic development and economic policy-making in the scenarios. However, linear relationships based on correlation coefficients are usually used in existing scenario generators. This could significantly underestimate the tail risk, where correlation is much higher than normal.

This research goes one level deeper than most existing ESGs to explore cause-and-effect relationships in the economic system and reflect them in economic scenarios. Macroeconomic models are used to describe the economic environment, reflect economic cycles and nonlinear relationships, and govern the generation of yield curves and asset returns for the systemic risk part. Dynamic stochastic general equilibrium (DSGE) models are introduced as a modern macroeconomic model type. DSGE models have been widely used by central banks when setting monetary policies. They include details such as households, good and service producers, investment, trade, monetary policies and fiscal policies. These details are helpful for explaining the economic system and economic scenarios that are generated according to economic conditions. Asset returns, yield curves and other variables of interest can be mapped to modeled macroeconomic factors.

However, the complexity of DSGE models makes them unlikely to be accessible outside the academic world. This report tries to achieve the following goals:

- Introducing DSGE models to actuaries in a nontechnical way
- Providing step-by-step model derivation in the appendices for modelers
- Building a sample ESG based on a medium-scale DSGE model and testing its reasonableness
- Developing open-source codes for a DSGE model-based ESG

The report proceeds as follows:

- Section 2 (DSGE Model) introduces DSGE models focusing on concept and structure.
- Section 3 (ESG) discusses a new ESG built upon DSGE models and how to map asset returns to macroeconomic factors generated by DSGE models.
- Section 4 (Example) illustrates the application of the new ESG to the U.S. economy. It will touch on data, model calibration, validation and results interpretation.
- Section 5 (Model Validation and Adjustment) discusses the challenges of using the new ESG and measures that can be taken to mitigate model risk. If users want to adjust or incorporate their own economic views, this section also discusses the possible ways to do that.
- Section 6 (Further Development) discusses potential research that can improve the ESG and make it more appropriate for insurance applications.
- Section 7 (Conclusion) summarizes the key points of this research and concludes the main body of the report.
- Appendix A contains a detailed derivation of the DSGE model used in this report for educational purposes.
- Appendix B provides details on the multifactor regression model used to simulate asset returns, bond yields and default rates.

Section 2: DSGE Model

DSGE models are a modern type of macroeconomic model. DSGE models use econometric models to explain observed economic phenomena and predict the impact of monetary policies. A typical DSGE model studies equilibrium conditions of multiple markets through supply, demand and prices by analyzing the behavior of market participants: producers, importers, exporters, households, central banks and others. While DSGE models vary by scope and complexity, they share a few features:

1. **DSGE models are dynamic models.** They study the evolving of economies. What is the current status of an economy? How do business cycles affect future business development? How will the economy develop from the current status to long-term equilibrium?

2. **DSGE models are stochastic models.** They study how an economy reacts to random shocks with respect to technology improvements, labor market conditions, monetary policies, international trades and changes in household preference.

3. **DSGE models follow the general equilibrium theory that studies the equilibrium states of multiple correlated markets rather than a single market.** This enables DSGE models to reflect cause-and-effect relationships within economic systems.

4. **DSGE models are built on microeconomic foundations.** An economic system is described with three pillars: demand, supply and monetary policies. These pillars are defined based on the behavior of economic agents: household, producers, importers/exporters, the central bank and the government. How does a household choose between work and leisure? How does a company set price to maximize profit? How does the central bank determine monetary policies?

This section introduces a medium-scale DSGE model for open economies in a nontechnical way. Model details are provided in Appendix A.

2.1 Model Structure

Unlike most exiting ESGs that model interested variables directly based on their historical values, autocorrelation, mean-reverting process and volatility, DSGE models go another level deeper to consider decision-making in economic activities and their interactions. This enables us to explain the volatility with causes: technology shock, labor market instability, supply, demand, work/leisure preference shock, producer markup shock, monetary policy shock, fiscal policy shock, exchange rate and so on.

Figure 1

Sample DSGE Model Structure

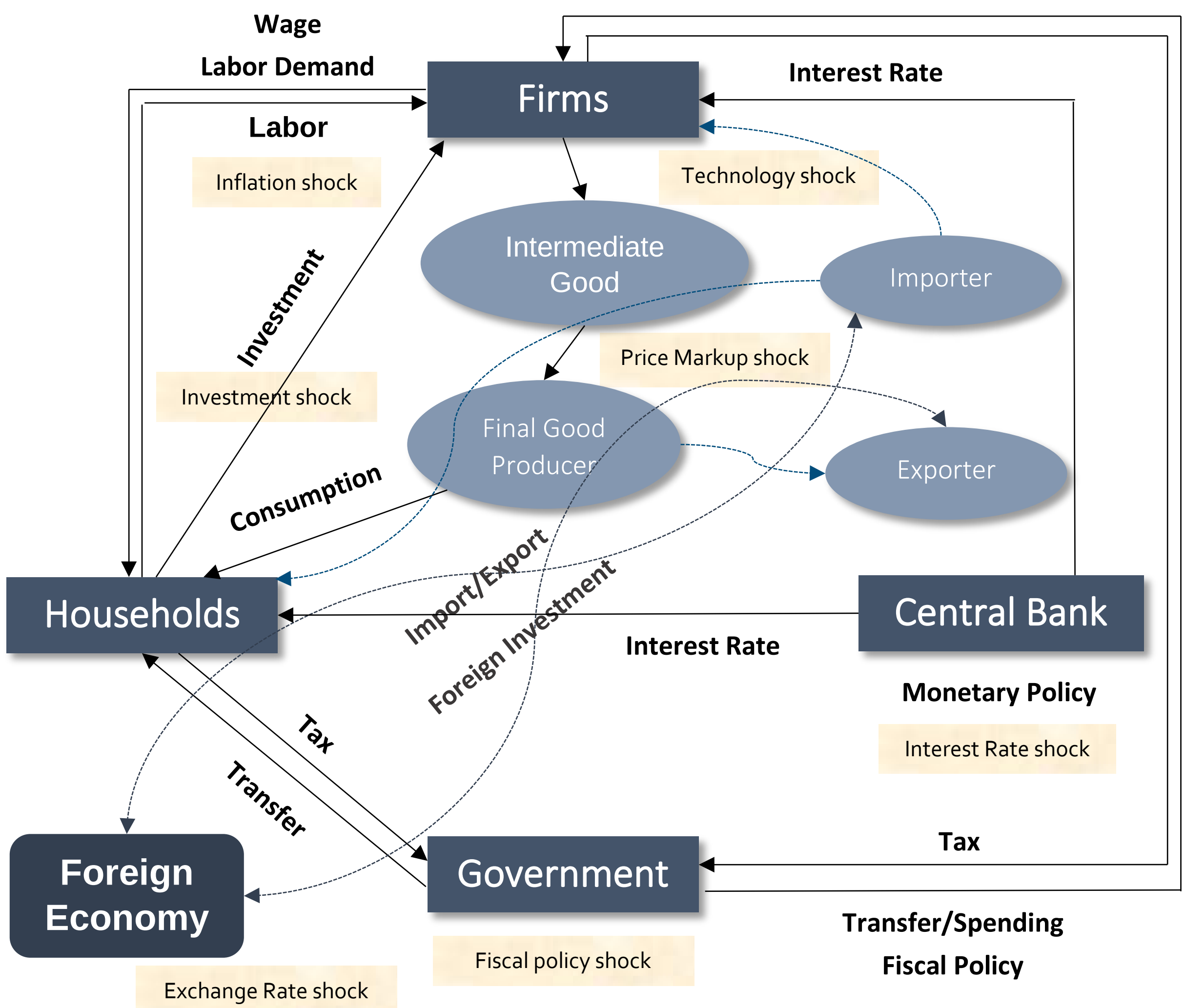


Figure 1 shows the economic agents and their interactions that are modeled in the DSGE model to be introduced in this report. The DSGE model follows the new Keynesian theory that emphasizes the role of central banks. The model includes the following economic agents:

1. **Final goods producer.** Final goods producers change immediate goods into final goods to be directly consumed by households. The goods produced can be consumption goods or investment goods. The final goods market is assumed to be a competitive market. The amount of goods that can be sold by an individual producer is determined by the price the producer charges and the level of market competition. The level of market competition is defined by the elasticity of substitution. If a producer's products can be easily produced by other producers, the elasticity of substitution is high.

   Given the demand function determined by the elasticity of substitution, individual producers set prices to maximize profit.

2. **Intermediate goods producer.** The intermediate goods market is assumed to be monopolistic competitive. In a monopolistic competitive market, many producers sell products that are differentiated from one another. Therefore, the intermediate goods market is assumed to be less competitive than the final goods market. The intermediate goods market is defined by four components: production function, cost function, choice of production inputs and price setting.

   - Production of intermediate goods utilizes three basic factors: capital, labor input and technology. The production level is determined by these factors and their relative importance in the production process. The impact of technology is separated further into permanent technology development and transitory technology change.

   - Producers need to pay the cost of using capital and labor for production. The cost is composed of wages paid to employees and cost of capital. In general, technology is assumed to be available to all producers. The potential cost of accessing advanced technology can be captured by transitory technology shock.

   - Unlike final goods producers whose production is solely driven by market demand based on the level of market competition, intermediate goods producers need to determine how much capital and labor they should use. To achieve the same amount production, a producer can use a different combination of capital and labor. The balance between capital and labor can be solved to minimize the cost of production. The optimization happens when the marginal return of capital equals that of labor.

   - Producers set the price of their products to maximize their profit. However, not all producers have the opportunity to set prices each time. For producers that cannot change prices to maximize profit, they are assumed to increase the price based on a weighted average of the actual inflation rate in the previous period and target the inflation rate set by the central bank. The profit optimization process leads to a version of a new Keynesian Phillips curve that describes the dynamic process of the inflation rate. The domestic inflation rate is a function of expected inflation, target inflation rate and marginal cost of intermediate goods production.

3. **Importer.** Importing firms are assumed to buy a homogeneous good at an international price and to turn the good into differentiated consumption goods or investment goods. Like final goods producers, the demand for individual importing firms depends on the level of market competition and goods prices. Like intermediate goods producers, importing firms may also change prices to maximize their profit with a certain probability. The optimization results in a new Keynesian Phillips curve for importing price inflation.

4. **Exporter.** Exporting firms purchase final goods and investment, differentiate them and export them to households in foreign countries. Like intermediate goods producers and importers, exporters can reset

prices with a certain probability. However, for exporters that cannot reset prices, the domestic inflation target is not considered when adjusting export goods and investments prices. This is different from domestic firms and importers.

5. **Household.** Domestic households consume domestic goods and imported goods, make investment, work and pay taxes. They need to make decisions on consumption, working and investment to maximize their utility. Utility is higher with more consumption, less work and more asset holdings:

    - Households provide labor services in a differentiated way. It is assumed that households have monopoly power setting a nominal wage with a certain probability. The optimal wage level is determined to maximize future wage income reduced by the disutility of working. For households that cannot reset a wage during a period, the wage is adjusted according to the actual inflation rate, target inflation rate and permanent technology development.

    - Households allocate wealth among cash, domestic bonds and foreign bonds. For simplicity, bonds are assumed to have a one-year maturity. Foreign bonds holding is adjusted with a risk premium that depends on the domestic economy's indebtedness in the international assets market and expected depreciation of domestic currency to reflect credit risk and foreign exchange risk. Households also make capital investment and own all the firms in return of profits. Capital stock grows with new investment and decreases by depreciation and investment adjustment cost.

    - Households face a constraint function based on their income, wealth and consumption. Households make consumption and investment decisions by maximizing their life-long utility subject to the constraint function.

6. **Government.** The government collects taxes, redistributes income through transfer payments and spends money. Fiscal policy is assumed to be an exogenous process given its uncertainty due to political reasons. A budget can be either a surplus or deficit.

7. **Central bank.** The central bank sets domestic short-term interest rates based on its recent history, target inflation rate, economic growth and foreign exchange rate. It is a rule-based process estimated based on historical data.

8. **Foreign economy.** The foreign economy is assumed to be exogeneous governed by three related variables: GDP growth, inflation and short-term interest rates.

To ensure the modeled economic system is consistent, some market-clearing conditions need to be met as well. Total output cannot be greater than consumption. The net position of the foreign bond market should be consistent with international trade and investment.

2.2 Source of Risk

DSGE-based ESGs take a very different approach from other ESGs when describing the sources of economic risk. Many ESGs used by insurance industry consider interest rate risk, equity risk, credit risk and volatility risk as the sources of economic risk. Although it is reasonable to decompose aggregate economic risk in this way, these are just categories of economic risk, rather than its causes. DSGE models want to better understand the causes at the macrolevel. Given the model structure described in the previous section, Table 1 lists the sources of risk that are included and used to generate stochastic economic scenarios.

Table 1
Sample Risk Sources

Source	Economic Agent	Model Component
Permanent technology	Producer	Production and wage level
Transitory technology	Producer	Production
Investment specific technology	Household	Capital stock change
Labor supply	Household	Disutility of working
Consumption preference	Household	Preference of consumption
Risk premium	Household	Foreign bond holding
Domestic markup*	Producer	Domestic price setting
Imported consumption markup	Importer	Importer price setting
Imported investment markup		
Exported consumption markup	Exporter	Exporter price setting
Exported investment markup		
Fiscal policy	Government	Government spending
Monetary policy	Central bank	Interest rate decision
Foreign output	Foreign economy	Foreign economy
Foreign inflation		
Foreign monetary policy		

*Markup is the difference between price and marginal cost of production.

By modeling sources of risk rather than their outcomes, DSGE models are expected to better capture cause-and-effect relationships in our economic systems. Economic scenarios generated with shocks to these sources are able to maintain relationships among economic variables. The relationships can be linear or nonlinear and contemporary or intertemporal.

2.3 Model Estimation

In general, historical data of observable economic variables can be used for model calibration. These economic variables are part of DSGE models and are publicly reported on a regular basis:

- Real gross domestic product (GDP)
- Private consumption
- Total fixed investment
- Total exports
- Total imports
- Nominal effective exchange rate
- Nonagricultural employment

- Compensation of employees
- Policy interest rate
- Fixed investment deflator
- Consumer price index (CPI)
- Producer price index (PPI)
- Target inflation rate of central bank
- Foreign real GDP
- Foreign CPI
- Foreign policy interest rate

Some economic variables are affected by changes in population. Real GDP, consumption, investment, imports, exports and employment may grow at the economy level but deteriorate at the per capita level. These variables are detrended to reflect the improvements in economic efficiency, not economic volume.

Maximum likelihood estimation (MLE) can be used to determine every model parameter by maximizing the likelihood of historical observations. However, DSGE models are not purely data-driven but have some behaviors of economic agents defined ahead. Given that the number of parameters is large, relying solely on MLE may lead to unstable model calibration results. In addition, some parameters such as disutility of working are behavior related and are better analyzed separately based on economic agents' characteristics. Therefore, additional data and analysis are needed for model calibration. For example, to determine how a central bank will react to changes in GDP growth rate, recent inflation rate and exchange rate movements, parameters may be calibrated using historical data of policy interest rates, GDP, inflation rates and exchange rates. Alternatively, the existing literature in economics can be relied on to set the values of these parameters.

After setting a subset of model parameters based on separate analysis, Bayesian Markov Chain Monte Carlo (MCMC) estimation is used to derive posterior distributions of model parameters that are not calibrated. Bayesian MCMC estimation has several advantages compared to other methods for DSGE models:

- Bayesian estimation ensures model consistence by fitting model parameters to historical observations in general. Relationships among economic variables implied from historical data can be kept in calibrated models. This makes DSGE models more suitable for practical uses.

- DSGE models are high-dimensional models with many local maxima and minima. Bayesian estimation is less prone to local optimization because it estimates the posterior distribution rather than a single point.

- Bayesian estimation allows users to provide prior distribution of model parameters. Prior distributions are additional inputs and can be helpful for optimization as well.

- Posterior distributions of parameters derived using Bayesian estimation can be used to model parameter risk separately from process risk. This is useful for model risk quantification and management.

Although in theory Bayesian estimation will be able to converge to the global maximum of likelihood, the optimization algorithm is not always effective. Setting appropriate priors, especially with a reasonable mean value, is important. Like parameter calibration, empirical analysis and the existing literature can be leveraged to set priors.

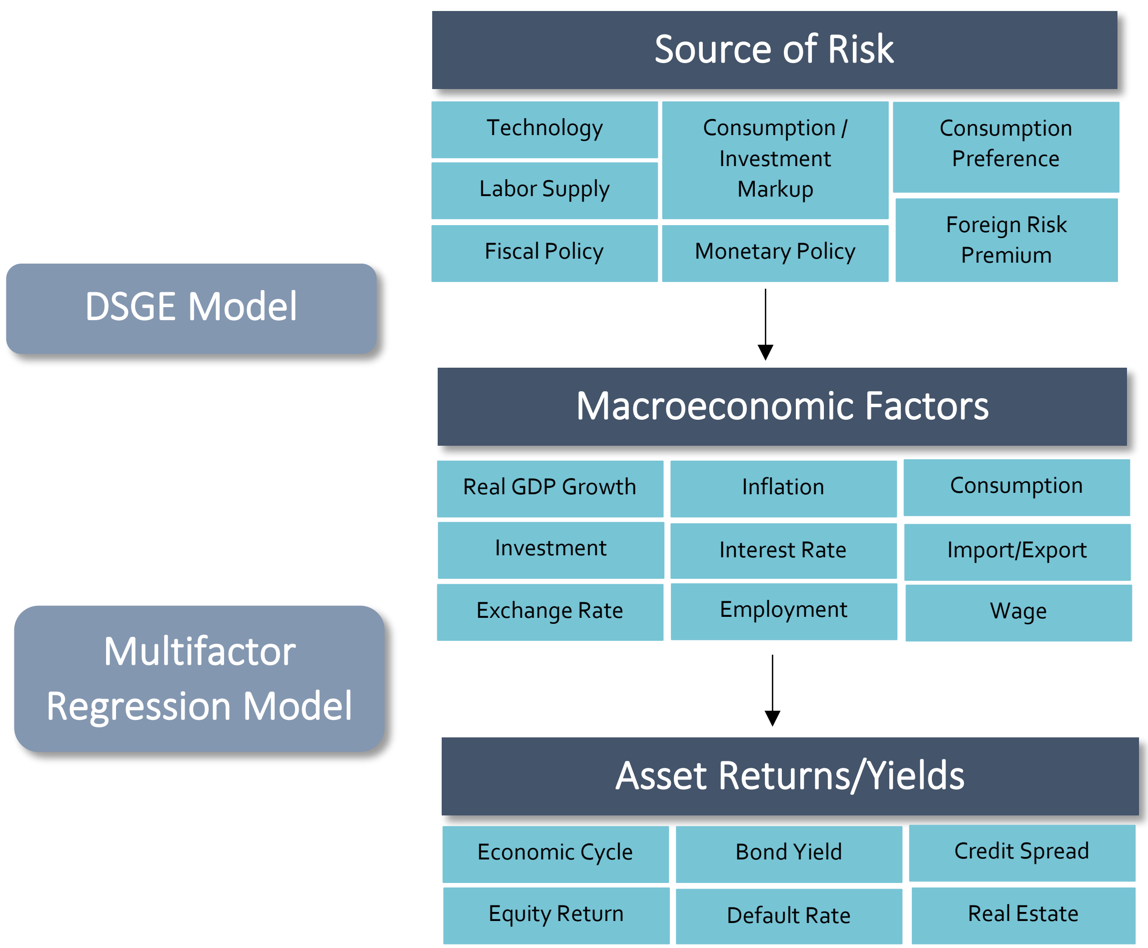
Modeled macroeconomic factors set economic conditions that can be used to govern the generation of asset returns and yield curves. This approach maintains the consistency of systemic risk among returns and yields within each economic scenario. Observable economic variables are usually chosen as DSGE model outputs or inputs to ESGs. These variables reflect basic macroeconomic conditions and are publicly available. Economic agents and investors use these variables to make informed decisions. Therefore, it is reasonable to use them as the driving factors behind systemic risk.

Section 3: ESG

With the outputs from a DSGE model, economic scenarios can be built using the relationships among macroeconomic factors, asset returns, bond yields and other modeled variables. Figure 2 shows the structure of a sample DSGE model-based ESG. The DSGE model generates macroeconomic factors driven by sources of economic risk. Based on these macroeconomic factors, multifactor models that capture both the systemic risk defined by macroeconomic factors and the idiosyncratic risk based on individual asset classes can be used to generate asset returns and yields that are readily used in stochastic analysis or stress testing.

Figure 2

Sample Structure of DSGE Model-Based ESG



The DSGE model has been discussed in Section 2. The remaining step is to use the macroeconomic factors generated by the DSGE model to simulate asset returns and bond yields. Different types of regression models may be used for this purpose, including linear regression, Lasso regression, ridge regression, classification and regression trees, K-nearest neighbors, gradient-boosting machines and artificial neural networks. An example of linear regression model is as follows:

$$\boldsymbol{y_t} = \boldsymbol{f}(\boldsymbol{y_{t-1}}, \boldsymbol{y_{t-2}}, \boldsymbol{F_t}, \boldsymbol{F_{t-1}}, \boldsymbol{F_{t-2}}) + \boldsymbol{e_t}$$

where

$\boldsymbol{y_t}$: A column vector containing asset returns during period t or yields at time t
$\boldsymbol{f}$: A regression function that is determined by the chosen model type
$\boldsymbol{F_t}$: A column vector including all the macroeconomic factors at time t or during period t generated by a DSGE model
$\boldsymbol{e_t}$: A column vector including random numbers at time t or during period t that represents idiosyncratic risk. The components in this vector are not necessarily independent but may show different correlations during different phases in an economic cycle.

In the function above, it is assumed that information on the current quarter and previous two quarters will be used for simulation. The data may imply otherwise, and more or less history can be used as inputs to the function.

Model selection depends on model accuracy, ease of explanation and model robustness. Some sophisticated nonlinear models have a high accuracy but face the problem of overfitting. The volume of historical macroeconomic data may not justify the use of complex nonlinear models in this research. A longer history may be used for calibration, but the model may become less relevant to current economic structure. Section 4.2 shows an example of model selection.

The dependency among modeled variables is governed by the ESG, which generates scenarios of macroeconomic factors that determine the phase of the economy: expansion or recession. Based on the economic phase, asset returns and other factors are generated. It is important that the ESG can maintain the dependency among all the economic variables observed in the historical data. It is achieved by the following:

- **Intertemporal dependency among macroeconomic factors is built into the DSGE model.** These macroeconomic factors jointly determine the phase of the economy: recession or expansion.
- **The macroeconomic factors in the DSGE model are not independent from each other.** Their contemporary correlations are reflected when generating the random part of macroeconomic factors.
- **Both contemporary and intertemporal dependency between asset returns and macroeconomic factors are governed by the regression models used to generate asset return scenarios.**
- **The error terms (idiosyncratic factor) of modeled variables are not independent from each other.** Their contemporary correlations are reflected when generating the random part of the asset returns.
- **To incorporate nonlinear relationship, the volatility of idiosyncratic factors, their interdependency and their dependency on macroeconomic factors vary by the status of the simulated economy.** Higher volatility and dependency are used when an economic recession is simulated.

The ESG is a real-world scenario generator that tries to model systemic risk based on macroeconomic models. Therefore, it is not mark to daily market conditions but reflects economic patterns that persist in a longer time horizon. That means it is not suitable for hedging, financial option pricing, market consistent valuation and tactic asset allocation. It is more suitable for real-world strategy analysis such as strategic asset allocation, business planning, economic forecasting and long-term capital management. It may be used for reserving and capital management with an ongoing view of the business. As a rule of thumb, the ESG is not preferred for any decision-making that considers patterns that persist only less than a quarter to be important.

Section 4: Example

The DSGE model and ESG are applied to the U.S. economy. The entire process, including data selection, model calibration, simulation and result interpretation, is discussed. This section focuses on practical implications rather than technical details, which are included in the appendices. Open-source programs used in this section are publicly available at: https://github.com/Society-of-actuaries-research-institute/LP152-Macroeconomics-based-ESG.

4.1 DSGE Model

The DSGE model is applied to the U.S. economy and calibrated to historical data from 1992 to 2018. Quarterly data are used because the quarter is the highest frequency at which most economic data are reported. This selected period includes three economic cycles and moved from high interest rates to low interest rates. Therefore, it contains some patterns and volatilities that the DSGE model can represent. As in most economic scenario generation models, data selection is important for DSGE models. A longer period of data may capture volatility and extreme events better, while a shorter period of data can reflect recent structural changes in the economy. Table 2 lists the economic data used for the estimation of the DSGE model.

Table 2

Data Source (2Q1992–2Q2018)

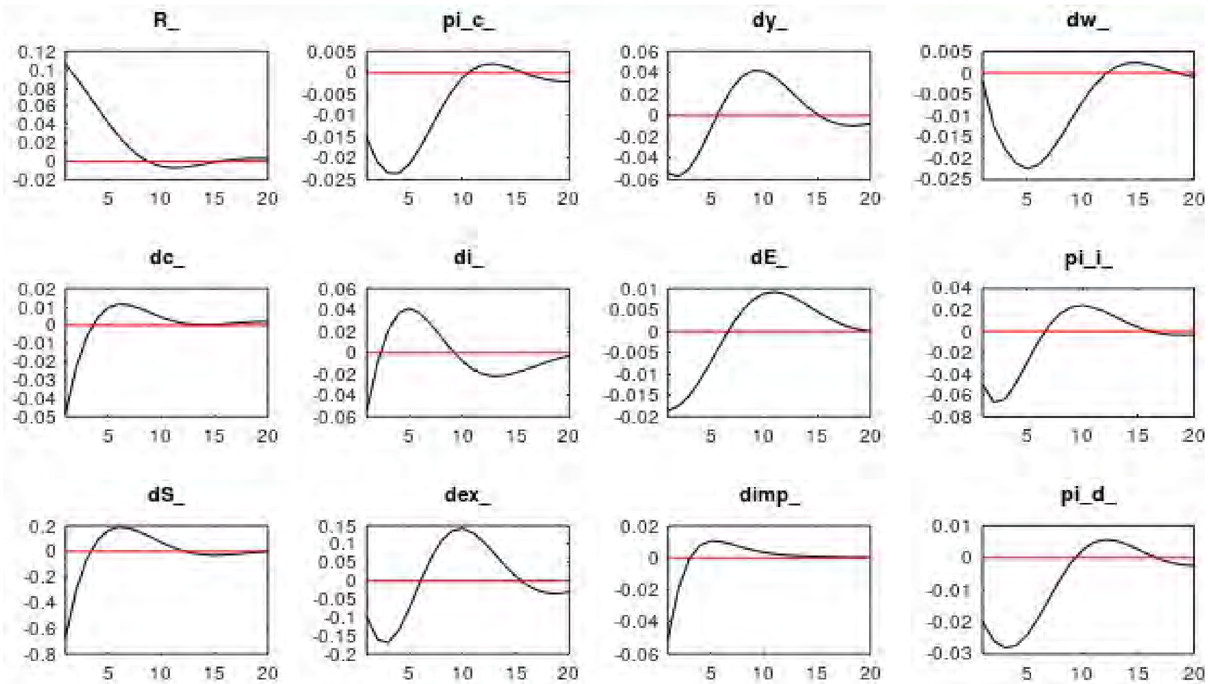
Variable	Note	Source
Real GDP	Real gross domestic product	Fed economic data
Private consumption	Personal consumption expenditures	Fed economic data
Total fixed investment	Private nonresidential fixed investment	Fed economic data
Total exports	Exports of goods and services	Fed economic data
Total imports	Imports of goods and services	Fed economic data
Nominal effective exchange rate	Broad effective exchange rate for the United States	Fed economic data
Nonagricultural employment	All employees, thousands, total nonfarm	Fed economic data
Compensation of employees	Employment cost index: wages and salaries, private industry workers	Fed economic data
Policy interest rate	Effective federal funds rate, percent, monthly	Fed economic data
Fixed investment deflator	Gross private domestic investment: fixed investment, nonresidential	Fed economic data
CPI	CPI, all urban consumers	Bureau of Labor Statistics
PPI	PPI industry group data for total manufacturing industries, not seasonally adjusted	Bureau of Labor Statistics
Foreign real GDP	Weighted average of 10 major U.S. trading partners	Fed economic data
Foreign CPI	Weighted average of 10 major U.S. trading partners	Bureau of Labor Statistics
Foreign policy interest rate	Weighted average of 10 major U.S. trading partners	Fed economic data
Population	U.S. population and foreign countries' population for per capita calculation	Fed economic data

Calibration and estimation details are summarized in Appendix A.13 Calibration. To have a glance of how the calibrated DSGE model works, the impulse response function (IRF) is a good starting point. An IRF describes the evolution of a variable after a shock of another variable. For example, what will the economy change if the central bank raises interest rate? Figure 3 shows the IRFs of monetary policy shock. By increasing the policy interest rate

(R_) by its standard deviation at time zero, the model predicts a decrease in inflation rates, real GDP growth rate, consumption, investment, employment, real exchange rate and imports and exports. The impact of the shock will dampen over time to a steady state. This is consistent with economic patterns observed in the past. A rising interest rate normally cools an economy.

Figure 3

Impulsive Response Functions: Monetary Policy Shock



Notes:

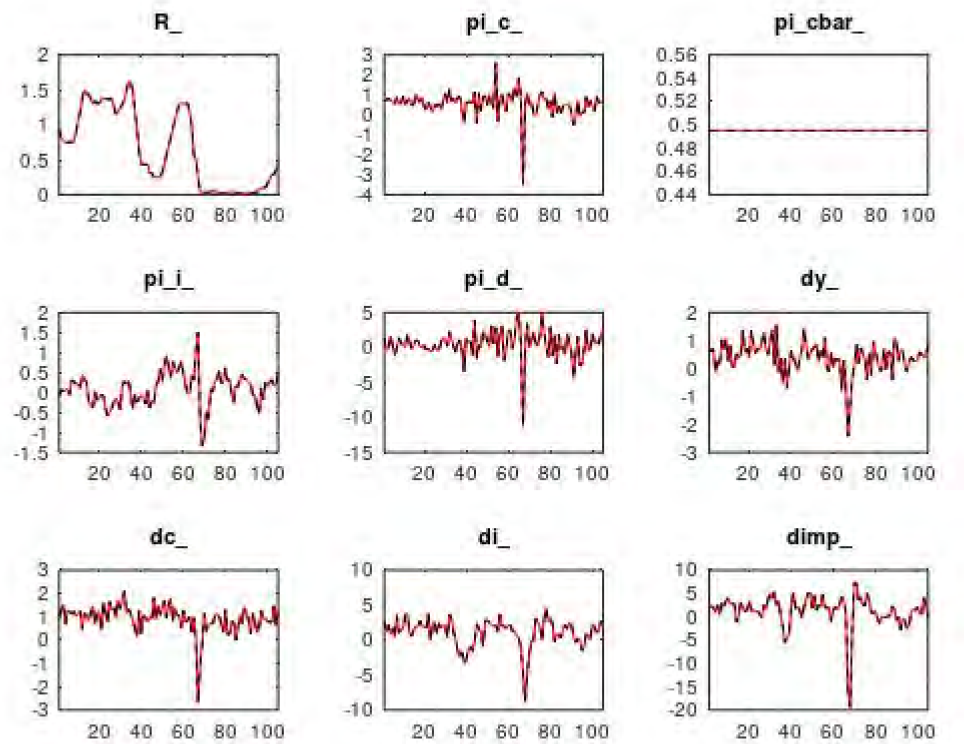
- R_: policy interest rate
- pi_c_: inflation rate based on CPI
- dy_: change in real GDP per capita
- dc_: change in consumption per capita
- di_: change in investment per capita
- dE_: change in employment per capita
- dS_: change in exchange rate
- dex_: change in export per capita
- dimp_: change in import per capita
- dw_: change in wage index
- pi_i_: inflation based on fixed investment
- pi_d_: inflation rate based on PPI

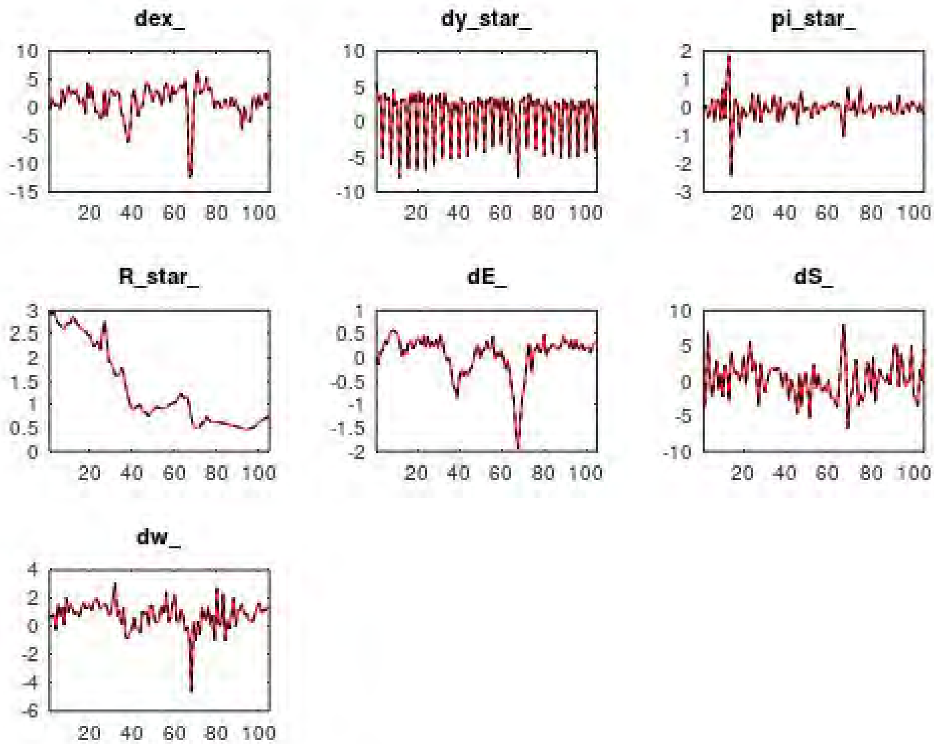
This analysis can be performed for all risk sources, such as labor supply, government spending, consumption, investment, risk premium and technology development, as listed in Appendix A.13 Calibration.

Statistical models usually compare actual values with estimated values to assess model accuracy. For example, R^2 indicates the percentage of variance that can be explained by a model. DSGE models are dynamic models that acknowledge that the path to the equilibrium state is abundant with shocks, and some shocks have a long-lasting impact on the path. Shocks are usually applied to endogenous variables that are not observable. The equilibrium state may be hardly achievable, but the economy tends to revert back to equilibrium if no further significant shocks occur. Figure 4 shows the historical data and model estimation. Model estimation uses calibrated models and shocks to reconstruct historical time series. For all the observable economic variables, model estimation matches historical experience perfectly.

Figure 4

DSGE Model Fitting: Historical and Estimated Value





Notes:

- pi_cbar_: central bank target inflation rate
- dy_star_: change in foreign real GDP per capita weighted by trade
- pi_star_: foreign inflation rate weighted by trade
- R_star_: foreign policy interest rate weighted by trade
- Other terms are the same as in Figure 3

Models can be compared using the log-likelihood function. By maximizing the log-likelihood, the observed data are most probable given the calibrated model. Bayesian estimation tries to maximize the log-likelihood function as well using the Markov Chain Monte Carlo (MCMC) method to simulate from posterior distributions and adjust these distributions to get a higher log-likelihood function. By running three chains with each chain containing 40,000 replications using the Metropolis-Hastings algorithm, the log-likelihood function reduces to –973.0. Here the log-likelihood function (log posterior) is adjusted by a multiple that equals the number of estimated parameters divided by the number of observable variables. The adjustment is made to facilitate comparison with other models.

The value of the log-likelihood function is not very meaningful if not compared to that of another model. A vector autoregressive (VAR) model is used as a benchmark to evaluate the performance of the DSGE model. The VAR model describes the intertemporal relationships among 16 macroeconomic factors of the U.S. economy, as in the DSGE model. The input data to both models are the same. The VAR model is as follows:

$$\mathbf{F}_t = \mathbf{c} + \boldsymbol{A}_1 \mathbf{F}_{t-1} + \mathbf{e}_t$$

where

$\mathbf{F}_t$ is a column vector with macroeconomic factors at time t or during period t
$\mathbf{c}$ is a column vector representing the constant terms of the macroeconomic factors

$\boldsymbol{A_1}$ is a square matrix containing the model parameters describing the linear dependence of macroeconomic factors and

$\mathbf{e}_t$ is a column vector to store the error terms of macroeconomic factors that cannot be explained by linear models

A similar VAR model was used in Shang and Hossen (2019) and showed some quite good prediction power. Table 3 shows the log-likelihood function, number of parameters to be estimated and the Akaike information criterion (AIC). The normal distribution is used to calibrate each macroeconomic factor individually. This can be seen as a lower bound of macroeconomic models because it considers that all the factors are random. The DSGE model has a relatively high log-likelihood function in this case.

Table 3

Log-Likelihood Function

Model	Log-Likelihood	No. of Parameters	AIC
Independent normal distribution	−2336	32	4736
VAR(1)	−1022	256	2555
DEGE	−976	59	2071

Simulated macroeconomic factors are compared to historical data to assess model reasonableness. Table 4 shows the mean and standard deviation of each observed variable based on historical data and simulated data. One thousand simulations of 120 quarters are used, and the difference in mean is not material in most cases. The simulated interest rates have a lower standard deviation than implied from the historical data. This is likely to be caused by the low interest environment at the beginning of projection.

Table 4

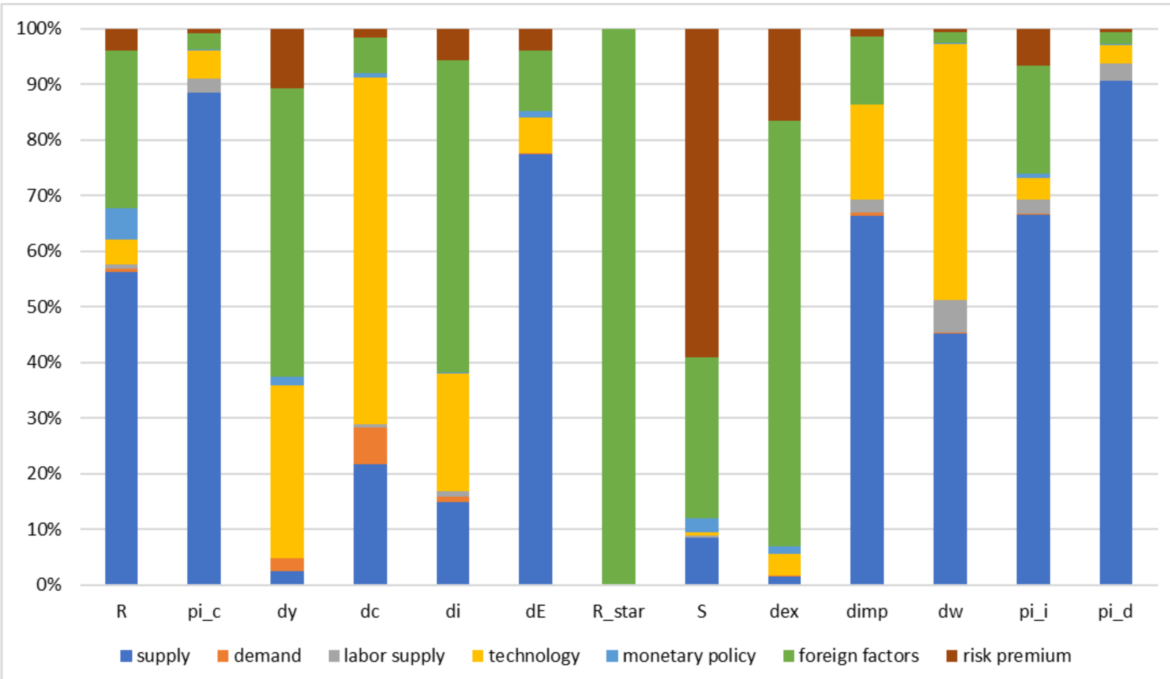
Moments of Observed Economic Variables

Variable	Historical Mean	Simulated Mean	Historical Standard Deviation	Simulated Standard Deviation
R_	0.66	0.68	0.55	0.12
pi_c_	0.56	0.54	0.59	0.53
pi_cbar_	0.50	0.53	0.00	0.03
pi_i_	0.09	0.08	0.42	0.69
pi_d_	0.51	0.56	1.97	0.83
dy_	0.4	0.46	0.57	0.66
dc_	0.93	1.09	0.59	0.41
di_	1.06	1.28	1.99	0.99
dimp_	1.28	1.55	3.48	1.27
dex_	1.12	1.21	2.89	2.33
dE_	0.07	0.03	0.43	0.22
dS_	0.33	0.27	2.67	2.80
dw_	0.81	1.06	0.99	0.48
dy_star_	0.94	1.16	3.68	1.71
pi_star_	-−0.02	0.01	0.44	0.84
R_star_	1.34	1.42	0.85	0.24

Understanding the sources of volatility and how they contribute to the violability of observable macroeconomic factor is also helpful. Figure 5 shows the variance decomposition of the calibrated DSGE model.

Figure 5

Variance Decomposition



In this model, employment, inflation rates (CPI and PPI) and imports are heavily affected by supply, which includes domestic firms, importers and transitory technology. Investment are driven by supply, permanent technology and foreign economies. Consumption is driven by technology development, supply and demand, which includes consumption, investment and government spending. Foreign factors have a big impact on exports, policy interest rate, real GDP per capita growth rate and real exchange rate. The risk premium is the main contributor to real exchange rate volatility.

4.2 Multifactor Model

With macroeconomic factors ready, the next step is to determine the economic status of each simulated period. Historical data of U.S. economic cycles (1991Q1–2016Q4) from the National Bureau of Economic Research (NBER) and economic factors are used to calibrate the logistic model, with parameters shown in Table 5. Current and the previous two quarters' values of macroeconomic factors are used to determine the current status of the economy. The calibrated logistic model has 100% accuracy in matching the history of the past 26 years. Changes from quarter to quarter are more important than the absolute value of economic factors for predicting the economic status. Other models such as K-nearest neighbors (KNN), classification and regression tree (CART), artificial neural network (ANN) and gradient-boosting machine (GBM) are tested as well but did not show higher accuracy than the logistic model.

Table 5

Economic Recession Prediction: Logistic Model Parameter

Variable	Period	Parameter (▯)
Intercept		−78.9
CPI growth rate	Current quarter *t*	58.4
	Previous quarter *t* − 1	4.1
	Previous quarter *t* − 2	50.8
Real GDP growth rate	Current quarter *t*	−85.7
	Previous quarter *t* − 1	−0.9
	Previous quarter *t* − 2	1.6
Consumption growth rate	Current quarter *t*	1.3
	Previous quarter *t* − 1	8.8
	Previous quarter *t* − 2	−75.0
Investment growth rate	Current quarter *t*	−3.8
	Previous quarter *t* − 1	−5.0
	Previous quarter *t* − 2	−17.2
Employment growth rate	Current quarter *t*	−158.0
	Previous quarter *t* − 1	−68.3
	Previous quarter *t* − 2	86.5

Asset returns, bond yields and other modeled variables are mapped to macroeconomic factors and idiosyncratic factors. Table 6 lists the historical data used for multifactor regression analysis.

Table 6

Asset Return Historical Data

Asset Class	Return Type	Time Period	Data Source
Treasury bond yield curve (terms: 1, 2, 3, 5, 7, 10, 20 and 30 years)	Yield	1991Q1–2016Q4 except for 20-year bond yield starting from 1993Q4	U.S. Department of the Treasury
AAA–, AA–, A– and BBB– rated corporate bonds	Credit spread	1996Q4–2016Q4	Bank of America Merrill Lynch U.S. Corporate AAA, AA, A, BBB Effective Yield (BAMLC0A3CAEY, BAMLC0A2CAAEY, BAMLC0A1CAAAEY, BAMLC0A4CBBBEY) *Federal Reserve Economic Data*
	Default rate	1991Q1–2016Q4	*2016 S&P Annual Global Corporate Default Study and Rating Transitions Report*
Public equity	Dividend yield	1991Q1–2016Q4	S&P 500 Index (^GSPC) *Yahoo Finance*
	Capital return		
REITs	Cap rate	1991Q1–2016Q4	FTSE NAREIT US Real Estate Index—All REITs
	Capital return		
Crude oil	Total return	1991Q1–2016Q4	WTI Crude Oil Price (DCOILWTICO) *Federal Reserve Economic Data*
Gold	Total return	1991Q1–2016Q4	London Bullion Market Association (LBMA) Gold Price (GOLDPMGBD228NLBM) *Federal Reserve Economic Data*

Eight model types are tested to build the relationships among modeled variables and macroeconomic factors: linear regression, Lasso regression, Ridge regression, Elastic Net regression, K-nearest neighbors (KNN), classification and regression tree (CART), artificial neural network (ANN) and gradient-boosting machine (GBM).

Linear regression assumes a linear relationship between macroeconomic factors and asset returns/bond yields. Lasso, Ridge regression and Elastic Net are variations of the linear regression model with different methods of regularization to prevent overfitting. In addition to minimizing the squared errors, Lasso models add the sum of the absolute value of parameters into the error function. Ridge regression uses the sum of squared parameters as the regularization term, and Elastic Net models use both. The weight of the regularization term is determined by cross-validation to get the highest model accuracy.

KNN is a type of nonparametric models that predicts the explained variable based on the values of the *k* closest neighbors. In this report, the closeness is measured by the Euclidean distance based on macroeconomic factors. Different numbers of neighbors are tested, and five nearest neighbors are used for all models for relatively better performance. The average return of five nearest neighbors in historical data is then used to predict the asset return.

CART models build trees to split the data based on macroeconomic factors. At each split, a factor is used to separate the data into two subgroups. The factor and split rule are chosen to provide the best split that improves the purity of the data in the subgroups. In the calibration of CART models for asset returns, two criteria are used to limit the size of the tree and avoid overfitting.

- The minimum number of data points in a node for splitting is 10
- The minimum improvement of data purity is 0.001 for each split

Each terminal node has an estimation based on the average value of the explained variable. For example, if the terminal node has 10 historical data records, the asset return is calculated as the average return of the 10 data records.

ANN models mimic biological neural networks to make predictions based on a large amount of data. Unlike traditional predictive models such as linear regression and logistic regression, the mathematical function that describes the relationship between the explained variable and explanatory variables is unknown. With enough data and appropriate training, ANN models are believed to mimic any complex relationships. The ANN model used in this study has four layers: the input layer that includes all the macroeconomic factors and the intercept, the first hidden layer with 10 neurons, the second hidden layer with five neurons and the output layer that is the modeled variable such as the asset return and bond yield. Neurons are linked with the tangent hyperbolic function. For crude oil, an extra hidden layer is used to improve model accuracy.

GBM is a decision tree–based ensemble method. Each tree is a weak estimator trying to estimate the residual error that the estimation of previous trees has caused. Gradually with a sufficient number of decision trees, the estimation error will decline to a very low level. GBM is usually proven to have better accuracy than many other methods when presented with nonlinear relationships. The model is also faster to train compared to neural networks. In this example, 70% of the training set were randomly selected each time to fit the next tree. The depth of the trees is set to be six, and 50 trees were used. The minimum observations in each node are set to be two, given the limited amount of training data.

For all the model testing, the data are randomly split into a training dataset (80%) and a validation dataset (20%). The training dataset is used to find the parameters that minimize the sum of squared prediction errors for the training data. The validation dataset is used to measure the prediction accuracy based on out-of-sample data. Table 7 shows the aggregate performance of the eight model types.

Table 7
Multifactor Model Aggregate Performance

Model	Training Data		Validation Data	
	RMSE	R^2	RMSE	R^2
LM	1.43	0.85	2.69	0.44
Lasso	2.09	0.68	2.42	0.61
Ridge	1.51	0.79	2.47	0.56
Elastic Net	1.77	0.75	1.90	0.70
KNN	1.98	0.56	2.83	0.24
CART	1.07	0.88	2.52	0.17
ANN	0.17	0.97	4.49	−4.57
GBM	0.30	0.99	2.16	0.57

Notes:
RMSE: Root-mean-square error
R^2: Coefficient of determination

Given the data volume, overfitting is a critical issue in this example. ANN models showed very good estimation on the training data but the worst estimation on the validation data. With only 103 data records and many more model parameters, the superficial performance of ANN models is caused by overfitting with many more parameters than other models. CART models face the issue of overfitting as well, but less severe. In addition, the possible values predicted by CART models are limited given that the estimation is based on the average of a subset of historical data. KNN models had similar issues as CART models with limited possible values and unsatisfactory prediction accuracy based on validation data. Linear regression models, especially Elastic Net models, did a good job predicting government bond yields, credit spreads and dividend yields. For public equity, REITs and commodities, idiosyncratic factors contribute much to the volatility. GBM models have better prediction accuracy than linear regression models for REIT equity returns. However, the improvement of accuracy may not justify a much more complicated model type. For succinctness, Elastic Net models are used for multifactor regression in this example.

The relationship between asset returns and fundamental economic factors is not always linear. During an economic recession, higher volatility and correlation are often observed. Asset return economic scenarios need to be further adjusted to reflect the nonlinear relationship. Table 8 shows the volatility of idiosyncratic factors (error terms) and the correlation between systemic factors (prediction by Elastic Net models) and idiosyncratic factors, using data either in expansion or in recession. Volatility and correlation behaved quite differently in recession. These values capture second-order impact in addition to the linear relationships built into linear regression models.

Table 8
Linear Model Idiosyncratic Factors: Volatility and Correlation

Asset Class		Idiosyncratic Factor Volatility (%)		Correlation With Systemic Factors	
		Expansion	Recession	Expansion	Recession
Treasury bond, 0 rate (for given term)	1 year	0.1	0.5	19%	−75%
	2 years	0.2	0.4	17%	−68%
	3 years	0.2	0.4	15%	−56%
	5 years	0.2	0.5	14%	−50%
	7 years	0.2	0.3	13%	−39%
	10 years	0.2	0.3	13%	−34%
	20 years	0.2	0.2	13%	−32%
	30 years	0.1	0.1	7%	−25%
AAA-rated corporate bonds	Credit spread	0.1	0.1	13%	−24%
	Default rate	0.0	0.0	0%	0%
AA-rated corporate bonds	Credit spread	0.1	0.3	21%	29%
	Default rate	0.0	0.0	25%	22%
A-rated corporate bonds	Credit spread	0.1	0.4	21%	23%
	Default rate	0.0	0.0	14%	−35%
BBB-rated corporate bonds	Credit spread	0.2	0.3	15%	38%
	Default rate	0.0	0.0	14%	−41%
Public equity	Dividend yield	0.0	0.2	18%	2%
	Capital return	43.9	62.8	23%	49%
REITs	Cap rate	0.0	0.1	17%	−26%
	Capital return	61.1	157.8	27%	61%
Crude oil	Total return	160.1	139.3	20%	45%
Gold	Total return	37.1	30.7	21%	28%

Details of the regression models and correlation adjustment can be found in Appendix B.

4.3 Scenario Generation

With the DSGE model and multifactor models, economic scenarios can be simulated in the following steps:

- Simulate macroeconomic factors using the DSGE model.
- With the simulated macroeconomic factors, economic conditions (expansion or recession) are determined for each period under each scenario.
- Depending on the economic condition, asset returns and bond yields are simulated based on the multifactor models.

Table 9 compares the simulated scenarios with historical data in terms of mean and standard deviation. One thousand simulations with each covering 30 years of data are used in the analysis.

Table 9
Moments of Asset Returns and Bond Yields

Asset Class		Mean (%)		Standard Deviation (%)	
		Historical	Simulated	Historical	Simulated
Treasury bond, 0 rate (for given term)	1 year	2.62	2.12	2.18	1.21
	2 years	2.93	2.39	2.24	1.27
	3 years	3.25	2.52	2.25	1.28
	5 years	3.49	2.98	2.18	1.20
	7 years	3.93	3.34	2.02	1.16
	10 years	4.26	3.60	1.90	1.15
	20 years	4.51	3.92	1.75	1.18
	30 years	4.82	4.11	1.56	1.12
AAA-rated corporate bonds	Credit spread	0.47	0.60	0.58	0.64
	Default rate	0.00	0.00	0.00	0.00
AA-rated corporate bonds	Credit spread	0.58	0.71	0.78	0.68
	Default rate	0.02	0.03	0.08	0.34
A-rated corporate bonds	Credit spread	1.04	1.08	0.93	0.65
	Default rate	0.06	0.05	0.11	0.35
BBB-rated corporate bonds	Credit spread	1.83	1.80	1.08	0.67
	Default rate	0.18	0.15	0.25	0.30
Public equity	Dividend yield	2.00	1.93	0.54	0.60
	Capital return	2.04	2.28	9.33	5.49
REITs	Cap rate	1.56	1.21	0.44	1.51
	Capital return	1.43	1.49	9.09	5.58
Crude oil	Total return	2.37	2.48	16.18	11.99
Gold	Total return	1.37	1.45	6.75	5.92

Simulated scenarios have similar means to historical data except that the bond yields are lower, which is likely to be caused by starting the simulation from a low interest rate environment. Therefore, the volatility of bond yields is lower than historical data as well. Volatility of simulated public equity capital returns and REIT capital returns is lower than historical data. Other differences are not material in this example. The differences may be attributed to the following reasons:

- **The DSGE model is a combination of predefined economic system and historical data.** While historical data play an important role in calibration, assumptions about behaviors of consumers, firms, the central bank and government and their interactions may indicate a different pattern than the historical pattern.
- **Less volatile inflation rates, investment activities and imports are generated by the calibrated DSGE model, compared to historical data.**

- **Multifactor models (linear regression) that generate asset returns based on macroeconomic factors do not have a perfect prediction power.** Sampling errors may contribute to the differences as well

With these differences in mind, more tests can be done to assess the reasonableness of generated scenarios and make adjustments if necessary, as discussed in the next section.

Section 5: Model Validation and Adjustment

DSGE models can provide a detailed analysis of an economy considering cause-and-effect relationships and business cycles in a structured way. However, their complexity leads to high model risk. The model requires specification of general preference and decision-making processes of economic agents. A good understanding of the behavior of economic agents is necessary for setting a reasonable model structure. Even with a correct model structure, model calibration is a challenge given so many parameters. Each parameter needs a separate analysis to determine its value or its prior distribution. The mapping from asset returns and yield curves to macroeconomic factors in a DSGE model also requires material modeling efforts.

In addition to model calibration, data collection and selection need to be considered carefully. Many economic variables are reported on only a quarterly, if not yearly, basis. For example, 20 years of data can provide only 80 data points per time series. The data volume may not be considered sufficient for a fully credible DSGE model. Longer-period historical data may cause a different issue. Economic structural changes in the past can make older data irrelevant for future prediction.

Model implementation requires robust optimization algorithms and extensive programming. Therefore, these models have a deep learning curve and are more error prone.

A few tests can be used to mitigate the high model risk of ESGs built on DSGE models:

1. **Sensitivities to economic shocks.** Responses of economic variables to certain economic shocks can be quantified by DSGE models. For example, what is the impact of a labor market shock on next year's real GDP growth rate? By comparing the model result with historical observations or forward-looking expectation, ESGs can be validated against existing economic patterns or views of future economic development. Impulsive response functions in Figure 3 and Figures A.3 to Figure A.7 show the impact of shocking an individual source of risk by one standard deviation at time zero. Table 10 lists the immediate impact on inflation rate, real GDP growth rate, wage and employment given a shock to an individual source of risk modeled in the DSGE model.

Table 10
Immediate Impact on Economic Factors

Source of Risk	Shock Size	Inflation Rate	Real GDP Growth	Wage	Employment
Consumption preference	0.83	0.003	0.05	−0.007	0.01
Investment-specific technology	0.17	0.0002	0.002	0.0004	0.00
Fiscal policy	0.98	0.02	0.15	0.004	0.007
Monetary policy	0.11	−0.02	−0.05	−0.002	−0.02
Transitory technology	1.01	−0.55	0.06	0.03	−0.19
Labor supply	0.72	0.09	−0.005	0.16	−0.002
Exported consumption markup	0.21	−0.0004	−0.003	0.0001	−0.0005
Domestic markup	1.07	0.43	−0.09	−0.04	−0.0004
Imported consumption markup	0.26	0.0003	0.00	0.00	−0.0001
Imported investment markup	0.23	0.00	0.0005	0.01	0.00
Permanent technology	0.48	0.07	0.44	0.45	0.02
Foreign monetary policy	0.11	−0.02	−0.15	−0.0007	−0.003
Foreign inflation	0.64	0.01	−0.04	−0.02	0.04
Foreign output	0.84	−0.09	−0.74	−0.01	−0.05

The shock size is the standard deviation of risk source based on its posterior distribution. The value of the shock size may not be meaningful by itself but needs to be viewed with the corresponding model parameter. Viewing the shock size as one standard deviation is more convenient. In this example, the calibrated model contains some expected relationships. For example, expansionary fiscal policy has a positive impact on real GDP growth. A shortage of labor supply leads to higher wages. When domestic producers have a higher profit margin, the inflation rate will increase. It may slow real GDP growth even though the nominal GDP growth rate is higher. Permanent technology improvement boosts GDP and wages materially. An increase in the foreign interest rate will cause more overseas investment and drags down the domestic economy. An increase in foreign output also means less domestic production and therefore a lower GDP growth rate. More in-depth analysis can be made to evaluate the magnitude of the impact and the speed of reverting back to the equilibrium based on the impulsive response function.

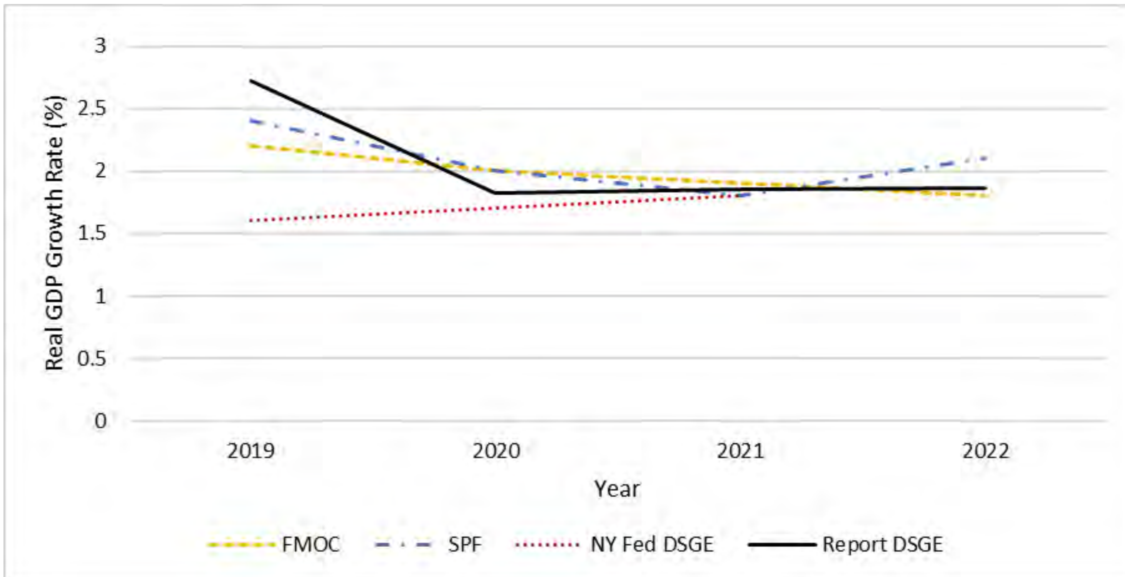
With the impact on macroeconomic factors determined, the impact of systemic risk on asset returns/bond yields can be quantified. Following the example in Table 10, the short-term impact on the 10-year bond yield, credit spread of AA-rated corporate bonds, S&P 500 capital returns and REIT cap rates is listed in Table 11. In general, some impacts are not very intuitive because the size of change is small and only the immediate impact is reflected in the model. For some risk sources such as permanent technology development and foreign GDP growth, the impact is consistent with that on economic factors. This can serve as a quick check to identify potential problems in the model. To understand the long-term impact, the ESG needs to be used to simulate scenarios based on these shocks.

Table 11
Immediate Impact on Asset Returns/Bond Yields

Source of Risk	Shock Size	10-Year Treasury Bond Yield (%)	AA-Rated Bond Credit Spread (%)	S&P 500 Capital Return (%)	REIT Cap Rate (%)
Consumption preference	0.83	0.009	−0.01	0.15	0.14
Investment-specific technology	0.17	0.003	−0.004	0.05	0.05
Fiscal policy	0.98	0.02	−0.02	0.41	0.03
Monetary policy	0.11	−0.01	−0.01	−0.03	−0.03
Transitory technology	1.01	−0.04	−0.02	0.31	−0.30
Labor supply	0.72	−0.01	−0.01	0.18	−0.01
Exported consumption markup	0.21	−0.005	−0.003	0.01	0.01
Domestic markup	1.07	0.03	0.02	−0.40	0.19
Imported consumption markup	0.26	0.01	0.01	−0.16	0.07
Imported investment markup	0.23	0.01	0.002	−0.05	0.02
Permanent technology	0.48	0.07	−0.04	1.02	0.60
Foreign monetary policy	0.11	0.01	0.01	−0.06	0.07
Foreign inflation	0.64	0.02	0.03	−0.24	−0.08
Foreign output	0.84	−0.08	0.05	−1.41	−0.61

2. **Forecast range assessment.** DSGE model results can be compared to forecasts by economists. If the estimates align with the range of expert estimates, that indicates a reasonable model. Figures 6, 7, and 8 compare the median of the simulated real GDP growth rate, inflation rate and Fed rate to some publicly available forecasts.

Figure 6
Real GDP Growth Rate Forecast



Notes:

FMOC: Forecasts by U.S. Federal Market Open Committee: Federal Reserve Bank of St. Louis, “Release Tables: Summary of Economic Projections,” https://fred.stlouisfed.org/release/tables?rid=326&eid=783029&od=#

SPF: Federal Reserve Bank of Philadelphia, “First Quarter 2019 Survey of Professional Forecasters,” https://www.philadelphiafed.org/research-and-data/real-time-center/survey-of-professional-forecasters/2019/survq119

NY Fed DSGE: Federal Reserve Bank of New York, “The New York Fed DSGE Model Forecast—January 2019,” https://libertystreeteconomics.newyorkfed.org/2019/02/the-new-york-fed-dsge-model-forecastjanuary-2019.html

Report DSGE: Example DSGE model in Section 4.

Figure 7
Inflation Rate Forecast

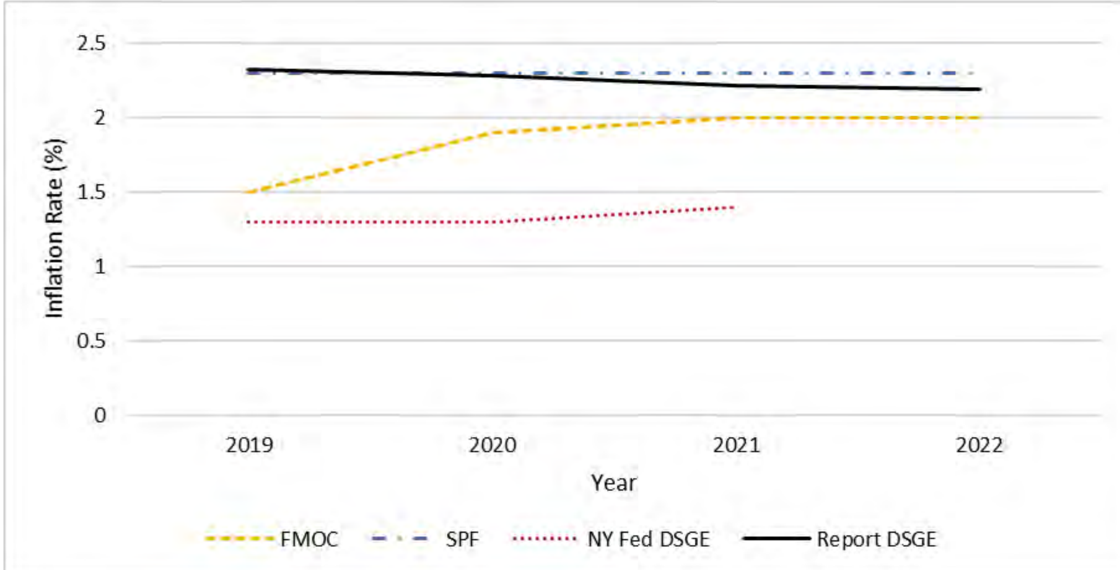
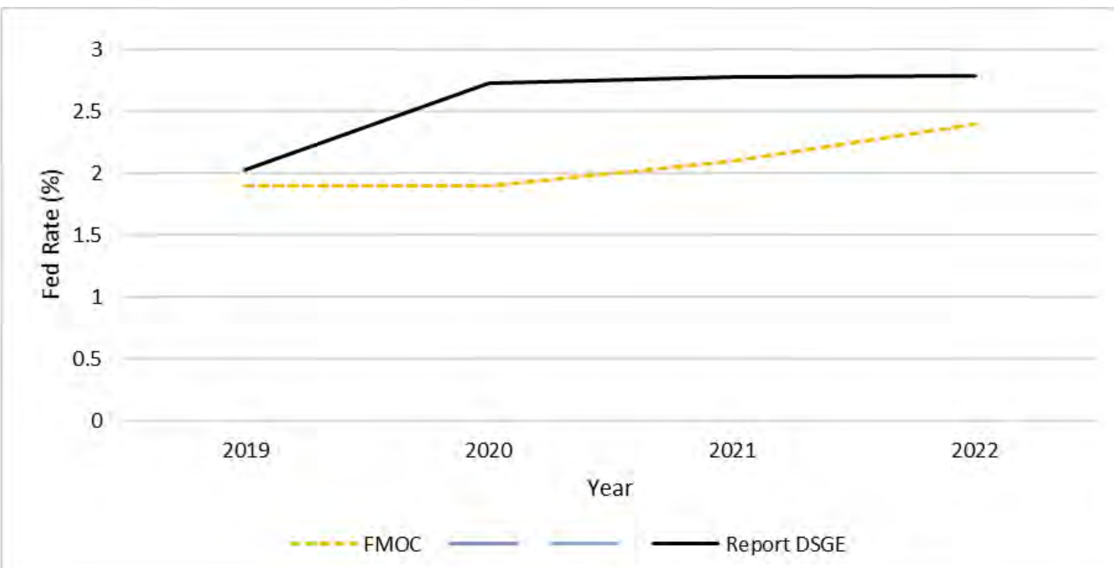


Figure 8
Fed Rate Forecast



The example DSGE model predicts a higher real GDP growth rate in 2019 than other forecasts but converges to the average forecast later. Forecasts of the inflation rate and Fed rate seem more aligned with other forecasts.

Table 12 shows the range of forecasts. The FMOC forecast range is small because its purpose is to get not the full distribution but the likely range. The example DSGE model has a similar real GDP forecast range compared to the New York Fed DSGE and the Survey of Professional Forecasters. The example DSGE model generated a wider range for the inflation rate.

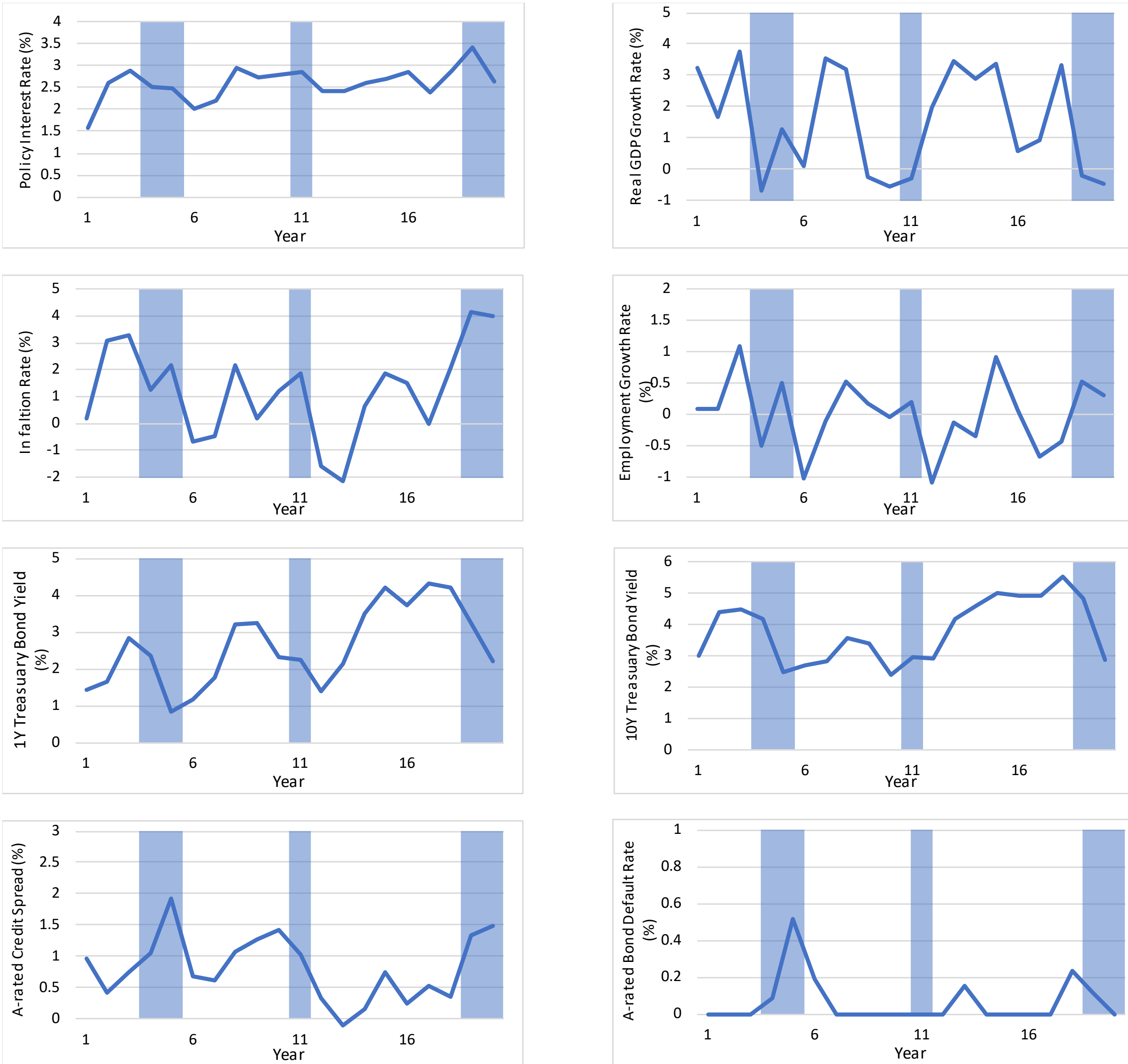
Table 12
Forecast Range Comparison

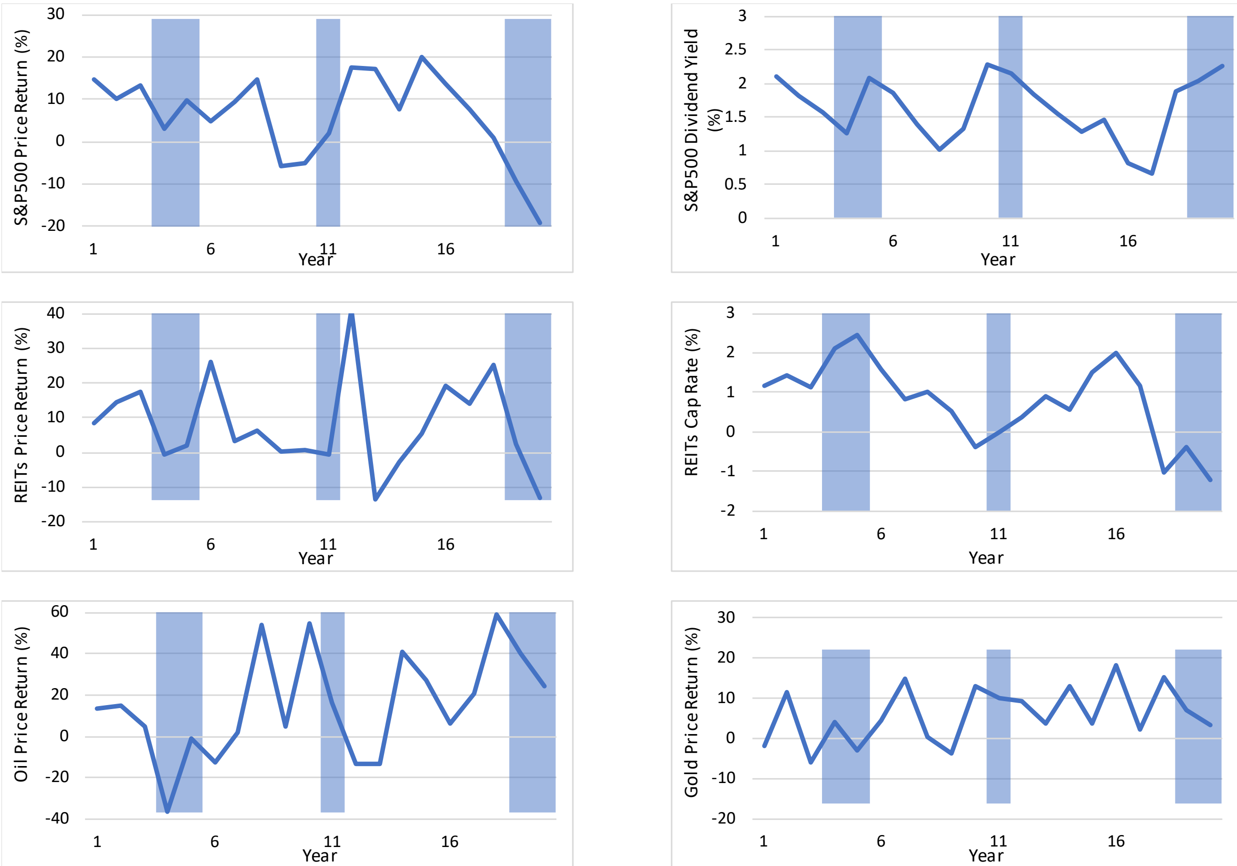
Variable	Year	FMOC		SPF		N.Y. Fed DSGE		Report DSGE	
		Low	High	Low	High	Low	High	Low	High
Real GDP growth rate	2019	2.1	2.4	−3	6	−3	6	−3.0	5.3
	2020	1.7	2.3	−3	6	−3	6	−3.2	4.9
	2021	1.7	2.1	−3	6	−3	6	−3.5	5.0
	2022	1.6	2.1	−3	6	−3	6	−3.4	6.1
Inflation rate	2019	1.4	1.7	0	4	−0.5	3.5	−2.2	4.5
	2020	1.7	2.1	0	4	−0.5	3.5	−2.0	5.0
	2021	1.8	2.3	0	4	−0.5	3.5	−2.6	4.5
	2022	1.8	2.2	0	4	−0.5	3.5	−2.4	4.4
Fed rate	2019	1.6	2.1					0.8	3.3
	2020	1.7	2.4					1.1	4.1
	2021	1.6	2.6					1.4	4.3
	2022	1.4	2.9					1.2	4.3

3. **Scrutiny of individual scenarios.** In a DSGE-based ESG, each scenario is expected to maintain economic patterns such as business cycles and nonlinear relationships among generated outcomes. By viewing sample scenarios to check the existence of these patterns, users can understand how well generated scenarios match model assumptions. Figure 9 shows a sample scenario generated by the example DSGE model. It is quite obvious that economic cycles exist in this scenario. During economic recessions, it shows lower GDP growth rates, lower employment growth, lower bond yields, higher credit spreads and default probabilities, lower equity returns, lower REIT returns, lower oil prices and higher gold prices. Sometimes the policy interest rate drops near the end of the recession period, reflecting that the monetary policy is reacting to economic data. Lower inflation rates are usually observed during and after recessions.

Figure 9

Sample Scenario





4. **Alternative models.** Although building a customized DSGE model gives flexibility and better understanding, existing models and calibrations used by central banks may be relied on to reduce model development efforts. Akinci et al. (2019) published quarterly updates of the New York Fed DSGE model forecast of key economic variables such as the GDP growth rate, inflation rate and natural rate of interest. Comparing the current interest rate with the natural rate of interest can help assess the position of monetary policies. These variables can be used as the macroeconomic factors based on which asset returns and bond yields are further generated using multifactor models. The New York Fed DSGE model is also open sourced using Julia as the programming language. The codes are hosted at: https://github.com/Society-of-actuaries-research-institute/LP152-Macroeconomics-based-ESG.

   Using the Fed model gives more credibility and less effort. It may also take the potential monetary policy decision-making logic into consideration when generating economic scenarios. Central bank models may serve as benchmarks to assess the credibility and reasonability of a DSGE model. Parameter values and patterns can be compared to validate results. However, it has some downsides as well. The Fed DSGE model is used for analyzing monetary policies, and it may omit factors that are important for the capital market. Efforts are still needed to understand the model and codes. To use the New York Fed DSGE model for economic scenario generation, users need to do recalibration themselves because only a summary of the quarterly forecast is published.

Even with these model risk mitigation methods, more efforts are likely to be needed to improve the DSGE models before they can become a serious candidate for economic scenario generation in the insurance domain. If the ESG is used as a tool for some regular tasks such as reserving and asset allocation, the calibrated model can be updated with new data or recalibrated if deemed necessary. Since the existing calibration may have already covered a long history, an additional quarter of data may not have material impact on the calibration, and, therefore, the recalibration frequency does not need to be quarterly but could be semiannual or yearly. However, with new data coming in each quarter, the starting point of the projection will be changed. The data input for the simulation part includes the latest two quarters of economic data and capital market data. They need to be updated before the scenario generation.

In some cases, the DSGE model-based ESG may not generate desired patterns, and it is not obvious how to make adjustments because model calibration is mainly data driven. In general, there are three ways to achieve this goal:

1. **Adjust data inputs.** The desired pattern may be present in recent historical data. During the ESG calibration process, the pattern may be dominated by older historical data. Removing some older data may be helpful. However, there is a risk that the data are not enough to get statistically credible conclusions. In addition, users may have economic views that never happened in history, such as a prolonged period of negative interest rates. In these difficult cases, predicted data inputs that reflect the economic views may be used together with historical data. For example, if a low interest rate and inflation environment is expected in the next three years, three years of future data may be created and used for DSGE calibration. The logic is to add more weight to data that reflect the new economic views so that the model can learn and reflect it.

2. **Adjust model parameters.** For experienced users, adjusting model parameters may be convenient in some cases. The calibrated DSGE model is a big linear system, as described in Appendix A.11. It is the sum of the steady-state value and the impact of any deviation from the steady state caused by state variables. It is possible to adjust the mean and volatility of some variables. For example, if the model generates higher interest rates and inflation rates than expected on average, we can adjust the steady-state value downward to reflect the trend. If higher volatility is observed in the simulated scenarios, we can lower the average deviation from the steady state. However, it is not recommended to change the parameters that control the interaction among modeled macroeconomic factors and state variables because they are complex and not always intuitive. The behaviors of economic agents modeled in the DSGE model are interrelated, and changing one parameter may have unexpected outcomes.

   Adjustment of parameters in multifactor regression models is easier if the regression model is in simple format. For example, the five-year Treasury bond yield is simulated using the following function:

   $$y_t = 0.14 + 0.9y_{t-1} + 0.14\pi_t^c - 0.002\pi_{t-2}^c + 0.07R_{t-2} + e_t$$

   where

   y_t: Five-year Treasury bond yield at time *t*
   π_t^c: Inflation rate based on CPI at time *t*
   R_{t-2}: Interest rate two quarters ago

   The intercept term may be adjusted to reflect a level shift of average bond yields in the future. The coefficient of the previous bond yield may also be adjusted if the autocorrelation is believed to be different from 0.9. Other parameters may be changed, and new variables may be added to the function. The idiosyncratic factor may be adjusted as well to reflect the different level of volatility.

3. **Adjust the generated scenarios.** Generated scenarios may be adjusted directly to reflect different views and to remove unreasonable scenarios. Default rates need to be floored at zero. Credit spreads may also be floored if sovereign risk is believed to be smaller than corporate credit risk. Interest rates and bond yields may be floored at zero or a negative level if a negative interest rate is deemed possible. Modeled variables

may also be capped at the historical maximum, and scenarios may also be adjusted to change the average value or add certain trends. But it is not straightforward to adjust the volatilities.

Although model adjustment is possible to embed a forward-looking view, it is important to understand the impact of the adjustment and avoid adjusting the state variables that are not observable.

Section 6: Further Development

The ESG introduced in this report is a simple version that can be enhanced to better capture the reality of our economies.

1. **The behavior of entrepreneurs and banks is not explicitly modeled.** The inclusion of the interest rate decision-making process and 2008 financial crisis data in the DSGE model implicitly reflects economic crises caused by credit events. For economies with an influential financial sector, modeling credit risk in terms of loans and financial leverage can be beneficial.

2. **It is assumed that economic agents make decisions to maximize their utility or profit.** Although in general and in the long term it is a reasonable assumption, irrational behaviors exist in our economies. Assuming only some economic agents can make decisions introduces some irrationality to the models to some extent. However, these models can be enhanced to reflect irrational behaviors such as herding during economic recessions.

3. **In this research, nonlinear relationships are captured in a discrete way using different sets of correlation assumptions for periods of economic expansion and periods of economic recession.** Given sufficient data, a copula may be used to better describe nonlinear relationships.

4. **Most of the relationships modeled in this research are contemporary.** This is reasonable given a quarterly time step because the reaction time of economic agents is not expected to be much longer than a quarter. For example, a central bank may change its monetary policy within a couple of months after the beginning a financial crisis. However, modeling an intertemporal relationship directly can capture the timing of economic events that lead to either an economic success or an economic crisis. This may better reflect cause-and-effect relationships in the ESG. However, the decision is used made based on data availability.

These potential improvements may address issues that are important for the adoption of more advanced but not necessarily more realistic macroeconomic models in economic scenario generation.

Section 7: Conclusion

DSGE models can be used to model a complex economic system with a predefined high-level economic structure and supporting historical data. Embedding DSGE models in economic scenario generation provides an opportunity to model cause-and-effect relationships in the economic environment explicitly. This can help maintain the “reality” of individual scenarios in addition to the reality of their aggregate impact. Economic agents’ behaviors, high tail risk correlation and economic cyclical patterns are something that can be observed in individual scenarios as well. This may help explain individual real-world scenarios, especially when questioned for their reasonableness for investment decisions.

Although potential benefits of using complicated macroeconomic models in ESGs are promising, they come with risks. These models require more data than simpler macroeconomic models, such as a VAR model. Being able to define the interdependency of economic agents and their behaviors means microeconomic analysis is needed as well. Efforts to learn and calibrate the models are not trivial. Model risk exists not only for DSGE models, but also for the multifactor regression models that generate asset returns and bond yields based on economic factors. Tests can be done to assess the reasonableness of generated scenarios.

Although challenges remain for using DSGE models in economic scenario generation, their adoption by economic policy-makers, their flexibility to include customized views of the economy and a more holistic view in each individual scenario may attract more users. This report provides the basics to understand, build and test such an advanced and complex ESG.

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Appendix A: DSGE Model Details

This appendix provides all the model specifications and step-by-step derivation of the DSGE model introduced in Section 2. It covers the behavior of each modeled economic agent and the model calibration process. The author is not aware of any other sources that provide such a detailed derivation. It is intended for educational purposes.

A.1 Final Goods Producers

Final goods producers change immediate goods into final goods that can be consumption or investment. The production function assumes constant elasticity of substitution (CES):

$$Y_t = \left[\int_0^1 {Y_{i,t}}^{\frac{1}{\lambda_{d,t}}} di\right]^{\lambda_{d,t}} \qquad (1)$$

where

$\lambda_{d,t}$: time-varying markup for domestic goods

$$\lambda_{d,t} = \left(1-\rho_{\lambda_d}\right)\lambda_d + \rho_{\lambda_d}\lambda_{d,t-1} + \epsilon_t^{\lambda_d} \qquad (2)$$

where

$$\epsilon_t^{\lambda_d} \sim N\left(0, \sigma^{\lambda_d}\right)$$

Each individual producer i wants to maximize profit:

$$\mathcal{L} = \max_{Y_{i,t}} \left\{P_t Y_t - \int_0^1 P_{i,t} Y_{i,t} di\right\}$$

where

$Y_{i,t}$: amount of immediate goods by producer i

$P_{i,t}$: price of immediate goods by producer i

P_t: price of the final goods

Y_t: demand function for intermediate goods

Substitute Y_t with (1) =>

$$\mathcal{L} = \max_{Y_{i,t}} P_t \left[\int_0^1 {Y_{i,t}}^{\frac{1}{\lambda_{d,t}}} di\right]^{\lambda_{d,t}} - \int_0^1 P_{i,t} Y_{i,t} di$$

First-Order Condition (F.O.C.):

$$\frac{\partial \mathcal{L}}{\partial Y_{i,t}} = P_t \lambda_{d,t} \left[\int_0^1 {Y_{i,t}}^{\frac{1}{\lambda_{d,t}}} di\right]^{\lambda_{d,t}-1} \frac{1}{\lambda_{d,t}} {Y_{i,t}}^{\frac{1}{\lambda_{d,t}}-1} - P_{i,t} = 0$$

$$\Rightarrow \qquad P_t\left[\int_0^1 Y_{i,t}^{\frac{1}{\lambda_{d,t}}} di\right]^{\lambda_{d,t}-1} Y_{i,t}^{\frac{1-\lambda_{d,t}}{\lambda_{d,t}}} = P_{i,t}$$

$$\Rightarrow \qquad \frac{P_{i,t}}{P_{j,t}} = \frac{Y_{i,t}^{\frac{1-\lambda_{d,t}}{\lambda_{d,t}}}}{Y_{j,t}^{\frac{1-\lambda_{d,t}}{\lambda_{d,t}}}}$$

$$\Rightarrow \qquad P_{j,t} = P_{i,t} Y_{j,t}^{\frac{1-\lambda_{d,t}}{\lambda_{d,t}}} Y_{i,t}^{\frac{-1+\lambda_{d,t}}{\lambda_{d,t}}}$$

$$\Rightarrow \qquad P_{j,t} Y_{j,t} = P_{i,t} Y_{j,t}^{\frac{1}{\lambda_{d,t}}} Y_{i,t}^{\frac{-1+\lambda_{d,t}}{\lambda_{d,t}}}$$

$$\Rightarrow \qquad \int_0^1 P_{j,t} Y_{j,t} dj = P_{i,t} Y_{i,t}^{\frac{-1+\lambda_{d,t}}{\lambda_{d,t}}} \int_0^1 Y_{j,t}^{\frac{1}{\lambda_{d,t}}} dj = P_{i,t} Y_{i,t}^{\frac{-1+\lambda_{d,t}}{\lambda_{d,t}}} Y_t^{\frac{1}{\lambda_{d,t}}}$$

Zero profit condition =>

$$\int_0^1 P_{j,t} Y_{j,t} dj = P_t Y_t = P_{i,t} Y_{i,t}^{\frac{-1+\lambda_{d,t}}{\lambda_{d,t}}} Y_t^{\frac{1}{\lambda_{d,t}}}$$

$$\Rightarrow \qquad P_t = \frac{P_{i,t} Y_{i,t}^{\frac{-1+\lambda_{d,t}}{\lambda_{d,t}}}}{Y_t^{\frac{-1+\lambda_{d,t}}{\lambda_{d,t}}}}$$

$$\Rightarrow \qquad Y_{i,t} = \left(\frac{P_t}{P_{i,t}}\right)^{\frac{\lambda_{d,t}}{\lambda_{d,t}-1}} Y_t \qquad (3)$$

$$P_t Y_t = \int_0^1 P_{i,t} Y_{i,t} di$$

Substitute $Y_{i,t}$ with (3):

$$P_t Y_t = \int_0^1 P_{i,t} \left(\frac{P_t}{P_{i,t}}\right)^{\frac{\lambda_{d,t}}{\lambda_{d,t}-1}} Y_t di$$

$$\Rightarrow \qquad P_t = P_t^{\frac{\lambda_{d,t}}{\lambda_{d,t}-1}} \int_0^1 P_{i,t}^{\frac{1}{1-\lambda_{d,t}}} di$$

$$\Rightarrow \qquad P_t^{\frac{-1}{\lambda_{d,t}-1}} = \int_0^1 P_{i,t}^{\frac{1}{1-\lambda_{d,t}}} di$$

$$\Rightarrow \qquad P_t = \left[\int_0^1 P_{i,t}{}^{\frac{1}{1-\lambda_{d,t}}} di\right]^{1-\lambda_{d,t}} \qquad (4)$$

A.2 Intermediate Goods Producers

A.2.1 Production

The intermediate goods market is assumed to be monopolistic competitive such that many producers sell products that are differentiated from one another and therefore are not perfect substitutes. Individual intermediate goods producer i has the following production function:

$$Y_{i,t} = \varepsilon_t \left(K_{i,t}^s\right)^\alpha \left(z_t H_{i,t}\right)^{1-\alpha} - z_t \phi \qquad (5)$$

where

$Y_{i,t}$: inputs for final goods producer

ε_t: transitory technology shock

$K_{i,t}^s$: capital services

z_t: permanent technology shock

$H_{i,t}$: homogenized labor input

ϕ: fixed cost

Permanent technology shock z_t follows the pattern in (6):

$$\frac{z_t}{z_{t-1}} = \mu_t^z = \left(1 - \rho_{\mu^z}\right)\mu^z + \rho_{\mu^z}\mu_{t-1}^z + \epsilon_t^{\mu^z} \qquad (6)$$

Transitory technology shock follows the pattern in (7):

$\hat{\varepsilon}_t = \rho_\varepsilon \hat{\varepsilon}_{t-1} + \epsilon_t^\varepsilon$ (7)

where

$\hat{\varepsilon}_t = \frac{\varepsilon_t - 1}{1}$

$E(\varepsilon_t) = 1$

$$K_t^s = u_t K_t \qquad (8)$$

where

u_t: capital utilization rate

K_t: available capital

A.2.2 Detrending

If the growth rate of permanent technology shock $\mu_t^z > 1$, all the economic variables will be nonstationary, which can lead to failed convergence to economic equilibrium. Some variables are linked to a price index. The nonstationary price index can also cause failed convergence. Therefore, economic variables need to be detrended to be stationary. For example,

$$k_{t+1} = \frac{K_{t+1}}{z_t}$$

$$w_t = \frac{W_t}{z_t P_t^d}$$

where

P_t^d is domestic price level, a nonstationary stochastic trend

W_t: labor cost

Variables will be detrend before a function is loglinearized.

Loglinearize (8):

$$\Rightarrow \quad \frac{K_t^s}{z_{t-1}} = u_t \frac{K_t}{z_{t-1}}$$

$$\Rightarrow \quad k_t^s = u_t k_t$$

$$\Rightarrow \quad \hat{k}_t^s = \hat{u}_t + \hat{k}_t \qquad (L.1)$$

Loglinearize (1):

$$Y_t = \left[\int_0^1 Y_{i,t}{}^{\frac{1}{\lambda_{d,t}}} di\right]^{\lambda_{d,t}} \qquad (1)$$

$$\Rightarrow \quad Y_t^{\frac{1}{\lambda_{d,t}}} = \int_0^1 Y_{i,t}{}^{\frac{1}{\lambda_{d,t}}} di$$

$$\Rightarrow \quad y_t^{\frac{1}{\lambda_{d,t}}} = \int_0^1 y_{i,t}{}^{\frac{1}{\lambda_{d,t}}} di$$

where

$y_t = \frac{Y_t}{z_t}$

$y_{i,t} = \frac{Y_{i,t}}{z_t}$

$$Y_{i,t} = \varepsilon_t \left(K_{i,t}^s\right)^{\alpha} \left(z_t H_{i,t}\right)^{1-\alpha} - z_t \phi \qquad (5)$$

$$\Rightarrow \quad y_t = (\mu_t^z)^{-\alpha} \varepsilon_t (k_t^s)^{\alpha} (H_t)^{1-\alpha} - \phi$$

$$\Rightarrow \qquad y_t^{\frac{1}{\lambda_{d,t}}} = \int_0^1 \left[\varepsilon_t \left(k_{i,t}^s\right)^\alpha (\mu_t^z)^{-\alpha} \left(H_{i,t}\right)^{1-\alpha} - \phi\right]^{\frac{1}{\lambda_{d,t}}} di$$

Left-Hand Side (L.H.S.):

$$\bar{y}^{\frac{1}{\lambda_d}} + \frac{1}{\lambda_d} \bar{y}^{\frac{1}{\lambda_d}-1}(y_t - \bar{y}) + \bar{y}^{\frac{1}{\lambda_d}} ln\bar{y} \left(-\frac{1}{{\lambda_d}^2}\right)(\lambda_{d,t} - \lambda_d)$$

Right-Hand Side (R.H.S.):

$$\int_0^1 \bar{y}_i^{\frac{1}{\lambda_d}} di + \int_0^1 \bar{y}_i^{\frac{1}{\lambda_d}} di \frac{1}{\lambda_d} \frac{\bar{y}_i + \phi}{\bar{y}_i} \left[\hat{\varepsilon}_t + \alpha\left(\hat{k}_t^s - \hat{\mu}_t^z\right) + (1-\alpha)\widehat{H}_t\right] + \int_0^1 \bar{y}_i^{\frac{1}{\lambda_d}} di \, ln\bar{y}_i \left(-\frac{1}{{\lambda_d}^2}\right)(\lambda_{d,t} - \lambda_d)$$

L.H.S. = R.H.S. =>

$$\hat{y}_t = \frac{\bar{y}_i + \phi}{\bar{y}_i}\left[\hat{\varepsilon}_t + \alpha\left(\hat{k}_t^s - \hat{\mu}_t^z\right) + (1-\alpha)\widehat{H}_t\right] + (ln\bar{y} - ln\bar{y}_i)\hat{\lambda}_{d,t}$$

In steady state, the real profit is zero:

$$\Pi^R = \lambda_d \bar{y} - \bar{y} - \phi = (\lambda_d - 1)\bar{y} - \phi = 0$$

$$\Rightarrow \qquad \phi = (\lambda_d - 1)\bar{y}$$

$$\Rightarrow \qquad \hat{y}_t = \lambda_d\left[\hat{\varepsilon}_t + \alpha\left(\hat{k}_t^s - \hat{\mu}_t^z\right) + (1-\alpha)\widehat{H}_t\right] \qquad (L.2)$$

A.2.3 Intermediate Goods Producer Cost Minimization

$$\mathcal{L} = \min_{K_{i,t}^s, H_{i,t}} W_t H_{i,t} + R_t^k K_{i,t}^s + \lambda_t P_{i,t}^d \left[Y_{i,t} - \varepsilon_t \left(K_{i,t}^s\right)^\alpha \left(z_t H_{i,t}\right)^{1-\alpha} + z_t \phi\right] \qquad (9)$$

where

R_t^k: rent cost for capital services $K_{i,t}^s$

F.O.C.:

$$\frac{\partial \mathcal{L}}{\partial K_{i,t}^s} = R_t^k + \lambda_t P_{i,t}^d (-\varepsilon_t) \alpha \left(K_{i,t}^s\right)^{\alpha-1} \left(z_t H_{i,t}\right)^{1-\alpha} = 0$$

$$\Rightarrow \qquad R_t^k = \alpha \lambda_t P_{i,t}^d {z_t}^{1-\alpha} \varepsilon_t \left(K_{i,t}^s\right)^{\alpha-1} {H_{i,t}}^{1-\alpha} = 0 \qquad (10)$$

$$\frac{\partial \mathcal{L}}{\partial H_{i,t}} = W_t + \lambda_t P_{i,t}^d (-\varepsilon_t) \left(K_{i,t}^s\right)^\alpha (1-\alpha) \left(z_t H_{i,t}\right)^{-\alpha} z_t = 0$$

$$\Rightarrow \qquad W_t = (1-\alpha) \lambda_t P_{i,t}^d {z_t}^{1-\alpha} \varepsilon_t \left(K_{i,t}^s\right)^\alpha {H_{i,t}}^{-\alpha} \qquad (11)$$

Marginal returns of capital/labor = cost of compensation

(10)/(11) =>

$$\frac{R_t^k}{W_t} = \frac{\alpha}{1-\alpha}\frac{H_{i,t}}{K_{i,t}^s}$$

$$\Rightarrow \qquad \frac{R_t^k}{P_t^d} = \frac{\alpha}{1-\alpha}\frac{W_t}{z_t P_t^d}\frac{H_{i,t}}{\frac{K_{i,t}^s}{z_{t-1}}\frac{z_{t-1}}{z_t}}$$

$$\Rightarrow \qquad r_t^k = \frac{\alpha}{1-\alpha} w_t \frac{H_{i,t}}{k_{i,t}^s}\mu_t^z$$

$$\Rightarrow \qquad r_t^k = \frac{\alpha}{1-\alpha} w_t \mu_t^z \frac{H_t}{k_t^s} \qquad (12)$$

Loglinearize (12):

$$\bar{r}e^{\hat{r}_t^k} = \frac{\alpha}{1-\alpha}\bar{w}e^{\hat{w}_t}\overline{\mu^z}e^{\hat{\mu}_t^z}\left(\frac{\bar{H}}{\bar{k}}\right)e^{\hat{H}_t - \hat{k}_t^s}$$

Steady state:

$$\bar{r} = \frac{\alpha}{1-\alpha}\bar{w}\overline{\mu^z}\left(\frac{\bar{H}}{\bar{k}}\right)$$

$$\Rightarrow \qquad \hat{r}_t^k = \hat{w}_t + \hat{\mu}_t^z + \hat{H}_t - \hat{k}_t^s \qquad (L.3)$$

Nominal marginal cost $MC_t = \lambda_t P_{i,t}^d$

Substitute $\lambda_t P_{i,t}^d$ with (10):

$$\Rightarrow \qquad MC_t = \lambda_t P_{i,t}^d = \frac{R_t^k}{\alpha} {z_t}^{\alpha-1}\frac{1}{\varepsilon_t}\left(K_{i,t}^s\right)^{1-\alpha} {H_{i,t}}^{\alpha-1}$$

$$\Rightarrow \qquad mc_t = \frac{MC_t}{P_{i,t}^d} = \lambda_t = \frac{R_t^k}{P_t^d}\frac{1}{\alpha}{z_t}^{\alpha-1}\frac{1}{\varepsilon_t}(K_t^s)^{1-\alpha}{H_t}^{\alpha-1}$$

$$\Rightarrow \qquad mc_t = r_t^k \frac{1}{\alpha}{z_t}^{\alpha-1}\frac{1}{\varepsilon_t}\left(\frac{H_t}{K_t^s}\right)^{\alpha-1}$$

$$\Rightarrow \qquad mc_t = (r_t^k)^{\alpha}(r_t^k)^{1-\alpha}\frac{1}{\alpha}{z_t}^{\alpha-1}\frac{1}{\varepsilon_t}\left(\frac{H_t}{K_t^s}\right)^{\alpha-1}$$

Substitute $(r_t^k)^{1-\alpha}$ with (12):

$$\Rightarrow \qquad mc_t = (r_t^k)^{\alpha}\frac{1}{\varepsilon_t}\left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha}(w_t)^{1-\alpha}(\mu_t^z)^{1-\alpha}\left(\frac{H_t}{k_t^s}\right)^{1-\alpha}\frac{1}{\alpha}{z_t}^{\alpha-1}\left(\frac{H_t}{K_t^s}\right)^{\alpha-1}$$

$$\Rightarrow \qquad mc_t = (r_t^k)^{\alpha}\frac{1}{\varepsilon_t}\left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha}(w_t)^{1-\alpha}(\mu_t^z)^{1-\alpha}\left(\frac{H_t}{k_t^s}\right)^{1-\alpha}\frac{1}{\alpha}{z_t}^{\alpha-1}\left(\frac{H_t}{k_t^s z_{t-1}}\right)^{\alpha-1}$$

$$\Rightarrow \qquad mc_t = \left(\frac{1}{1-\alpha}\right)^{1-\alpha}\left(\frac{1}{\alpha}\right)^{\alpha}\frac{1}{\varepsilon_t}(r_t^k)^{\alpha}(w_t)^{1-\alpha} \qquad (13)$$

Loglinearize (13):

$$\overline{mc}^d e^{\widehat{mc}_t^d} = \left(\frac{1}{1-\alpha}\right)^{1-\alpha}\left(\frac{1}{\alpha}\right)^{\alpha}\frac{1}{\bar{\varepsilon}}e^{-\hat{\varepsilon}_t}\left(\bar{\gamma}^k e^{\hat{r}_t^k}\right)^{\alpha}\left(\bar{w}e^{\widehat{w}_t}\right)^{1-\alpha}$$

Steady state:

$$\overline{mc}^d = \left(\frac{1}{1-\alpha}\right)^{1-\alpha}\left(\frac{1}{\alpha}\right)^{\alpha}\frac{1}{\bar{\varepsilon}}\left(\bar{\gamma}^k\right)^{\alpha}(\bar{w})^{1-\alpha}$$

$$e^{\widehat{mc}_t^d} = e^{-\hat{\varepsilon}_t+\alpha\hat{r}_t^k+(1-\alpha)\widehat{w}_t}$$

$$\Rightarrow \qquad \widehat{mc}_t^d = \alpha\hat{r}_t^k + (1-\alpha)\widehat{w}_t - \hat{\varepsilon}_t \qquad (L.4)$$

A.2.4 Domestic Price Setting

It is assumed that intermediate goods producers set goods prices in a staggered manner as suggested by Calvo (1983). Not all firms can set prices. A firm's probability of price adjustment is $1-\theta_d$. The probability of being a price follower is θ_d. Price followers will set the next-period price as follows:

$$P_{t+1}^d = (\pi_t^d)^{\chi_d}(\bar{\pi}_{t+1}^c)^{1-\chi_d}P_t^d \qquad (14)$$

where

$\pi_t^d = \frac{P_t^d}{P_{t-1}^d}$ is the inflation rate

$\bar{\pi}_{t+1}^c$: current inflation target

χ_d: degree of indexation to past inflation

For price setters, the optimal price $\tilde{P}_t$ is determined based on the following intertemporal price optimization problem:

$$\max_{\tilde{P}_t} \mathbb{E}_t \sum_{s=0}^{\infty}(\beta\theta_d)^s v_{t+s}\left\{\left[\left(\prod_{k=1}^{s}\pi_{t+k-1}\right)^{\chi_d}\left(\prod_{k=1}^{s}\bar{\pi}_{t+k}^c\right)^{1-\chi_d}\tilde{P}_t\right]Y_{i,t+s} - MC_{i,t+s}\left(Y_{i,t+s}+z_{t+s}\phi\right)\right\} \quad (15)$$

According to (4)

$$P_t = \left[\theta_d\left((\pi_{t-1})^{\chi_d}(\bar{\pi}_t^c)^{1-\chi_d}P_{t-1}\right)^{\frac{1}{1-\lambda_{d,t}}} + (1-\theta_d)\left(\tilde{P}_t\right)^{\frac{1}{1-\lambda_{d,t}}}\right]^{1-\lambda_{d,t}} \qquad (16)$$

where

$\pi_t = \frac{P_t}{P_{t-1}}$

$$\Rightarrow \qquad P_t^{\frac{1}{1-\lambda_{d,t}}} = \theta_d\left((\pi_{t-1})^{\chi_d}(\bar{\pi}_t^c)^{1-\chi_d}P_{t-1}\right)^{\frac{1}{1-\lambda_{d,t}}} + (1-\theta_d)\left(\tilde{P}_t\right)^{\frac{1}{1-\lambda_{d,t}}}$$

$$\Rightarrow \qquad \left(\frac{P_t}{P_{t-1}}\right)^{\frac{1}{1-\lambda_{d,t}}} = \theta_d\left((\pi_{t-1})^{\chi_d}(\bar{\pi}_t^c)^{1-\chi_d}P_{t-1}\right)^{\frac{1}{1-\lambda_{d,t}}} + (1-\theta_d)\left(\frac{\tilde{P}_t}{P_{t-1}}\right)^{\frac{1}{1-\lambda_{d,t}}}$$

Loglinearize (16):

$$(\pi_t)^{\frac{1}{1-\lambda_{d,t}}} = \theta_d\left((\pi_{t-1})^{\chi_d}(\bar{\pi}_t^c)^{1-\chi_d}\right)^{\frac{1}{1-\lambda_{d,t}}} + (1-\theta_d)\left(\frac{\tilde{P}_t}{P_{t-1}}\right)^{\frac{1}{1-\lambda_{d,t}}}$$

Steady state:

$\pi_t = \pi_{t-1} = \bar{\pi}_t^c = \pi = 1; \quad \tilde{P}_t = P_{t-1} = P; \quad \lambda_{d,t} = \lambda_d$

$$\begin{aligned}\Rightarrow \qquad \pi + \frac{1}{1-\lambda_d}(\pi)^{\frac{1}{1-\lambda_d}-1}\hat{\pi}_t &= \left[\theta_d + (1-\theta_d)\left(\frac{P}{P}\right)^{\frac{1}{1-\lambda_d}}\right] + (1-\theta_d)\left(\frac{1}{P}\right)^{\frac{1}{1-\lambda_d}}\frac{1}{1-\lambda_d}(P)^{\frac{1}{1-\lambda_d}-1}\left(\tilde{P}_t - P\right) \\ &+ (1-\theta_d)(P)^{\frac{1}{1-\lambda_d}}\left(-\frac{1}{1-\lambda_d}\right)(P)^{-\frac{1}{1-\lambda_d}-1}(P_{t-1}-P) \\ &+ \theta_d(\pi)^{\frac{1-\chi_d}{1-\lambda_d}}\left(\frac{\chi_d}{1-\lambda_d}\right)(\pi)^{\frac{\chi_d}{1-\lambda_d}-1}(\pi_{t-1}-\pi) + \theta_d\left(\frac{1-\chi_d}{1-\lambda_d}\right)(\pi)^{\frac{1-\chi_d}{1-\lambda_d}-1}(\pi)^{\frac{\chi_d}{1-\lambda_d}}(\bar{\pi}_t^c - \bar{\pi}^c)\end{aligned}$$

$$\Rightarrow \qquad \frac{1}{1-\lambda_d}\hat{\pi}_t = \frac{1-\theta_d}{1-\lambda_d}(\tilde{p}_t - p) - \left(\frac{1-\theta_d}{1-\lambda_d}\right)(p_{t-1}-p) + \left(\frac{\theta_d\chi_d}{1-\lambda_d}\right)\hat{\pi}_{t-1} + \theta_d\left(\frac{1-\chi_d}{1-\lambda_d}\right)\hat{\bar{\pi}}_t^c$$

$$\Rightarrow \qquad \hat{\pi}_t = (1-\theta_d)(\tilde{p}_t - p_{t-1}) + \theta_d\chi_d\hat{\pi}_{t-1} + \theta_d(1-\chi_d)\hat{\bar{\pi}}_t^c \qquad (L.5)$$

where

$\hat{\bar{\pi}}_t^c = \rho_\pi\hat{\bar{\pi}}_{t-1}^c + \varepsilon_t^{\bar{\pi}^c}$

To derive the optimal price $\tilde{P}_t$, substitute $Y_{i,t+s}$ with (3) in (15):

$$\max_{\tilde{P}_t}\mathbb{E}_t\sum_{s=0}^{\infty}(\beta\theta_d)^s v_{t+s}\left\{\left[\left(\prod_{k=1}^{s}\pi_{t+k-1}\right)^{\chi_d}\left(\prod_{k=1}^{s}\bar{\pi}_{t+k}^c\right)^{1-\chi_d}\tilde{P}_t - MC_{i,t+s}\right]\left(\frac{P_{t+s}}{P_{i,t+s}}\right)^{\frac{\lambda_{d,t}}{\lambda_{d,t}-1}}Y_{t+s} - MC_{i,t+s}z_{t+s}\phi\right\}$$

$P_{i,t+s} = (\pi_t\pi_{t+1}\dots\pi_{t+s-1})^{\chi_d}(\bar{\pi}_{t+1}^c\bar{\pi}_{t+2}^c\dots\bar{\pi}_{t+s}^c)^{1-\chi_d}\tilde{P}_t$

$$\begin{aligned}\Rightarrow \qquad \mathcal{L} = \mathbb{E}_t\sum_{s=0}^{\infty}(\beta\theta_d)^s v_{t+s}\Bigg\{&\left[\frac{(\prod_{k=1}^{s}\pi_{t+k-1})^{\chi_d}(\prod_{k=1}^{s}\bar{\pi}_{t+k}^c)^{1-\chi_d}\tilde{P}_t}{P_{t+s}}\right]^{1-\frac{\lambda_{d,t}}{\lambda_{d,t}-1}}P_{t+s} \\ &- MC_{i,t+s}\left[\frac{(\prod_{k=1}^{s}\pi_{t+k-1})^{\chi_d}(\prod_{k=1}^{s}\bar{\pi}_{t+k}^c)^{1-\chi_d}\tilde{P}_t}{P_{t+s}}\right]^{-\frac{\lambda_{d,t}}{\lambda_{d,t}-1}}\Bigg\}Y_{t+s} - MC_{i,t+s}z_{t+s}\phi\end{aligned}$$

F.O.C.:

$$\frac{\partial \mathcal{L}}{\partial \tilde{P}_t}$$

$$= \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_d)^s v_{t+s} \Bigg\{ \Bigg(1$$

$$- \frac{\lambda_{d,t}}{\lambda_{d,t}-1} \Bigg) \left[\frac{(\prod_{k=1}^{s} \pi_{t+k-1})^{\chi_d} (\prod_{k=1}^{s} \bar{\pi}_{t+k}^{c})^{1-\chi_d} \tilde{P}_t}{P_{t+s}} \right]^{-\frac{\lambda_{d,t}}{\lambda_{d,t}-1}} \frac{(\prod_{k=1}^{s} \pi_{t+k-1})^{\chi_d} (\prod_{k=1}^{s} \bar{\pi}_{t+k}^{c})^{1-\chi_d}}{P_{t+s}} P_{t+s}$$

$$- MC_{i,t+s} \left(-\frac{\lambda_{d,t}}{\lambda_{d,t}-1} \right) \left[\frac{(\prod_{k=1}^{s} \pi_{t+k-1})^{\chi_d} (\prod_{k=1}^{s} \bar{\pi}_{t+k}^{c})^{1-\chi_d} \tilde{P}_t}{P_{t+s}} \right]^{-\frac{\lambda_{d,t}}{\lambda_{d,t}-1}-1} \frac{(\prod_{k=1}^{s} \pi_{t+k-1})^{\chi_d} (\prod_{k=1}^{s} \bar{\pi}_{t+k}^{c})^{1-\chi_d}}{P_{t+s}} \Bigg\} Y_{t+s}$$

$$= 0$$

$$\Rightarrow \quad \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_d)^s v_{t+s} \Bigg\{ \left[\frac{(\prod_{k=1}^{s} \pi_{t+k-1})^{\chi_d} (\prod_{k=1}^{s} \bar{\pi}_{t+k}^{c})^{1-\chi_d}}{P_{t+s}} \right]^{-\frac{\lambda_{d,t}}{\lambda_{d,t}-1}+1} \tilde{P}_t^{-\frac{\lambda_{d,t}}{\lambda_{d,t}-1}} Y_{t+s} P_{t+s}$$

$$- \lambda_{d,t} MC_{i,t+s} \left[\frac{(\prod_{k=1}^{s} \pi_{t+k-1})^{\chi_d} (\prod_{k=1}^{s} \bar{\pi}_{t+k}^{c})^{1-\chi_d}}{P_{t+s}} \right]^{-\frac{\lambda_{d,t}}{\lambda_{d,t}-1}} \tilde{P}_t^{-\frac{\lambda_{d,t}}{\lambda_{d,t}-1}-1} Y_{t+s} \Bigg\} = 0$$

$$\Rightarrow \quad \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_d)^s v_{t+s} \left[\frac{\left(\frac{P_{t+s-1}}{P_{t-1}}\right)^{\chi_d} (\prod_{k=1}^{s} \bar{\pi}_{t+k}^{c})^{1-\chi_d}}{\frac{P_{t+s}}{P_t}} \right]^{-\frac{\lambda_{d,t}}{\lambda_{d,t}-1}+1} \frac{\tilde{P}_t}{P_t} Y_{t+s} P_{t+s} \qquad (L.H.S.)$$

$$= \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_d)^s v_{t+s} \lambda_{d,t} MC_{i,t+s} \left[\frac{\left(\frac{P_{t+s-1}}{P_{t-1}}\right)^{\chi_d} (\prod_{k=1}^{s} \bar{\pi}_{t+k}^{c})^{1-\chi_d}}{\frac{P_{t+s}}{P_t}} \right]^{-\frac{\lambda_{d,t}}{\lambda_{d,t}-1}} Y_{t+s} \qquad (R.H.S.)$$

$\theta_d = 0 \quad \Rightarrow \quad P = \lambda_d MC^n = \lambda_d MC^r P$

where

MC^n: nominal marginal cost

MC^r: real marginal cost

$$MC^r = \frac{1}{\lambda_d}$$

$$MC^n = \frac{P}{\lambda_d}$$

$$YP = \lambda_d MC^n Y$$

Loglinearize (15):

L.H.S.:

$$\mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_d)^s YP + \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_d)^s P(Y_{t+s} - Y) + \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_d)^s Y(\tilde{P}_t - P)$$

$$+\mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_d)^s \left(\frac{1}{P}\right)^{\chi_d\left(-\frac{\lambda_d}{\lambda_d-1}+1\right)} YP\chi_d \left(-\frac{\lambda_d}{\lambda_d - 1} + 1\right) P^{\chi_d\left(-\frac{\lambda_d}{\lambda_d-1}+1\right)} \left(\frac{1}{P}\right)(P_{t+s-1} - P)$$

$$+\mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_d)^s P^{-\frac{\lambda_d}{\lambda_d-1}+1} Y \left(\frac{\lambda_d}{\lambda_d - 1}\right) P^{\frac{\lambda_d}{\lambda_d-1}-1}(P_{t+s} - P)$$

$$+\mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_d)^s P^{\chi_d\left(-\frac{\lambda_d}{\lambda_d-1}+1\right)} YP(-\chi_d) \left(-\frac{\lambda_d}{\lambda_d - 1} + 1\right) P^{(-\chi_d)\left(-\frac{\lambda_d}{\lambda_d-1}+1\right)} \left(\frac{1}{P}\right)(P_{t-1} - P)$$

$$+\mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_d)^s \left(\frac{1}{P}\right)^{-\frac{\lambda_d}{\lambda_d-1}+1} YP^2 \left(-\frac{\lambda_d}{\lambda_d - 1}\right) P^{\frac{\lambda_d}{\lambda_d-1}-1}(P_t - P)$$

$$+\mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_d)^s YP(1-\chi_d) \left(-\frac{\lambda_d}{\lambda_d - 1} + 1\right) \sum_{j=1}^{s} \hat{\hat{\pi}}_{t+j}^c$$

$$+\mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_d)^s YP(ln1) \left(-\frac{1}{(1-\lambda_d)^2}\right)(-1)\left(\lambda_{d,t} - \lambda_d\right)$$

$$= \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_d)^s YP \left\{ 1 + (y_{t+s} - y) + (\tilde{p}_t - p) + \left(-\frac{\chi_d}{\lambda_d - 1}\right)(p_{t+s-1} - p) + \left(\frac{\lambda_d}{\lambda_d - 1}\right)(p_{t+s} - p) \right.$$
$$\left. + \left(\frac{\chi_d}{\lambda_d - 1}\right)(p_{t-1} - p) + \left(-\frac{\lambda_d}{\lambda_d - 1}\right)(p_t - p) + (1-\chi_d)\left(-\frac{\lambda_d}{\lambda_d - 1} + 1\right) \sum_{j=1}^{s} \hat{\hat{\pi}}_{t+j}^c \right\}$$

R.H.S.:

$$\mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_d)^s \lambda_d MC^n Y + \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_d)^s \lambda_d MC^n (Y_{t+s} - Y)$$

$$+\mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_d)^s \lambda_d MC^n Y \left(\frac{1}{P}\right)^{\chi_d\left(-\frac{\lambda_d}{\lambda_d-1}\right)} \chi_d \left(-\frac{\lambda_d}{\lambda_d - 1}\right) P^{\chi_d\left(-\frac{\lambda_d}{\lambda_d-1}\right)} \left(\frac{1}{P}\right)(P_{t+s-1} - P)$$

$$+\mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_d)^s \lambda_d MC^n Y P^{-\frac{\lambda_d}{\lambda_d-1}} \left(\frac{\lambda_d}{\lambda_d - 1}\right) P^{\frac{\lambda_d}{\lambda_d-1}-1}(P_{t+s} - P)$$

$$+\mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_d)^s \lambda_d MC^n Y P^{\chi_d\left(-\frac{\lambda_d}{\lambda_d-1}\right)}(-\chi_d) \left(-\frac{\lambda_d}{\lambda_d - 1}\right) P^{(-\chi_d)\left(-\frac{\lambda_d}{\lambda_d-1}\right)} \left(\frac{1}{P}\right)(P_{t-1} - P)$$

$$+\mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_d)^s \lambda_d MC^n Y \left(\frac{1}{P}\right)^{-\frac{\lambda_d}{\lambda_d-1}} \left(-\frac{\lambda_d}{\lambda_d-1}\right) P^{\frac{\lambda_d}{\lambda_d-1}-1}(P_t - P)$$

$$+\mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_d)^s \lambda_d MC^n Y (1-\chi_d) \left(-\frac{\lambda_d}{\lambda_d-1}\right) \sum_{j=1}^{s} \hat{\hat{\pi}}_{t+j}^{c}$$

$$+\mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_d)^s \lambda_d Y \left(MC_{i,t+s} - MC^n\right)$$

$$+\mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_d)^s Y MC^n \left(1 + \lambda_d (ln1)\left(-\frac{1}{(1-\lambda_d)^2}\right)(-1)\right)\left(\lambda_{d,t} - \lambda_d\right)$$

$$= \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_d)^s \lambda_d MC^n Y \left\{1 + (y_{t+s} - y) + \left(-\frac{\chi_d\lambda_d}{\lambda_d-1}\right)(p_{t+s-1} - p) + \left(\frac{\lambda_d}{\lambda_d-1}\right)(p_{t+s} - p) \right.$$
$$+ \left(\frac{\chi_d\lambda_d}{\lambda_d-1}\right)(p_{t-1} - p) + \left(-\frac{\lambda_d}{\lambda_d-1}\right)(p_t - p) + (1-\chi_d)\left(-\frac{\lambda_d}{\lambda_d-1}\right)\sum_{j=1}^{s}\hat{\hat{\pi}}_{t+j}^{c} + \left(mc_{i,t+s} - mc^n\right)$$
$$\left. + \left(\frac{\lambda_{d,t} - \lambda_d}{\lambda_d}\right)\right\}$$

L.H.S. = R.H.S. and $YP = \lambda_d MC^n Y$

$$\Rightarrow \quad \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_d)^s (\tilde{p}_t - p)$$
$$= \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_d)^s \left\{(-\chi_d)(p_{t+s-1} - p) + \chi_d(p_{t-1} - p) - (1-\chi_d)\sum_{j=1}^{s}\hat{\hat{\pi}}_{t+j}^{c} + \widehat{mc}_{t+s} + \hat{\lambda}_{d,t+s}\right\}$$

$$\Rightarrow \quad \frac{1}{1-\beta\theta_d}(\tilde{p}_t - p) = \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_d)^s \left\{(-\chi_d)(p_{t+s-1} - p_{t-1}) - (1-\chi_d)\sum_{j=1}^{s}\hat{\hat{\pi}}_{t+j}^{c} + \widehat{mc}_{t+s} + \hat{\lambda}_{d,t+s}\right\}$$

$$\Rightarrow \quad \tilde{p}_t = (1-\beta\theta_d)\mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_d)^s \left\{-\chi_d \sum_{j=1}^{s}\hat{\pi}_{t+j-1} - (1-\chi_d)\sum_{j=1}^{s}\hat{\hat{\pi}}_{t+j}^{c} + \widehat{mc}_{t+s} + \hat{\lambda}_{d,t+s} + p\right\}$$

$$(1-\beta\theta_d)\mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_d)^s \left[-\chi_d \sum_{j=1}^{s}\hat{\pi}_{t+j-1}\right]$$

$$= (-\chi_d)[\beta\theta_d\hat{\pi}_t + (\beta\theta_d)^2(\hat{\pi}_t + \hat{\pi}_{t+1}) + (\beta\theta_d)^3(\hat{\pi}_t + \hat{\pi}_{t+1} + \hat{\pi}_{t+2}) + \cdots$$

$$-(\beta\theta_d)^2\hat{\pi}_t - (\beta\theta_d)^3(\hat{\pi}_t + \hat{\pi}_{t+1}) - (\beta\theta_d)^4(\hat{\pi}_t + \hat{\pi}_{t+1} + \hat{\pi}_{t+2}) - \cdots]$$

$$= -\chi_d \beta\theta_d \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_d)^s \hat{\pi}_{t+s}$$

$$(1-\beta\theta_d)\mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_d)^s \left[-(1-\chi_d) \sum_{j=1}^{s} \hat{\hat{\pi}}_{t+j}^c \right]$$

$$= -(1-\chi_d)[\beta\theta_d \hat{\hat{\pi}}_{t+1}^c + (\beta\theta_d)^2(\hat{\hat{\pi}}_{t+1}^c + \hat{\hat{\pi}}_{t+2}^c) + (\beta\theta_d)^3(\hat{\hat{\pi}}_{t+1}^c + \hat{\hat{\pi}}_{t+2}^c + \hat{\hat{\pi}}_{t+3}^c) + \cdots$$

$$-(\beta\theta_d)^2 \hat{\hat{\pi}}_{t+1}^c - (\beta\theta_d)^3(\hat{\hat{\pi}}_{t+1}^c + \hat{\hat{\pi}}_{t+2}^c) - (\beta\theta_d)^4(\hat{\hat{\pi}}_{t+1}^c + \hat{\hat{\pi}}_{t+2}^c + \hat{\hat{\pi}}_{t+3}^c) - \cdots]$$

$$= -(1-\chi_d)\beta\theta_d \sum_{s=0}^{\infty} (\beta\theta_d)^s \hat{\hat{\pi}}_{t+s+1}^c$$

$$\Rightarrow \quad \tilde{p}_t = (1-\beta\theta_d)\mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_d)^s \left(\widehat{mc}_{t+s} + \hat{\lambda}_{d,t+s} + p\right) - \chi_d \beta\theta_d \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_d)^s \hat{\pi}_{t+s} - (1-\chi_d)\beta\theta_d \sum_{s=0}^{\infty} (\beta\theta_d)^s \hat{\hat{\pi}}_{t+s+1}^c$$

$$\Rightarrow \quad \tilde{p}_t - p_{t-1} = (1-\beta\theta_d)\mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_d)^s \left(\widehat{mc}_{t+s} + \hat{\lambda}_{d,t+s} + p - p_{t-1}\right) - \chi_d \beta\theta_d \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_d)^s \hat{\pi}_{t+s} - (1-\chi_d)\beta\theta_d \sum_{s=0}^{\infty} (\beta\theta_d)^s \hat{\hat{\pi}}_{t+s+1}^c$$

$$= (1-\beta\theta_d)\left(\widehat{mc}_t + \hat{\lambda}_{d,t}\right) + (1-\beta\theta_d)\frac{1}{1-\beta\theta_d}(p - p_{t-1}) - \chi_d\beta\theta_d\hat{\pi}_t - (1-\chi_d)\beta\theta_d\hat{\hat{\pi}}_{t+1}^c + (1-\beta\theta_d)\mathbb{E}_t \sum_{s=1}^{\infty} (\beta\theta_d)^s \left(\widehat{mc}_{t+s} + \hat{\lambda}_{d,t+s}\right) - \chi_d \beta\theta_d \mathbb{E}_t \sum_{s=1}^{\infty} (\beta\theta_d)^s \hat{\pi}_{t+s} - (1-\chi_d)\beta\theta_d \sum_{s=1}^{\infty} (\beta\theta_d)^s \hat{\hat{\pi}}_{t+s+1}^c$$

$$= (1-\beta\theta_d)\left(\widehat{mc}_t + \hat{\lambda}_{d,t}\right) + (p - p_{t-1}) - \chi_d\beta\theta_d\hat{\pi}_t - (1-\chi_d)\beta\theta_d\hat{\hat{\pi}}_{t+1}^c + \beta\theta_d \left[(1-\beta\theta_d)\mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_d)^s \left(\widehat{mc}_{t+s+1} + \hat{\lambda}_{d,t+s+1}\right) - \chi_d \beta\theta_d \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_d)^s \hat{\pi}_{t+s+1} - (1-\chi_d)\beta\theta_d \sum_{s=0}^{\infty} (\beta\theta_d)^s \hat{\hat{\pi}}_{t+s+2}^c \right]$$

$$= (1-\beta\theta_d)\left(\widehat{mc}_t + \hat{\lambda}_{d,t}\right) + (p - p_{t-1}) - \chi_d\beta\theta_d\hat{\pi}_t - (1-\chi_d)\beta\theta_d\mathbb{E}_t\hat{\hat{\pi}}_{t+1}^c + \beta\theta_d\mathbb{E}_t(\tilde{p}_{t+1} - p_t) - \beta\theta_d(p - p_t)$$

$$\hat{\pi}_t = (1-\theta_d)(\tilde{p}_t - p_{t-1}) + \theta_d\chi_d\hat{\pi}_{t-1} + \theta_d(1-\chi_d)\hat{\hat{\pi}}_t^c$$

$$= (1-\theta_d)(1-\beta\theta_d)\left(\widehat{mc}_t + \hat{\lambda}_{d,t}\right) + (1-\theta_d)(p - p_{t-1}) - (1-\theta_d)\chi_d\beta\theta_d\hat{\pi}_t - (1-\theta_d)(1-\chi_d)\beta\theta_d\mathbb{E}_t\hat{\hat{\pi}}_{t+1}^c + \beta\theta_d\mathbb{E}_t\hat{\pi}_{t+1} - (1-\theta_d)\beta\theta_d(p - p_t) - \beta\theta_d\theta_d\chi_d\hat{\pi}_t - \beta\theta_d\theta_d(1-\chi_d)\hat{\hat{\pi}}_{t+1}^c + \theta_d\chi_d\hat{\pi}_{t-1} + \theta_d(1-\chi_d)\hat{\hat{\pi}}_t^c$$

$$= \beta\theta_d \mathbb{E}_t\hat{\pi}_{t+1} - \beta\theta_d\chi_d\hat{\pi}_t - \beta\theta_d(1-\chi_d)\mathbb{E}_t\hat{\bar{\pi}}^c_{t+1} + \theta_d\chi_d\hat{\pi}_{t-1} + \theta_d(1-\chi_d)\hat{\bar{\pi}}^c_t + (1-\theta_d)(1-\beta\theta_d)\left(\widehat{mc}_t + \hat{\lambda}_{d,t}\right)$$
$$+ (1-\theta_d)(p - p_{t-1}) - (1-\theta_d)\beta\theta_d(p - p_t)$$

$$= \beta\theta_d \mathbb{E}_t\hat{\pi}_{t+1} - \beta\theta_d\chi_d\hat{\pi}_t + (1-\theta_d)(p_t - p_{t-1}) + (1-\theta_d)(p - p_t) - (1-\theta_d)\beta\theta_d(p - p_t)$$
$$+ (1-\theta_d)(1-\beta\theta_d)\left(\widehat{mc}_t + \hat{\lambda}_{d,t}\right) - \beta\theta_d(1-\chi_d)\mathbb{E}_t\hat{\bar{\pi}}^c_{t+1} + \theta_d\chi_d\hat{\pi}_{t-1} + \theta_d(1-\chi_d)\hat{\bar{\pi}}^c_t$$

$$= \beta\theta_d \mathbb{E}_t\hat{\pi}_{t+1} - \beta\theta_d\chi_d\hat{\pi}_t + (1-\theta_d)\hat{\pi}_t + (1-\theta_d)(1-\beta\theta_d)(p - p_t) + (1-\theta_d)(1-\beta\theta_d)\left(\widehat{mc}_t + \hat{\lambda}_{d,t}\right)$$
$$- \beta\theta_d(1-\chi_d)\mathbb{E}_t\hat{\bar{\pi}}^c_{t+1} + \theta_d\chi_d\hat{\pi}_{t-1} + \theta_d(1-\chi_d)\hat{\bar{\pi}}^c_t$$

$$\approx \beta\theta_d \mathbb{E}_t\hat{\pi}_{t+1} + (1-\theta_d-\theta_d\beta\chi_d)\hat{\pi}_t + (1-\theta_d)(1-\beta\theta_d)\left(\widehat{mc}_t + \hat{\lambda}_{d,t}\right) - \beta\theta_d(1-\chi_d)\mathbb{E}_t\hat{\bar{\pi}}^c_{t+1} + \theta_d\chi_d\hat{\pi}_{t-1}$$
$$+ \theta_d(1-\chi_d)\hat{\bar{\pi}}^c_t$$

$$\Rightarrow \quad \theta_d(1+\beta\chi_d)\hat{\pi}_t$$
$$= \beta\theta_d \mathbb{E}_t\hat{\pi}_{t+1} + (1-\theta_d)(1-\beta\theta_d)\left(\widehat{mc}_t + \hat{\lambda}_{d,t}\right) - \beta\theta_d(1-\chi_d)\mathbb{E}_t\hat{\bar{\pi}}^c_{t+1} + \theta_d\chi_d\hat{\pi}_{t-1}$$
$$+ \theta_d(1-\chi_d)\hat{\bar{\pi}}^c_t$$

$$\Rightarrow \quad \hat{\pi}_t = \frac{\beta}{1+\beta\chi_d}\mathbb{E}_t\hat{\pi}_{t+1} - \frac{\beta(1-\chi_d)}{1+\beta\chi_d}\mathbb{E}_t\hat{\bar{\pi}}^c_{t+1} + \frac{\chi_d}{1+\beta\chi_d}\hat{\pi}_{t-1} + \frac{1-\chi_d}{1+\beta\chi_d}\hat{\bar{\pi}}^c_t$$
$$+ \frac{(1-\theta_d)(1-\beta\theta_d)}{\theta_d(1+\beta\chi_d)}\left(\widehat{mc}_t + \hat{\lambda}_{d,t}\right) \qquad (17)$$

***Coefficient of $\hat{\bar{\pi}}^c_t$ in (17)

$$-\frac{\beta(1-\chi_d)}{1+\beta\chi_d}\mathbb{E}_t\hat{\bar{\pi}}^c_{t+1} + \frac{1-\chi_d}{1+\beta\chi_d}\hat{\bar{\pi}}^c_t = \frac{-\rho_\pi\beta + \rho_\pi\beta\chi_d + 1 - \chi_d}{1+\beta\chi_d}\hat{\bar{\pi}}^c_t$$

$$\hat{\bar{\pi}}^c_{t+1} = \rho_\pi\hat{\bar{\pi}}^c_t + \varepsilon_{\hat{\bar{\pi}}^c_t} \qquad \Rightarrow \qquad \mathbb{E}_t\hat{\bar{\pi}}^c_{t+1} = \rho_\pi\hat{\bar{\pi}}^c_t$$

=> New Keynesian Phillips Curve (NKPC)

$$\hat{\pi}_t - \hat{\bar{\pi}}^c_t = \frac{\beta}{1+\beta\chi_d}\left(\mathbb{E}_t\hat{\pi}_{t+1} - \rho_\pi\hat{\bar{\pi}}^c_t\right) + \frac{\chi_d}{1+\beta\chi_d}\left(\hat{\pi}_{t-1} - \hat{\bar{\pi}}^c_t\right) - \frac{\chi_d\beta(1-\rho_\pi)}{1+\beta\chi_d}\hat{\bar{\pi}}^c_t$$
$$+ \frac{(1-\theta_d)(1-\beta\theta_d)}{\theta_d(1+\beta\chi_d)}\left(\widehat{mc}_t + \hat{\lambda}_{d,t}\right) \qquad (L.6)$$

***Coefficient of $\hat{\bar{\pi}}^c_t$ in (L.6) equals coefficient of $\hat{\bar{\pi}}^c_t$ in (17)

$$\hat{\bar{\pi}}^c_t - \frac{\beta\rho_\pi}{1+\beta\chi_d}\hat{\bar{\pi}}^c_t - \frac{\chi_d}{1+\beta\chi_d}\hat{\bar{\pi}}^c_t - \frac{\chi_d\beta(1-\rho_\pi)}{1+\beta\chi_d}\hat{\bar{\pi}}^c_t = \frac{-\rho_\pi\beta + \rho_\pi\beta\chi_d + 1 - \chi_d}{1+\beta\chi_d}\hat{\bar{\pi}}^c_t$$

A.3 Importers

Importing firms buy a homogeneous good at international price P^*_t. Import i turns the homogeneous good into a differentiated consumption good $C^m_{i,t}$ or investment good $I^m_{i,t}$.

$J_t \in \{C^m_t, I^m_t\}$ is defined as aggregated imported consumption and investment.

The final imported good is an aggregation of differentiated import goods based on CES:

$$J_t = \left[\int_0^1 J_{i,t}^{\frac{1}{\lambda_t^{m,j}}} di\right]^{\lambda_t^{m,j}} \qquad (18.\mathrm{a})$$

where

$\lambda_t^{m,j} \epsilon [1, \infty)$: markup shock

$j \in \{c, i\}$

Demand function for importing firm i:

$$J_{i,t} = \left(\frac{P_{i,t}^{m.j}}{P_t^{m.j}}\right)^{-\frac{\lambda_t^{m,j}}{\lambda_t^{m,j}-1}} J_t \qquad (18.\mathrm{b})$$

where

$\lambda_t^{m,j}$: time-varying markup for imported goods

$$\lambda_t^{m,j} = \left(1 - \rho_{\lambda^{m,j}}\right)\lambda^{m,j} + \rho_{\lambda^{m,j}}\lambda_{t-1}^{m,j} + \epsilon_t^{\lambda^{m,j}} \qquad (19)$$

Like domestic firms, importing firms face a Calvo probability when setting prices. Importing consumption firms change prices with a probability of $\left(1 - \theta_{m,c}\right)$. Importing investment firms change prices with a probability of $\left(1 - \theta_{m,i}\right)$. For firms that cannot change prices, the price is set as follows:

$$P_{t+1}^{m.j} = \left(\pi_t^{m,j}\right)^{\chi_{m.j}} (\bar{\pi}_{t+1}^c)^{1-\chi_{m.j}} P_t^{m,j} \qquad (20)$$

Importing firms' optimization problem becomes

$$\max_{\tilde{P}_t^{m,j}} \mathbb{E}_t \sum_{s=0}^{\infty} \left(\beta\theta_{m,j}\right)^s v_{t+s} \left\{ \left[\left(\prod_{k=1}^{s} \pi_{t+k-1}^{m,j} \right)^{\chi_{m,j}} \left(\prod_{k=1}^{s} \bar{\pi}_{t+k}^c \right)^{1-\chi_{m,j}} \tilde{P}_t^{m,j} J_{i,t+s} \right] - S_{t+s} P_{t+s}^* \left(J_{i,t+s} + z_{t+s}\phi^{m,j}\right) \right\} \qquad (21)$$

where

S_t: nominal exchange rate (number of domestic currency units to buy one unit of foreign currency)

Aggregate imported goods price index:

$$P_t^{m,j} = \left[\int_0^1 P_{i,t}^{m,j\,\frac{1}{1-\lambda_t^{m,j}}} di\right]^{1-\lambda_t^{m,j}}$$

$$P_t^{m,j} = \left[\theta_{m,j}\left(P_{t-1}^{m,j}\left(\pi_{t-1}^{m,j}\right)^{\chi_{m,j}}\left(\bar{\pi}_t^c\right)^{1-\chi_{m,j}}\right)^{\frac{1}{1-\lambda_t^{m,j}}} + \left(1-\theta_{m,j}\right)\left(\tilde{P}_t^{m,j}\right)^{\frac{1}{1-\lambda_t^{m,j}}}\right]^{1-\lambda_t^{m,j}} \qquad (22)$$

Like price optimization of domestic good producers, importing firm price optimization leads to its NKPC based on the loglinearization of (21) and (22):

$$\begin{aligned}\hat{\pi}_t^{m,j} - \hat{\bar{\pi}}_t^c = {} & \frac{\beta}{1+\beta\chi_{m,j}}\left(\mathbb{E}_t\hat{\pi}_{t+1}^{m,j} - \rho_\pi\hat{\bar{\pi}}_t^c\right) + \frac{\chi_{m.j}}{1+\beta\chi_{m,j}}\left(\hat{\pi}_{t-1}^{m,j} - \hat{\bar{\pi}}_t^c\right) - \frac{\chi_{m,j}\beta(1-\rho_\pi)}{1+\beta\chi_{m,j}}\hat{\bar{\pi}}_t^c \\ & + \frac{\left(1-\theta_{m,j}\right)\left(1-\beta\theta_{m,j}\right)}{\theta_{m,j}\left(1+\beta\chi_{m,j}\right)}\left(\widehat{mc}_t^{m,j} + \hat{\lambda}_t^{m,j}\right) \qquad (L.7)\end{aligned}$$

where

$$\widehat{mc}_t^{m,j} = \hat{s}_t + \hat{p}_t^* - \hat{p}_t^{m,j}$$

A.4 Exporters

Exporting firms purchase final goods and investment, differentiate them and export them to households in foreign countries. Exporting firm i has a demand function $\tilde{X}_{i,t}$ as follows:

$$\tilde{X}_{i,t} = \left(\frac{P_{i,t}^x}{P_t^x}\right)^{-\frac{\lambda_t^x}{\lambda_t^x-1}}\tilde{X}_t \qquad (23)$$

where

P_t^x: foreign currency price of exports

$$\lambda_t^x = (1-\rho_{\lambda^x})\lambda^x + \rho_{\lambda^x}\lambda_{t-1}^x + \epsilon_t^{\lambda^x} \qquad (24)$$

Like domestic producers and importers, exporters face a Calvo probability when setting prices. Exporting firms set prices with a probability of $1-\theta_x$. For firms that cannot change prices, the price is set as follows:

$$P_{t+1}^x = \pi_t^x P_t^x \qquad (25)$$

Unlike domestic firms and importers, the domestic inflation target $\hat{\bar{\pi}}_t^c$ is not considered when setting export goods and investment. The export price $\tilde{P}_t^x$ is determined by maximizing the profit of exporters:

$$\max_{\tilde{P}_t^x}\mathbb{E}_t\sum_{s=0}^{\infty}(\beta\theta_x)^s v_{t+s}\left\{\left(\prod_{k=1}^{s}\pi_{t+k-1}^x\tilde{P}_t^x\right)\tilde{X}_{i,t+s} - \frac{P_{t+s}}{S_{t+s}}\left(\tilde{X}_{i,t+s} + z_{t+s}\phi^x\right)\right\} \qquad (26)$$

where

${P_{t+s}}/{S_{t+s}}$: nominal marginal cost

The aggregate export price index is determined as follows:

$$P_t^x = \left[\int_0^1 P_{i,t}^{x\,\frac{1}{1-\lambda_t^x}} di\right]^{1-\lambda_t^x}$$

$$P_t^x = \left[\theta_x \left(P_{t-1}^x \pi_{t-1}^x\right)^{\frac{1}{1-\lambda_t^{m,j}}} + (1-\theta_x)\left(\tilde{P}_t^x\right)^{\frac{1}{1-\lambda_t^x}}\right]^{1-\lambda_t^x} \qquad (27)$$

Like domestic goods producer price optimization, exporting firm price optimization leads to its NKPC based on the loglinearization of (26) and (27).

$$\hat{\pi}_t^x = \frac{\beta}{1+\beta}\mathbb{E}_t\hat{\pi}_{t+1}^x + \frac{1}{1+\beta}\hat{\pi}_{t-1}^x + \frac{(1-\theta_x)(1-\beta\theta_x)}{(1+\beta)\theta_x}\left(\widehat{mc}_t^x + \hat{\lambda}_t^x\right) \qquad (L.8)$$

where

$$\widehat{mc}_t^x = \hat{p}_t^d - \hat{s}_t - \hat{p}_t^x$$

In a foreign economy, exported goods are used for consumption or investment by households in foreign countries.

Foreign demand for exported consumption and investment goods is determined as

$$C_t^x = \left[\frac{P_t^x}{P_t^*}\right]^{-\eta_f} C_t^* \qquad (28)$$

$$I_t^x = \left[\frac{P_t^x}{P_t^*}\right]^{-\eta_f} I_t^* \qquad (29)$$

where

C_t^*: foreign consumption

I_t^*: foreign investment

η_f: elasticity for foreign consumption and investment

Aggregated demand for exports is then determined as

$$\tilde{X}_t = C_t^x + I_t^x = \left[\frac{P_t^x}{P_t^*}\right]^{-\eta_f}(C_t^* + I_t^*) = \left[\frac{P_t^x}{P_t^*}\right]^{-\eta_f} Y_t^* \qquad (30)$$

A.5 Households

A.5.1 Utility Function

Households consume domestic goods and imported goods, hold cash, make investment, work and pay taxes. They make decisions on consumption, working and investment to maximize their utility. Household j's expected lifetime utility can be described as follows:

$$\mathbb{E}_t^j \sum_{t=0}^{\infty} \beta^t \left[\xi_t^c \ln\left(C_{j,t} - bC_{j,t-1}\right) - \xi_t^h A_L \frac{\left(h_{j,t}\right)^{1+\sigma_L}}{1+\sigma_L} + A_q \frac{\left(\frac{Q_{j,t}}{z_t P_t}\right)^{1-\sigma_q}}{1-\sigma_q} \right] \qquad (31)$$

where

ξ_t^c: consumption preference shock. $\xi_t^c = \rho_c \xi_{t-1}^c + \varepsilon_t^c$

$C_{j,t}$: consumption

b: degree of habit formation in consumption

ξ_t^h: labor supply shock. $\xi_t^h = \rho_h \xi_{t-1}^h + \varepsilon_t^h$

A_L: disutility of supplying labor

$h_{j,t}$: labor

σ_L: inverted Frisch elasticity of labor supply

A_q: utility of real asset holdings

$Q_{j,t}$: non-interest-bearing real asset

σ_q: elasticity of real asset holdings

$\mathbb{E}(\xi_t^i) = 1 \qquad \hat{\xi}_t^i = \frac{\xi_t^i - 1}{1}$

A.5.2 Consumption

Households have access to the following aggregate consumption:

$$C_t = \left[(1-\vartheta_c)^{\frac{1}{\eta_c}} (C_t^d)^{\frac{\eta_c - 1}{\eta_c}} + \vartheta_c^{\frac{1}{\eta_c}} (C_t^m)^{\frac{\eta_c - 1}{\eta_c}} \right]^{\frac{\eta_c}{\eta_c - 1}} \qquad (32)$$

where

C_t^d: consumption of domestic goods

C_t^m: consumption of imported goods

ϑ_c: share of imported goods

Accordingly, the demand for domestic goods and imported goods is given below:

$$C_t^d = (1-\vartheta_c) \left(\frac{P_t^d}{P_t^c} \right)^{-\eta_c} C_t \qquad (33)$$

$$C_t^m = \vartheta_c \left(\frac{P_t^{m,c}}{P_t^c}\right)^{-\eta_c} C_t \qquad (34)$$

The inflation rate for aggregate consumption goods (CPI) is determined as

$$P_t^c = [(1-\vartheta_c)(P_t^d)^{1-\eta_c} + \vartheta_c (P_t^{m,c})^{1-\eta_c}]^{\frac{1}{1-\eta_c}} \qquad (35)$$

A.5.3 Labor Aggregation

Each household provides labor services in a differentiated way. The aggregate labor supply to intermediate firms is determined as follows:

$$H_t = \left[\int_0^1 (h_{j,t})^{\frac{1}{\lambda_w}} dj\right]^{\lambda_w} \qquad (36)$$

where

$h_{j,t}$: labor provided by household j

By supplying a differentiated labor service, each household has monopoly power setting nominal wage $W_{j,t}$. The demand for each household's labor service is determined based on the wage level accordingly:

$$h_{j,t} = \left(\frac{W_{j,t}}{W_t}\right)^{\frac{\lambda_w}{1-\lambda_w}} H_t \qquad (37)$$

$$W_t = \left[\int_0^1 (W_{j,t})^{\frac{1}{1-\lambda_w}} dj\right]^{1-\lambda_w} \qquad (38)$$

where

W_t: aggregate wage level

A.5.4 Wage Setting

Households face a Calvo probability when setting wages. Households set wages with a probability of $1-\theta_w$. The remaining households set wages as follows:

$$W_{j,t+1} = (\pi_t^c)^{\chi_w} (\bar{\pi}_{t+1}^c)^{1-\chi_w} \mu_{t+1}^z W_{j,t} \qquad (39)$$

where

$\mu_{t+1}^z = \frac{z_{t+1}}{z_t}$ is permanent technology growth

Optimal wage $\widetilde{W}_{j,t}$ is set by maximizing the following equation:

$$\max_{\widetilde{W}_{j,t}} \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_w)^s \Bigg\{ -\xi_{t+s}^h A_L \frac{\left(h_{j,t+s}\right)^{1+\sigma_L}}{1+\sigma_L} \\ + \nu_{t+s} \left(\frac{1-\iota_t^y}{1+\iota_t^w}\right) h_{j,t+s} \left[\left(\prod_{k=1}^{s} \pi_{t+k-1}^c\right)^{\chi_w} \left(\prod_{k=1}^{s} \bar{\pi}_{t+k}^c\right)^{1-\chi_w} \left(\prod_{k=1}^{s} \mu_{t+k}^z\right) \widetilde{W}_{j,t} \right] \Bigg\} \quad (40)$$

where

ι_t^w: aggregate payroll tax rate

ι_t^y: aggregate income tax rate

Insert (37) into (40):

$$\mathcal{L} = \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_w)^s \left\{ -\xi_{t+s}^h A_L \frac{\left[\left(\frac{W_{j,t+s}}{W_{t+s}}\right)^{\frac{\lambda_w}{1-\lambda_w}} H_{t+s}\right]^{1+\sigma_L}}{1+\sigma_L} \right.$$

$$\left. + \nu_{t+s} \left(\frac{1-\iota_t^y}{1+\iota_t^w}\right) \left(\frac{W_{j,t+s}}{W_{t+s}}\right)^{\frac{\lambda_w}{1-\lambda_w}} H_{t+s} \left[\left(\prod_{k=1}^{s} \pi_{t+k-1}^c\right)^{\chi_w} \left(\prod_{k=1}^{s} \bar{\pi}_{t+k}^c\right)^{1-\chi_w} \left(\prod_{k=1}^{s} \mu_{t+k}^z\right) \widetilde{W}_{j,t} \right] \right\}$$

$$W_{j,t+s} = \left(\prod_{k=1}^{s} \pi_{t+k-1}^c\right)^{\chi_w} \left(\prod_{k=1}^{s} \bar{\pi}_{t+k}^c\right)^{1-\chi_w} \left(\prod_{k=1}^{s} \mu_{t+k}^z\right) \widetilde{W}_{j,t}$$

F.O.C.:

$$\frac{\partial \mathcal{L}}{\partial \widetilde{W}_{j,t}} = \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_w)^s \Bigg\{ -\xi_{t+s}^h A_L \frac{(1+\sigma_L) {h_{j,t+s}}^{\sigma_L}}{1+\sigma_L} H_{t+s} \left(\frac{\lambda_w}{1-\lambda_w}\right) \left(\frac{W_{j,t+s}}{W_{t+s}}\right)^{\frac{\lambda_w}{1-\lambda_w}-1} \frac{1}{W_{t+s}} \\ \cdot \left[\left(\prod_{k=1}^{s} \pi_{t+k-1}^c\right)^{\chi_w} \left(\prod_{k=1}^{s} \bar{\pi}_{t+k}^c\right)^{1-\chi_w} \left(\prod_{k=1}^{s} \mu_{t+k}^z\right) \right] \\ + \nu_{t+s} \left(\frac{1-\iota_{t+s}^y}{1+\iota_{t+s}^w}\right) \left(\frac{\lambda_w}{1-\lambda_w}+1\right) \left(\frac{W_{j,t+s}}{W_{t+s}}\right)^{\frac{\lambda_w}{1-\lambda_w}} H_{t+s} \\ \cdot \left[\left(\prod_{k=1}^{s} \pi_{t+k-1}^c\right)^{\chi_w} \left(\prod_{k=1}^{s} \bar{\pi}_{t+k}^c\right)^{1-\chi_w} \left(\prod_{k=1}^{s} \mu_{t+k}^z\right) \right] \Bigg\}$$

$$= \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_w)^s h_{j,t+s} \Bigg\{ -\xi_{t+s}^h A_L h_{j,t+s}{}^{\sigma_L} \left(\frac{\lambda_w}{1-\lambda_w}\right) \frac{1}{\widetilde{W}_{j,t}}$$

$$+ \nu_{t+s} \left(\frac{1-\iota_{t+s}^y}{1+\iota_{t+s}^w}\right) \left(\frac{1}{1-\lambda_w}\right) \left[\left(\frac{P_{t+s-1}^c}{P_{t-1}^c}\right)^{\chi_w} \left(\prod_{k=1}^{s} \bar{\pi}_{t+k}^c\right)^{1-\chi_w} \frac{z_{t+s}}{z_t} \right] \Bigg\} = 0$$

$$\Rightarrow \qquad \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_w)^s h_{j,t+s} \left\{ -\lambda_w \xi_{t+s}^h A_L h_{j,t+s}{}^{\sigma_L} \frac{1}{\widetilde{W}_{j,t}} + \nu_{t+s} \left(\frac{1-\iota_{t+s}^y}{1+\iota_{t+s}^w}\right) \left[\left(\frac{P_{t+s-1}^c}{P_{t-1}^c}\right)^{\chi_w} \left(\prod_{k=1}^{s} \bar{\pi}_{t+k}^c\right)^{1-\chi_w} \frac{z_{t+s}}{z_t} \right] \right\} = 0$$

$$\Rightarrow \qquad \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_w)^s h_{j,t+s} \left\{ -\xi_{t+s}^h A_L h_{j,t+s}{}^{\sigma_L} + \frac{\widetilde{W}_{j,t}}{z_t} \frac{z_{t+s}\nu_{t+s}}{\lambda_w} \left(\frac{1-\iota_{t+s}^y}{1+\iota_{t+s}^w}\right) \left[\left(\frac{P_{t+s-1}^c}{P_{t-1}^c}\right)^{\chi_w} \left(\prod_{k=1}^{s} \bar{\pi}_{t+k}^c\right)^{1-\chi_w} \right] \right\} = 0$$

$$\Rightarrow \qquad \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_w)^s h_{j,t+s} \left\{ -\xi_{t+s}^h A_L h_{j,t+s}{}^{\sigma_L} + \frac{\widetilde{W}_{j,t}}{z_t P_t} \frac{z_{t+s}\nu_{t+s}P_{t+s}}{\lambda_w} \left(\frac{1-\iota_{t+s}^y}{1+\iota_{t+s}^w}\right) \frac{\left(\frac{P_{t+s-1}^c}{P_{t-1}^c}\right)^{\chi_w} \left(\prod_{k=1}^{s} \bar{\pi}_{t+k}^c\right)^{1-\chi_w}}{\frac{P_{t+s}}{P_t}} \right\} = 0$$

$$\Rightarrow \qquad \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_w)^s \xi_{t+s}^h A_L h_{j,t+s}{}^{1+\sigma_L}$$

$$= \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_w)^s \; h_{j,t+s} \frac{\widetilde{W}_{j,t}}{z_t P_t} \frac{z_{t+s}\nu_{t+s}P_{t+s}}{\lambda_w} \left(\frac{1-\iota_{t+s}^y}{1+\iota_{t+s}^w}\right) \frac{\left(\frac{P_{t+s-1}^c}{P_{t-1}^c}\right)^{\chi_w} \left(\prod_{k=1}^{s} \bar{\pi}_{t+k}^c\right)^{1-\chi_w}}{\frac{P_{t+s}}{P_t}}$$

$$h_{j,t+s} = \left(\frac{W_{j,t+s}}{W_{t+s}}\right)^{\frac{\lambda_w}{1-\lambda_w}} H_{t+s} = \left(\frac{\left(\frac{P_{t+s-1}^c}{P_{t-1}^c}\right)^{\chi_w} \left(\prod_{k=1}^{s} \bar{\pi}_{t+k}^c\right)^{1-\chi_w} \frac{z_{t+s}}{z_t} \frac{\widetilde{W}_t}{P_t}}{\frac{W_{t+s}}{P_{t+s}}} \frac{P_t}{P_{t+s}} \right)^{\frac{\lambda_w}{1-\lambda_w}} H_{t+s}$$

$$= \left(\frac{\left(\frac{P_{t+s-1}^c}{P_{t-1}^c}\right)^{\chi_w} \left(\prod_{k=1}^{s} \bar{\pi}_{t+k}^c\right)^{1-\chi_w} \widetilde{w}_t}{w_{t+s} \left(\frac{P_{t+s}}{P_t}\right)} \right)^{\frac{\lambda_w}{1-\lambda_w}} H_{t+s}$$

Let $\widetilde{W}_{j,t} = \widetilde{W}_t$ so that all households choose the same optimal wage:

$$\Rightarrow \qquad \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_w)^s \xi_{t+s}^h A_L \left\{ \left(\frac{\left(\frac{P_{t+s-1}^c}{P_{t-1}^c}\right)^{\chi_w} \left(\prod_{k=1}^{s} \bar{\pi}_{t+k}^c\right)^{1-\chi_w} \widetilde{w}_t}{w_{t+s}\left(\frac{P_{t+s}}{P_t}\right)} \right)^{\frac{\lambda_w}{1-\lambda_w}} H_{t+s} \right\}^{1+\sigma_L}$$

$$= \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_w)^s \left(\frac{\left(\frac{P_{t+s-1}^c}{P_{t-1}^c}\right)^{\chi_w} \left(\prod_{k=1}^{s} \bar{\pi}_{t+k}^c\right)^{1-\chi_w} \widetilde{w}_t}{w_{t+s}\left(\frac{P_{t+s}}{P_t}\right)} \right)^{\frac{\lambda_w}{1-\lambda_w}}$$

$$\times H_{t+s}\widetilde{w}_t \frac{z_{t+s}\nu_{t+s}P_{t+s}}{\lambda_w} \left(\frac{1-\iota_{t+s}^y}{1+\iota_{t+s}^w}\right) \frac{\left(\frac{P_{t+s-1}^c}{P_{t-1}^c}\right)^{\chi_w} \left(\prod_{k=1}^{s} \bar{\pi}_{t+k}^c\right)^{1-\chi_w}}{\frac{P_{t+s}}{P_t}} \qquad (41)$$

Loglinearize (41):

L.H.S.:

$$\mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_w)^s \xi^h A_L H^{1+\sigma_L} \left\{ 1 + \left(\frac{1}{w}\right)^{\frac{\lambda_w}{1-\lambda_w}(1+\sigma_L)} \frac{\lambda_w}{1-\lambda_w}(1+\sigma_L) w^{\frac{\lambda_w}{1-\lambda_w}(1+\sigma_L)-1}(\widetilde{w}_t - w) \right.$$
$$+ (w)^{\frac{\lambda_w}{1-\lambda_w}(1+\sigma_L)} \frac{-\lambda_w}{1-\lambda_w}(1+\sigma_L) w^{\frac{-\lambda_w}{1-\lambda_w}(1+\sigma_L)-1}(w_{t+s} - w)$$
$$+ \left(\frac{1}{P}\right)^{\chi_w\frac{\lambda_w}{1-\lambda_w}(1+\sigma_L)} \chi_w \frac{\lambda_w}{1-\lambda_w}(1+\sigma_L) P^{\frac{\chi_w\lambda_w}{1-\lambda_w}(1+\sigma_L)-1}(P_{t+s-1}^c - P)$$
$$+ (P)^{\chi_w\frac{\lambda_w}{1-\lambda_w}(1+\sigma_L)} (-\chi_w) \frac{\lambda_w}{1-\lambda_w}(1+\sigma_L) P^{\frac{-\chi_w\lambda_w}{1-\lambda_w}(1+\sigma_L)-1}(P_{t-1}^c - P)$$
$$+ (P)^{\frac{\lambda_w}{1-\lambda_w}(1+\sigma_L)} \frac{-\lambda_w}{1-\lambda_w}(1+\sigma_L) P^{\frac{-\lambda_w}{1-\lambda_w}(1+\sigma_L)-1}(P_{t+s} - P)$$
$$+ \left(\frac{1}{P}\right)^{\frac{\lambda_w}{1-\lambda_w}(1+\sigma_L)} \frac{\lambda_w}{1-\lambda_w}(1+\sigma_L) P^{\frac{\lambda_w}{1-\lambda_w}(1+\sigma_L)-1}(P_t - P) + (1-\chi_w)\frac{\lambda_w}{1-\lambda_w}(1+\sigma_L)\sum_{k=1}^{s} \hat{\bar{\pi}}_{t+k}^c$$
$$\left. + \frac{1+\sigma_L}{H}(H_{t+s} - H) + \frac{1}{\xi^h}(\xi_{t+s}^h - \xi^h) \right\}$$

$$= \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_w)^s \xi^h A_L H^{1+\sigma_L} \left\{ 1 + \frac{\lambda_w(1+\sigma_L)}{1-\lambda_w}\hat{\widetilde{w}}_t + \frac{-\lambda_w(1+\sigma_L)}{1-\lambda_w}\hat{w}_{t+s} + \frac{\chi_w\lambda_w(1+\sigma_L)}{1-\lambda_w}(p_{t+s-1}^c - p) \right.$$
$$+ \frac{-\chi_w\lambda_w(1+\sigma_L)}{1-\lambda_w}(p_{t-1}^c - p) + \frac{-\lambda_w(1+\sigma_L)}{1-\lambda_w}(p_{t+s} - p) + \frac{\lambda_w(1+\sigma_L)}{1-\lambda_w}(p_t - p)$$
$$\left. + \frac{\lambda_w(1-\chi_w)(1+\sigma_L)}{1-\lambda_w}\sum_{k=1}^{s} \hat{\bar{\pi}}_{t+k}^c + (1+\sigma_L)\hat{h}_{t+s} + \hat{\xi}_{t+s}^h \right\}$$

R.H.S.:

$$\mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_w)^s \Bigg\{ Hw \frac{P}{\lambda_w} \left(\frac{1-\iota^y}{1+\iota^w}\right) + \left(\frac{1-\iota^y}{1+\iota^w}\right) \left(\frac{1}{w}\right)^{\frac{\lambda_w}{1-\lambda_w}} H \frac{P}{\lambda_w} \left(\frac{\lambda_w}{1-\lambda_w} + 1\right) w^{\frac{\lambda_w}{1-\lambda_w}} (\widetilde{w}_t - w)$$

$$+ \left(\frac{1-\iota^y}{1+\iota^w}\right) (w)^{\frac{\lambda_w}{1-\lambda_w}+1} H \frac{P}{\lambda_w} \frac{-\lambda_w}{1-\lambda_w} w^{\frac{-\lambda_w}{1-\lambda_w}-1} (w_{t+s} - w)$$

$$+ \left(\frac{1-\iota^y}{1+\iota^w}\right) \left(\frac{1}{P}\right)^{\chi_w\left(\frac{\lambda_w}{1-\lambda_w}+1\right)} Hw \frac{P}{\lambda_w} P^{\chi_w\left(\frac{\lambda_w}{1-\lambda_w}+1\right)} \frac{1}{P} \chi_w \left(\frac{\lambda_w}{1-\lambda_w} + 1\right) (P^c_{t+s-1} - P)$$

$$+ \left(\frac{1-\iota^y}{1+\iota^w}\right) (P)^{\chi_w\left(\frac{\lambda_w}{1-\lambda_w}+1\right)} Hw \frac{P}{\lambda_w} \left(\frac{1}{P}\right)^{\chi_w\left(\frac{\lambda_w}{1-\lambda_w}+1\right)} \frac{1}{P} (-\chi_w) \left(\frac{\lambda_w}{1-\lambda_w} + 1\right) (P^c_{t-1} - P)$$

$$+ \left(\frac{1-\iota^y}{1+\iota^w}\right) (P)^{\frac{\lambda_w}{1-\lambda_w}+1} Hw \frac{1}{\lambda_w} \frac{-\lambda_w}{1-\lambda_w} P^{\frac{-\lambda_w}{1-\lambda_w}-1} (P_{t+s} - P)$$

$$+ \left(\frac{1-\iota^y}{1+\iota^w}\right) \left(\frac{1}{P}\right)^{\frac{\lambda_w}{1-\lambda_w}} Hw \frac{1}{\lambda_w} \left(\frac{\lambda_w}{1-\lambda_w} + 1\right) P^{\frac{\lambda_w}{1-\lambda_w}} (P_t - P)$$

$$+ \left(\frac{1-\iota^y}{1+\iota^w}\right) Hw \frac{P}{\lambda_w} (1-\chi_w) \left(1 + \frac{\lambda_w}{1-\lambda_w}\right) \sum_{k=1}^{s} \hat{\hat{\pi}}^c_{t+k} + \left(\frac{1-\iota^y}{1+\iota^w}\right) w \frac{P}{\lambda_w} (H_{t+s} - H)$$

$$+ \left(\frac{1-\iota^y}{1+\iota^w}\right) Hw \frac{P}{\lambda_w} (z_{t+s} - z) + Hw \frac{P}{\lambda_w} \left(\frac{-1}{1+\iota^w}\right) (\iota^y_{t+s} - \iota^y)$$

$$+ Hw \frac{P}{\lambda_w} \left(-\frac{1-\iota^y}{(1+\iota^w)^2}\right) (\iota^w_{t+s} - \iota^w) \Bigg\}$$

$$= \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_w)^s Hw \frac{P}{\lambda_w} \left(\frac{1-\iota^y}{1+\iota^w}\right) \Bigg\{ 1 + \left(\frac{\lambda_w}{1-\lambda_w} + 1\right) \hat{\widetilde{w}}_t + \frac{-\lambda_w}{1-\lambda_w} \widehat{w}_{t+s} + \chi_w \left(\frac{\lambda_w}{1-\lambda_w} + 1\right) (p^c_{t+s-1} - p)$$

$$+ (-\chi_w) \left(\frac{\lambda_w}{1-\lambda_w} + 1\right) (p^c_{t-1} - p) + \left(\frac{-\lambda_w}{1-\lambda_w}\right) (p_{t+s} - p) + \left(\frac{\lambda_w}{1-\lambda_w} + 1\right) (p_t - p)$$

$$+ (1-\chi_w) \left(1 + \frac{\lambda_w}{1-\lambda_w}\right) \sum_{k=1}^{s} \hat{\hat{\pi}}^c_{t+k} + \hat{h}_{t+s} + \hat{z}_{t+s} + \left(\frac{-\iota^y}{1-\iota^y}\right) \hat{\iota}^y_{t+s} + \left(\frac{-\iota^w}{1+\iota^w}\right) \hat{\iota}^w_{t+s} \Bigg\}$$

Steady state: $\xi^h A_L H^{1+\sigma_L} = Hw \frac{P}{\lambda_w} \left(\frac{1-\iota^y}{1+\iota^w}\right)$ => $\xi^h A_L H^{\sigma_L} = w \frac{P}{\lambda_w} \left(\frac{1-\iota^y}{1+\iota^w}\right)$

L.H.S. = R.H.S.:

$$\mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_w)^s (\lambda_w + \lambda_w \sigma_L - 1)\hat{\tilde{w}}_t$$

$$= \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_w)^s \Big\{ \lambda_w \sigma_L \hat{w}_{t+s} + \big(\chi_w - \chi_w \lambda_w (1+\sigma_L)\big)(p^c_{t+s-1} - p)$$
$$+ (\chi_w \lambda_w (1+\sigma_L) - \chi_w)(p^c_{t-1} - p) + \lambda_w \sigma_L (p_{t+s} - p) + \big(1 - \lambda_w (1+\sigma_L)\big)(p_t - p)$$
$$- (\lambda_w + \lambda_w \sigma_L - 1)(1 - \chi_w) \sum_{k=1}^{s} \hat{\bar{\pi}}^c_{t+k} - \sigma_L (1-\lambda_w)\hat{h}_{t+s} - (1-\lambda_w)\hat{\xi}^h_{t+s} + (1-\lambda_w)\hat{z}_{t+s}$$
$$- \iota^y \left(\frac{1-\lambda_w}{1-\iota^y}\right) \hat{\iota}^y_{t+s} - \iota^w \left(\frac{1-\lambda_w}{1+\iota^w}\right) \hat{\iota}^w_{t+s} \Big\}$$

$$(1-\beta\theta_w)\mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_w)^s \sum_{k=1}^{s} \hat{\bar{\pi}}^c_{t+k}$$
$$= \beta\theta_w \hat{\bar{\pi}}^c_{t+1} + (\beta\theta_w)^2 (\hat{\bar{\pi}}^c_{t+1} + \hat{\bar{\pi}}^c_{t+2}) + (\beta\theta_w)^3 (\hat{\bar{\pi}}^c_{t+1} + \hat{\bar{\pi}}^c_{t+2} + \hat{\bar{\pi}}^c_{t+3}) + \cdots - (\beta\theta_w)^2 \hat{\bar{\pi}}^c_{t+1}$$
$$- (\beta\theta_w)^3 (\hat{\bar{\pi}}^c_{t+1} + \hat{\bar{\pi}}^c_{t+2}) - (\beta\theta_w)^4 (\hat{\bar{\pi}}^c_{t+1} + \hat{\bar{\pi}}^c_{t+2} + \hat{\bar{\pi}}^c_{t+3}) - \cdots = \beta\theta_w \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_w)^s \hat{\bar{\pi}}^c_{t+s+1}$$

$$\Rightarrow \quad \hat{\tilde{w}}_t = \frac{1-\beta\theta_w}{\lambda_w + \lambda_w \sigma_L - 1} \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_w)^s \Big\{ \lambda_w \sigma_L \hat{w}_{t+s} + \big(\chi_w - \chi_w \lambda_w (1+\sigma_L)\big)(p^c_{t+s-1} - p)$$
$$+ (\chi_w \lambda_w (1+\sigma_L) - \chi_w)(p^c_{t-1} - p) + \lambda_w \sigma_L (p_{t+s} - p) + \big(1 - \lambda_w (1+\sigma_L)\big)(p_t - p)$$
$$- \sigma_L (1-\lambda_w)\hat{h}_{t+s} - (1-\lambda_w)\hat{\xi}^h_{t+s} + (1-\lambda_w)\hat{z}_{t+s} - \iota^y \left(\frac{1-\lambda_w}{1-\iota^y}\right) \hat{\iota}^y_{t+s} - \iota^w \left(\frac{1-\lambda_w}{1+\iota^w}\right) \hat{\iota}^w_{t+s} \Big\}$$
$$- \beta\theta_w (1-\chi_w)\mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_w)^s \hat{\bar{\pi}}^c_{t+s+1}$$

$$\Rightarrow \quad \hat{\tilde{w}}_t = \frac{1-\beta\theta_w}{\lambda_w + \lambda_w \sigma_L - 1} \Big\{ \lambda_w \sigma_L \hat{w}_t + (1-\lambda_w)(p_t - p) - \sigma_L (1-\lambda_w)\hat{h}_t - (1-\lambda_w)\hat{\xi}^h_t + (1-\lambda_w)\hat{z}_t$$
$$- \iota^y \left(\frac{1-\lambda_w}{1-\iota^y}\right) \hat{\iota}^y_t - \iota^w \left(\frac{1-\lambda_w}{1+\iota^w}\right) \hat{\iota}^w_t \Big\} - \beta\theta_w (1-\chi_w)\mathbb{E}_t \hat{\bar{\pi}}^c_{t+1} + \beta\theta_w \mathbb{E}_t \hat{\tilde{w}}_{t+1}$$

Aggregate wage index W_t:

$$W_t = \left[\theta_w \left((\pi^c_{t-1})^{\chi_w} (\bar{\pi}^c_t)^{1-\chi_w} \mu^z_t W_{t-1} \right)^{\frac{1}{1-\lambda_w}} + (1-\theta_w) \tilde{W}_t^{\frac{1}{1-\lambda_w}} \right]^{1-\lambda_w} \qquad (42)$$

Loglinearize (42):

Real wage $w_t = \frac{W_t}{P_t z_t}$

$$\Rightarrow \quad (W_t)^{\frac{1}{1-\lambda_w}} = \theta_w \left((\pi^c_{t-1})^{\chi_w} (\bar{\pi}^c_t)^{1-\chi_w} \mu^z_t W_{t-1} \right)^{\frac{1}{1-\lambda_w}} + (1-\theta_w) \tilde{W}_t^{\frac{1}{1-\lambda_w}}$$

$$\Rightarrow \quad \left(\frac{W_t}{P_t z_t}\right)^{\frac{1}{1-\lambda_w}} = \theta_w \left(\frac{(\pi^c_{t-1})^{\chi_w} (\bar{\pi}^c_t)^{1-\chi_w} \mu^z_t W_{t-1}}{P_t z_t} \right)^{\frac{1}{1-\lambda_w}} + (1-\theta_w) \left(\frac{\tilde{W}_t}{P_t z_t}\right)^{\frac{1}{1-\lambda_w}}$$

$$\Rightarrow \qquad (w_t)^{\frac{1}{1-\lambda_w}} = \theta_w \left(\frac{(\pi_{t-1}^c)^{\chi_w} (\bar{\pi}_t^c)^{1-\chi_w} w_{t-1}}{\pi_t} \right)^{\frac{1}{1-\lambda_w}} + (1-\theta_w)(\tilde{w}_t)^{\frac{1}{1-\lambda_w}}$$

L.H.S.:

$$(w)^{\frac{1}{1-\lambda_w}} + \left(\frac{1}{1-\lambda_w}\right) w^{\frac{1}{1-\lambda_w}-1}(w_t - w)$$

R.H.S.:

$$\begin{aligned}
&(w)^{\frac{1}{1-\lambda_w}} + (1-\theta_w)\left(\frac{1}{1-\lambda_w}\right) w^{\frac{1}{1-\lambda_w}-1}(\tilde{w}_t - w) + \theta_w\left(\frac{1}{1-\lambda_w}\right) w^{\frac{1}{1-\lambda_w}-1}(w_{t-1} - w) \\
&\qquad + \theta_w\left(\frac{1}{1-\lambda_w}\right) w^{\frac{1}{1-\lambda_w}-1} w\chi_w(\bar{\pi}^c)^{\chi_w - 1}(\pi_{t-1}^c - \bar{\pi}) \\
&\qquad + \theta_w\left(\frac{1}{1-\lambda_w}\right) w^{\frac{1}{1-\lambda_w}-1} w(1-\chi_w)(\bar{\pi}^c)^{1-\chi_w - 1}(\bar{\pi}_t^c - \bar{\pi}) \\
&\qquad + \theta_w\left(\frac{1}{1-\lambda_w}\right) w^{\frac{1}{1-\lambda_w}-1} w\left(\frac{-1}{\pi^2}\right)(\pi_t - \bar{\pi})
\end{aligned}$$

L.H.S. = R.H.S.:

$$\hat{w}_t = (1-\theta_w)\hat{\tilde{w}}_t + \theta_w\chi_w\hat{\pi}_{t-1}^c + \theta_w(1-\chi_w)\hat{\bar{\pi}}_t^c - \theta_w\hat{\pi}_t + \theta_w\hat{w}_{t-1}$$

$$\begin{aligned}
\Rightarrow \qquad \hat{w}_t = (1-\theta_w)\frac{1-\beta\theta_w}{\lambda_w + \lambda_w\sigma_L - 1}\Big[&\lambda_w\sigma_L\hat{w}_t + (1-\lambda_w)(p_t - p) - \sigma_L(1-\lambda_w)\hat{h}_t - (1-\lambda_w)\hat{\xi}_t^h + (1-\lambda_w)\hat{z}_t \\
&- \iota^y\left(\frac{1-\lambda_w}{1-\iota^y}\right)\hat{\iota}_t^y - \iota^w\left(\frac{1-\lambda_w}{1+\iota^w}\right)\hat{\iota}_t^w\Big] - \beta\theta_w(1-\theta_w)(1-\chi_w)\mathbb{E}_t\hat{\bar{\pi}}_{t+1}^c + \beta\theta_w(1-\theta_w)\mathbb{E}_t\hat{\tilde{w}}_{t+1} \\
&+ \theta_w\chi_w\hat{\pi}_{t-1}^c + \theta_w(1-\chi_w)\hat{\bar{\pi}}_t^c - \theta_w\hat{\pi}_t + \theta_w\hat{w}_{t-1}
\end{aligned}$$

$$\mathbb{E}_t\hat{w}_{t+1} = \mathbb{E}_t\left((1-\theta_w)\hat{\tilde{w}}_{t+1} + \theta_w\chi_w\hat{\pi}_t^c + \theta_w(1-\chi_w)\hat{\bar{\pi}}_{t+1}^c - \theta_w\hat{\pi}_{t+1} + \theta_w\hat{w}_t\right)$$

$$\Rightarrow \qquad (1-\theta_w)\mathbb{E}_t\hat{\tilde{w}}_{t+1} = \mathbb{E}_t(\hat{w}_{t+1} - \theta_w\chi_w\hat{\pi}_t^c - \theta_w(1-\chi_w)\hat{\bar{\pi}}_{t+1}^c + \theta_w\hat{\pi}_{t+1} - \theta_w\hat{w}_t)$$

$$\Rightarrow \qquad \beta\theta_w(1-\theta_w)\mathbb{E}_t\hat{\tilde{w}}_{t+1} = \mathbb{E}_t(\beta\theta_w\hat{w}_{t+1} - \beta\theta_w^2\chi_w\hat{\pi}_t^c - \beta\theta_w^2(1-\chi_w)\hat{\bar{\pi}}_{t+1}^c + \beta\theta_w^2\hat{\pi}_{t+1} - \beta\theta_w^2\hat{w}_t)$$

$$\begin{aligned}
\Rightarrow \qquad \hat{w}_t = (1-\theta_w)\frac{1-\beta\theta_w}{\lambda_w + \lambda_w\sigma_L - 1}\Big[&\lambda_w\sigma_L\hat{w}_t + (1-\lambda_w)(p_t - p) - \sigma_L(1-\lambda_w)\hat{h}_t - (1-\lambda_w)\hat{\xi}_t^h + (1-\lambda_w)\hat{z}_t \\
&- \iota^y\left(\frac{1-\lambda_w}{1-\iota^y}\right)\hat{\iota}_t^y - \iota^w\left(\frac{1-\lambda_w}{1+\iota^w}\right)\hat{\iota}_t^w\Big] - \beta\theta_w(1-\theta_w)(1-\chi_w)\mathbb{E}_t\hat{\bar{\pi}}_{t+1}^c \\
&+ [\beta\theta_w\mathbb{E}_t\hat{w}_{t+1} - \beta\theta_w^2\chi_w\hat{\pi}_t^c - \beta\theta_w^2(1-\chi_w)\mathbb{E}_t\hat{\bar{\pi}}_{t+1}^c + \beta\theta_w^2\mathbb{E}_t\hat{\pi}_{t+1} - \beta\theta_w^2\hat{w}_t] + \theta_w\chi_w\hat{\pi}_{t-1}^c \\
&+ \theta_w(1-\chi_w)\hat{\bar{\pi}}_t^c - \theta_w\hat{\pi}_t + \theta_w\hat{w}_{t-1}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \qquad &\left(1 + \beta\theta_w^2 - \frac{\lambda_w\sigma_L(1-\theta_w)(1-\beta\theta_w)}{\lambda_w + \lambda_w\sigma_L - 1}\right)\hat{w}_t \\
&= \frac{(1-\theta_w)(1-\beta\theta_w)}{\lambda_w + \lambda_w\sigma_L - 1}\Big[(1-\lambda_w)(p_t - p) - \sigma_L(1-\lambda_w)\hat{h}_t - (1-\lambda_w)\hat{\xi}_t^h + (1-\lambda_w)\hat{z}_t \\
&\quad - \iota^y\left(\frac{1-\lambda_w}{1-\iota^y}\right)\hat{\iota}_t^y - \iota^w\left(\frac{1-\lambda_w}{1+\iota^w}\right)\hat{\iota}_t^w\Big] + \theta_w\hat{w}_{t-1} + \beta\theta_w\mathbb{E}_t\hat{w}_{t+1} + \theta_w\chi_w\hat{\pi}_{t-1}^c - \beta\theta_w^2\chi_w\hat{\pi}_t^c - \theta_w\hat{\pi}_t \\
&\quad + \beta\theta_w^2\mathbb{E}_t\hat{\pi}_{t+1} - \beta\theta_w^2(1-\chi_w)\mathbb{E}_t\hat{\bar{\pi}}_{t+1}^c + \theta_w(1-\chi_w)\hat{\bar{\pi}}_t^c
\end{aligned}$$

$$\Rightarrow \qquad \widehat{w}_t = -\frac{1}{\phi_0}\left[\phi_1 \widehat{w}_{t-1} + \phi_2 \widehat{w}_{t+1} + \phi_3(\widehat{\pi}_t - \widehat{\bar{\pi}}_t^c) + \phi_4(\mathbb{E}_t \widehat{\pi}_{t+1} - \rho_\pi \widehat{\bar{\pi}}_t^c) + \phi_5(\widehat{\pi}_{t-1}^c - \widehat{\bar{\pi}}_t^c) + \phi_6(\widehat{\pi}_t^c - \rho_\pi \widehat{\bar{\pi}}_t^c) + \phi_7 \widehat{\psi}_t^z + \phi_8 \widehat{\iota}_t^y + \phi_9 \widehat{\iota}_t^w + \phi_{10} \widehat{H}_t + \phi_{11} \hat{\xi}_t^h\right] \qquad (L.9)$$

where

$\psi_t^z \equiv v_t P_t$ stationary Lagrange multiplier

$$b_w = \frac{\lambda_w + \lambda_w \sigma_L - 1}{(1-\theta_w)(1-\beta\theta_w)}$$

$$\phi_0 = \lambda_w \sigma_L - b_w(1 + \beta\theta_w^2)$$

$$\phi_1 = b_w \theta_w$$

$$\phi_2 = b_w \beta \theta_w$$

$$\phi_3 = -b_w \theta_w$$

$$\phi_4 = b_w \beta \theta_w^2$$

$$\phi_5 = b_w \theta_w \chi_w$$

$$\phi_6 = -b_w \beta \theta_w^2 \chi_w$$

$$\phi_7 = 1 - \lambda_w$$

$$\phi_8 = -\iota^y \left(\frac{1-\lambda_w}{1-\iota^y}\right)$$

$$\phi_9 = -\iota^w \left(\frac{1-\lambda_w}{1+\iota^w}\right)$$

$$\phi_{10} = -(1-\lambda_w)\sigma_L$$

$$\phi_{11} = -(1-\lambda_w)$$

A.5.5 Bond Investment

Households allocate wealth among cash Q_t, domestic bonds $B_t - Q_t$ and foreign bonds B_t^* with a one-year maturity. The corresponding nominal interest rates are R_t for B_t and R_t^* for B_t^*.

The purchase of foreign bonds is adjusted with a risk premium that depends on the domestic economy's indebtedness in the international asset market and expected depreciation of domestic currency: $\frac{S_{t+1}}{S_{t-1}}$:

$$\Phi\left(\frac{A_t}{z_t}, S_t, \tilde{\phi}_t\right) = e^{\left\{-\tilde{\phi}_a(a_t - a) - \tilde{\phi}_s\left[\frac{\mathbb{E}_t S_{t+1}}{S_t}\frac{S_t}{S_{t-1}} - \left(\frac{\pi}{\pi^*}\right)^2\right] + \tilde{\phi}_t\right\}} \qquad (43)$$

where

$A_t \equiv \frac{S_t B_{t+1}^*}{P_t^d}$ $\qquad (44)$: net foreign asset position

$a_t - a$ measures foreign asset position

$\frac{\pi}{\pi^*} = 1$ is steady state of inflation differential

$\tilde{\phi}_t$: shock component that follows AR(1), a first-order autoregressive model.

$\Phi(0,1,0) = 1$

Equation (43) indicates that if domestic economy is a net foreign borrower, investors need to pay a premium for foreign bond investment. If domestic currency is expected to depreciate, investment is also willing to pay a premium because the expected investment return in domestic currency is higher.

A.5.6 Equity Investment

Households own the firms and make investment in domestic and foreign equity. They decide how much to invest I_t and may purchase domestic (I_t^d) or imported investment goods (I_t^m):

$$I_t = \left[(1-\vartheta_i)^{\frac{1}{\eta_i}}(I_t^d)^{\frac{\eta_i-1}{\eta_i}} + (\vartheta_i)^{\frac{1}{\eta_i}}(I_t^m)^{\frac{\eta_i-1}{\eta_i}}\right]^{\frac{\eta_i}{\eta_i-1}} \qquad (45)$$

where

η_i: substitution elasticity between I_t^d and I_t^m

ϑ_i: share of imports in aggregate investment

The demand for domestic investment goods and imported investment goods can be determined as follows:

$$I_t^d = (1-\vartheta_i)\left(\frac{P_t^d}{P_t^i}\right)^{-\eta_i} I_t \qquad (46)$$

$$I_t^m = \vartheta_i\left(\frac{P_t^{m,i}}{P_t^i}\right)^{-\eta_i} I_t \qquad (47)$$

The inflation rate for aggregate investment goods is determined as

$$P_t^i = \left[(1-\vartheta_i)(P_t^d)^{1-\eta_i} + \vartheta_i\left(P_t^{m,i}\right)^{1-\eta_i}\right]^{\frac{1}{1-\eta_i}} \qquad (48)$$

Given households' investment decision, capital stock K_t follows

$$K_{t+1} = (1-\delta)K_t + \xi_t^i F(I_t, I_{t-1}) + \Delta_t \qquad (49)$$

where

ξ_t^i: investment specific technology shock. $\mathbb{E}(\xi_t^i) = 1$ $\quad \hat{\xi}_t^i = \rho_c \hat{\xi}_{t-1}^i + \varepsilon_t^i \quad \hat{\xi}_t^i = \frac{\xi_t^i - 1}{1}$

$F(I_t, I_{t-1})$: investment adjustment cost paid by households when the rate of change is not equal to μ^z.

Δ_t: installed capital that households may purchase in the secondary market from other households. In equilibrium $\Delta_t = 0$.

The investment adjustment cost function is assumed to take the following form:

$$F(I_t, I_{t-1}) = \left[1 - S\left(\frac{I_t}{I_{t-1}}\right)\right] I_t \qquad (50)$$

where

$$S\left(\frac{I_t}{I_{t-1}}\right) = \frac{\phi_i}{2}\left(\frac{I_t}{I_{t-1}} - \mu^z\right)^2$$

$$S(\mu^z) = S'(\mu^z)=0$$

$$S''(\mu^z) = \phi_i \quad \phi_i > 0$$

As (8) indicates, not all capital stock may be rented to firms. Utilization rate u_t determines the effective capital stock $K_t^s = u_t K_t$. Households are also required to pay capital adjustment cost $a(u_t)$ that meets the following conditions:

$$a(1) = 0 \quad a'(1) = (1 - \iota^k)\gamma^k \quad a''(1) \geq 0 \quad \sigma_a = \frac{a''(1)}{a'(1)} \geq 0$$

where

ι^k: capital gain tax rate

Loglinearize (49):

$$K_{t+1} = (1-\delta)K_t + \xi_t^i F(I_t, I_{t-1}) + \Delta_t = (1-\delta)K_t + \xi_t^i \left(1 - \frac{\phi_i}{2}\left(\frac{I_t}{I_{t-1}} - \mu^z\right)^2\right) I_t + \Delta_t$$

$$\Rightarrow \quad \frac{K_{t+1}}{z_t} = (1-\delta)\frac{K_t}{z_{t-1}}\frac{z_{t-1}}{z_t} + \xi_t^i\left(1 - \frac{\phi_i}{2}\left(\frac{i_t z_t}{i_{t-1} z_{t-1}} - \mu^z\right)^2\right) i_t + \frac{\Delta_t}{z_t}$$

$$\Rightarrow \quad k_{t+1} = (1-\delta)\frac{k_t}{\mu_t^z} + \xi_t^i\left(1 - \frac{\phi_i}{2}\left(\frac{i_t}{i_{t-1}}\mu_t^z - \mu^z\right)^2\right) i_t + \frac{\Delta_t}{z_t}$$

Steady state: $k = (1-\delta)\frac{k}{\mu^z} + i \quad \Rightarrow \quad i = k - (1-\delta)\frac{k}{\mu^z}$

$$\begin{aligned} k_{t+1} - k = {} & \frac{1-\delta}{\mu^z}(k_t - k) + i(\xi_t^i - \xi^i) + \left[-\frac{(1-\delta)k}{(\mu^z)^2} - \xi^i\frac{\phi_i}{2}2i(\mu^z - \mu^z)\right](\mu_t^z - \mu^z) \\ & + \xi^i\left[1 + i\left(-\frac{\phi_i}{2}2\frac{\mu^z}{i}(\mu^z - \mu^z)\right)\right](i_t - i) + \xi^i\left[i\left(-\frac{\phi_i}{2}2\left(-\frac{i\mu^z}{i^2}\right)(\mu^z - \mu^z)\right)\right](i_{t-1} - i) + \widetilde{\Delta}_t \end{aligned}$$

$$\Rightarrow \quad \hat{k}_{t+1} = \frac{1-\delta}{\mu^z}(\hat{k}_t - \hat{\mu}_t^z) + \frac{i}{k}(\hat{\xi}_t^i + \hat{\imath}_t) + \frac{\widetilde{\Delta}_t}{k}$$

When $\widetilde{\Delta}_t = 0$, which is expected in an equilibrium state,

$$\hat{k}_{t+1} = \frac{1-\delta}{\mu^z}\left(\hat{k}_t - \hat{\mu}_t^z\right) + \frac{i}{k}\left(\hat{\xi}_t^i + \hat{\imath}_t\right) \qquad (L.10)$$

A.5.7 Budget Constraint

Each household faces its own budget constraints that contains employment income, consumption, investment, tax and income and expense. The budget constraint is formalized in the following way:

$$\begin{aligned}
&\left(B_{j,t+1} - Q_{j,t+1}\right) + Q_{j,t+1} + S_t B_{j,t+1}^* + P_t^c C_{j,t}(1+\iota_t^c) + P_t^i I_{j,t} + P_t^d\left[a\left(u_{j,t}\right)K_{j,t} + P_t^{k'}\Delta_t\right] \\
&\quad = R_t\left(B_{j,t} - Q_{j,t}\right) + Q_{j,t} + R_t^*\Phi\left(\frac{A_{t-1}}{z_{t-1}}, S_{t-1}, \tilde{\phi}_{t-1}\right) S_t B_{j,t}^* + \frac{1-\iota_t^y}{1+\iota_t^w} W_{j,t} h_{j,t} + (1-\iota_t^k) R_t^k u_{j,t} K_{j,t} \\
&\quad + (1-\iota_t^k)\Pi_t + TR_t \\
&\quad - \iota_t^k\left((R_t - 1)\left(B_{j,t} - Q_{j,t}\right) + \left(R_t^*\Phi\left(\frac{A_{t-1}}{z_{t-1}}, S_{t-1}, \tilde{\phi}_{t-1}\right) - 1\right) S_t B_{j,t}^* + B_{j,t}^*(S_t - S_{t-1})\right) \qquad (51)
\end{aligned}$$

where

ι_t^c: aggregate consumption tax rate

ι_t^w: aggregate payroll tax rate

ι_t^y: aggregate income tax rate

ι_t^k: aggregate capital gain tax rate

$P_t^{k'}$: price of capital stock in the secondary market

Π_t: profit from equity investment in firms

TR_t: transfer payment

L.H.S. is the allocation of current available resources. R.H.S. carries forward the wealth of previous period and includes the income during the period.

A.5.8 Optimization

Households want to maximize the expected lifetime utility (31) subject to the constraints defined by capital stock flow (49) and budget planning (51):

$$\mathcal{L} = \mathbb{E}_0^j \sum_{t=0}^{\infty} \beta^t \left[\xi_t^c \ln\left(C_{j,t} - bC_{j,t-1}\right) - \xi_t^h A_L \frac{\left(h_{j,t}\right)^{1+\sigma_L}}{1+\sigma_L} + A_q \frac{\left(\frac{Q_{j,t}}{z_t P_t}\right)^{1-\sigma_q}}{1-\sigma_q} \right]$$
$$+ \sum_{t=0}^{\infty} \beta^t v_t \Bigg\{ R_{t-1}\left(B_{j,t} - Q_{j,t}\right) + Q_{j,t} + R_{t-1}^* \Phi\left(\frac{A_{t-1}}{z_{t-1}}, S_{t-1}, \tilde{\phi}_{t-1}\right) S_t B_{j,t}^* + \frac{1-\iota_t^y}{1+\iota_t^w} W_{j,t} h_{j,t}$$
$$+ (1-\iota_t^k) R_t^k u_{j,t} K_{j,t} + (1-\iota_t^k)\Pi_t + TR_t$$
$$- \iota_t^k \left((R_{t-1} - 1)\left(B_{j,t} - Q_{j,t}\right) + \left(R_{t-1}^* \Phi\left(\frac{A_{t-1}}{z_{t-1}}, S_{t-1}, \tilde{\phi}_{t-1}\right) - 1\right) S_t B_{j,t}^* + B_{j,t}^*(S_t - S_{t-1}) \right)$$
$$- \left[\left(B_{j,t+1} - Q_{j,t+1}\right) + Q_{j,t+1} + S_t B_{j,t+1}^* + P_t^c C_{j,t}(1+\iota_t^c) + P_t^i I_{j,t} + P_t^d\left[a\left(u_{j,t}\right)K_{j,t} + P_t^{k'} \Delta_t\right] \right] \Bigg\}$$
$$+ \sum_{t=0}^{\infty} \beta^t \omega_t \left[(1-\delta)K_t + \xi_t^i F(I_t, I_{t-1}) + \Delta_t - K_{t+1}\right] \qquad (52)$$

F.O.C. for C_t:

$$\frac{\partial \mathcal{L}}{\partial C_t} = \beta^t \xi_t^c \frac{1}{C_t - bC_{t-1}} + \beta^{t+1} \xi_t^c \mathbb{E}_t \left(\frac{1}{C_{t+1} - bC_t} (-b) \right) - v_t P_t^c (1+\iota_t^c) = 0$$

$$\Rightarrow \qquad \xi_t^c \frac{1}{\frac{C_t}{z_t} - b \frac{C_{t-1}}{z_{t-1}} \frac{z_{t-1}}{z_t}} + \beta \xi_t^c \mathbb{E}_t \frac{1}{\frac{C_{t+1}}{z_{t+1}} \frac{z_{t+1}}{z_t} - b \frac{C_t}{z_t}} (-b) - (v_t P_t) \frac{P_t^c}{P_t} z_t (1+\iota_t^c) = 0$$

$$\Rightarrow \qquad \frac{\xi_t^c}{c_t - bc_{t-1}\frac{1}{\mu_t^z}} - \beta b \mathbb{E}_t \frac{\xi_t^c}{c_{t+1}\mu_{t+1}^z - bc_t} - \psi_t^z \frac{P_t^c}{P_t}(1+\iota_t^c) = 0 \qquad (53)$$

where

$$\psi_t^z = v_t P_t z_t = \psi_t z_t$$

Steady state:

$$\frac{1}{c - \frac{bc}{\mu^z}} - \beta b \frac{1}{c\mu^z - bc} - \psi^z \frac{P}{P}(1+\iota_t^c) = 0$$

$$\Rightarrow \qquad \psi^z = \frac{\frac{1}{c - \frac{bc}{\mu^z}} - \beta b \frac{1}{c\mu^z - bc}}{1+\iota^c} = \frac{\mu^z - \beta b}{(c\mu^z - bc)(1+\iota^c)}$$

Loglinearize (53):

$$\mathbb{E}_t\left[\frac{1}{c-\frac{bc}{\mu^z}}(\xi_t^c-\xi^c)-\frac{\beta b}{c\mu^z-bc}(\xi_{t+1}^c-\xi^c)+\left(-\frac{1}{\left(c-\frac{bc}{\mu^z}\right)^2}\right)(-bc)\left(-\frac{1}{(\mu^z)^2}\right)(\mu_t^z-\mu^z)\right.$$

$$-\beta b\left(-\frac{1}{(c\mu^z-bc)^2}c\right)(\mu_{t+1}^z-\mu^z)+\left(-\frac{1}{\left(c-\frac{bc}{\mu^z}\right)^2}\left(-\frac{b}{\mu^z}\right)\right)(c_{t-1}-c)$$

$$-\beta b\left(-\frac{1}{(c\mu^z-bc)^2}\mu^z\right)(c_{t+1}-c)+\left(-\frac{1}{\left(c-\frac{bc}{\mu^z}\right)^2}-\beta b\left(-\frac{1}{(c\mu^z-bc)^2}\right)(-b)\right)(c_t-c)$$

$$\left.-(\psi_t^z-\psi^z)(1+\iota^c)-\frac{\psi^z}{P}(P_t^c-P)(1+\iota^c)-\psi^z P\left(-\frac{1}{P^2}\right)(P_t-P)(1+\iota^c)-\psi^z(\iota_t^c-\iota^c)\right]=0$$

$$\Rightarrow \quad \mathbb{E}_t\left[\frac{\mu^z}{c(\mu^z-b)}\hat{\xi}_t^c-\frac{\beta b}{c(\mu^z-b)}\hat{\xi}_{t+1}^c-\frac{bc\mu^z}{c^2(\mu^z-b)^2}\hat{\mu}_t^z-\frac{\beta bc\mu^z}{c^2(\mu^z-b)^2}\hat{\mu}_{t+1}^z+\frac{bc\mu^z}{c^2(\mu^z-b)^2}\hat{c}_{t-1}\right.$$
$$-\frac{\mu^{z2}+\beta b^2}{c(\mu^z-b)^2}\hat{c}_t+\frac{\beta bc\mu^z}{c^2(\mu^z-b)^2}\hat{c}_{t+1}-\frac{(\psi_t^z-\psi^z)}{\psi^z}\psi^z(1+\iota^c)-\psi^z(p_t^c-p_t)(1+\iota^c)$$
$$\left.-\psi^z\iota^c\frac{(\iota_t^c-\iota^c)}{\iota^c}\right]=0$$

$$\Rightarrow \quad \mathbb{E}_t\left[\frac{\mu^z}{c(\mu^z-b)}\hat{\xi}_t^c-\frac{\beta b}{c(\mu^z-b)}\hat{\xi}_{t+1}^c-\frac{bc\mu^z}{c^2(\mu^z-b)^2}\hat{\mu}_t^z-\frac{\beta bc\mu^z}{c^2(\mu^z-b)^2}\hat{\mu}_{t+1}^z+\frac{bc\mu^z}{c^2(\mu^z-b)^2}\hat{c}_{t-1}\right.$$
$$\left.-\frac{\mu^{z2}+\beta b^2}{c(\mu^z-b)^2}\hat{c}_t+\frac{\beta bc\mu^z}{c^2(\mu^z-b)^2}\hat{c}_{t+1}-\psi^z(1+\iota^c)\left(\hat{\psi}_t^z+\hat{\gamma}_t^{c,d}\right)-\psi^z\iota^c\hat{\iota}_t^c\right]=0$$

$$\Rightarrow \quad \hat{c}_t=\frac{\mu^z b}{(\mu^z)^2+\beta b^2}\hat{c}_{t-1}+\frac{\beta\mu^z b}{(\mu^z)^2+\beta b^2}\mathbb{E}_t\hat{c}_{t+1}-\frac{\mu^z b}{(\mu^z)^2+\beta b^2}(\hat{\mu}_t^z-\beta\mathbb{E}_t\hat{\mu}_{t+1}^z)$$
$$+\frac{\mu^z-b}{(\mu^z)^2+\beta b^2}\left(\mu^z\hat{\xi}_t^c-\beta b\mathbb{E}_t\hat{\xi}_{t+1}^c\right)-\frac{\psi^z c(\mu^z-b)^2(1+\iota^c)}{(\mu^z)^2+\beta b^2}\left(\hat{\psi}_t^z+\hat{\gamma}_t^{c,d}\right)-\frac{\psi^z c(\mu^z-b)^2\iota^c}{(\mu^z)^2+\beta b^2}\hat{\iota}_t^c$$

$$\psi^z=\frac{\mu^z-\beta b}{(c\mu^z-bc)(1+\iota^c)}$$

$$\Rightarrow \quad \psi^z c(\mu^z-b)^2=\frac{\mu^z-\beta b}{(c\mu^z-bc)(1+\iota^c)}c(\mu^z-b)^2=\frac{(\mu^z-\beta b)(\mu^z-b)}{1+\iota^c}$$

$$\Rightarrow \quad \hat{c}_t=\frac{\mu^z b}{(\mu^z)^2+\beta b^2}\hat{c}_{t-1}+\frac{\beta\mu^z b}{(\mu^z)^2+\beta b^2}\mathbb{E}_t\hat{c}_{t+1}-\frac{\mu^z b}{(\mu^z)^2+\beta b^2}(\hat{\mu}_t^z-\beta\mathbb{E}_t\hat{\mu}_{t+1}^z)$$
$$+\frac{\mu^z-b}{(\mu^z)^2+\beta b^2}\left(\mu^z\hat{\xi}_t^c-\beta b\mathbb{E}_t\hat{\xi}_{t+1}^c\right)-\frac{(\mu^z-\beta b)(\mu^z-b)}{(\mu^z)^2+\beta b^2}\left(\hat{\psi}_t^z+\hat{\gamma}_t^{c,d}\right)$$
$$-\frac{(\mu^z-\beta b)(\mu^z-b)}{(\mu^z)^2+\beta b^2}\frac{\iota^c}{1+\iota^c}\hat{\iota}_t^c \qquad (L.11)$$

F.O.C. for Δ_t:

$$\frac{\partial \mathcal{L}}{\partial \Delta_t} = \omega_t + v_t\left(-P_t^d P_t^{k'}\right) = 0$$

$$\Rightarrow \qquad \omega_t - \psi_t P_t^{k'} = 0 \qquad (54)$$

F.O.C. for I_t:

$$\frac{\partial \mathcal{L}}{\partial I_t} = -v_t P_t^i + \omega_t \xi_t^i F_1(I_t, I_{t-1}) + \beta \mathbb{E}_t \omega_{t+1} \xi_{t+1}^i F_2(I_{t+1}, I_t) = 0$$

Replace ω_t with $\psi_t P_t^{k'}$ as in (54):

$$\Rightarrow \qquad -v_t P_t \frac{P_t^i}{P_t} + \psi_t P_t^{k'} \xi_t^i F_1(I_t, I_{t-1}) + \beta \mathbb{E}_t \psi_{t+1} P_{t+1}^{k'} \xi_{t+1}^i F_2(I_{t+1}, I_t) = 0$$

$$\Rightarrow \qquad -(v_t P_t z_t) \frac{P_t^i}{P_t} + (\psi_t z_t) P_t^{k'} \xi_t^i F_1(I_t, I_{t-1}) + \beta \mathbb{E}_t (\psi_{t+1} z_{t+1}) \frac{z_t}{z_{t+1}} P_{t+1}^{k}{}' \xi_{t+1}^i F_2(I_{t+1}, I_t) = 0$$

$$\Rightarrow \qquad -\psi_t^z \frac{P_t^i}{P_t} + \psi_t^z P_t^{k'} \xi_t^i F_1(I_t, I_{t-1}) + \beta \mathbb{E}_t \frac{\psi_{t+1}^z}{\mu_{t+1}^z} P_{t+1}^{k}{}' \xi_{t+1}^i F_2(I_{t+1}, I_t) = 0 \qquad (55)$$

$$F(I_t, I_{t-1}) = \left[1 - \frac{\phi_i}{2}\left(\frac{I_t}{I_{t-1}} - \mu^z\right)^2\right] I_t$$

$$\begin{aligned} F_1(I_t, I_{t-1}) &= 1 - I_t \phi_i \left(\frac{I_t}{I_{t-1}} - \mu^z\right) \frac{1}{I_{t-1}} - \frac{\phi_i}{2}\left(\frac{I_t}{I_{t-1}} - \mu^z\right)^2 \\ &= 1 - \phi_i \left(\frac{i_t}{i_{t-1}} \mu_t^z\right)^2 + \phi_i \frac{i_t}{i_{t-1}} \mu_t^z \mu^z - \frac{\phi_i}{2}\left(\frac{i_t}{i_{t-1}} \mu_t^z\right)^2 + \phi_i \frac{i_t}{i_{t-1}} \mu_t^z \mu^z - \frac{\phi_i}{2} (\mu^z)^2 \\ &= 1 - \frac{3}{2} \phi_i \left(\frac{i_t}{i_{t-1}} \mu_t^z\right)^2 + 2\phi_i \frac{i_t}{i_{t-1}} \mu_t^z \mu^z - \frac{\phi_i}{2} (\mu^z)^2 \end{aligned}$$

$$F_2(I_{t+1}, I_t) = -I_{t+1} \phi_i \left(\frac{I_{t+1}}{I_t} - \mu^z\right) I_{t+1} \left(-\frac{1}{I_t^2}\right) = \phi_i \left(\frac{i_{t+1}}{i_t} \mu_{t+1}^z\right)^3 - \phi_i \left(\frac{i_{t+1}}{i_t} \mu_{t+1}^z\right)^2 \mu^z$$

Replace $F_1(I_t, I_{t-1})$ and $F_2(I_{t+1}, I_t)$ in (55):

$$\begin{aligned} \Rightarrow \qquad -\psi_t^z \frac{P_t^i}{P_t} &+ \psi_t^z P_t^{k'} \xi_t^i \left(1 - \frac{3}{2} \phi_i \left(\frac{i_t}{i_{t-1}} \mu_t^z\right)^2 + 2\phi_i \frac{i_t}{i_{t-1}} \mu_t^z \mu^z - \frac{\phi_i}{2} (\mu^z)^2\right) \\ &+ \beta \mathbb{E}_t \frac{\psi_{t+1}^z}{\mu_{t+1}^z} P_{t+1}^{k}{}' \xi_{t+1}^i \phi_i \left(\left(\frac{i_{t+1}}{i_t} \mu_{t+1}^z\right)^3 - \left(\frac{i_{t+1}}{i_t} \mu_{t+1}^z\right)^2 \mu^z\right) = 0 \end{aligned}$$

Loglinearize (55):

$$\left\{\psi^z P^{k'}\left[-\frac{1}{i^2}3i\phi_i(\mu^z)^2+\frac{1}{i}2\phi_i(\mu^z)^2\right]+\beta\frac{\psi^z}{\mu^z}P^{k'}\phi_i\left[i^3(\mu^z)^3(-3)\frac{1}{i^4}-i^2(\mu^z)^3(-2)\frac{1}{i^3}\right]\right\}(i_t-i)$$
$$+\beta\frac{\psi^z}{\mu^z}P^{k'}\phi_i\left[\frac{1}{i^3}(\mu^z)^3 3i^2-\frac{1}{i^2}(\mu^z)^3 2i\right](i_{t+1}-i)$$
$$+\psi^z P^{k'}\left[-\frac{3}{2}\phi_i i^2(\mu^z)^2(-2)\frac{1}{i^3}+2\phi_i i(\mu^z)^2\left(-\frac{1}{i^2}\right)\right](i_{t-1}-i)$$
$$+\psi^z P^{k'}\phi_i(-3\mu^z+2\mu^z)(\mu_t^z-\mu^z)+\beta\psi^z P^{k'}\phi_i[3\mu^z-2\mu^z](\mu_{t+1}^z-\mu^z)+\psi^z\left(P_t^{k'}-P^{k'}\right)$$
$$+0\left(P_{t+1}^{k}{}'-P^{k'}\right)+\psi^z P^{k'}\left(\xi_t^i-\xi^i\right)+0\left(\xi_{t+1}^i-\xi^i\right)-\psi^z\left(\gamma_t^{i,d}-\gamma^{i,d}\right)=0$$

$$\Rightarrow\qquad \left\{\psi^z P^{k'}\left[-\frac{1}{i}\phi_i(\mu^z)^2\right]+\beta\psi^z P^{k'}\phi_i\left[-\frac{1}{i}(\mu^z)^2\right]\right\}(i_t-i)+\beta\psi^z P^{k'}\phi_i\left[\frac{1}{i}(\mu^z)^2\right](i_{t+1}-i)$$
$$+\psi^z P^{k'}\phi_i\left[(\mu^z)^2\frac{1}{i}\right](i_{t-1}-i)+\psi^z P^{k'}\phi_i(-\mu^z)(\mu_t^z-\mu^z)+\beta\psi^z P^{k'}\phi_i[\mu^z](\mu_{t+1}^z-\mu^z)$$
$$+\psi^z P^{k'}\frac{P_t^{k'}-P^{k'}}{P^{k'}}+\psi^z P^{k'}\left(\xi_t^i-\xi^i\right)-\psi^z\left(\gamma_t^{i,d}-\gamma^{i,d}\right)=0$$

$$\Rightarrow\qquad \psi^z P^{k'}\phi_i(\mu^z)^2[-(1+\beta)\hat{\imath}_t+\beta\hat{\imath}_{t+1}+\hat{\imath}_{t-1}-\hat{\mu}_t^z+\beta\hat{\mu}_{t+1}^z]+\psi^z P^{k'}\left(\hat{p}_t^{k'}+\hat{\xi}_t^i-\hat{\gamma}_t^{i,d}\right)=0 \qquad (L.12)$$

F.O.C. for K_{t+1}:

$$\frac{\partial\mathcal{L}}{\partial K_{t+1}}=\beta v_{t+1}(1-\iota_{t+1}^k)R_{t+1}^k u_{t+1}-\beta v_{t+1}P_{t+1}^d a(u_{t+1})-\omega_t+\beta\omega_{t+1}(1-\delta)=0$$

$$\Rightarrow\qquad \mathbb{E}_t\left[\beta v_{t+1}(1-\iota_{t+1}^k)R_{t+1}^k u_{t+1}-\beta v_{t+1}P_{t+1}^d a(u_{t+1})-\psi_t P_t^{k'}+\beta\psi_{t+1}P_{t+1}^{k}{}'(1-\delta)\right]=0$$

$$\Rightarrow\qquad -\psi_t P_t^{k'}z_t+\beta\mathbb{E}_t\left[\frac{v_{t+1}z_{t+1}(1-\iota_{t+1}^k)R_{t+1}^k u_{t+1}-\psi_{t+1}z_{t+1}P_{t+1}^d a(u_{t+1})+\psi_{t+1}z_{t+1}P_{t+1}^{k}{}'(1-\delta)}{\frac{z_{t+1}}{z_t}}\right]=0$$

$$\Rightarrow\qquad -\psi_t^z P_t^{k'}+\beta\mathbb{E}_t\frac{\psi_{t+1}^z}{\mu_{t+1}^z}\left[(1-\iota_{t+1}^k)r_{t+1}^k u_{t+1}-a(u_{t+1})+P_{t+1}^{k}{}'(1-\delta)\right]=0 \qquad (56)$$

where

$$u=1 \qquad a(1)=0 \qquad a'(1)=(1-\iota^k)r^k \qquad a''(1)=\sigma_a r^k$$

Steady state:

$$-\psi^z P^{k'}+\beta\frac{\psi^z}{\mu^z}\left((1-\iota^k)r^k+P^{k'}(1-\delta)\right)=0$$

$$\Rightarrow\qquad \beta\frac{\psi^z}{\mu^z}(1-\iota^k)r^k=\psi^z P^{k'}-\psi^z P^{k'}\frac{\beta(1-\delta)}{\mu^z}=\psi^z P^{k'}\left(1-\frac{\beta(1-\delta)}{\mu^z}\right)$$

Loglinearize (56):

$$
\begin{aligned}
\mathbb{E}_t \Big\{ & -\psi^z\left(P_t^{k'} - P^{k'}\right) + \beta(1-\delta)\frac{\psi^z}{\mu^z}\left(P_{t+1}^{k}{}' - P^{k'}\right) - P^{k'}(\psi_t^z - \psi^z) \\
& + \beta\frac{1}{\mu^z}\Big((1-\iota^k)r^k + P^{k'}(1-\delta)\Big)(\psi_{t+1}^z - \psi^z) + \beta\frac{\psi^z}{\mu^z}(1-\iota^k)(r_{t+1}^k - r^k) \\
& + \beta\frac{\psi^z}{\mu^z}\big((1-\iota^k)r^k - a'(1)\big)(u_{t+1} - u) + \beta\psi^z\left((1-\iota^k)r^k + P^{k'}(1-\delta)\right)\left(-\frac{1}{(\mu^z)^2}\right)(\mu_t^z - \mu^z) \\
& - \beta\frac{\psi^z}{\mu^z}r^k(\iota_{t+1}^k - \iota^k)\Big\} = 0
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \quad \psi^z P^{k'}\hat{p}_t^{k'} = \mathbb{E}_t \Big\{ & \psi^z P^{k'}\Big[\frac{\beta(1-\delta)}{\mu^z}\hat{p}_{t+1}^{k}{}' - \hat{\psi}_t^z + \hat{\psi}_{t+1}^z - \hat{\mu}_{t+1}^z + \frac{\mu^z - \beta(1-\delta)}{\mu^z}\hat{r}_{t+1}^k \\
& - \frac{\mu^z - \beta(1-\delta)}{\mu^z}\frac{\iota^k}{1-\iota^k}\hat{\iota}_{t+1}^k\Big]\Big\}
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \quad \hat{p}_t^{k'} = \mathbb{E}_t \Big[& \frac{\beta(1-\delta)}{\mu^z}\hat{p}_{t+1}^{k}{}' - \hat{\psi}_t^z + \hat{\psi}_{t+1}^z - \hat{\mu}_{t+1}^z + \frac{\mu^z - \beta(1-\delta)}{\mu^z}\hat{r}_{t+1}^k \\
& - \frac{\mu^z - \beta(1-\delta)}{\mu^z}\frac{\iota^k}{1-\iota^k}\hat{\iota}_{t+1}^k\Big] \qquad (L.13)
\end{aligned}
$$

F.O.C. for u_t:

$$
\frac{\partial \mathcal{L}}{\partial u_t} = v_t\left((1-\iota_t^k)R_t^k K_t - P_t^d a'(u_t)K_t\right) = 0
$$

$$
\Rightarrow \quad v_t P_t^d\left(\frac{(1-\iota_t^k)R_t^k}{P_t^d} - a'(u_t)\right) = 0
$$

$$
\Rightarrow \quad \psi_t\left((1-\iota_t^k)r_t^k - a'(u_t)\right) = 0 \qquad (57)
$$

Loglinearize (57):

$$
\psi(1-\iota^k)(r_t^k - r^k) - \psi r^k(\iota_t^k - \iota^k) - \psi(1-\iota^k)a''(u)(u_t - u) = 0
$$

$$
\Rightarrow \quad \psi r^k(1-\iota^k)\left(\hat{r}_t^k - \frac{\iota^k}{1-\iota^k}\hat{\iota}_t^k - \frac{a''(u)}{r^k}\hat{u}_t\right) = 0
$$

$$
a''(u) = \sigma_a r^k
$$

$$
\Rightarrow \quad \hat{u}_t = \frac{1}{\sigma_a}\left(\hat{r}_t^k - \frac{\iota^k}{1-\iota^k}\hat{\iota}_t^k\right) \qquad (L.14)
$$

F.O.C. for Q_t:

$$
\frac{\partial \mathcal{L}}{\partial Q_t} = \mathbb{E}_t\left(A_q\left(\frac{Q_{j,t}}{z_t P_t}\right)^{-\sigma_q}\frac{1}{z_t P_t} + v_t\left(1 - R_{t-1} + \iota_t^k(R_{t-1} - 1)\right)\right) = 0
$$

$$
\Rightarrow \quad A_q(q_t)^{-\sigma_q} + v_t z_t P_t(1 - R_{t-1})(1 - \iota_t^k) = 0
$$

$$\Rightarrow \qquad A_q(q_t)^{-\sigma_q} - \psi_t^z(R_{t-1}-1)(1-\iota_t^k) = 0 \qquad (58)$$

Steady state:

$$A_q(q)^{-\sigma_q} - \psi^z(R-1)(1-\iota^k) = 0$$

$$\Rightarrow \qquad A_q(q)^{-\sigma_q} = \psi^z(R-1)(1-\iota^k)$$

$$\Rightarrow \qquad q = \left(\frac{A_q}{\psi^z(R-1)(1-\iota^k)}\right)^{\frac{1}{\sigma_q}}$$

Loglinearize (58):

$$A_q\left(-\sigma_q\right)(q)^{-\sigma_q-1}(q_t-q) - (R-1)(1-\iota^k)(\psi_t^z-\psi^z) + \psi^z(R-1)(\iota_t^k-\iota^k) - \psi^z(1-\iota^k)(R_{t-1}-R) = 0$$

$$\Rightarrow \qquad A_q(q)^{-\sigma_q}\left(-\sigma_q\right)\hat{q}_t = \psi^z(R-1)(1-\iota^k)\left(-\hat{\psi}_t^z + \frac{\iota^k}{1-\iota^k}\hat{\iota}_t^k - \frac{R}{R-1}\hat{R}_{t-1}\right)$$

$$\Rightarrow \qquad \hat{q}_t = \frac{1}{\sigma_q}\left(-\hat{\psi}_t^z + \frac{\iota^k}{1-\iota^k}\hat{\iota}_t^k - \frac{R}{R-1}\hat{R}_{t-1}\right) \qquad (L.15)$$

F.O.C. for B_{t+1}:

$$\frac{\partial \mathcal{L}}{\partial B_{t+1}} = \mathbb{E}_t\left(\beta v_{t+1}\left(R_t - \iota_{t+1}^k(R_t-1)\right) - v_t\right) = 0$$

$$\Rightarrow \qquad -v_tP_tz_t\frac{1}{P_tz_t} + \mathbb{E}_t\frac{\beta v_{t+1}P_{t+1}z_{t+1}}{P_{t+1}z_{t+1}}\left(R_t - \iota_{t+1}^k(R_t-1)\right) = 0$$

$$\Rightarrow \qquad -\psi_t^z + \mathbb{E}_t\frac{\beta\psi_{t+1}^z}{\pi_{t+1}\mu_{t+1}^z}\left(R_t - \iota_{t+1}^k(R_t-1)\right) = 0$$

$$\Rightarrow \qquad -\psi_t^z + \beta\mathbb{E}_t\left[\frac{\psi_{t+1}^z}{\mu_{t+1}^z\pi_{t+1}}\left(R_t - \iota_{t+1}^k(R_t-1)\right)\right] = 0 \qquad (59)$$

Steady state: $-\psi^z + \beta\frac{\psi^z}{\mu^z}\left(R - \iota^k(R-1)\right) = 0 \quad \Rightarrow \quad \psi^z = \beta\frac{\psi^z}{\mu^z}\left(R - \iota^k(R-1)\right)$

Loglinearize (59):

$$\mathbb{E}_t\Bigg(-(\psi_t^z-\psi^z) + \frac{\beta\left(R-\iota^k(R-1)\right)}{\mu^z}(\psi_{t+1}^z-\psi^z) + \beta\frac{\psi^z}{\mu^z}(1-\iota^k)(R_t-R) - \beta\frac{\psi^z}{\mu^z}(R-1)(\iota_{t+1}^k-\iota^k)$$
$$+ \beta\psi^z\left(R-\iota^k(R-1)\right)\left(-\frac{1}{(\mu^z)^2}\right)(\mu_{t+1}^z-\mu^z) + \beta\frac{\psi^z}{\mu^z}\left(R-\iota^k(R-1)\right)\left(-\frac{1}{\pi^2}\right)(\pi_{t+1}-\pi)\Bigg) = 0$$

$$\Rightarrow \qquad \psi^z\hat{\psi}_t^z = \frac{\beta\psi^z\left(R-\iota^k(R-1)\right)}{\mu^z}\mathbb{E}_t\left(\hat{\psi}_{t+1}^z + \frac{R(1-\iota^k)}{R-\iota^k(R-1)}\hat{R}_t - \frac{\iota^k(R-1)}{R-\iota^k(R-1)}\hat{\iota}_{t+1}^k - \hat{\mu}_{t+1}^z - \hat{\pi}_{t+1}\right)$$

$$\Rightarrow \qquad \hat{\psi}_t^z = \mathbb{E}_t\left(\hat{\psi}_{t+1}^z + \frac{R(1-\iota^k)}{R-\iota^k(R-1)}\hat{R}_t - \frac{\iota^k(R-1)}{R-\iota^k(R-1)}\hat{\iota}_{t+1}^k - \hat{\mu}_{t+1}^z - \hat{\pi}_{t+1}\right) \qquad (L.16)$$

F.O.C. for B_{t+1}^*:

$$\frac{\partial\mathcal{L}}{\partial B_{t+1}^*} = \mathbb{E}_t\left\{\beta v_{t+1}\left[R_t^*\Phi\left(a_t,S_t,\tilde{\phi}_t\right)S_{t+1} - \iota_{t+1}^k\left(\left(R_t^*\Phi\left(a_t,S_t,\tilde{\phi}_t\right)-1\right)S_{t+1} + (S_{t+1}-S_t)\right)\right]\right\} - v_tS_t = 0$$

$$\Rightarrow \qquad \mathbb{E}_t\Bigg(\beta(v_{t+1}P_{t+1}z_{t+1})\left[R_t^*\Phi\left(a_t,S_t,\tilde{\phi}_t\right)S_{t+1} - \iota_{t+1}^k\left(\left(R_t^*\Phi\left(a_t,S_t,\tilde{\phi}_t\right)-1\right)S_{t+1} + (S_{t+1}-S_t)\right)\right]$$
$$- (v_tP_tz_t)S_t\frac{P_{t+1}}{P_t}\frac{z_{t+1}}{z_t}\Bigg) = 0$$

$$\Rightarrow \qquad \mathbb{E}_t\left(\frac{\beta\psi_{t+1}^z}{\mu_{t+1}^z\pi_{t+1}}\left[R_t^*\Phi\left(a_t,S_t,\tilde{\phi}_t\right)S_{t+1} - \iota_{t+1}^k\left(\left(R_t^*\Phi\left(a_t,S_t,\tilde{\phi}_t\right)-1\right)S_{t+1} + (S_{t+1}-S_t)\right)\right] - \psi_t^zS_t\right)$$
$$= 0 \qquad (60)$$

where

$$\Phi\left(a_t,S_t,\tilde{\phi}_t\right) = e^{\left\{-\tilde{\phi}_a(a_t-a)-\tilde{\phi}_s\left[\frac{\mathbb{E}_tS_{t+1}}{S_t}\frac{S_t}{S_{t-1}}-\left(\frac{\pi}{\pi^*}\right)^2\right]+\tilde{\phi}_t\right\}} = e^{\left\{-\tilde{\phi}_a(a_t-a)-\tilde{\phi}_s\left[\frac{\mathbb{E}_tS_{t+1}}{S_{t-1}}-\left(\frac{\pi}{\pi^*}\right)^2\right]+\tilde{\phi}_t\right\}}$$

Steady state: $\beta\frac{\psi^z}{\mu^z}\left(R-\iota^k(R-1)\right) - \psi^z = 0 \quad \Rightarrow \quad \psi^z = \beta\frac{\psi^z}{\mu^z}\left(R-\iota^k(R-1)\right)$

Loglinearize (60):

$$\mathbb{E}_t\Bigg(-S(\psi_t^z-\psi^z) + \frac{\beta(RS-\iota^k(R-1)S)}{\mu^z}(\psi_{t+1}^z-\psi^z) + \beta\frac{\psi^z}{\mu^z}S(1-\iota^k)(R_t^*-R) - \beta\frac{\psi^z}{\mu^z}(R-1)S(\iota_{t+1}^k-\iota^k)$$
$$+ \beta\psi^z(RS-\iota^k(R-1)S)\left(-\frac{1}{(\mu^z)^2}\right)(\mu_{t+1}^z-\mu^z) + \beta\frac{\psi^z}{\mu^z}(RS-\iota^k(R-1)S)\left(-\frac{1}{\pi^2}\right)(\pi_{t+1}-\pi)$$
$$+ \beta\frac{\psi^z}{\mu^z}\left(RS\left(-\tilde{\phi}_a\right) - \iota^kRS\left(-\tilde{\phi}_a\right)\right)(a_t-a) + \beta\frac{\psi^z}{\mu^z}(RS-\iota^kRS)\left(\tilde{\phi}_t-\tilde{\phi}\right)$$
$$+ \beta\frac{\psi^z}{\mu^z}\left(R + RS\left(-\frac{\tilde{\phi}_s}{S}\right) - \iota^k - \iota^k(R-1) - \iota^kSR\left(-\frac{\tilde{\phi}_s}{S}\right)\right)(S_{t+1}-S)$$
$$+ \left(\beta\frac{\psi^z}{\mu^z}\iota^k - \psi^z\right)(S_t-S) + \beta\frac{\psi^z}{\mu^z}\left(RS\left(-\tilde{\phi}_s\right)S\left(-\frac{1}{S^2}\right) - \iota^kRS\left(-\tilde{\phi}_s\right)S\left(-\frac{1}{S^2}\right)\right)(S_{t-1}-S)\Bigg) = 0$$

$$\Rightarrow \qquad \mathbb{E}_t\Bigg(-(\psi_t^z-\psi^z) + \frac{\beta\left(R-\iota^k(R-1)\right)}{\mu^z}(\psi_{t+1}^z-\psi^z) + \beta\frac{\psi^z}{\mu^z}(1-\iota^k)(R_t^*-R) - \beta\frac{\psi^z}{\mu^z}(R-1)(\iota_{t+1}^k-\iota^k)$$
$$+ \beta\psi^z\left(R-\iota^k(R-1)\right)\left(-\frac{1}{(\mu^z)^2}\right)(\mu_{t+1}^z-\mu^z) + \beta\frac{\psi^z}{\mu^z}\left(R-\iota^k(R-1)\right)\left(-\frac{1}{\pi^2}\right)(\pi_{t+1}-\pi)$$
$$+ \beta\frac{\psi^z}{\mu^z}R\left(-\tilde{\phi}_a\right)(1-\iota^k)(a_t-a) + \beta\frac{\psi^z}{\mu^z}R(1-\iota^k)\left(\tilde{\phi}_t-\tilde{\phi}\right) + \beta\frac{\psi^z}{\mu^z}R\left(1-\tilde{\phi}_s\right)(1-\iota^k)\frac{S_{t+1}-S}{S}$$
$$+ \left(\beta\frac{\psi^z}{\mu^z}\iota^k - \psi^z\right)\frac{S_t-S}{S} + \beta\frac{\psi^z}{\mu^z}R\tilde{\phi}_s(1-\iota^k)\frac{S_{t-1}-S}{S}\Bigg) = 0$$

$$\Rightarrow \quad \psi^z\hat{\psi}_t^z = \frac{\beta\psi^z\big(R-\iota^k(R-1)\big)}{\mu^z}\mathbb{E}_t\Bigg(\hat{\psi}_{t+1}^z + \frac{R(1-\iota^k)}{R-\iota^k(R-1)}\hat{R}_t^* - \frac{\iota^k(R-1)}{R-\iota^k(R-1)}\hat{\iota}_{t+1}^k - \hat{\mu}_{t+1}^z - \hat{\pi}_{t+1} + \frac{R\big(-\tilde{\phi}_a\big)(1-\iota^k)a}{R-\iota^k(R-1)}\hat{a}_t + \frac{R(1-\iota^k)\tilde{\phi}}{R-\iota^k(R-1)}\hat{\tilde{\phi}}_t + \frac{R\big(1-\tilde{\phi}_s\big)(1-\iota^k)}{R-\iota^k(R-1)}\hat{S}_{t+1} - \frac{R(1-\iota^k)}{R-\iota^k(R-1)}\hat{S}_t + \frac{R\tilde{\phi}_s(1-\iota^k)}{R-\iota^k(R-1)}\hat{S}_{t-1}\Bigg)$$

$$\Rightarrow \quad \hat{\psi}_t^z = \mathbb{E}_t\Bigg(\hat{\psi}_{t+1}^z + \frac{R(1-\iota^k)}{R-\iota^k(R-1)}\hat{R}_t^* - \frac{\iota^k(R-1)}{R-\iota^k(R-1)}\hat{\iota}_{t+1}^k - \hat{\mu}_{t+1}^z - \hat{\pi}_{t+1} + \frac{R\big(-\tilde{\phi}_a\big)(1-\iota^k)a}{R-\iota^k(R-1)}\hat{a}_t + \frac{R(1-\iota^k)\tilde{\phi}}{R-\iota^k(R-1)}\hat{\tilde{\phi}}_t + \frac{R\big(1-\tilde{\phi}_s\big)(1-\iota^k)}{R-\iota^k(R-1)}\hat{S}_{t+1} - \frac{R(1-\iota^k)}{R-\iota^k(R-1)}\hat{S}_t + \frac{R\tilde{\phi}_s(1-\iota^k)}{R-\iota^k(R-1)}\hat{S}_{t-1}\Bigg) \qquad (L.17)$$

Equations (L.16) and (L.17) lead to uncovered interest rate parity (UIP) in the model. UIP means that interest rate difference between two countries should be compensated by the expected change in exchange rate. It is an arbitrage-free condition:

$$\hat{\psi}_t^z = \mathbb{E}_t\left(\hat{\psi}_{t+1}^z + \frac{R(1-\iota^k)}{R-\iota^k(R-1)}\hat{R}_t - \frac{\iota^k(R-1)}{R-\iota^k(R-1)}\hat{\iota}_{t+1}^k - \hat{\mu}_{t+1}^z - \hat{\pi}_{t+1}\right) \qquad (L.16)$$

$$\hat{\psi}_t^z = \mathbb{E}_t\Bigg(\hat{\psi}_{t+1}^z + \frac{R(1-\iota^k)}{R-\iota^k(R-1)}\hat{R}_t^* - \frac{\iota^k(R-1)}{R-\iota^k(R-1)}\hat{\iota}_{t+1}^k - \hat{\mu}_{t+1}^z - \hat{\pi}_{t+1} + \frac{R\big(-\tilde{\phi}_a\big)(1-\iota^k)a}{R-\iota^k(R-1)}\hat{a}_t + \frac{R(1-\iota^k)\tilde{\phi}}{R-\iota^k(R-1)}\hat{\tilde{\phi}}_t + \frac{R\big(1-\tilde{\phi}_s\big)(1-\iota^k)}{R-\iota^k(R-1)}\hat{S}_{t+1} - \frac{R(1-\iota^k)}{R-\iota^k(R-1)}\hat{S}_t + \frac{R\tilde{\phi}_s(1-\iota^k)}{R-\iota^k(R-1)}\hat{S}_{t-1}\Bigg) \qquad (L.17)$$

$$\Rightarrow \quad \hat{R}_t - \hat{R}_t^* = \big(1-\tilde{\phi}_s\big)\mathbb{E}_t(\hat{s}_{t+1} - \hat{s}_t) - \tilde{\phi}_s(\hat{s}_t - \hat{s}_{t-1}) - \tilde{\phi}_a\hat{a}_t + \hat{\tilde{\phi}}_t = 0 \qquad (L.18)$$

A.6 Government

Government collects taxes, redistribute income through transfer payment and spend money. Its budget may have a surplus or deficit:

$$P_tG_t + TR_t = R_t(B_{t+1} - B_t) + \iota_t^c P_t^c C_t + \frac{\iota_t^y + \iota_t^w}{1+\iota_t^w}W_tH_t + \iota_t^k\left((R_t-1)(B_t-Q_t) + \left(R_t^*\Phi\left(\frac{A_t}{z_t}, S_t, \tilde{\phi}_t\right) - 1\right)S_tB_{j,t}^* + \Pi_t + R_t^k u_t K_t\right) \qquad (61)$$

where

G_t: Government expenditure

Fiscal policy is assumed to be an exogenous process given its uncertainty due to political reasons. The exogenous variables $EG_t = \begin{bmatrix}\hat{\iota}_t^k & \hat{\iota}_t^c & \hat{\iota}_t^y & \hat{\iota}_t^w & \hat{g}_t\end{bmatrix}'$ are assumed to follow a vector autoregressive (VAR) model as follows:

$$EG_t = C + A \times EG_{t-1} + \varepsilon_t^\iota \qquad (62)$$

where

$$C = \begin{bmatrix} c_{l^k} & c_{l^c} & c_{l^y} & c_{l^w} & c_g \end{bmatrix}'$$

$$A = \begin{bmatrix} \rho_{l^k}^k & \cdots & \rho_{l^k}^g \\ \vdots & \ddots & \vdots \\ \rho_g^k & \cdots & \rho_g^g \end{bmatrix}$$

$\varepsilon_t^\iota \sim N(0, \Sigma_\iota)$ contains the random component of the exogenous variables:

$$\Sigma_\iota = \begin{bmatrix} \varepsilon^{l^k} & & \\ & \ddots & \\ & & \varepsilon^g \end{bmatrix}$$

In cases where the tax rate has rarely changed, it can be assumed as a constant. For example, Canada's sales tax rates may be modeled as a constant value because they were not changed frequently to actively balance the budget.

A.7 Central Bank

The central bank set domestic short-term interest rate R_t. It is assumed that the monetary policy follows the rule specified in (63):

$$\hat{R}_t = \rho_R \hat{R}_{t-1} + (1-\rho_R)\left[\hat{\bar{\pi}}_t^c + \phi_\pi\left(\hat{\pi}_{t+1}^{c,4} - \hat{\bar{\pi}}_t^c\right) + \phi_{\Delta\pi}\Delta\hat{\pi}_t^c + \phi_{\mathrm{y}}\hat{y}_t + \phi_{\Delta\mathrm{y}}\Delta\hat{y}_t + \phi_{\mathrm{x}}\hat{x}_{t-1}\right] + \varepsilon_t^R \qquad (63)$$

where

$$\hat{\pi}_{t+1}^{c,4} = \frac{1}{4}\prod_{j=1}^{4}\pi_{t+1-j}$$

$$\hat{x}_t = \hat{S}_t + \hat{P}_t^* - \hat{P}_t^c$$

$$\hat{\pi}_t^c = (1-\vartheta_c)(\gamma^{d,c})^{1-\eta_c}\hat{\pi}_t^d + \vartheta_c(\gamma^{mc,c})^{1-\eta_c}\hat{\pi}_t^{m,c} \qquad (64)$$

$$\gamma^{d,c} \equiv \frac{P^d}{P^c}$$

$$\gamma^{mc,c} \equiv \frac{P^{m,c}}{P^c}$$

The short-term interest rate is changed based on the deviation of actual inflation rate $\hat{\pi}_t^c$ from its target level $\hat{\bar{\pi}}_t^c$, the output gap $\hat{y}_t$ and changes in exchange rate $\hat{x}_{t-1}$.

A.8 Market Clearing

In a state of equilibrium, the goods market and foreign bond market must clear, which means supply equals demand.

Final Goods Market

$$C_t^d + I_t^d + G_t + C_t^x + I_t^x \leq \varepsilon_t (K_t^s)^\alpha (z_t H_t)^{1-\alpha} - z_t\phi - a(u_t)K_t \qquad (65)$$

where

G_t: government spending, which is exogeneous.

Divide (65) by z_t:

$$\Rightarrow \quad (1-\vartheta_c)\left(\frac{P_t^c}{P_t^d}\right)^{\eta_c} c_t + (1-\vartheta_i)\left(\frac{P_t^i}{P_t^d}\right)^{\eta_i} i_t + g_t + \left(\frac{P_t^x}{P_t^*}\right)^{\eta_f} y_t^* \frac{z_t^*}{z_t} \le \varepsilon_t \left(\frac{k_t^s}{\mu_t^z}\right)^{\alpha} (H_t)^{1-\alpha} - \phi - a(u_t)\left(\frac{k_t}{\mu_t^z}\right) \qquad (66)$$

where

$\frac{z_t^*}{z_t}$: relative technology progress between foreign and domestic economy

$\tilde{z}_t^* = \frac{z_t^*}{z_t} \qquad \hat{\tilde{z}}_t^* = \rho_{\tilde{z}^*}\hat{\tilde{z}}_{t-1}^* + \varepsilon_t^{\tilde{z}^*} \qquad \hat{\tilde{z}}_t^* = {(\hat{\tilde{z}}_t^* - 1)}/{1}$

$Y_t^* = C_t^* + I_t^*$

$Y_{i,t} = \varepsilon_t \left(K_{i,t}^s\right)^{\alpha} \left(z_t H_{i,t}\right)^{1-\alpha} - z_t\phi \qquad (5)$

$$\Rightarrow \quad y_t = \varepsilon_t \left(\frac{k_t^s}{\mu_t^z}\right)^{\alpha} (H_t)^{1-\alpha} - \phi$$

$$\Rightarrow \quad y_t = (1-\vartheta_c)\left(\frac{P_t^c}{P_t^d}\right)^{\eta_c} c_t + (1-\vartheta_i)\left(\frac{P_t^i}{P_t^d}\right)^{\eta_i} i_t + g_t + \left(\frac{P_t^x}{P_t^*}\right)^{-\eta_f} y_t^* \frac{z_t^*}{z_t} + a(u_t)\left(\frac{k_t}{\mu_t^z}\right) \qquad (67)$$

Loglinearize (67):

$$\begin{aligned} y\hat{y}_t = {} & (1-\vartheta_c)\left(\frac{P^c}{P^d}\right)^{\eta_c} (c_t - c) + (1-\vartheta_c)\left(\frac{1}{P^d}\right)^{\eta_c} c\eta_c (P^c)^{\eta_c - 1}(P_t^c - P^c) \\ & + (1-\vartheta_c)(P^c)^{\eta_c} c(-\eta_c)(P^d)^{-\eta_c-1}(P_t^d - P^d) + (1-\vartheta_i)\left(\frac{P^i}{P^d}\right)^{\eta_i} (i_t - i) \\ & + (1-\vartheta_i)\left(\frac{1}{P^d}\right)^{\eta_i} i\eta_i (P^i)^{\eta_i-1}\left(P_t^i - P^i\right) + (1-\vartheta_i)(P^i)^{\eta_i} i(-\eta_i)(P^i)^{-\eta_i-1}(P_t^d - P^d) + (g_t - g) \\ & + \left(\frac{P^x}{P^*}\right)^{-\eta_f} \frac{z^*}{z}(y_t^* - y^*) + (P^x)^{-\eta_f}\frac{z^*}{z} y^*\left(\eta_f\right)(P^*)^{\eta_f - 1}(P_t^* - P^*) \\ & + \left(\frac{1}{P^*}\right)^{-\eta_f} \frac{z^*}{z} y^*\left(-\eta_f\right)(P^x)^{-\eta_f-1}(P_t^x - P^x) + \left(\frac{P^x}{P^*}\right)^{\eta_f} \frac{y^*}{z}(z_t^* - z^*) + \left(\frac{P^x}{P^*}\right)^{\eta_f} \frac{y^* z^*}{-z^2}(z_t - z) \\ & + a'(u)\left(\frac{k}{\mu^z}\right)(u_t - u) + \frac{a(u)}{\mu^z}(k_t - k) + a(u)k\left(-\frac{1}{(\mu^z)^2}\right)(\mu_t^z - \mu^z) \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad \hat{y}_t = {} & (1-\vartheta_c)(\gamma^{c,d})^{\eta_c}\frac{c}{y}\left(\hat{c}_t + \eta_c\hat{\gamma}_t^{c,d}\right) + (1-\vartheta_i)(\gamma^{i,d})^{\eta_i}\frac{i}{y}\left(\hat{\imath}_t + \eta_i\hat{\gamma}_t^{i,d}\right) + g_y\hat{g}_t \\ & + \frac{y^*}{y}\left(\hat{y}_t^* - \eta_f\hat{\gamma}_t^{x,*} + \hat{z}_t^* - \hat{z}_t\right) + \frac{(1-\iota^k)r^k}{y}\left(\frac{k}{\mu^z}\right)\hat{u}_t \end{aligned}$$

$$\Rightarrow \quad \hat{y}_t = (1-\vartheta_c)(\gamma^{c,d})^{\eta_c}\frac{c}{y}\left(\hat{c}_t + \eta_c\hat{\gamma}_t^{c,d}\right) + (1-\vartheta_i)(\gamma^{i,d})^{\eta_i}\frac{i}{y}\left(\hat{\imath}_t + \eta_i\hat{\gamma}_t^{i,d}\right) + g_y\hat{g}_t + \frac{y^*}{y}\left(\hat{y}_t^* - \eta_f\hat{\gamma}_t^{x,*} + \hat{\tilde{z}}_t^*\right)$$
$$+ \frac{(1-\iota^k)r^k}{y}\left(\frac{k}{\mu^z}\right)\left(\hat{k}_t^s - \hat{k}_t\right) \qquad (L.19)$$

Foreign Bond Market

$$S_tB_{j,t+1}^* = S_tP_t^x(C_t^x + I_t^x) - S_tP_t^*(C_t^m + I_t^m) + R_t^*\Phi\left(\frac{A_t}{z_t}, S_t, \tilde{\phi}_t\right)S_tB_t^* \qquad (68)$$

$$A_t \equiv \frac{S_tB_{t+1}^*}{P_t^d} \qquad (44)$$

$$\Rightarrow \quad a_t \equiv \frac{S_tB_{t+1}^*}{P_t^d z_t}$$

$$\Rightarrow \quad a_t = \frac{S_tP_t^x}{P_t^d}\left(\frac{C_t^x}{z_t} + \frac{I_t^x}{z_t}\right) - \frac{S_tP_t^*}{P_t^d}\left(\frac{C_t^m}{z_t} + \frac{I_t^m}{z_t}\right) + a_{t-1}\frac{S_t}{S_{t-1}}\frac{P_{t-1}^d}{P_t^d}\frac{z_{t-1}}{z_t}R_t^*\Phi\left(a_t, S_t, \tilde{\phi}_t\right)$$

$$\Rightarrow \quad a_t = (mc_t^x)^{-1}(c_t^x + i_t^x) - \left(r_t^f\right)^{-1}(c_t^m + i_t^m) + \frac{a_{t-1}}{\pi_t^d\mu_t^z}\frac{S_t}{S_{t-1}}R_t^*\Phi\left(a_t, S_t, \tilde{\phi}_t\right) \qquad (69)$$

where

$$mc_t^x = \frac{P_t^d}{S_tP_t^x}$$

$$r_t^f = \frac{P_t^d}{S_tP_t^*}$$

$$c_t^x + i_t^x = \left(\frac{P_t^x}{P_t^*}\right)^{-\eta_f}\frac{Y_t^*}{z_t^*}\frac{z_t^*}{z_t}$$

$$\Rightarrow \quad a_t = (mc_t^x)^{-1}(\gamma_t^{x,*})^{-\eta_f}y_t^*\tilde{z}_t^* - \left(r_t^f\right)^{-1}(c_t^m + i_t^m) + \frac{a_{t-1}}{\pi_t^d\mu_t^z}\frac{S_t}{S_{t-1}}R_t^*\Phi\left(a_t, S_t, \tilde{\phi}_t\right)$$

$$c_t^m = \vartheta_c\left(\frac{P_t^{m,c}}{P_t^c}\right)^{-\eta_c}c_t = \vartheta_c\left(\frac{\gamma_t^{mc,d}}{\gamma_t^{c,d}}\right)^{-\eta_c}c_t = c_m\left(\hat{c}_t + (-\eta_c)\hat{\gamma}_t^{mc,d} + \eta_c\hat{\gamma}_t^{c,d}\right)$$

$$\hat{\gamma}_t^{c,d} = \hat{\pi}_t^c - \hat{\pi}_t^d = (1-\vartheta_c)\left(\frac{1}{\gamma^{c,d}}\right)^{1-\eta_c}\hat{\pi}_t^d - \hat{\pi}_t^d + \vartheta_c(\gamma^{mc,c})^{1-\eta_c}\hat{\pi}_t^{m,c} = \vartheta_c(\gamma^{c,d})^{-(1-\eta_c)}(\hat{\pi}_t^{m,c} - \hat{\pi}_t^d)$$
$$= \vartheta_c(\gamma^{c,d})^{-(1-\eta_c)}\hat{\gamma}_t^{mc,d}$$

$$c_t^m = c^m\left(\hat{c}_t + (-\eta_c)(1-\vartheta_c)(\gamma^{c,d})^{-(1-\eta_c)}\hat{\gamma}_t^{mc,d}\right)$$

$$\Rightarrow \quad \hat{a}_t = -y_t^*\widehat{mc}_t^x - \eta_f y_t^*\hat{\gamma}_t^{x,*} + y_t^*\hat{y}_t^* + y_t^*\hat{\tilde{z}}_t^* + (c_t^m + i_t^m)\hat{r}_t^f - c^m\left(\hat{c}_t + (-\eta_c)(1-\vartheta_c)(\gamma^{c,d})^{-(1-\eta_c)}\hat{\gamma}_t^{mc,d}\right)$$
$$- i^m\left(\hat{\imath}_t + (-\eta_i)(1-\vartheta_i)(\gamma^{c,d})^{-(1-\eta_i)}\hat{\gamma}_t^{mi,d}\right) + \frac{R^*}{\pi\mu^z}\hat{a}_{t-1} \qquad (L.20)$$

A.9 Relative Prices

The model includes four price indices: domestic price index, imported consumption goods price index, imported investment goods price index and foreign goods price index. In this section, relative prices are defined accordingly, and marginal costs of producers, importers and exporters are presented using relative prices.

$$\gamma_t^{c,d} \equiv \frac{P_t^c}{P_t^d} \quad (70) \quad \Rightarrow \quad \hat{\gamma}_t^{c,d} = \hat{\gamma}_{t-1}^{c,d} + \hat{\pi}_t^c - \hat{\pi}_t^d \quad (L.21)$$

$$\gamma_t^{i,d} \equiv \frac{P_t^i}{P_t^d} \quad (71) \quad \Rightarrow \quad \hat{\gamma}_t^{i,d} = \hat{\gamma}_{t-1}^{i,d} + \hat{\pi}_t^i - \hat{\pi}_t^d \quad (L.22)$$

$$\gamma_t^{mc,d} \equiv \frac{P_t^{m,c}}{P_t^d} \quad (72) \quad \Rightarrow \quad \hat{\gamma}_t^{mc,d} = \hat{\gamma}_{t-1}^{mc,d} + \hat{\pi}_t^{m,c} - \hat{\pi}_t^d \quad (L.23)$$

$$\gamma_t^{mi,d} \equiv \frac{P_t^{m,i}}{P_t^d} \quad (73) \quad \Rightarrow \quad \hat{\gamma}_t^{mi,d} = \hat{\gamma}_{t-1}^{mi,d} + \hat{\pi}_t^{m,i} - \hat{\pi}_t^d \quad (L.24)$$

$$\gamma_t^{x,*} \equiv \frac{P_t^x}{P_t^*} \quad (74) \quad \Rightarrow \quad \hat{\gamma}_t^{x,*} = \hat{\gamma}_{t-1}^{x,*} + \hat{\pi}_t^x - \hat{\pi}_t^* \quad (L.25)$$

$$\gamma_t^f \equiv \frac{P_t^d}{S_t P_t^*} \quad (75) \quad \text{domestic} - \text{foreign relative price}$$

$$mc_t^{m,c} \equiv \frac{S_t P_t^*}{P_t^{m,c}} = \left(\gamma_t^f \gamma_t^{mc,d}\right)^{-1} \quad (76) \quad \Rightarrow \quad \widehat{mc}_t^{m,c} = -\hat{\gamma}_t^f - \hat{\gamma}_t^{mc,d} \quad (L.26)$$

$$mc_t^{m,i} \equiv \frac{S_t P_t^*}{P_t^{m,i}} = \left(\gamma_t^f \gamma_t^{mi,d}\right)^{-1} \quad (77) \quad \Rightarrow \quad \widehat{mc}_t^{m,i} = -\hat{\gamma}_t^f - \hat{\gamma}_t^{mi,d} \quad (L.27)$$

$$mc_t^x \equiv \frac{\gamma_t^f}{\gamma_t^{x,*}} \quad (78) \quad \Rightarrow \quad \hat{\gamma}_t^f = \widehat{mc}_t^x + \hat{\gamma}_t^{x,*} \quad (L.28)$$

(74), (75) and (78) =>

$$mc_t^x = \frac{P_t^d}{S_t P_t^*} \frac{P_t^*}{P_t^x} = mc_{t-1}^x \frac{P_t^d}{S_t P_t^x} \frac{S_{t-1} P_{t-1}^x}{P_{t-1}^d} = mc_{t-1}^x \frac{\pi_t^d}{\pi_t^* \left({S_t}/{S_{t-1}}\right)} \quad (79)$$

$$\Rightarrow \quad \widehat{mc}_t^x = \widehat{mc}_{t-1}^x + \hat{\pi}_t^d - \hat{\pi}_t^* - \Delta \hat{S}_t \quad (L.29)$$

$$\gamma_t^s = x_t = \frac{S_t P_t^*}{P_t^c} \quad (80)$$

$$mc_t^x = \frac{P_t^d}{S_t P_t^x} \quad \Rightarrow \quad x_t = \frac{P_t^d P_t^*}{mc_t^x P_t^c P_t^x} = \frac{P_t^d}{P_t^c} \frac{1}{mc_t^x \gamma_t^{x,*}} \quad (81)$$

$$\frac{P_t^d}{P_t^c} = \frac{P_t^d}{\left[(1-\vartheta_c)(P_t^d)^{1-\eta_c} + \vartheta_c\left(P_t^{m,c}\right)^{1-\eta_c}\right]^{\frac{1}{1-\eta_c}}} \quad (82)$$

Loglinearize (82):

$$\begin{aligned}&\left\{\frac{1}{P^c}+P^d\left(-\frac{1}{1-\eta_c}\right)[(1-\vartheta_c)(P^d)^{1-\eta_c}+\vartheta_c(P^{m,c})^{1-\eta_c}]^{-\frac{1}{1-\eta_c}-1}(1-\vartheta_c)(1-\eta_c)(P^d)^{-\eta_c}\right\}(P_t^d-P^d)\\&\quad+\left\{P^d\left(-\frac{1}{1-\eta_c}\right)[(1-\vartheta_c)(P^d)^{1-\eta_c}+\vartheta_c(P^{m,c})^{1-\eta_c}]^{-\frac{1}{1-\eta_c}-1}\vartheta_c(1-\eta_c)(P^{m,c})^{-\eta_c}\right\}(P_t^{m,c}\\&\quad-P^{m,c})=\frac{P^d}{P^c}\left\{1-(1-\vartheta_c)\frac{(P^d)^{1-\eta_c}}{(P^c)^{1-\eta_c}}\right\}(p_t^d-p^d)+\frac{P^d}{P^c}(-\vartheta_c)\frac{(P^{m,c})^{1-\eta_c}}{(P^c)^{1-\eta_c}}(p_t^{m,c}-p^{m,c})\\&\quad=\frac{P^d}{P^c}(-\vartheta_c)\left(\frac{1}{\gamma^{mc,c}}\right)^{\eta_c-1}\gamma_t^{mc,d}\end{aligned}$$

$$\Rightarrow \qquad \hat{\gamma}_t^s=\hat{x}_t=-\vartheta_c(\gamma^{c,mc})^{1-\eta_c}\hat{\gamma}_t^{mc,d}-\hat{\gamma}_t^{x,*}-\widehat{mc}_t^x \qquad (L.30)$$

A.10 Foreign Economy

A foreign economy is assumed to be exogeneous governed by three equations:

Output

$$\hat{y}_t^*=\mathbb{E}_t\hat{y}_{t+1}^*-\frac{1}{\sigma^*}\left(\hat{R}_t^*-\mathbb{E}_t\hat{\pi}_{t+1}^*+\xi_t^{y,*}\right) \qquad (83)$$

Inflation

$$\hat{\pi}_t^*=\beta\mathbb{E}_t\hat{\pi}_{t+1}^*+\chi^*\hat{y}_t^*+\xi_t^{\pi,*} \qquad (84)$$

Interest Rate

$$\hat{R}_t^*=\rho_R^*\hat{R}_{t-1}^*+(1-\rho_R^*)\left(\phi_{\hat{\pi}}^*\hat{\pi}_t^*+\phi_y^*\hat{y}_t^*\right)+\varepsilon_t^{R,*} \qquad (85)$$

$\xi_t^{y,*}$ and $\xi_t^{\pi,*}$ follow the AR(1) shock process.

A.11 Data

The following observable data fields are used to estimate the DSGE model:

$\Delta \ln \tilde{Y}_t$	Real GDP
$\Delta \ln \tilde{C}_t$	Private consumption
$\Delta \ln \tilde{I}_t$	Total fixed investment
$\Delta \ln \tilde{X}_t$	Total exports
$\Delta \ln \widetilde{M}_t$	Total imports
$\Delta \ln \tilde{S}_t$	Nominal effective exchange rate
$\Delta \ln \widetilde{H}_t$	Nonagricultural employment (aggregate hours worked)
$\Delta \ln \widetilde{W}_t$	Compensation of employees

$\tilde{R}_t$	Repo rate
$\tilde{\pi}_t^i$	Fixed investment deflator
$\tilde{\pi}_t^c$	CPI
$\tilde{\pi}_t^d$	PPI
$\tilde{\tilde{\pi}}_{t+1}^c$	Inflation target median
$\Delta \ln \tilde{Y}_t^*$	Foreign real GDP
$\tilde{\pi}_t^*$	Foreign CPI
$\tilde{R}_t^*$	Foreign repo rate

Because the DSGE model is based on stationary time series, the data are detrended (first-order difference) before used for model calibration.

In addition, the observable data may be different from the variables defined in the DSGE model. For example, observable consumption data $\tilde{C}_t = C_t^m + C_t^d$. In the DSGE model, the consumption is measured as follows:

$$\tilde{C}_t = \left((1-\vartheta_c)\left(\frac{P_t^d}{P_t^c}\right)^{-\eta_c} + \vartheta_c\left(\frac{P_t^{m,c}}{P_t^c}\right)^{-\eta_c} \right) C_t \qquad (86)$$

Therefore, measurement errors exist. They may also include extreme shocks that are not persistent in the data. The data need to be adjusted before used for model estimation, as shown in the following measurement equations:

$$\tilde{I}_t = I_t^m + I_t^d = \left((1-\vartheta_i)\left(\frac{P_t^d}{P_t^i}\right)^{-\eta_i} + \vartheta_i\left(\frac{P_t^{m,i}}{P_t^i}\right)^{-\eta_i} \right) I_t \qquad (87)$$

$$\widetilde{M}_t = C_t^m + I_t^m = \vartheta_c\left(\frac{P_t^{m,c}}{P_t^c}\right)^{-\eta_c} C_t + \vartheta_i\left(\frac{P_t^{m,i}}{P_t^i}\right)^{-\eta_i} I_t \qquad (88)$$

$$\tilde{X}_t = C_t^x + I_t^x = \left[\frac{P_t^x}{P_t^*}\right]^{-\eta_f} Y_t^* \qquad (89)$$

Equation (65) =>

$$(C_t^d + C_t^m) + (I_t^d + I_t^m) + G_t + (C_t^x + I_t^x) - (C_t^m + I_t^m) \le \varepsilon_t (K_t^s)^\alpha (z_t H_t)^{1-\alpha} - z_t\phi - a(u_t)K_{t-1} \qquad (90)$$

where

C_t^m *and* I_t^m are measurement errors.

$\tilde{\pi}_t^c, \tilde{\pi}_t^d, \tilde{\pi}_t^*, \tilde{R}_t, \tilde{R}_t^*, \Delta \ln \tilde{S}_t$ are assumed to have zero measurement error.

Measurement equations are loglinearized as well to embed them into the DSGE model.

Loglinearize $\Delta \ln \tilde{Y}_t = \ln \tilde{Y}_t - \ln \tilde{Y}_{t-1}$:

$$\begin{aligned}\Delta \ln \tilde{Y}_t = \ln \tilde{Y}_t - \ln \tilde{Y}_{t-1} &= \ln(\tilde{y}_t z_t) - \ln(\tilde{y}_{t-1} z_{t-1}) = \ln(\tilde{y}_t z_{t-1}\mu_t^z) - \ln(\tilde{y}_{t-1} z_{t-1}) \\ &= \ln(\tilde{y}_t) + \ln(z_{t-1}) + \ln(\mu_t^z) - \ln(\tilde{y}_{t-1}) - \ln(z_{t-1}) \\ &= (\ln(\tilde{y}_t) - \ln(y)) - (\ln(\tilde{y}_{t-1}) - \ln(y)) + (\ln(\mu_t^z) - \ln(\mu^z)) + \ln(\mu^z)\end{aligned}$$

$$\Rightarrow \qquad \Delta \ln \tilde{Y}_t = \hat{y}_t - \hat{y}_{t-1} + \hat{\mu}_t^z + \ln(\mu^z) \qquad (L.31)$$

Loglinearize $\Delta \ln \tilde{C}_t = \ln \tilde{C}_t - \ln \tilde{C}_{t-1}$:

$$\tilde{C}_t = \left((1-\vartheta_c)\left(\frac{P_t^d}{P_t^c}\right)^{-\eta_c} + \vartheta_c \left(\frac{P_t^{m,c}}{P_t^c}\right)^{-\eta_c} \right) C_t$$

$$\begin{aligned}\Rightarrow \qquad \Delta \ln \tilde{C}_t &= \ln C_t - \ln C_{t-1} + \ln\left((1-\vartheta_c)\left(\frac{P_t^d}{P_t^c}\right)^{-\eta_c} + \vartheta_c \left(\frac{P_t^{m,c}}{P_t^c}\right)^{-\eta_c} \right) \\ &- \ln\left((1-\vartheta_c)\left(\frac{P_{t-1}^d}{P_{t-1}^c}\right)^{-\eta_c} + \vartheta_c \left(\frac{P_{t-1}^{m,c}}{P_{t-1}^c}\right)^{-\eta_c} \right) \\ &= \ln(c_t z_{t-1}\mu_t^z) - \ln(c_{t-1} z_{t-1}) \\ &+ \frac{1}{(1-\vartheta_c)\left(\frac{P^d}{P^c}\right)^{-\eta_c} + \vartheta_c\left(\frac{P^{m,c}}{P^c}\right)^{-\eta_c}} \left\{ (1-\vartheta_c)(-\eta_c)\left(\frac{1}{P^c}\right)^{-\eta_c} (P^d)^{-\eta_c-1}(P_t^d - P^d) \right. \\ &+ \vartheta_c(-\eta_c)\left(\frac{1}{P^c}\right)^{-\eta_c} (P^{m,c})^{-\eta_c-1}(P_t^{m,c} - P^{m,c}) \\ &\left. + [(1-\vartheta_c)\eta_c (P^d)^{-\eta_c}(P^c)^{\eta_c-1} + \vartheta_c\eta_c(P^{m,c})^{-\eta_c}(P^c)^{\eta_c-1}](P_t^c - P^c) \right\} \\ &- \frac{1}{(1-\vartheta_c)\left(\frac{P^d}{P^c}\right)^{-\eta_c} + \vartheta_c\left(\frac{P^{m,c}}{P^c}\right)^{-\eta_c}} \left\{ (1-\vartheta_c)(-\eta_c)\left(\frac{1}{P^c}\right)^{-\eta_c} (P^d)^{-\eta_c-1}(P_{t-1}^d - P^d) \right. \\ &+ \vartheta_c(-\eta_c)\left(\frac{1}{P^c}\right)^{-\eta_c} (P^{m,c})^{-\eta_c-1}(P_{t-1}^{m,c} - P^{m,c}) \\ &\left. + [(1-\vartheta_c)\eta_c (P^d)^{-\eta_c}(P^c)^{\eta_c-1} + \vartheta_c\eta_c(P^{m,c})^{-\eta_c}(P^c)^{\eta_c-1}](P_{t-1}^c - P^c) \right\} \\ &= \hat{c}_t - \hat{c}_{t-1} + \hat{\mu}_t^z + \ln(\mu^z) \\ &+ \frac{c}{c^d + c^m}\left\{ \frac{c^d}{c}\frac{(-\eta_c)}{P^d}(P_t^d - P^d) + \frac{c^m}{c}\frac{(-\eta_c)}{P^{m,c}}(P_t^{m,c} - P^{m,c}) + \frac{c^d + c^m}{c}\frac{\eta_c}{P^c}(P_t^c - P^c) \right. \\ &\left. - \frac{c^d}{c}\frac{-\eta_c}{P^d}(P_{t-1}^d - P^d) - \frac{c^m}{c}\frac{(-\eta_c)}{P^{m,c}}(P_{t-1}^{m,c} - P^{m,c}) - \frac{c^d + c^m}{c}\frac{\eta_c}{P^c}(P_{t-1}^c - P^c) \right\} \\ &= \hat{c}_t - \hat{c}_{t-1} + \hat{\mu}_t^z + \ln(\mu^z) + \frac{\eta_c}{c^d + c^m}(-c^d\hat{\pi}_t^d - c^m\hat{\pi}_t^{m,c} + (c^d + c^m)\hat{\pi}_t^c)\end{aligned}$$

Replace $\hat{\pi}_t^c$ with Equation (64): $\hat{\pi}_t^c = (1-\vartheta_c)(\gamma^{d,c})^{1-\eta_c}\hat{\pi}_t^d + \vartheta_c(\gamma^{mc,c})^{1-\eta_c}\hat{\pi}_t^{m,c}$

$$\begin{aligned}\Rightarrow \qquad \Delta \ln \tilde{C}_t &= \hat{c}_t - \hat{c}_{t-1} + \hat{\mu}_t^z + \ln(\mu^z) \\ &+ \frac{\eta_c}{c^d + c^m}\Big((c^d(1-\vartheta_c)(\gamma^{c,d})^{\eta_c-1} - c^d)\hat{\pi}_t^d + (c^m(1-\vartheta_c)(\gamma^{c,d})^{\eta_c-1})\hat{\pi}_t^d \\ &+ (c^d\vartheta_c(\gamma^{c,mc})^{\eta_c-1})\hat{\pi}_t^{m,c} + (c^m\vartheta_c(\gamma^{c,mc})^{\eta_c-1} - c^m)\hat{\pi}_t^{m,c} \Big)\end{aligned}$$

$\gamma^{c,mc} = \gamma^{c,d} = 1$

$$\Rightarrow \qquad \Delta \ln \tilde{C}_t = \frac{\eta_c}{c^d + c^m}(c^d \vartheta_c (\gamma^{c,mc})^{\eta_c - 1} - c^m (1 - \vartheta_c)(\gamma^{c,d})^{\eta_c - 1})(\hat{\pi}_t^{m,c} - \hat{\pi}_t^d) + \hat{c}_t - \hat{c}_{t-1} + \hat{\mu}_t^z + \ln(\mu^z) \qquad (L.32)$$

The loglinearization of $\Delta \ln \tilde{I}_t = \ln \tilde{I}_t - \ln \tilde{I}_{t-1}$ takes a similar approach as that of $\Delta \ln \tilde{C}_t$:

$$\tilde{I}_t = I_t^m + I_t^d = \left((1 - \vartheta_i) \left(\frac{P_t^d}{P_t^i} \right)^{-\eta_i} + \vartheta_i \left(\frac{P_t^{m,i}}{P_t^i} \right)^{-\eta_i} \right) I_t$$

$$\hat{\pi}_t^i = (1 - \vartheta_i)(\gamma^{d,i})^{1-\eta_i} \hat{\pi}_t^d + \vartheta_i (\gamma^{mi,i})^{1-\eta_i} \hat{\pi}_t^{m,i}$$

$$\Rightarrow \qquad \Delta \ln \tilde{I}_t = \frac{\eta_i}{i^d + i^m}(i^d \vartheta_i (\gamma^{i,mi})^{\eta_i - 1} - i^m (1 - \vartheta_i)(\gamma^{i,d})^{\eta_i - 1})\left(\hat{\pi}_t^{m,i} - \hat{\pi}_t^d\right) + \hat{\imath}_t - \hat{\imath}_{t-1} + \hat{\mu}_t^z + \ln(\mu^z) \qquad (L.33)$$

Loglinearize $\Delta \ln \tilde{X}_t = \ln \tilde{X}_t - \ln \tilde{X}_{t-1}$:

$$\tilde{X}_t = \left[\frac{P_t^x}{P_t^*} \right]^{-\eta_f} Y_t^*$$

$$\begin{aligned}
\Rightarrow \qquad \Delta \ln \tilde{X}_t = \ln \tilde{X}_t - \ln \tilde{X}_{t-1} &= -\eta_f \ln \frac{P_t^x}{P_t^*} + \ln Y_t^* - \left(-\eta_f \ln \frac{P_{t-1}^x}{P_{t-1}^*} + \ln Y_{t-1}^* \right) \\
&= -\eta_f (\ln P_t^x - \ln P_t^*) + \ln Y_t^* + \eta_f (\ln P_{t-1}^x - \ln P_{t-1}^*) - \ln Y_{t-1}^* \\
&= -\eta_f \ln \pi_t^x + \eta_f \ln \pi_t^* + \ln(y_t^* z_t^*) - \ln(y_{t-1}^* z_{t-1}^*) \\
&= -\eta_f \ln \pi_t^x + \eta_f \ln \pi_t^* + \ln(y_t^* \tilde{z}_t z_{t-1} \mu_t^z) - \ln(y_{t-1}^* \tilde{z}_{t-1} z_{t-1}) \\
&= -\eta_f (\hat{\pi}_t^x - \hat{\pi}_t^*) + \hat{y}_t^* - \hat{y}_{t-1}^* + \hat{\tilde{z}}_t - \hat{\tilde{z}}_{t-1} + \hat{\mu}_t^z + \ln(\mu^z) \qquad (L.34)
\end{aligned}$$

Loglinearize $\Delta \ln \widetilde{M}_t = \ln \widetilde{M}_t - \ln \widetilde{M}_{t-1}$:

$$\widetilde{M}_t = \vartheta_c \left(\frac{P_t^{m,c}}{P_t^c} \right)^{-\eta_c} C_t + \vartheta_i \left(\frac{P_t^{m,i}}{P_t^i} \right)^{-\eta_i} I_t$$

$$\begin{aligned}
\Rightarrow \qquad \Delta \ln \widetilde{M}_t &= \ln \widetilde{M}_t - \ln \widetilde{M}_{t-1} \\
&= \ln\left[\vartheta_c \left(\frac{P_t^{m,c}}{P_t^c}\right)^{-\eta_c} c_t + \vartheta_i \left(\frac{P_t^{m,i}}{P_t^i}\right)^{-\eta_i} i_t\right] + \ln z_{t-1} + \ln \mu_t^z \\
&- \ln\left[\vartheta_c \left(\frac{P_{t-1}^{m,c}}{P_{t-1}^c}\right)^{-\eta_c} c_{t-1} + \vartheta_i \left(\frac{P_{t-1}^{m,i}}{P_{t-1}^i}\right)^{-\eta_i} i_{t-1}\right] - \ln z_{t-1} \\
&= \frac{1}{c_m + i_m}\left\{\vartheta_c \left(\frac{P^{m,c}}{P^c}\right)^{-\eta_c} (c_t - c) + \vartheta_c c \left(\frac{1}{P^c}\right)^{-\eta_c} (-\eta_c)(P^{m,c})^{-\eta_c - 1}(P_t^{m,c} - P^{m,c})\right. \\
&+ \vartheta_c c (P^{m,c})^{-\eta_c}(\eta_c)(P^c)^{\eta_c - 1}(P_t^c - P^c) + \vartheta_i \left(\frac{P^{m,i}}{P^i}\right)^{-\eta_i} (i_t - i) \\
&\left.+ \vartheta_i i \left(\frac{1}{P^i}\right)^{-\eta_i} (-\eta_i)(P^{m,i})^{-\eta_i - 1}\left(P_t^{m,i} - P^{m,i}\right) + \vartheta_i i (P^{m,i})^{-\eta_i}(\eta_i)(P^i)^{\eta_i - 1}\left(P_t^i - P^i\right)\right\} \\
&- \frac{1}{c_m + i_m}\left\{\vartheta_c \left(\frac{P^{m,c}}{P^c}\right)^{-\eta_c} (c_{t-1} - c) + \vartheta_c c \left(\frac{1}{P^c}\right)^{-\eta_c} (-\eta_c)(P^{m,c})^{-\eta_c - 1}(P_{t-1}^{m,c} - P^{m,c})\right. \\
&+ \vartheta_c c (P^{m,c})^{-\eta_c}(\eta_c)(P^c)^{\eta_c - 1}(P_{t-1}^c - P^c) + \vartheta_i \left(\frac{P^{m,i}}{P^i}\right)^{-\eta_i} (i_{t-1} - i) \\
&\left.+ \vartheta_i i \left(\frac{1}{P^i}\right)^{-\eta_i} (-\eta_i)(P^{m,i})^{-\eta_i - 1}\left(P_{t-1}^{m,i} - P^{m,i}\right) + \vartheta_i i (P^{m,i})^{-\eta_i}(\eta_i)(P^i)^{\eta_i - 1}\left(P_{t-1}^i - P^i\right)\right\} + \hat{\mu}_t^z \\
&+ \ln(\mu^z) \\
&= \frac{c_m}{c_m + i_m}\left(\hat{c}_t - \hat{c}_{t-1} + \eta_c(\hat{\pi}_t^c - \hat{\pi}_t^{m,c})\right) + \frac{i_m}{c_m + i_m}\left(\hat{\imath}_t - \hat{\imath}_{t-1} + \eta_i\left(\hat{\pi}_t^i - \hat{\pi}_t^{m,i}\right)\right) + \hat{\mu}_t^z + \ln(\mu^z)
\end{aligned}$$

$$\hat{\pi}_t^c - \hat{\pi}_t^{m,c} = (1 - \vartheta_c)(\gamma^{d,c})^{1-\eta_c}\hat{\pi}_t^d + \vartheta_c(\gamma^{mc,c})^{1-\eta_c}\hat{\pi}_t^{m,c} - \hat{\pi}_t^{m,c} = (1 - \vartheta_c)(\gamma^{c,d})^{\eta_c - 1}(\hat{\pi}_t^d - \hat{\pi}_t^{m,c})$$

$$\hat{\pi}_t^i - \hat{\pi}_t^{m,i} = (1 - \vartheta_i)(\gamma^{d,i})^{1-\eta_i}\hat{\pi}_t^d + \vartheta_i(\gamma^{mi,i})^{1-\eta_i}\hat{\pi}_t^{m,i} - \hat{\pi}_t^{m,i} = (1 - \vartheta_i)(\gamma^{i,d})^{\eta_i - 1}\left(\hat{\pi}_t^d - \hat{\pi}_t^{m,i}\right)$$

$$\begin{aligned}
\Rightarrow \qquad \Delta \ln \widetilde{M}_t &= \frac{c_m}{c_m + i_m}\left(\hat{c}_t - \hat{c}_{t-1} + \eta_c(1 - \vartheta_c)(\gamma^{c,d})^{\eta_c - 1}(\hat{\pi}_t^d - \hat{\pi}_t^{m,c})\right) \\
&+ \frac{i_m}{c_m + i_m}\left(\hat{\imath}_t - \hat{\imath}_{t-1} + \eta_i(1 - \vartheta_i)(\gamma^{i,d})^{\eta_i - 1}\left(\hat{\pi}_t^d - \hat{\pi}_t^{m,i}\right)\right) + \hat{\mu}_t^z + \ln(\mu^z) \qquad (L.35)
\end{aligned}$$

Loglinearize $\Delta \ln \tilde{Y}_t^* = \ln \tilde{Y}_t^* - \ln \tilde{Y}_{t-1}^*$:

$$\Delta \ln \tilde{Y}_t^* = \ln \tilde{Y}_t^* - \ln \tilde{Y}_{t-1}^* = \ln \tilde{y}_t^* z_t^* - \ln \tilde{y}_{t-1}^* z_{t-1}^* = \ln \tilde{y}_t^* \tilde{z}_t z_{t-1} \mu_t^z - \ln \tilde{y}_{t-1}^* \tilde{z}_{t-1} z_{t-1}$$

$$\Rightarrow \qquad \Delta \ln \tilde{Y}_t^* = \hat{y}_t^* - \hat{y}_{t-1}^* + \hat{\tilde{z}}_t - \hat{\tilde{z}}_{t-1} + \hat{\mu}_t^z + \ln \mu^z \qquad (L.36)$$

Loglinearize $\Delta \ln \widetilde{W}_t = \ln \widetilde{W}_t - \ln \widetilde{W}_{t-1}$:

$$w_t = \frac{W_t}{z_t P_t^d}$$

$$\begin{aligned}
\Rightarrow \qquad \Delta \ln \widetilde{W}_t &= \ln \widetilde{W}_t - \ln \widetilde{W}_{t-1} = \ln(w_t z_{t-1} \mu_t^z P_t^d) - \ln(w_{t-1} P_{t-1}^d z_{t-1}) \\
&= \ln(w_t) - \ln(w_{t-1}) + \ln\left(\frac{P_t^d}{P_{t-1s}^d}\right) + \ln \mu^z
\end{aligned}$$

$$\Rightarrow \qquad \Delta \ln \widetilde{W}_t = \widehat{w}_t - \widehat{w}_{t-1} + \hat{\pi}_t^d + \hat{\mu}_t^z + \ln \mu^z \qquad (L.37)$$

Loglinearize $\Delta \ln \tilde{E}_t = \ln E_t - \ln E_{t-1}$:

$$\Delta \ln \tilde{E}_t = \ln E_t - \ln E_{t-1} = \hat{E}_t - \hat{E}_{t-1} \qquad (L.38)$$

$$\tilde{\pi}_t^c = \hat{\pi}_t^c + \ln \pi \qquad (L.39)$$

$$\tilde{\pi}_t^d = \hat{\pi}_t^d + \ln \pi \qquad (L.40)$$

$$\tilde{\pi}_t^i = \hat{\pi}_t^i + \ln \pi \qquad (L.41)$$

$$\tilde{\pi}_t^* = \hat{\pi}_t^* + \ln \pi \qquad (L.42)$$

$$\tilde{\bar{\pi}}_t^c = \hat{\bar{\pi}}_t^c + \ln \pi \qquad (L.43)$$

$$\Delta \ln \tilde{S}_t = \ln \tilde{S}_t - \ln \tilde{S}_{t-1} = \Delta \hat{S}_t + \ln\left(\frac{\pi}{\pi^*}\right) \qquad (L.44)$$

$$\tilde{R}_t = \hat{R}_t + \ln R \qquad (L.45)$$

$$\tilde{R}_t^* = \hat{R}_t^* + \ln R^* \qquad (L.46)$$

The following exogeneous variables in the DSGE model are assumed to follow AR(1) processes:

$$\Xi_t = \rho \Xi_{t-1} + \Gamma_t \qquad (91)$$

where

$$\Xi_t = \begin{bmatrix} \hat{\xi}_t^c & \hat{\xi}_t^i & \hat{\tilde{\phi}}_t & \hat{\varepsilon}_t & \hat{\xi}_t^h & \hat{\lambda}_t^x & \hat{\lambda}_t^d & \hat{\lambda}_t^{m,c} & \hat{\lambda}_t^{m,i} & \hat{\tilde{z}}_t^* & \hat{\mu}_t^z & \hat{\bar{\pi}}_t^c & \varepsilon_t^R \end{bmatrix}'$$

$$\rho = \begin{bmatrix} \rho_c & \rho_i & \rho_{\tilde{\phi}} & \rho_\varepsilon & \rho_h & \rho_{\lambda^x} & \rho_d & \rho_{\lambda^{m,c}} & \rho_{\lambda^{m,i}} & \rho_{\tilde{z}^*} & \rho_{\mu^z} & \rho_\pi & \rho_R \end{bmatrix}'$$

$$\Gamma_t = \begin{bmatrix} \varepsilon_t^c & \varepsilon_t^i & \varepsilon_t^{\tilde{\phi}} & \varepsilon_t^\varepsilon & \varepsilon_t^h & \varepsilon_t^x & \varepsilon_t^d & \varepsilon_t^{m,c} & \varepsilon_t^{m,i} & \varepsilon_t^{\tilde{z}^*} & \varepsilon_t^{\mu^z} & \varepsilon_t^{\bar{\pi}^c} & \varepsilon_t^{\mathrm{r}} \end{bmatrix}'$$

By solving the system of loglinearized equations and exogeneous processes, the model can be represented in a state space form as follows:

$$S_t = FS_{t-1} + Q\epsilon_t \qquad (92.\mathrm{a})$$

$$Y_t = M + HS_t + \eta_t \qquad (92.\mathrm{b})$$

where

S_t: endogenous processes

F, Q: model parameters

Y_t: observable data

M: steady-state information

H: mapping of endogenous variables to the data

$\begin{bmatrix} \epsilon_t \\ \eta_t \end{bmatrix} \sim N\left(0, \begin{bmatrix} \sigma & 0 \\ 0 & R \end{bmatrix}\right)$

ϵ_t: structural shocks

η_t: measurement error

$R = \mathbb{E}(\eta_t \eta_t')$

Bayesian methods are used to estimate the model, which can incorporate prior information of model parameters such as their distributions. With the estimated model, Y_t can be simulated with random values of ϵ_t and η_t.

A.12 Steady State

The steady state in the model can be derived analytically:

$$\pi = \frac{\mu_b}{\mu_z} \qquad (S.1)$$

where

μ_b: steady money growth rate.

$$(59) \quad \Rightarrow \quad -\psi^z + \beta \frac{\psi^z}{\mu^z \pi}\left(R - \iota^k (R-1)\right) = 0 \quad \Rightarrow \quad \psi^z = \beta \frac{\psi^z}{\mu^z \pi}\left(R - \iota^k (R-1)\right)$$

$$\Rightarrow \quad 1 = \beta \frac{1}{\mu^z \pi}\left(R - \iota^k (R-1)\right) \quad \Rightarrow \quad R = \frac{\mu^z \pi - \beta \iota^k}{\beta(1-\iota^k)} \qquad (S.2)$$

The markup of imported consumption $\lambda^{m,c}$ and of imported investment $\lambda^{m,i}$ can be used to represent the substitution elasticity among imported consumption goods and investment goods, respectively.

$$\lambda^{m,c} = \frac{\eta^{m,c}}{\eta^{m,c} - 1} \qquad (S.3) \qquad \Rightarrow \quad \eta^{m,c} = \frac{\lambda^{m,c}}{\lambda^{m,c} - 1}$$

$$\lambda^{m,i} = \frac{\eta^{m,i}}{\eta^{m,i} - 1} \qquad (S.4) \qquad \Rightarrow \quad \eta^{m,i} = \frac{\lambda^{m,i}}{\lambda^{m,i} - 1}$$

Assuming a steady state where $R^* = R$, $\pi^* = \pi$, and the initial domestic price and foreign price are the same ($P_0^* = P_0^d$), this leads to the following steady-state condition:

$$\frac{SP^*}{P^d} = 1$$

$$(35) \quad \Rightarrow \quad \gamma_t^{c,d} \equiv \frac{P_t^c}{P_t^d} = \left[(1-\vartheta_c) + \vartheta_c \left(\frac{P_t^{m,c}}{P_t^d}\right)^{1-\eta_c}\right]^{\frac{1}{1-\eta_c}}$$

$$\frac{P_t^{m,c}}{P_t^d} = \frac{\eta^{m,c}}{\eta^{m,c}-1}\frac{SP^*}{P^d} = \frac{\eta^{m,c}}{\eta^{m,c}-1}$$

$$\Rightarrow \quad \gamma^{c,d} = \left[(1-\vartheta_c) + \vartheta_c \left(\frac{\eta^{m,c}}{\eta^{m,c}-1}\right)^{1-\eta_c}\right]^{\frac{1}{1-\eta_c}} \qquad (S.5)$$

$$(35) \quad \Rightarrow \quad \gamma_t^{c,mc} \equiv \frac{P_t^c}{P_t^{m,c}} = \left[(1-\vartheta_c)\left(\frac{P_t^d}{P_t^{m,c}}\right)^{1-\eta_c} + \vartheta_c\right]^{\frac{1}{1-\eta_c}}$$

$$\Rightarrow \quad \gamma^{c,mc} = \left[(1-\vartheta_c)\left(\frac{\eta^{m,c}-1}{\eta^{m,c}}\right)^{1-\eta_c} + \vartheta_c\right]^{\frac{1}{1-\eta_c}} \qquad (S.6)$$

$$\frac{(S.5)}{(S.6)} = \frac{P^{m,c}}{P} = \left[\frac{(1-\vartheta_c) + \vartheta_c \left(\frac{\eta^{m,c}}{\eta^{m,c}-1}\right)^{1-\eta_c}}{(1-\vartheta_c)\left(\frac{\eta^{m,c}-1}{\eta^{m,c}}\right)^{1-\eta_c} + \vartheta_c}\right]^{\frac{1}{1-\eta_c}} = \frac{\eta^{m,c}}{\eta^{m,c}-1}$$

Similarly, for imported investment goods, the following steady-state conditions can be derived:

$$\gamma^{i,d} = \left[(1-\vartheta_i) + \vartheta_i \left(\frac{\eta^{m,i}}{\eta^{m,i}-1}\right)^{1-\eta_i}\right]^{\frac{1}{1-\eta_i}} \qquad (S.7)$$

$$\gamma^{i,mi} = \left[(1-\vartheta_i)\left(\frac{\eta^{m,i}-1}{\eta^{m,i}}\right)^{1-\eta_i} + \vartheta_i\right]^{\frac{1}{1-\eta_i}} \qquad (S.8)$$

$$\frac{P^{m,i}}{P} = \left[\frac{(1-\vartheta_i) + \vartheta_i \left(\frac{\eta^{m,i}}{\eta^{m,i}-1}\right)^{1-\eta_i}}{(1-\vartheta_i)\left(\frac{\eta^{m,i}-1}{\eta^{m,i}}\right)^{1-\eta_i} + \vartheta_i}\right]^{\frac{1}{1-\eta_i}} = \frac{\eta^{m,i}}{\eta^{m,i}-1}$$

$$(55) \quad \Rightarrow \quad P^{k'} = \frac{P^i}{P} = \gamma^{i,d}$$

$$(56) \quad \Rightarrow \quad -\psi^z P^{k'} + \beta\frac{\psi^z}{\mu^z}\Big((1-\iota^k)r^k + P^{k'}(1-\delta)\Big) = 0$$

$$\Rightarrow \qquad \beta\frac{1}{\mu^z}\Big((1-\iota^k)r^k + P^{k'}(1-\delta)\Big) = P^{k'} \quad \Rightarrow \quad r^k = \frac{P^{k'}\big(\mu^z - \beta(1-\delta)\big)}{\beta(1-\iota^k)}$$

$$\Rightarrow \quad r^k = \frac{\gamma^{i,d}\big(\mu^z - \beta(1-\delta)\big)}{\beta(1-\iota^k)} \qquad (S.9)$$

$$(49) \quad \Rightarrow \quad k = (1-\delta)\frac{k}{\mu^z} + i \quad \Rightarrow \quad i = k - (1-\delta)\frac{k}{\mu^z} \qquad (93)$$

$$(53) \quad \Rightarrow \quad \frac{1}{c - \frac{bc}{\mu^z}} - \beta b \frac{1}{c\mu^z - bc} - \psi^z \frac{P^c}{P}(1 + \iota_t^c) = 0$$

$$\Rightarrow \quad \psi^z = \frac{\frac{1}{c - \frac{bc}{\mu^z}} - \beta b \frac{1}{c\mu^z - bc}}{1 + \iota^c} \frac{P}{P^c} = \frac{\mu^z - \beta b}{(c\mu^z - bc)(1 + \iota^c)} \frac{1}{\gamma^{c,d}} \qquad (94)$$

$$(13) \quad \Rightarrow \quad mc = \left(\frac{1}{1-\alpha}\right)^{1-\alpha} \left(\frac{1}{\alpha}\right)^{\alpha} (r^k)^{\alpha} (\overline{w})^{1-\alpha}$$

$$P = \lambda^d MC \quad \Rightarrow \quad \frac{MC}{P} = \frac{1}{\lambda^d} \quad \Rightarrow \quad mc = \frac{1}{\lambda^d}$$

$$\Rightarrow \quad \frac{1}{\lambda^d} = \left(\frac{1}{1-\alpha}\right)^{1-\alpha} \left(\frac{1}{\alpha}\right)^{\alpha} (r^k)^{\alpha} (\overline{w})^{1-\alpha}$$

$$\Rightarrow \quad \overline{w} = (1-\alpha)\left(\frac{1}{\lambda^d}\right)^{\frac{1}{1-\alpha}} (\alpha)^{\frac{\alpha}{1-\alpha}} (r^k)^{\frac{\alpha}{1-\alpha}} \qquad (S.10)$$

$$(12) \quad \Rightarrow \quad r = \frac{\alpha}{1-\alpha} \overline{w} \mu^z \left(\frac{H}{k}\right) \quad \Rightarrow \quad \frac{k}{H} = \frac{\alpha}{1-\alpha} \frac{\overline{w}\mu^z}{r} \qquad (S.11)$$

$$(65) \quad \Rightarrow \quad c^d + i^d + c^x + i^x = (1 - g_y)\left[\left(\frac{1}{\mu^z}\right)^{\alpha} \left(\frac{k}{H}\right)^{\alpha} H - \phi\right]$$

$$\Rightarrow \quad (1-\vartheta_c)\left(\frac{P^c}{P}\right)^{\eta_c} c + (1-\vartheta_i)\left(\frac{P^i}{P}\right)^{\eta_i} i + c^x + i^x = (1 - g_y)\left[\left(\frac{1}{\mu^z}\right)^{\alpha} \left(\frac{k}{H}\right)^{\alpha} H - \phi\right]$$

In steady state, the real profit of intermediate goods producers is zero:

$$\Pi^R = \lambda_d y - y - \phi = (\lambda_d - 1)y - \phi = 0$$

$$(5) \quad \Rightarrow \quad y = \left(\frac{1}{\mu^z}\right)^{\alpha} \left(\frac{k}{H}\right)^{\alpha} H - \phi \qquad (95)$$

$$\Rightarrow \quad \phi = (\lambda_d - 1)y = (\lambda_d - 1)\left[\left(\frac{1}{\mu^z}\right)^{\alpha} \left(\frac{k}{H}\right)^{\alpha} H - \phi\right]$$

$$\Rightarrow \quad \phi = (\lambda_d - 1)y = (\lambda_d - 1)\left[\left(\frac{1}{\mu^z}\right)^{\alpha} \left(\frac{k}{H}\right)^{\alpha} H - \phi\right]$$

$$\Rightarrow \quad \phi = (\lambda_d - 1)y = \frac{\lambda_d - 1}{\lambda_d}\left(\frac{1}{\mu^z}\right)^{\alpha} \left(\frac{k}{H}\right)^{\alpha} H \qquad (96)$$

$$\Rightarrow \qquad (1-\vartheta_c)\left(\frac{P^c}{P}\right)^{\eta_c} c + (1-\vartheta_i)\left(\frac{P^i}{P}\right)^{\eta_i} i + c^x + i^x = \frac{1-g_y}{\lambda_d}\left(\frac{1}{\mu^z}\right)^{\alpha}\left(\frac{k}{H}\right)^{\alpha} H$$

In a steady state with zero net imports, zero foreign debt and unchanged nominal exchange rate,

$$c^x + i^x = c^m + i^m = \vartheta_c\left(\frac{P^{m,c}}{P^c}\right)^{-\eta_c} c + \vartheta_i\left(\frac{P^{m,i}}{P^i}\right)^{-\eta_i} i$$

$$\Rightarrow \qquad \left[(1-\vartheta_c)\left(\frac{P^c}{P}\right)^{\eta_c} + \vartheta_c\left(\frac{P^c}{P^{m,c}}\right)^{\eta_c}\right] c + \left[(1-\vartheta_i)\left(\frac{P^i}{P}\right)^{\eta_i} + \vartheta_i\left(\frac{P^i}{P^{m,i}}\right)^{\eta_i}\right] i = \frac{1-g_y}{\lambda_d}\left(\frac{1}{\mu^z}\right)^{\alpha}\left(\frac{k}{H}\right)^{\alpha} H$$

Replacing i with (93):

$$\left[(1-\vartheta_c)\left(\frac{P^c}{P}\right)^{\eta_c} + \vartheta_c\left(\frac{P^c}{P^{m,c}}\right)^{\eta_c}\right] c + \left[(1-\vartheta_i)\left(\frac{P^i}{P}\right)^{\eta_i} + \vartheta_i\left(\frac{P^i}{P^{m,i}}\right)^{\eta_i}\right]\left(1-\frac{1-\delta}{\mu^z}\right)\left(\frac{k}{H}\right) H = \frac{1-g_y}{\lambda_d}\left(\frac{1}{\mu^z}\right)^{\alpha}\left(\frac{k}{H}\right)^{\alpha} H \qquad (97)$$

F.O.C. of (40):

$$\Rightarrow \qquad \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_w)^s h_{j,t+s}\left\{-\lambda_w \xi_{t+s}^h A_L h_{j,t+s}{}^{\sigma_L}\frac{1}{\widetilde{W}_{j,t}} + \nu_{t+s}\left(\frac{1-\iota_{t+s}^y}{1+\iota_{t+s}^w}\right)\left[\left(\frac{P_{t+s-1}^c}{P_{t-1}^c}\right)^{\chi_w}\left(\prod_{k=1}^{s}\bar{\pi}_{t+k}^c\right)^{1-\chi_w}\frac{z_{t+s}}{z_t}\right]\right\} = 0$$

$if\ \theta_w = 0$

$$\begin{aligned}\mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_w)^s h_{j,t+s}&\left\{-\lambda_w \xi_{t+s}^h A_L h_{j,t+s}{}^{\sigma_L}\frac{1}{\widetilde{W}_{j,t}} + \nu_{t+s}\left(\frac{1-\iota_{t+s}^y}{1+\iota_{t+s}^w}\right)\left[\left(\frac{P_{t+s-1}^c}{P_{t-1}^c}\right)^{\chi_w}\left(\prod_{k=1}^{s}\bar{\pi}_{t+k}^c\right)^{1-\chi_w}\frac{z_{t+s}}{z_t}\right]\right\} \\ &= h_{j,t}\left\{-\lambda_w \xi_t^h A_L h_{j,t}{}^{\sigma_L}\frac{1}{\widetilde{W}_{j,t}} + \nu_t\left(\frac{1-\iota_t^y}{1+\iota_t^w}\right)\left[\left(\frac{P_{t-1}^c}{P_{t-1}^c}\right)^{\chi_w}\frac{z_t}{z_t}\right]\right\} \\ &= h_{j,t}\left\{-\lambda_w \xi_t^h A_L h_{j,t}{}^{\sigma_L}\frac{1}{\widetilde{W}_{j,t}} + \nu_t\left(\frac{1-\iota_t^y}{1+\iota_t^w}\right)\right\} = 0\end{aligned}$$

$$\Rightarrow \qquad -\xi_t^h A_L h_{j,t}{}^{\sigma_L} + \frac{\widetilde{W}_{j,t}}{z_t P^t}\frac{z_t P^t \nu_t}{\lambda_w}\left(\frac{1-\iota_t^y}{1+\iota_t^w}\right) = 0$$

$$\Rightarrow \qquad -\xi_t^h A_L h_{j,t}{}^{\sigma_L} + \bar{w}_{j,t}\frac{\psi_t^z}{\lambda_w}\left(\frac{1-\iota_t^y}{1+\iota_t^w}\right) = 0$$

In a steady state where the wage is fully flexible,

$$-A_L h_j{}^{\sigma_L} + \bar{w}_j\frac{\psi^z}{\lambda_w}\left(\frac{1-\iota^y}{1+\iota^w}\right) = 0 \qquad \Rightarrow \qquad -A_L H^{\sigma_L} + \bar{w}\frac{\psi^z}{\lambda_w}\left(\frac{1-\iota^y}{1+\iota^w}\right) = 0$$

$$\Rightarrow \quad H = \left(\frac{1-\iota^y}{1+\iota^w}\frac{\overline{w}\psi^z}{A_L\lambda_w}\right)^{\frac{1}{\sigma_L}} \qquad (98)$$

Equations (94) , (97) and (98) can be rearranged as the following equations of ψ^z, c and H:

$$\psi^z = \frac{1}{c}T_1$$

$$T_2 c = T_3 H$$

$$H = T_4(\psi^z)^{\frac{1}{\sigma_L}}$$

where

$$T_1 = \frac{\mu^z - \beta b}{(\mu^z - b)(1+\iota^c)}\frac{1}{\gamma^{c,d}}$$

$$T_2 = (1-\vartheta_c)\left(\frac{P^c}{P}\right)^{\eta_c} + \vartheta_c\left(\frac{P^c}{P^{m,c}}\right)^{\eta_c}$$

$$T_3 = \frac{1-g_y}{\lambda_d}\left(\frac{1}{\mu^z}\right)^{\alpha}\left(\frac{k}{H}\right)^{\alpha} - \left[(1-\vartheta_i)\left(\frac{P^i}{P}\right)^{\eta_i} + \vartheta_i\left(\frac{P^i}{P^{m,i}}\right)^{\eta_i}\right]\left(1-\frac{1-\delta}{\mu^z}\right)\left(\frac{k}{H}\right)$$

$$T_4 = \left(\frac{1-\iota^y}{1+\iota^w}\frac{\overline{w}}{A_L\lambda_w}\right)^{\frac{1}{\sigma_L}}$$

$$H = T_4(\psi^z)^{\frac{1}{\sigma_L}} = T_4\left(\frac{1}{c}T_1\right)^{\frac{1}{\sigma_L}} = T_4\left(\frac{T_2}{T_3 H}T_1\right)^{\frac{1}{\sigma_L}} \quad \Rightarrow \quad H^{\frac{1+\sigma_L}{\sigma_L}} = T_4\left(\frac{T_2}{T_3}T_1\right)^{\frac{1}{\sigma_L}}$$

$$\Rightarrow \quad H = \left[T_4\left(\frac{T_1 T_2}{T_3}\right)^{\frac{1}{\sigma_L}}\right]^{\frac{\sigma_L}{1+\sigma_L}} \qquad (S.12)$$

$$c = \frac{T_3}{T_2}H \qquad (S.13)$$

$$\psi^z = \frac{1}{c}T_1 \qquad (S.14)$$

Equations (95) and (96)

$$\Rightarrow \quad y = \frac{1}{\lambda_d}\left(\frac{1}{\mu^z}\right)^{\alpha}\left(\frac{k}{H}\right)^{\alpha} H \qquad (S.15)$$

$$(58) \qquad \Rightarrow \qquad q = \left(\frac{A_q}{\psi^z(R-1)(1-\iota^k)}\right)^{\frac{1}{\sigma_q}} \qquad (S.16)$$

Steady state:

$$A_q(q)^{-\sigma_q} - \psi^z(R-1)(1-\iota^k) = 0$$

$$\Rightarrow \qquad A_q(q)^{-\sigma_q} = \psi^z(R-1)(1-\iota^k)$$

$$\Rightarrow \qquad q = \left(\frac{A_q}{\psi^z(R-1)(1-\iota^k)}\right)^{\frac{1}{\sigma_q}}$$

A.13 Calibration

Bayesian analysis is used to estimate the specified economic system given observable data. However, not all model parameters can be estimated based on observable data. Some model parameters are calibrated to be a fixed value. Others are assumed to follow certain prior distributions and are estimated based on observable data. If the observable data can provide enough information about the parameter, the parameter can be estimated by updating its prior distribution. Otherwise, the parameter needs to be calibrated separately from the Bayesian estimation process. Sometimes, two parameters may be highly correlated and difficult to separate. One of them needs to be calibrated, and the other can be estimated using the Bayesian method. Parameters may also need to be calibrated to maintain the relationships in a steady state derived from the model.

Table A.1 lists the parameters that are calibrated for the U.S. economy.

Table A.1

Parameter Calibration: U.S. Economy

Parameter	Description	Value	Note
β	Discount factor	0.9975	Low interest rate environment: 1%
δ	Capital depreciation rate	0.013	Annual depreciation of 5.2%
α	Capital share in production	0.23	Historical average
A_L	Disutility of supplying labor	7.5	30% of time working
σ_L	Inverted Frisch elasticity of labor supply	1	Christiano, Eichenbaum and Evans (2005)
A_q	Utility of real asset holdings	0.436	Cash to money (M1/M3)
σ_q	Elasticity of real asset holdings	10.62	Christiano, Eichenbaum and Evans (2005)
σ_a	$\frac{a''(1)}{a'(1)}$ where a is capital adjustment cost	1,000,000	High capital utilization, follows Adolfson et al. (2007)
θ_w	Probability of households not setting wages	0.69	Wage contract renewed every 3.3 quarters
χ_w	Indexation of wages	0.5	Half indexation
ϑ_c	Share of imports in aggregate consumption	0.36	Historical average
ϑ_i	Share of imports in aggregate investment	0.48	Historical average
η_c	Substitution elasticity between domestic goods and imported goods	1.5	Usually between 1 and 2, follows Chari et al. (2002)
η_i	Substitution elasticity between domestic investment and imported investment	1.5	Usually between 1 and 2, follows Chari et al. (2002)
η_f	Elasticity for foreign consumption and investment	1.25	Usually between 1 and 2, follows Chari et al. (2002)

ρ_π	Inflation target persistency	0	A constant annual inflation rate of 2% is used
ρ_g	Government spending persistency	0.815	Historical experience
g_y	Government spending/GDP	0.197	Government expenditure/output
μ^z	Stable permanent technology growth	1.004	Real GDP growth rate 1.6%
μ^b	M2 growth rate	1.013	Historical experience
λ_w	Wage markup	1.05	Christiano, Eichenbaum and Evans (2005)
λ_d	Domestic price markup	1.1	Christiano, Eichenbaum and Evans (2005)
$\lambda_{m,c}$	Imported consumption price markup	1.1	Christiano, Eichenbaum and Evans (2005)
$\lambda_{m,i}$	Imported investment price markup	1.1	Christiano, Eichenbaum and Evans (2005)
σ^*	Sensitivity of foreign inflation rate to foreign real GDP growth rate	1.3	Historical experience
χ^*	Sensitivity of foreign real GDP growth rate to foreign interest rate and inflation rate	0.2	Historical experience
π^*	Steady foreign inflation index	0.9998	Historical average
c_{l^k}	Capital tax rate	0.3918	Historical average
c_{l^c}	Consumption tax rate	0.0651	Historical average
c_{l^y}	Profit tax rate	0.1242	Historical average
c_{l^w}	Wage tax rate	0.0765	Historical average
$\rho_{l^{*,*}}$	Autocorrelation of tax rates	0	Historical experience

Table A.2 lists the prior and posterior distribution of each parameter. By comparing the prior and posterior distributions of model parameters, it shows the impact of observed data on model estimation. It may also give a better initial value to improve Bayesian estimation.

Table A.2

Parameter Estimation: U.S. Economy

Parameter	Description	Prior			Posterior	
		Distribution	Mean	Std. Dev.	Mean	90% Interval
ϕ_i	$S''(\mu^z)$ as in investment adjustment cost function	Normal	8	1.5	7.74	[7.67, 7.84]
b	Degree of habit formation in consumption	Beta	0.7	0.1	0.77	[0.75, 0.79]
θ_d	Probability of intermediate goods producers not setting goods price	Beta	0.9	0.05	0.4	[0.39, 0.41]

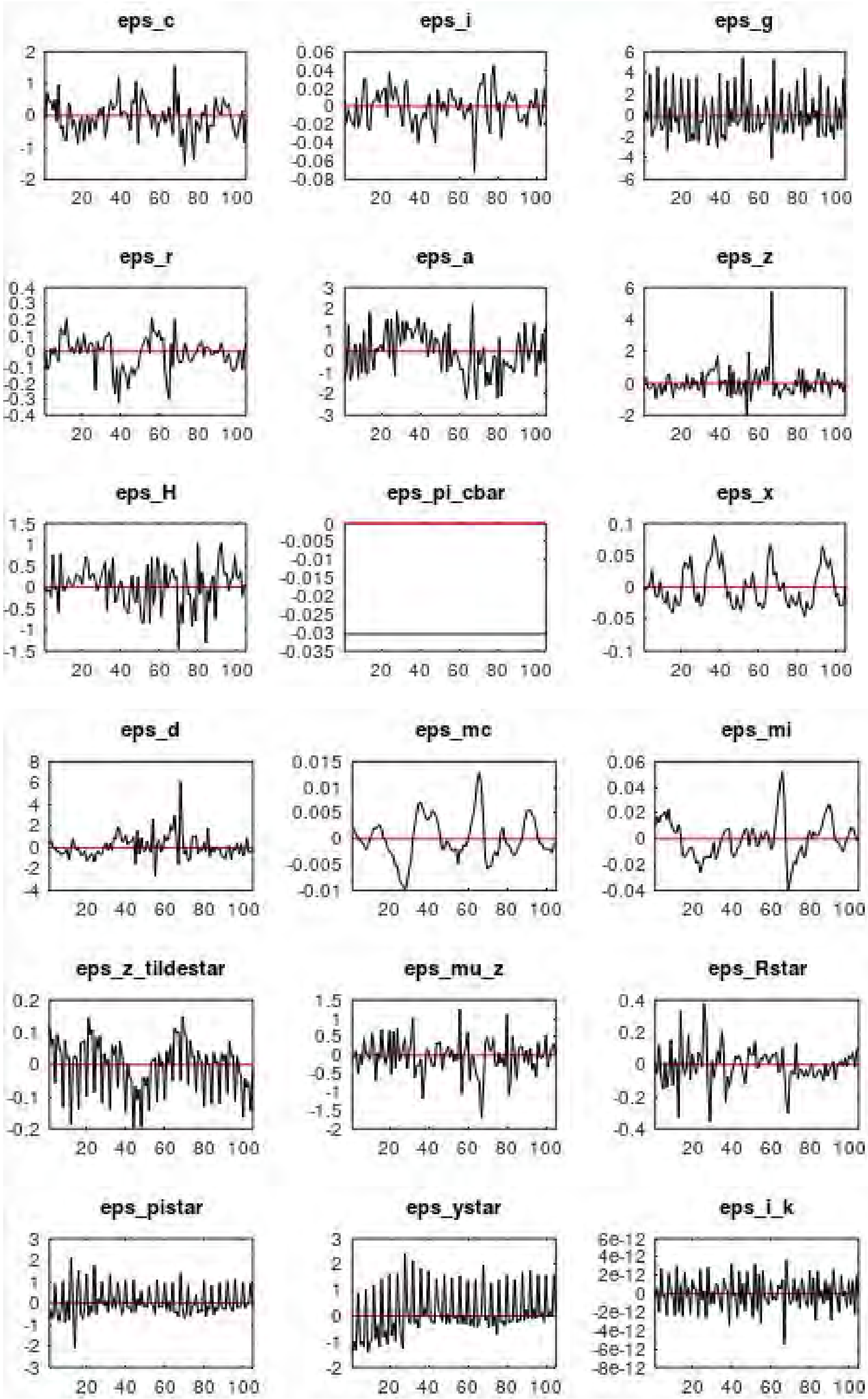
θ_{mc}	Probability of importing consumption firms not setting consumption goods price	Beta	0.9	0.05	0.96	[0.95, 0.98]
θ_{mi}	Probability of importing consumption firms not setting investment goods price	Beta	0.8	0.05	0.83	[0.80, 0.87]
θ_{x}	Probability of exporters not setting goods price	Beta	0.5	0.05	0.79	[0.787, 0.790]
χ_{d}	Indexation of domestic goods price	Beta	0.6	0.1	0.12	[0.08, 0.16]
χ_{mc}	Indexation of imported consumption goods price	Beta	0.6	0.1	0.90	[0.86, 0.94]
χ_{mi}	Indexation of imported investment goods price	Beta	0.6	0.1	0.60	[0.53, 0.64]
$\tilde{\phi}_{a}$	Sensitivity of foreign bond investment risk premium to net asset position	Inverse Gamma	0.04	1	0.08	[0.05, 0.11]
$\tilde{\phi}_{s}$	Sensitivity of foreign bond investment risk premium to uncovered interest rate parity	Inverse Gamma	0.6	1	0.40	[0.38, 0.45]
ρ_{R}	Policy rate persistency	Beta	0.8	0.05	0.90	[0.88, 0.93]
ϕ_{π}	Coefficient for inflation	Gamma	1	0.15	0.98	[0.83, 1.14]
$\phi_{\Delta\pi}$	Coefficient for change in inflation	Gamma	0.1	0.05	0.22	[0.13, 0.33]
ϕ_{y}	Coefficient for production	Gamma	0.5	0.1	0.48	[0.42, 0.54]
$\phi_{\Delta y}$	Coefficient for change in production	Gamma	0.25	0.1	0.16	[0.11, 0.24]
ϕ_{x}	Coefficient for real exchange rate	Normal	0.01	0.05	-0.01	[-0.03, 0.02]
ρ_{c}	Consumption preference shock persistency	Beta	0.7	0.1	0.89	[0.79, 0.95]

ρ_i	Investment technology shock persistency	Beta	0.7	0.1	0.70	[0.63, 0.77]
$\rho_{\tilde{\phi}}$	Risk premium shock persistency	Beta	0.7	0.1	0.70	[0.67, 0.74]
ρ_h	Labor supply shock persistency	Beta	0.7	0.1	0.63	[0.52, 0.74]
ρ_{λ^x}	Export price markup persistency	Beta	0.7	0.1	0.60	[0.53, 0.71]
$\rho_{\lambda^{m,c}}$	Imported consumption markup persistency	Beta	0.7	0.1	0.63	[0.54, 0.72]
$\rho_{\lambda^{m,i}}$	Imported investment markup persistency	Beta	0.7	0.1	0.78	[0.66, 0.86]
ρ_d	Domestic price markup persistency	Beta	0.7	0.1	0.97	[0.96, 0.98]
ρ_ε	Transitory technology persistency	Beta	0.7	0.1	0.98	[0.97, 0.99]
$\rho_{\tilde{z}^*}$	Asymmetric technology persistency	Beta	0.7	0.1	0.91	[0.88, 0.96]
ρ_{μ^z}	Permanent technology persistency	Beta	0.7	0.1	0.70	[0.64, 0.76]
ρ_R^*	Foreign policy rate persistency	Beta	0.7	0.1	0.96	[0.95, 0.97]
ρ_{π^*}	Foreign inflation persistency	Beta	0.7	0.1	0.86	[0.83, 0.91]
ρ_{y^*}	Foreign production persistency	Beta	0.7	0.1	0.84	[0.82, 0.88]
σ_c	Consumption preference shock persistency	Inverse Gamma	0.3	Inf	0.83	[0.75, 0.90]
σ_i	Investment technology shock persistency	Inverse Gamma	0.3	Inf	0.17	[0.07, 0.28]
$\sigma_{\tilde{\phi}}$	Risk premium shock persistency	Inverse Gamma	0.3	Inf	0.89	[0.82, 0.96]
σ_ε	Transitory technology persistency	Inverse Gamma	0.3	Inf	1.01	[0.94, 1.09]
σ_h	Labor supply shock persistency	Inverse Gamma	0.3	Inf	0.72	[0.57, 0.89]
σ_r	Central bank interest rate shock persistency	Inverse Gamma	0.3	Inf	0.11	[0.09, 0.12]
σ_{λ^x}	Export price markup persistency	Inverse Gamma	0.3	Inf	0.21	[0.13, 0.29]

$\sigma_{\lambda^{m,c}}$	Imported consumption markup persistency	Inverse Gamma	0.3	Inf	0.26	[0.18, 0.37]
$\sigma_{\lambda^{m,i}}$	Imported investment markup persistency	Inverse Gamma	0.3	Inf	0.23	[0.10, 0.36]
σ_d	Export price markup persistency	Inverse Gamma	0.3	Inf	1.07	[1.00, 1.15]
$\sigma_{\tilde{z}^*}$	Asymmetric technology persistency	Inverse Gamma	0.3	Inf	0.34	[0.15, 0.58]
σ_{μ^z}	Permanent technology persistency	Inverse Gamma	0.3	Inf	0.48	[0.41, 0.55]

Shocks derived from historical data are helpful to assess the reasonableness of the model. For example, during the 2008 financial crisis (*x*-axis value from 63 to 69) as shown in Figure A.1, significant shocks in consumption, investment, interest rate, technology and inflation happened. Items starting with “eps” are the shocks defined in Γ_t in equation (91). Items starting with “me” are the measurement errors.

Figure A.1
Historical Shocks and Measurement Errors



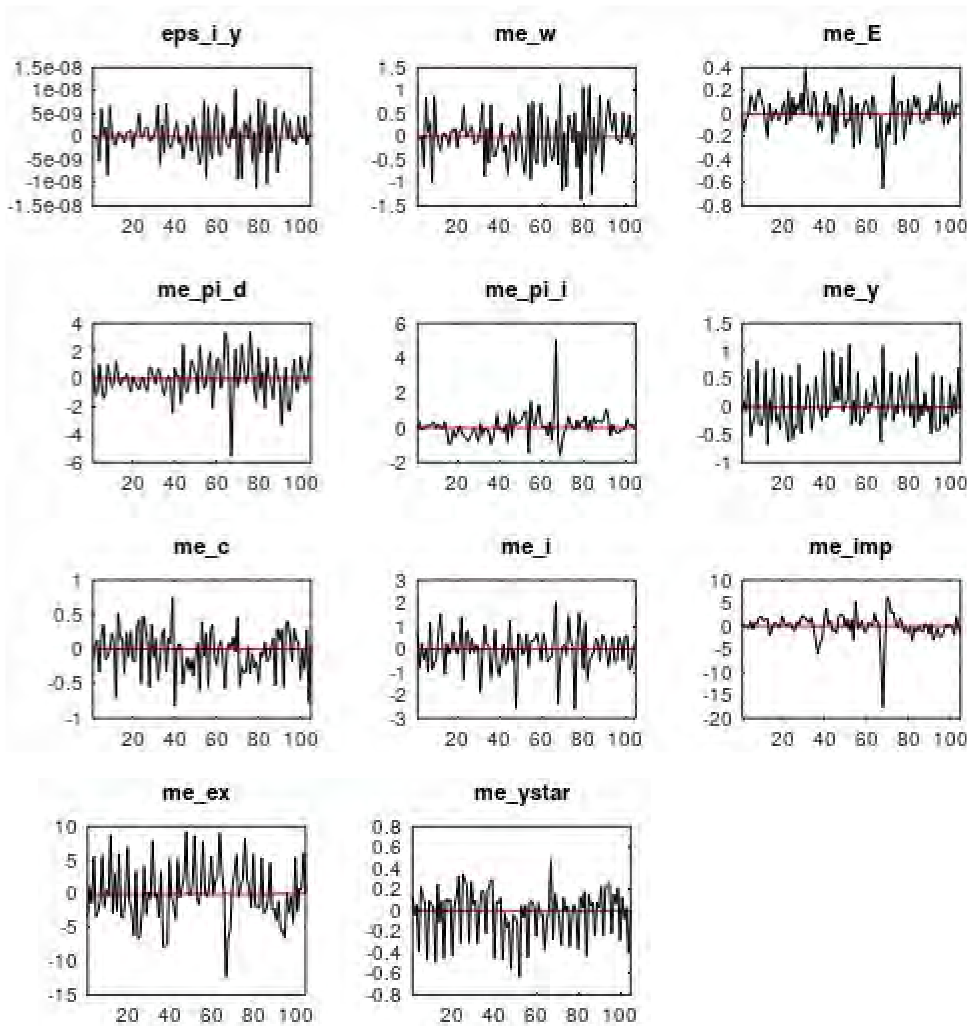


Figure A.2 shows that a technology improvement boosts the economy everywhere except that the policy interest rate will be increased to cool the expansion. Such analysis can be used for other risk sources based on their impulsive response functions. Impulsive response functions in Figures A.3 to Figure A.7 show the impact of shocking the labor supply, risk premium, consumption, investment and government expenditure respectively by one standard deviation at time zero.

Figure A.2

Impulsive Response Function: Permanent Technology Shock

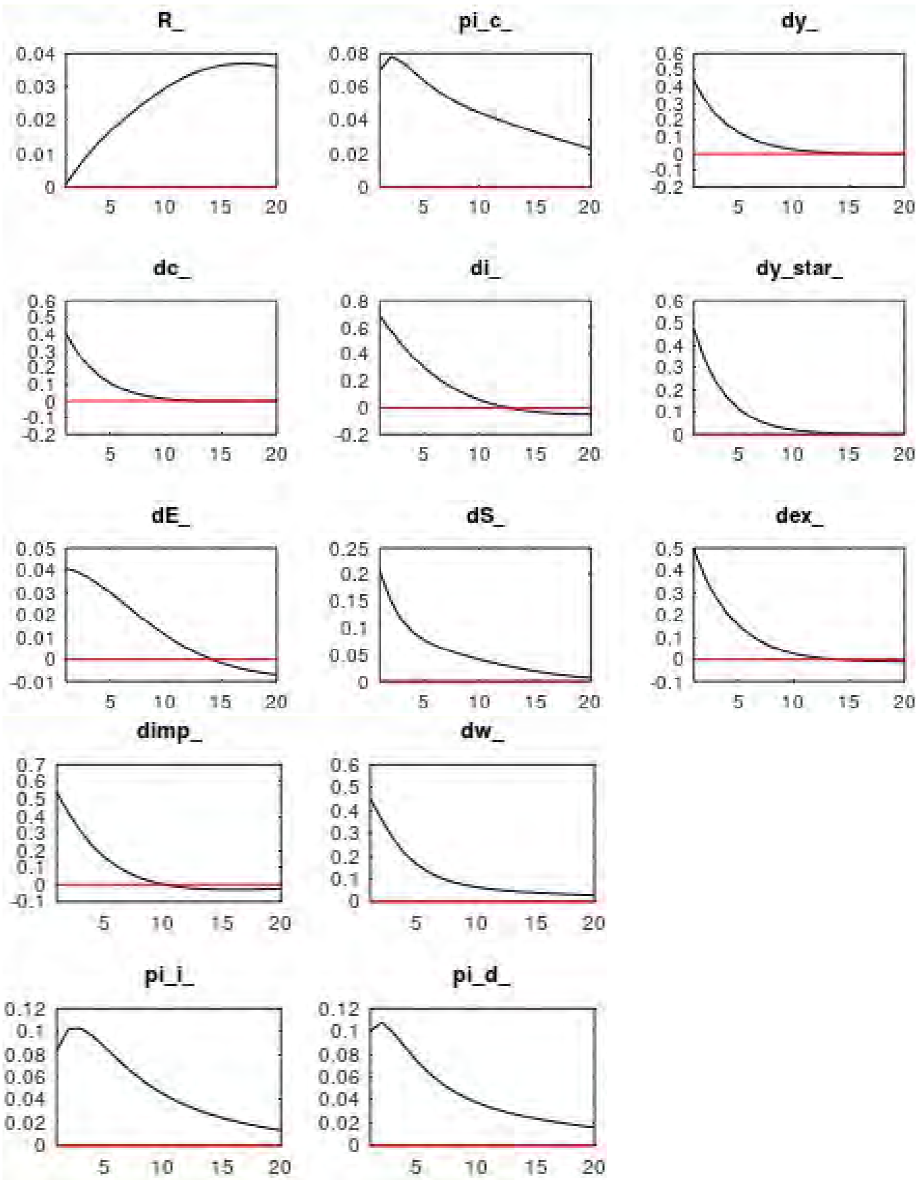


Figure A.3

Impulsive Response Function: Labor Supply Shock

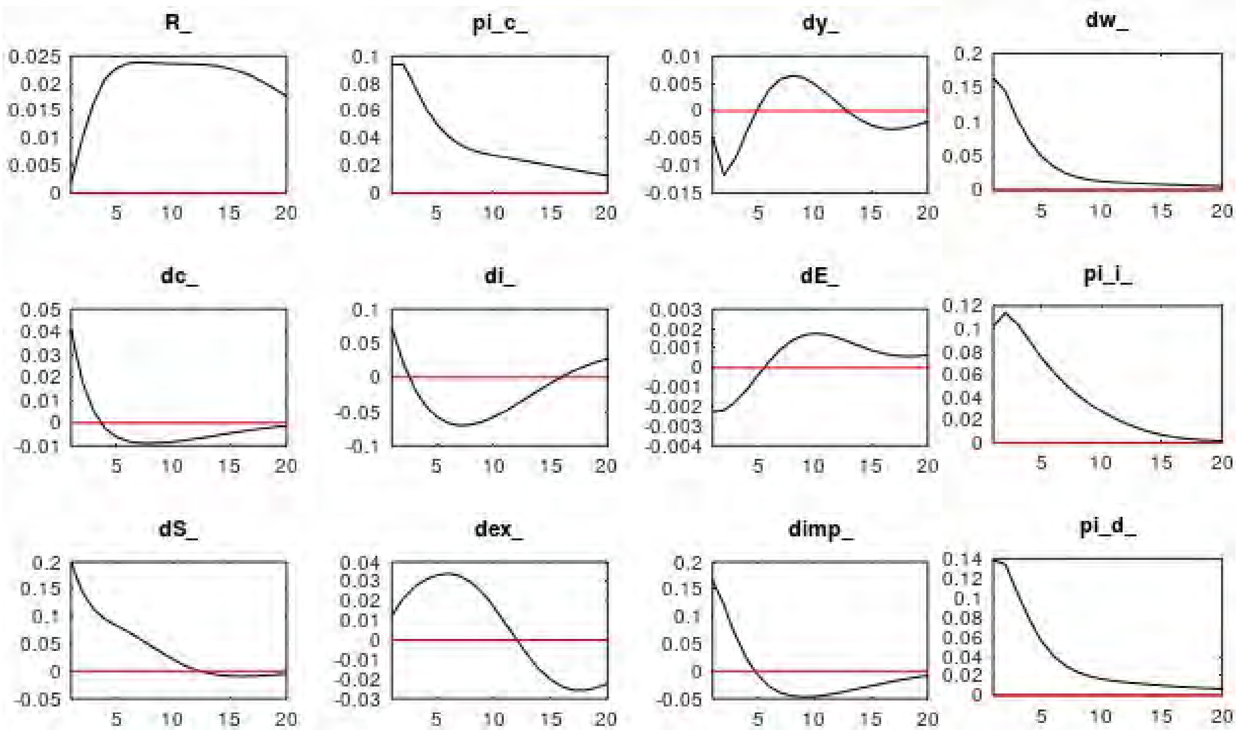


Figure A.4

Impulsive Response Function: Risk Premium Shock

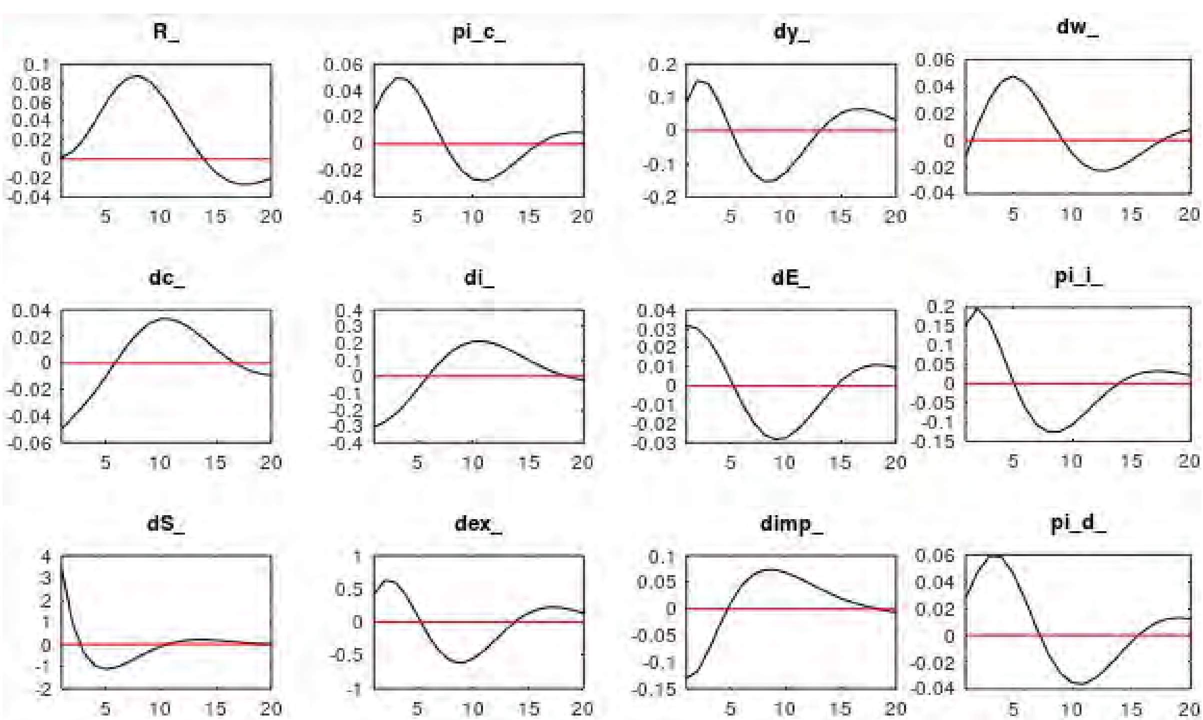


Figure A.5

Impulsive Response Function: Consumption Shock

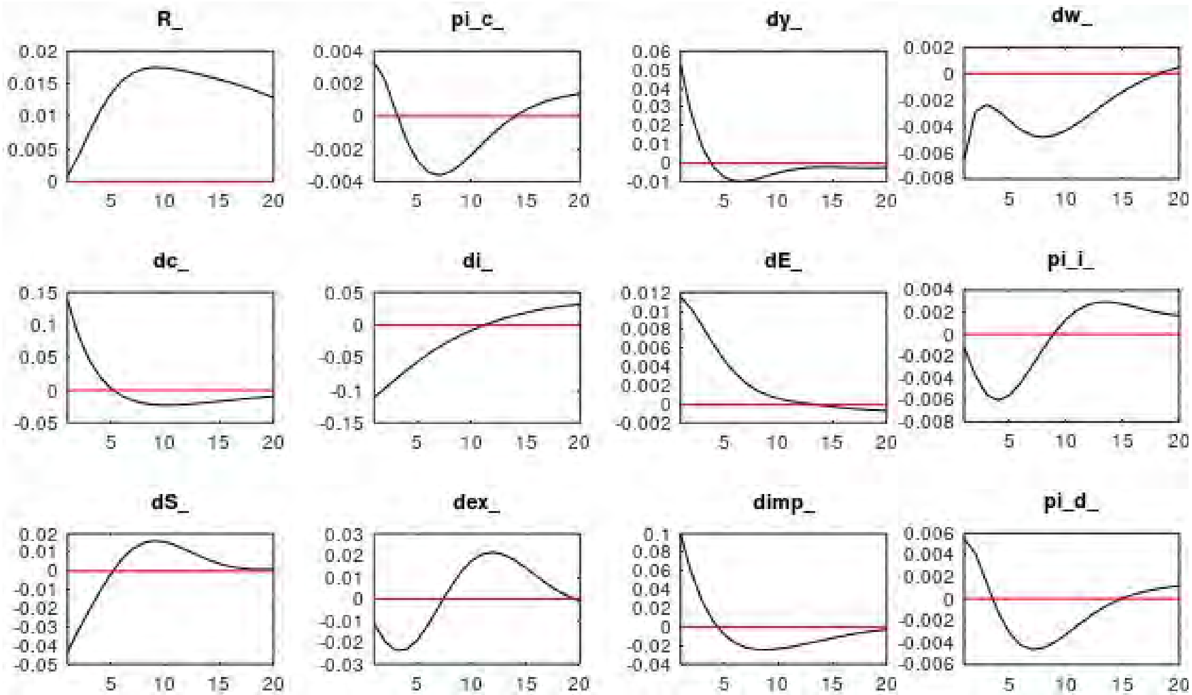


Figure A.6

Impulsive Response Function: Investment Shock

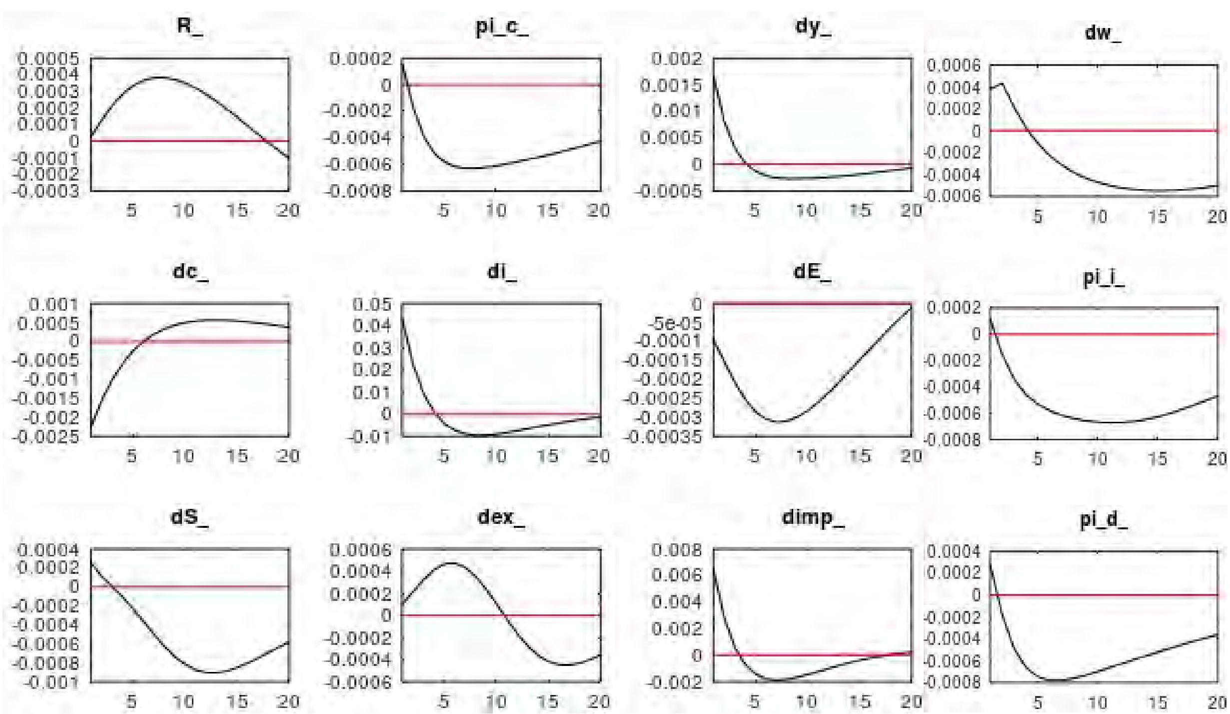
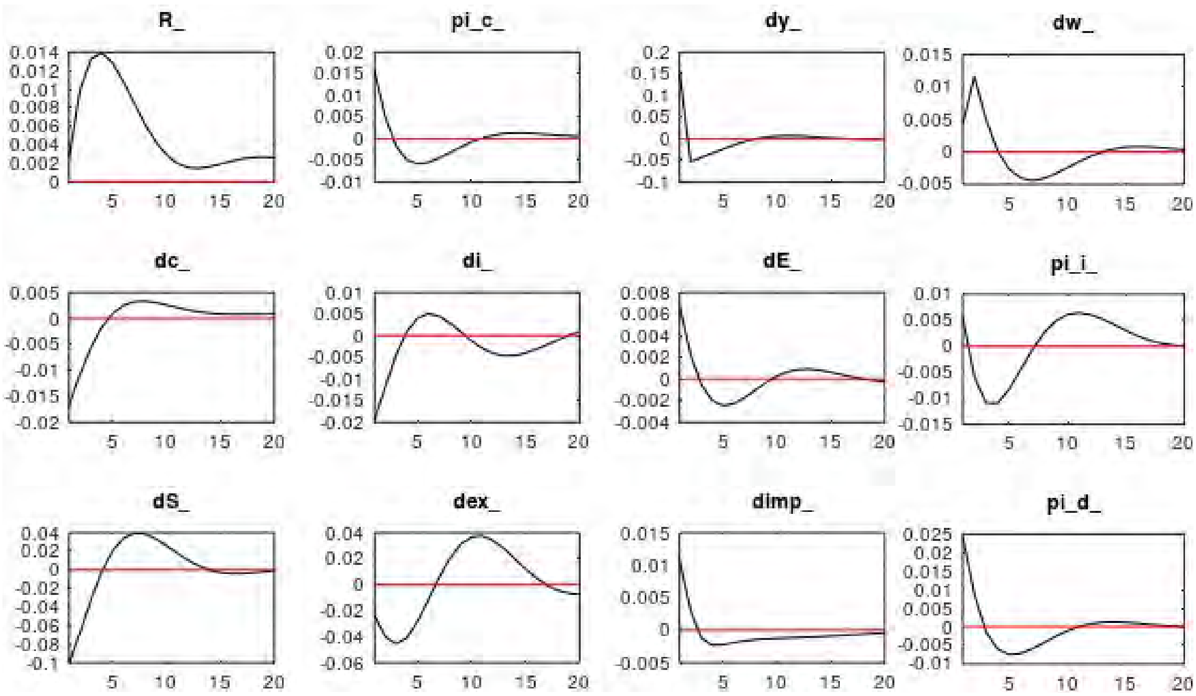


Figure A.7

Impulsive Response Function: Government Expenditure Shock



Appendix B: Multifactor Model Details

This appendix provides some explanation of the multifactor models used and tested in Section 3 and Section 4.

B.1 Regression Model

In this report, both linear and nonlinear models are tested, as described in Section 4.2. The Elastic Net model is chosen as one of many possible multifactor regression models. The resulting simulation function takes the form of a vector autoregressive (VAR) model. Modeled asset returns and yields are generated using linear models with macroeconomic factors as explanatory variables:

$$\boldsymbol{y_t} = \boldsymbol{\alpha} + \boldsymbol{\phi_1 y_{t-1}} + \boldsymbol{\phi_2 y_{t-2}} + \boldsymbol{B_0 F_t} + \boldsymbol{B_1 F_{t-1}} + \boldsymbol{B_2 F_{t-2}} + \boldsymbol{e_t}$$

where

$\boldsymbol{y_t}$: A column vector containing asset returns during period t or yields at time t
$\boldsymbol{\alpha}$: A column vector containing the constant terms
$\boldsymbol{\phi_i}$: A column vector containing parameters to govern the relationship between current return and return for all asset classes during period $t-i$
$\boldsymbol{F_t}$: A column vector including all the macroeconomic factors at time t or during period t generated by a DSGE model
$\boldsymbol{B_i}$: A matrix that contains model parameters for the macroeconomic factors during period $t-i$
$\boldsymbol{e_t}$: A column vector including all the idiosyncratic risk component at time t or during period t

The Elastic Net model is chosen based on model accuracy for validation data. However, limited data volume may imply that all data need to be used for model calibration and scenario simulation. The final multifactor model parameters are determined to achieve high model accuracy for the entire dataset.

Components of e_t are not independent with each other but contain nonlinear correlation, as explained in the following section.

B.2 Correlation Adjustment

In this report, the following approach is used to generate correlated asset returns and other modeled variables:

- **Step 1.** Predict whether or not the economy is in recession for each period under each scenario, based on macroeconomic factors. This can differentiate between economic recession and expansion. During an economic recession, higher volatility and correlation are usually observed and need to be reflected in stochastic asset returns. Real GDP growth rate, employment rate, consumption growth rate and investment growth rate are used in the following logistic model to predict whether the economy is in recession or not:

$$R_t = \frac{1}{1 + e^{-\boldsymbol{\beta}\mathbf{F}_t}}$$

where

R_t = the probability that the economy is in recession during period t
$\mathbf{F}_t = \begin{pmatrix} 1, \mathrm{dy\_}_t, \mathrm{dy\_}_{t-1}, \mathrm{dy\_}_{t-2}, \mathrm{dc\_}_t, \mathrm{dc\_}_{t-1}, \mathrm{dc\_}_{t-2}, \mathrm{di\_}_t, \mathrm{di\_}_{t-1}, \mathrm{di\_}_{t-2}, \\ \mathrm{dE\_}_t, \mathrm{dE\_}_{t-1}, \mathrm{dE\_}_{t-2}, \mathrm{pi\_c\_}_t, \mathrm{pi\_c\_}_{t-1}, \mathrm{pi\_c\_}_{t-2} \end{pmatrix}^T$, a column vector with 16 elements containing the constant term and fundamental economic factors during periods t, $t-1$ and $t-2$

dy_: change in real GDP growth
dc_: change in consumption
di_: change in investment
dE_: change in employment
pi_c_: consumer inflation rate
$\boldsymbol{\beta}$ is a row vector with 13 elements containing the model parameters for variables in $\mathbf{F}_t$

- Step 2. Construct two correlation matrices from the error terms of asset return models. The first correlation matrix, $\mathbf{CM}_{\mathrm{All}}$, describes the general relationships among error terms of all asset classes, using historical data during economic expansions. The second correlation matrix, $\mathrm{CM}_{\mathrm{Res}}$, describes the relationships among error terms only in economic recessions. Cholesky decomposition then can be performed to get the lower triangular matrices $\mathbb{L}_{\mathrm{All}}$ and $\mathbb{L}_{\mathrm{Res}}$ to generate correlated idiosyncratic factors.

- Step 3. Generate correlated idiosyncratic factors $\mathbf{I}_t^i$ for all asset returns during period t under scenario i:

$$\mathbf{I}_t^i = \begin{cases} \boldsymbol{\sigma}_{\mathbf{All}} \cdot \boldsymbol{L}_{\mathbf{All}} \boldsymbol{\varepsilon}_t^i & \text{if the economy is not in recession during period } t \text{ under scenario } i \\ \boldsymbol{\sigma}_{\mathbf{Res}} \cdot \boldsymbol{L}_{\mathbf{Res}} \boldsymbol{\varepsilon}_t^i & \text{otherwise} \end{cases}$$

where

$\boldsymbol{\sigma}_{\mathbf{All}}$ is a column vector containing the standard deviation of error terms of all asset return models in normal periods, as shown in column 3 of Table A.9
$\boldsymbol{\sigma}_{\mathbf{Res}}$ is a column vector containing the standard deviation of error terms of all asset return models in normal periods, as shown in column 4 of Table A.9
$\boldsymbol{\varepsilon}_t^i$ is a column vector containing independent random variables following a standard normal distribution for all asset return models
$\boldsymbol{L}_{\mathbf{All}}$ is a lower triangular matrix so that the error term correlation matrix $\mathbf{CM}_{\mathbf{All}}$ can be decomposed as $\boldsymbol{L}_{\mathbf{All}} \times \boldsymbol{L}_{\mathbf{All}}{}^T$ and
$\boldsymbol{L}_{\mathbf{Res}}$ is a lower triangular matrix so that the error term correlation matrix $\mathbf{CM}_{\mathbf{Res}}$ can be decomposed as $\boldsymbol{L}_{\mathbf{Res}} \times \boldsymbol{L}_{\mathbf{Res}}{}^T$

Through this step, nonconstant volatility and nonlinear relationships among idiosyncratic factors of asset return models have been accounted for. The economic scenario generation formula then becomes

$$\mathbf{y}_t = \boldsymbol{\alpha} + \boldsymbol{\phi}_1 \mathbf{y}_{t-1} + \boldsymbol{\phi}_2 \mathbf{y}_{t-2} + \boldsymbol{B}_0 \mathbf{F}_t + \boldsymbol{B}_1 \mathbf{F}_{t-1} + \boldsymbol{B}_2 \mathbf{F}_{t-2} + \mathbf{I}_t^i$$

$$\text{Let } \mathbf{y}_t^p = \boldsymbol{\alpha} + \boldsymbol{\phi}_1 \mathbf{y}_{t-1} + \boldsymbol{\phi}_2 \mathbf{y}_{t-2} + \boldsymbol{B}_0 \mathbf{F}_t + \boldsymbol{B}_1 \mathbf{F}_{t-1} + \boldsymbol{B}_2 \mathbf{F}_{t-2}$$

$$\mathbf{y}_t = \mathbf{y}_t^p + \mathbf{I}_t^i$$

- Step 4. Adjust $\mathbf{I}_t^i$ to reflect nonzero correlation between idiosyncratic factors and systemic factors during recessions:

$$\mathbf{J}_t^i = \begin{cases} \left(\boldsymbol{\rho}_r \cdot \mathbf{y}_t^p + \sqrt{\mathbf{1} - \boldsymbol{\rho}_r{}^2} \cdot \mathbf{I}_t^i\right) \dfrac{\boldsymbol{\sigma}_{\mathbf{Res}}}{\sqrt{\boldsymbol{\rho}_r{}^2 \left(\boldsymbol{\sigma}_{\mathbf{Res}}^p\right)^2 + (\mathbf{1} - \boldsymbol{\rho}_r{}^2)(\boldsymbol{\sigma}_{\mathbf{Res}})^2}} & \text{if in recession during period } t \text{ under scenario } i \\ \mathbf{I}_t^i & \text{otherwise} \end{cases}$$

where $\boldsymbol{\rho}_r$ is a column vector containing the nonzero correlation between idiosyncratic factors and systemic factors.

The economic scenario generation formula becomes

$$\mathbf{y}_t = \boldsymbol{\alpha} + \boldsymbol{\phi}_1 \mathbf{y}_{t-1} + \boldsymbol{\phi}_2 \mathbf{y}_{t-2} + \boldsymbol{B}_0 \mathbf{F}_t + \boldsymbol{B}_1 \mathbf{F}_{t-1} + \boldsymbol{B}_2 \mathbf{F}_{t-2} + \mathbf{J}_t^i$$

Nonlinearity in the relationships among modeled variables is driven by economic status in the ESG. This simple yet effective approach is one of many approaches that can be used. Other models such as regime-switching models and a copula may be used to model nonlinear relationships. However, quarterly data may be insufficient to achieve the desired statistical credibility.

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