SOCIETY OF ACTUARIES

EXAM MFE MODELS FOR FINANCIAL ECONOMICS

EXAM MFE SAMPLE QUESTIONS AND SOLUTIONS
ADVANCED DERIVATIVES

These questions and solutions are from McDonald Chapters 9-14, 18-19, 23, and 25 only and are identical to questions from the former set of MFE sample questions.

These questions are representative of the types of questions that might be asked of candidates sitting for Exam MFE. These questions are intended to represent the depth of understanding required of candidates. The distribution of questions by topic is not intended to represent the distribution of questions on future exams.

In this version, standard normal distribution values are obtained by using the Cumulative Normal Distribution Calculator and Inverse CDF Calculator

For extra practice on material from Chapter 9 or later in McDonald, also see the actual Exam MFE questions and solutions from May 2007 and May 2009

May 2007: Questions 1-11, 14-15, 17, and 19

Note: Questions 12-13, 16, and 18 do not apply to the new MFE curriculum

May 2009: Questions 1-5, 7, 9, 12-14, 16-17, and 19-20

Note: Questions 6, 8, 10-11, 15, and 18 do not apply to the new MFE curriculum

Note that some of these remaining items (from May 2007 and May 2009) may refer to "stock prices following geometric Brownian motion." In such instances, use the following phrase instead: "stock prices are lognormally distributed."

MFE-03-17

1. Consider a European call option and a European put option on a nondividend-paying stock. You are given:

(i) The current price of the stock is 60.

(ii) The call option currently sells for 0.15 more than the put option.

(iii) Both the call option and put option will expire in 4 years.

(iv) Both the call option and put option have a strike price of 70.

Calculate the continuously compounded risk-free interest rate.

(A) 0.039

(B) 0.049

(C) 0.059

(D) 0.069

(E) 0.079

Solution to (1) Answer: (A)

The put-call parity formula (for a European call and a European put on a stock with the same strike price and maturity date) is

$$C - P = F^P_{0,T}(S) - F^P_{0,T}(K)$$

$$= F^P_{0,T}(S) - \mathrm{PV}_{0,T}(K)$$

$$= F^P_{0,T}(S) - Ke^{-rT}$$

$$= S_0 - Ke^{-rT},$$

because the stock pays no dividends

We are given that $C - P = 0.15$, $S_0 = 60$, $K = 70$ and $T = 4$. Then, $r = 0.039$.

Remark 1: If the stock pays n dividends of fixed amounts $D_1, D_2, \ldots, D_n$ at fixed times $t_1, t_2, \ldots, t_n$ prior to the option maturity date, T, then the put-call parity formula for European put and call options is

$$C - P = F^P_{0,T}(S) - Ke^{-rT}$$
$$= S_0 - \mathrm{PV}_{0,T}(\mathrm{Div}) - Ke^{-rT},$$

where $\mathrm{PV}_{0,T}(\mathrm{Div}) = \sum_{i=1}^{n} D_i e^{-rt_i}$ is the present value of all dividends up to time T. The difference, $S_0 - \mathrm{PV}_{0,T}(\mathrm{Div})$, is the *prepaid forward price* $F^P_{0,T}(S)$.

Remark 2: The put-call parity formula above does not hold for American put and call options. For the American case, the parity relationship becomes

$$S_0 - \mathrm{PV}_{0,T}(\mathrm{Div}) - K \le C - P \le S_0 - Ke^{-rT}.$$

This result is given in Appendix 9A of McDonald (2013) but is not required for Exam MFE. Nevertheless, you may want to try proving the inequalities as follows:
For the first inequality, consider a portfolio consisting of a European call plus an amount of cash equal to $\mathrm{PV}_{0,T}(\mathrm{Div}) + K$.

For the second inequality, consider a portfolio of an American put option plus one share of the stock.

2. Near market closing time on a given day, you lose access to stock prices, but some European call and put prices for a stock are available as follows:

Strike Price	Call Price	Put Price
$40	$11	$3
$50	$6	$8
$55	$3	$11

All six options have the same expiration date.

After reviewing the information above, John tells Mary and Peter that no arbitrage opportunities can arise from these prices.

Mary disagrees with John. She argues that one could use the following portfolio to obtain arbitrage profit: Long one call option with strike price 40; short three call options with strike price 50; lend $1; and long some calls with strike price 55.

Peter also disagrees with John. He claims that the following portfolio, which is different from Mary's, can produce arbitrage profit: Long 2 calls and short 2 puts with strike price 55; long 1 call and short 1 put with strike price 40; lend $2; and short some calls and long the same number of puts with strike price 50.

Which of the following statements is true?

(A) Only John is correct.

(B) Only Mary is correct.

(C) Only Peter is correct.

(D) Both Mary and Peter are correct.

(E) None of them is correct.

Solution to (2) Answer: (D)

The prices are not arbitrage-free. To show that Mary's portfolio yields arbitrage profit, we follow the analysis in Table 9.7 on page 285 of McDonald (2013).

	Time 0	Time T			
		$S_T < 40$	$40 \le S_T < 50$	$50 \le S_T < 55$	$S_T \ge 55$
Buy 1 call Strike 40	– 11	0	$S_T - 40$	$S_T - 40$	$S_T - 40$
Sell 3 calls Strike 50	+ 18	0	0	$-3(S_T - 50)$	$-3(S_T - 50)$
Lend $1	– 1	e^{rT}	e^{rT}	e^{rT}	e^{rT}
Buy 2 calls strike 55	– 6	0	0	0	$2(S_T - 55)$
Total	0	$e^{rT} > 0$	$e^{rT} + S_T - 40 > 0$	$e^{rT} + 2(55 - S_T) > 0$	$e^{rT} > 0$

Peter's portfolio makes arbitrage profit, because:

	Time-0 cash flow	Time-T cash flow
Buy 2 calls & sells 2 puts Strike 55	$2(-3 + 11) = 16$	$2(S_T - 55)$
Buy 1 call & sell 1 put Strike 40	$-11 + 3 = -8$	$S_T - 40$
Lend $2	-2	$2e^{rT}$
Sell 3 calls & buy 3 puts Strike 50	$3(6 - 8) = -6$	$3(50 - S_T)$
Total	0	$2e^{rT}$

Remarks: Note that Mary's portfolio has no put options. The call option prices are not arbitrage-free; they do not satisfy the convexity condition (9.19) on page 282 of McDonald (2013). The time-T cash flow column in Peter's portfolio is due to the identity

$$\max[0, S - K] \; - \; \max[0, K - S] \; = \; S - K.$$

In *Loss Models*, the textbook for Exam C, max[0, α] is denoted as α_+. It appears in the context of stop-loss insurance, $(S - d)_+$, with S being the claim random variable and d the deductible. The identity above is a particular case of

$$x \; = \; x_+ \; - \; (-x)_+,$$

which says that every number is the difference between its positive part and negative part.

3. An insurance company sells single premium deferred annuity contracts with return linked to a stock index, the time-t value of one unit of which is denoted by $S(t)$. The contracts offer a minimum guarantee return rate of $g\%$. At time 0, a single premium of amount π is paid by the policyholder, and $\pi \times y\%$ is deducted by the insurance company. Thus, at the contract maturity date, T, the insurance company will pay the policyholder

$$\pi \times (1 - y\%) \times \text{Max}[S(T)/S(0), (1 + g\%)^T].$$

You are given the following information:

(i) The contract will mature in one year.

(ii) The minimum guarantee rate of return, $g\%$, is 3%.

(iii) Dividends are incorporated in the stock index. That is, the stock index is constructed with all stock dividends reinvested.

(iv) $S(0) = 100$.

(v) The price of a one-year European put option, with strike price of \$103, on the stock index is \$15.21.

Determine $y\%$, so that the insurance company does not make or lose money on this contract.

Solution to (3)

The payoff at the contract maturity date is

$$\pi \times (1 - y\%)\times\text{Max}[S(T)/S(0), (1 + g\%)^T]$$
$$= \pi \times (1 - y\%)\times\text{Max}[S(1)/S(0), (1 + g\%)^1] \quad \text{because } T = 1$$
$$= [\pi/S(0)](1 - y\%)\text{Max}[S(1), S(0)(1 + g\%)]$$
$$= (\pi/100)(1 - y\%)\text{Max}[S(1), 103] \quad \text{because } g = 3 \text{ \& } S(0)=100$$
$$= (\pi/100)(1 - y\%)\{S(1) + \text{Max}[0, 103 - S(1)]\}.$$

Now, Max[0, 103 – $S(1)$] is the payoff of a one-year European put option, with strike price \$103, on the stock index; the time-0 price of this option is given to be is \$15.21. Dividends are incorporated in the stock index (i.e., $\delta = 0$); therefore, $S(0)$ is the time-0 price for a time-1 payoff of amount $S(1)$. Because of the no-arbitrage principle, the time-0 price of the contract must be

$$(\pi/100)(1 - y\%)\{S(0) + 15.21\}$$
$$= (\pi/100)(1 - y\%) \times 115.21.$$

Therefore, the "break-even" equation is

$$\pi = (\pi/100)(1 - y\%)\times 115.21,$$

or

$$y\% = 100 \times (1 - 1/1.1521)\% = 13.202\%$$

Remarks:

(i) Many stock indexes, such as S&P500, do not incorporate dividend reinvestments. In such cases, the time-0 cost for receiving $S(1)$ at time 1 is the prepaid forward price $F_{0,1}^P(S)$, which is less than $S(0)$.

(ii) The identities

$$\text{Max}[S(T), K] = K + \text{Max}[S(T) - K, 0] = K + (S(T) - K)_+$$

and

$$\text{Max}[S(T), K] = S(T) + \text{Max}[0, K - S(T)] = S(T) + (K - S(T))_+$$

can lead to a derivation of the put-call parity formula. Such identities are useful for understanding Section 14.6 *Exchange Options* in McDonald (2013).

4. For a two-period binomial model, you are given:

(i) Each period is one year.

(ii) The current price for a nondividend-paying stock is 20.

(iii) $u = 1.2840$, where u is one plus the rate of capital gain on the stock per period if the stock price goes up.

(iv) $d = 0.8607$, where d is one plus the rate of capital loss on the stock per period if the stock price goes down.

(v) The continuously compounded risk-free interest rate is 5%.

Calculate the price of an American call option on the stock with a strike price of 22.

(A) 0

(B) 1

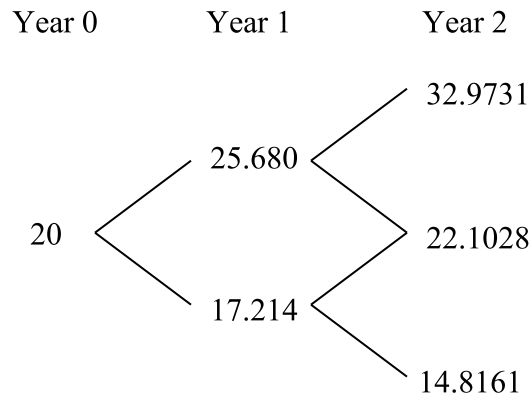
(C) 2

(D) 3

(E) 4

Solution to (4) Answer: (C)

First, we construct the two-period binomial tree for the stock price.



The calculations for the stock prices at various nodes are as follows:

$S_u = 20 \times 1.2840 = 25.680$
$S_d = 20 \times 0.8607 = 17.214$
$S_{uu} = 25.68 \times 1.2840 = 32.9731$
$S_{ud} = S_{du} = 17.214 \times 1.2840 = 22.1028$
$S_{dd} = 17.214 \times 0.8607 = 14.8161$

The risk-neutral probability for the stock price to go up is

$$p^* = \frac{e^{rh} - d}{u - d} = \frac{e^{0.05} - 0.8607}{1.2840 - 0.8607} = 0.4502.$$

Thus, the risk-neutral probability for the stock price to go down is 0.5498.

If the option is exercised at time 2, the value of the call would be
$C_{uu} = (32.9731 - 22)_+ = 10.9731$
$C_{ud} = (22.1028 - 22)_+ = 0.1028$
$C_{dd} = (14.8161 - 22)_+ = 0$

If the option is European, then $C_u = e^{-0.05}[0.4502C_{uu} + 0.5498C_{ud}] = 4.7530$ and $C_d = e^{-0.05}[0.4502C_{ud} + 0.5498C_{dd}] = 0.0440$.
But since the option is American, we should compare C_u and C_d with the value of the option if it is exercised at time 1, which is 3.68 and 0, respectively. Since $3.68 < 4.7530$ and $0 < 0.0440$, it is not optimal to exercise the option at time 1 whether the stock is in the up or down state. Thus the value of the option at time 1 is either 4.7530 or 0.0440.

Finally, the value of the call is
$C = e^{-0.05}[0.4502(4.7530) + 0.5498(0.0440)] = 2.0585.$

Remark: Since the stock pays no dividends, the price of an American call is the same as that of a European call. See pages 277-278 of McDonald (2013). The European option price can be calculated using the binomial probability formula. See formula (11.12) on page 335 and formula (19.2) on page 574 of McDonald (2013). The option price is

$$e^{-r(2h)}\left[\binom{2}{2}p^{*2}\,C_{uu} + \binom{2}{1}p^{*}(1-p^{*})C_{ud} + \binom{2}{0}(1-p^{*})^{2}C_{dd}\right]$$

$= e^{-0.1}\,[(0.4502)^2\times 10.9731 + 2\times 0.4502\times 0.5498\times 0.1028 + 0]$

$= 2.0507$

5. Consider a 9-month dollar-denominated American put option on British pounds. You are given that:

(i) The current exchange rate is 1.43 US dollars per pound.

(ii) The strike price of the put is 1.56 US dollars per pound.

(iii) The volatility of the exchange rate is $\sigma = 0.3$.

(iv) The US dollar continuously compounded risk-free interest rate is 8%.

(v) The British pound continuously compounded risk-free interest rate is 9%.

Using a three-period binomial model, calculate the price of the put.

Solution to (5)

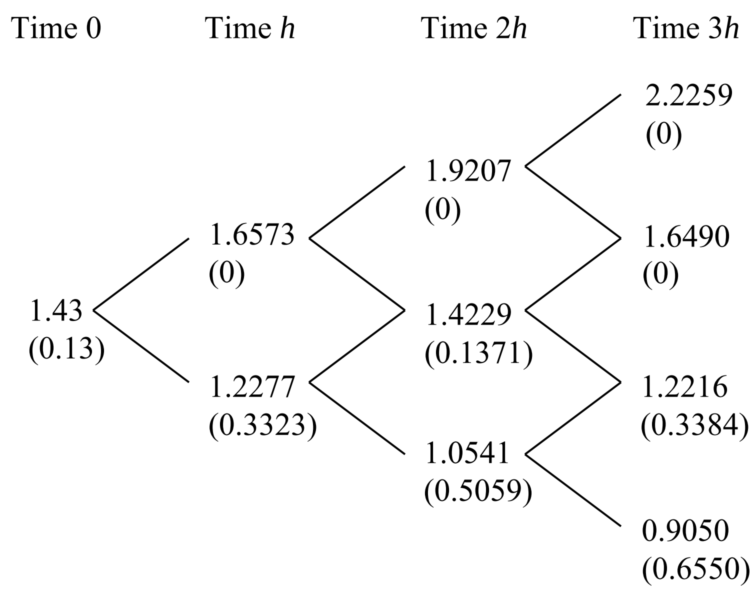
Each period is of length $h = 0.25$. Using the last two formulas on page 312 of McDonald (2013):

$$u = \exp[-0.01\times 0.25 + 0.3\times\sqrt{0.25}\,] = \exp(0.1475) = 1.158933,$$
$$d = \exp[-0.01\times 0.25 - 0.3\times\sqrt{0.25}\,] = \exp(-0.1525) = 0.858559.$$

Using formula (10.13), the risk-neutral probability of an up move is

$$p^* = \frac{e^{-0.01\times 0.25} - 0.858559}{1.158933 - 0.858559} = 0.4626\,.$$

The risk-neutral probability of a down move is thus 0.5374. The 3-period binomial tree for the exchange rate is shown below. The numbers within parentheses are the payoffs of the put option if exercised.



The payoffs of the put at maturity (at time $3h$) are
$P_{uuu} = 0$, $P_{uud} = 0$, $P_{udd} = 0.3384$ and $P_{ddd} = 0.6550$.

Now we calculate values of the put at time $2h$ for various states of the exchange rate.

If the put is European, then
$P_{uu} = 0$,
$P_{ud} = e^{-0.02}[0.4626P_{uud} + 0.5374P_{udd}] = 0.1783$,
$P_{dd} = e^{-0.02}[0.\ 4626P_{udd} + 0.5374P_{ddd}] = 0.4985$.
But since the option is American, we should compare P_{uu}, P_{ud} and P_{dd} with the values of the option if it is exercised at time $2h$, which are 0, 0.1371 and 0.5059, respectively. Since $0.4985 < 0.5059$, it is optimal to exercise the option at time $2h$ if the exchange rate has gone down two times before. Thus the values of the option at time $2h$ are $P_{uu} = 0$, $P_{ud} = 0.1783$ and $P_{dd} = 0.5059$.

Now we calculate values of the put at time h for various states of the exchange rate.

If the put is European, then
$P_u = e^{-0.02}[0.4626P_{uu} + 0.5374P_{ud}] = 0.0939$,
$P_d = e^{-0.02}[0.4626P_{ud} + 0.5374P_{dd}] = 0.3474$.
But since the option is American, we should compare P_u and P_d with the values of the option if it is exercised at time h, which are 0 and 0.3323, respectively. Since $0.3474 > 0.3323$, it is not optimal to exercise the option at time h. Thus the values of the option at time h are $P_u = 0.0939$ and $P_d = 0.3474$.

Finally, discount and average P_u and P_d to get the time-0 price,

$$P = e^{-0.02}[0.4626P_u + 0.5374P_d] = 0.2256.$$

Since it is greater than 0.13, it is not optimal to exercise the option at time 0 and hence the price of the put is 0.2256.

Remarks:

(i) Because $\dfrac{e^{(r-\delta)h} - e^{(r-\delta)h-\sigma\sqrt{h}}}{e^{(r-\delta)h+\sigma\sqrt{h}} - e^{(r-\delta)h-\sigma\sqrt{h}}} = \dfrac{1-e^{-\sigma\sqrt{h}}}{e^{\sigma\sqrt{h}} - e^{-\sigma\sqrt{h}}} = \dfrac{1}{1+e^{\sigma\sqrt{h}}}$, we can also calculate the risk-neutral probability p^* as follows:

$$p^* = \frac{1}{1+e^{\sigma\sqrt{h}}} = \frac{1}{1+e^{0.3\sqrt{0.25}}} = \frac{1}{1+e^{0.15}} = 0.46257.$$

(ii) $$1 - p^* = 1 - \frac{1}{1+e^{\sigma\sqrt{h}}} = \frac{e^{\sigma\sqrt{h}}}{1+e^{\sigma\sqrt{h}}} = \frac{1}{1+e^{-\sigma\sqrt{h}}}.$$

(iii) Because $\sigma > 0$, we have the inequalities

$$p^* < ½ < 1 - p^*.$$

6. You are considering the purchase of 100 units of a 3-month 25-strike European call option on a stock.

You are given:

(i) The Black-Scholes framework holds.

(ii) The stock is currently selling for 20.

(iii) The stock's volatility is 24%.

(iv) The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 3%.

(v) The continuously compounded risk-free interest rate is 5%.

Calculate the price of the block of 100 options.

(A) 0.04

(B) 1.93

(C) 3.63

(D) 4.22

(E) 5.09

Solution to (6) Answer: (C)

$$C(S, K, \sigma, r, T, \delta) = Se^{-\delta T} N(d_1) - Ke^{-rT} N(d_2) \tag{12.1}$$

with

$$d_1 = \frac{\ln(S/K) + (r - \delta + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \tag{12.2a}$$

$$d_2 = d_1 - \sigma\sqrt{T} \tag{12.2b}$$

Because $S = 20$, $K = 25$, $\sigma = 0.24$, $r = 0.05$, $T = 3/12 = 0.25$, and $\delta = 0.03$, we have

$$d_1 = \frac{\ln(20/25) + (0.05 - 0.03 + \frac{1}{2}0.24^2)0.25}{0.24\sqrt{0.25}} = -1.75786$$

and

$$d_2 = -1.75786 - 0.24\sqrt{0.25} = -1.87786$$

Using the Cumulative Normal Distribution Calculator, we obtain $N(-1.75786) = 0.03939$ and $N(-1.87786) = 0.03020$.

Hence, formula (12.1) becomes

$$C = 20e^{-(0.03)(0.25)}(0.03939) - 25e^{-(0.05)(0.25)}(0.03020) = 0.036292362$$

Cost of the block of 100 options = 100 × 0.0363 = \$3.63.

7. Company A is a U.S. international company, and Company B is a Japanese local company. Company A is negotiating with Company B to sell its operation in Tokyo to Company B. The deal will be settled in Japanese yen. To avoid a loss at the time when the deal is closed due to a sudden devaluation of yen relative to dollar, Company A has decided to buy at-the-money dollar-denominated yen put of the European type to hedge this risk.

You are given the following information:

(i) The deal will be closed 3 months from now.

(ii) The sale price of the Tokyo operation has been settled at 120 billion Japanese yen.

(iii) The continuously compounded risk-free interest rate in the U.S. is 3.5%.

(iv) The continuously compounded risk-free interest rate in Japan is 1.5%.

(v) The current exchange rate is 1 U.S. dollar = 120 Japanese yen.

(vi) The daily volatility of the yen per dollar exchange rate is 0.261712%.

(vii) 1 year = 365 days; 3 months = ¼ year.

Calculate Company A's option cost.

Solution to (7)

Let $X(t)$ be the exchange rate of U.S. dollar per Japanese yen at time t. That is, at time t,

$$¥1 = \$X(t).$$

We are given that $X(0) = 1/120$.

At time ¼, Company A will receive ¥ 120 billion, which is exchanged to \$[120 billion × $X(¼)$]. However, Company A would like to have

$$\$ \text{ Max}[1 \text{ billion}, \ 120 \text{ billion} \times X(¼)],$$

which can be decomposed as

$$\$120 \text{ billion} \times X(¼) + \$ \text{ Max}[1 \text{ billion} - 120 \text{ billion} \times X(¼), 0],$$

or

$$\$120 \text{ billion} \times \{X(¼) + \text{Max}[120^{-1} - X(¼), 0]\}.$$

Thus, Company A purchases 120 billion units of a put option whose payoff three months from now is

$$\$ \text{ Max}[120^{-1} - X(¼), 0].$$

The exchange rate can be viewed as the price, in US dollar, of a traded asset, which is the Japanese yen. The continuously compounded risk-free interest rate in Japan can be interpreted as δ, the dividend yield of the asset. See also page 355 of McDonald (2013) for the *Garman-Kohlhagen model*. Then, we have

$$r = 0.035, \delta = 0.015, S = X(0) = 1/120, K = 1/120, T = ¼.$$

It remains to determine the value of σ, which is given by the equation

$$\sigma\sqrt{\frac{1}{365}} = 0.261712\ \%.$$

Hence,

$$\sigma = 0.05.$$

Therefore,

$$d_1 = \frac{(r - \delta + \sigma^2/2)T}{\sigma\sqrt{T}} = \frac{(0.035 - 0.015 + 0.05^2/2)/4}{0.05\sqrt{1/4}} = 0.2125$$

and

$$d_2 = d_1 - \sigma\sqrt{T} = 0.2125 - 0.05/2 = 0.1875.$$

By (12.4) of McDonald (2013), the time-0 price of 120 billion units of the put option is

$$\$120 \text{ billion} \times [Ke^{-rT}N(-d_2) - X(0)e^{-\delta T}N(-d_1)]$$

$$= \$ [e^{-rT}N(-d_2) - e^{-\delta T}N(-d_1)] \text{ billion} \qquad \text{because } K = X(0) = 1/120$$

Using the Cumulative Normal Distribution Calculator, we obtain $N(-0.1875) = 0.42563$ and $N(-0.2125) = 0.41586$.

Thus, Company A's option cost is

$$e^{-0.035/4}\times 0.42563 - e^{-0.015/4}\times 0.41586$$

$$= 0.007618538 \text{ billion} \approx 7.62 \text{ million}.$$

Remarks:

(i) Suppose that the problem is to be solved using options on the exchange rate of Japanese yen per US dollar, i.e., using yen-denominated options. Let

$$\$1 = ¥U(t)$$

at time t, i.e., $U(t) = 1/X(t)$.

Because Company A is worried that the dollar may increase in value with respect to the yen, it buys 1 billion units of a 3-month yen-denominated European *call* option, with exercise price ¥120. The payoff of the option at time ¼ is

$$¥\ \text{Max}[U(¼) - 120, 0].$$

To apply the Black-Scholes call option formula (12.1) to determine the time-0 price in yen, use

$$r = 0.015, \delta = 0.035, S = U(0) = 120, K = 120, T = ¼, \text{ and } \sigma = 0.05.$$

Then, divide this price by 120 to get the time-0 option price in dollars. We get the same price as above, because d_1 here is $-d_2$ of above.

The above is a special case of formula (9.9) on page 275 of McDonald (2013).

(ii) There is a cheaper solution for Company A. At time 0, borrow

$$¥\ 120\times\exp(-\ ¼\ r_{¥}) \text{ billion},$$

and immediately convert this amount to US dollars. The loan is repaid with interest at time ¼ when the deal is closed.

On the other hand, with the option purchase, Company A will benefit if the yen increases in value with respect to the dollar.

8. You are considering the purchase of a 3-month 41.5-strike American call option on a nondividend-paying stock.

You are given:

(i) The Black-Scholes framework holds.

(ii) The stock is currently selling for 40.

(iii) The stock's volatility is 30%.

(iv) The current call option delta is 0.5.

Determine the current price of the option.

(A) $20 - 20.453\int_{-\infty}^{0.15} e^{-x^2/2}\mathrm{d}x$

(B) $20 - 16.138\int_{-\infty}^{0.15} e^{-x^2/2}\mathrm{d}x$

(C) $20 - 40.453\int_{-\infty}^{0.15} e^{-x^2/2}\mathrm{d}x$

(D) $16.138\int_{-\infty}^{0.15} e^{-x^2/2}\mathrm{d}x - 20.453$

(E) $40.453\int_{-\infty}^{0.15} e^{-x^2/2}\mathrm{d}x - 20.453$

Solution to (8) Answer: (D)

Since it is never optimal to exercise an American call option before maturity if the stock pays no dividends, we can price the call option using the European call option formula

$$C = SN(d_1) - Ke^{-rT} N(d_2),$$

where $d_1 = \dfrac{\ln(S/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$ and $d_2 = d_1 - \sigma\sqrt{T}$.

Because the call option delta is $N(d_1)$ and it is given to be 0.5, we have $d_1 = 0$.
Hence,

$$d_2 = -0.3 \times \sqrt{0.25} = -0.15 .$$

To find the continuously compounded risk-free interest rate, use the equation

$$d_1 = \frac{\ln(40/41.5) + (r + \frac{1}{2} \times 0.3^2) \times 0.25}{0.3\sqrt{0.25}} = 0,$$

which gives $r = 0.1023$.

Thus,

$$\begin{aligned} C &= 40N(0) - 41.5e^{-0.1023 \times 0.25}N(-0.15) \\ &= 20 - 40.453[1 - N(0.15)] \\ &= 40.453N(0.15) - 20.453 \\ &= \frac{40.453}{\sqrt{2\pi}} \int_{-\infty}^{0.15} e^{-x^2/2} \mathrm{d}x - 20.453 \\ &= 16.138 \int_{-\infty}^{0.15} e^{-x^2/2} \mathrm{d}x - 20.453 \end{aligned}$$

9. Consider the Black-Scholes framework. A market-maker, who delta-hedges, sells a three-month at-the-money European call option on a nondividend-paying stock.

You are given:

(i) The continuously compounded risk-free interest rate is 10%.

(ii) The current stock price is 50.

(iii) The current call option delta is 0.61791.

(iv) There are 365 days in the year.

If, after one day, the market-maker has zero profit or loss, determine the stock price move over the day.

(A) 0.41

(B) 0.52

(C) 0.63

(D) 0.75

(E) 1.11

Solution to (9)

According to the second paragraph on page 395 of McDonald (2013), such a stock price move is given by plus or minus of

$$\sigma S(0)\sqrt{h}\,,$$

where h = 1/365 and $S(0)$ = 50. It remains to find σ.

Because the stock pays no dividends (i.e., $\delta = 0$), it follows from the bottom of page 357 that $\Delta = N(d_1)$. Thus,

$$\begin{aligned} d_1 &= N^{-1}(\Delta) \\ &= N^{-1}(0.61791) \\ &= 0.3 \end{aligned}$$

by using the Inverse CDF Calculator.

Because $S = K$ and $\delta = 0$, formula (12.2a) is

$$d_1 = \frac{(r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

or

$$\tfrac{1}{2}\sigma^2 - \frac{d_1}{\sqrt{T}}\sigma + r = 0.$$

With $d_1 = 0.3$, $r = 0.1$, and $T = 1/4$, the quadratic equation becomes

$$\tfrac{1}{2}\sigma^2 - 0.6\sigma + 0.1 = 0,$$

whose roots can be found by using the quadratic formula or by factorization,

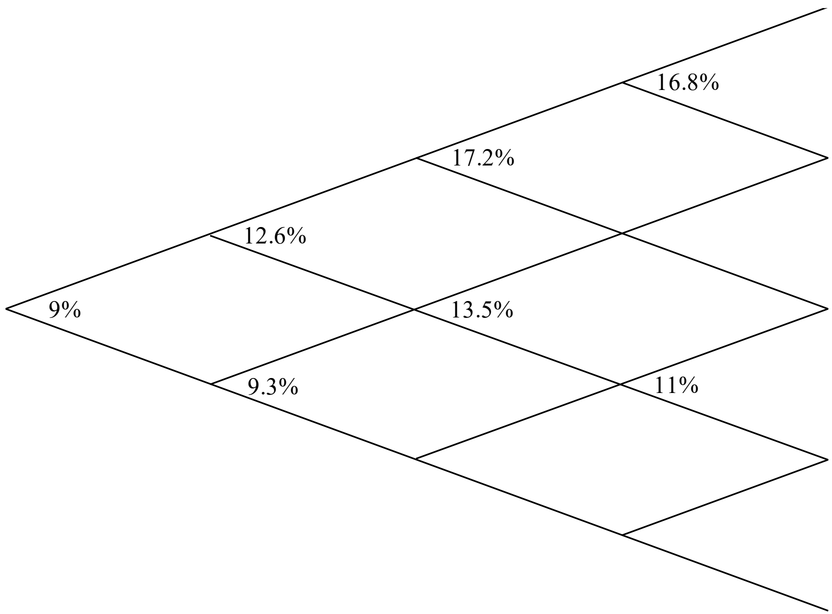
$$\tfrac{1}{2}(\sigma - 1)(\sigma - 0.2) = 0.$$

We reject $\sigma = 1$ because such a volatility seems too large (and none of the five answers fit). Hence,

$$\sigma S(0)\sqrt{h} = 0.2 \times 50 \times 0.052342 \approx 0.52.$$

10-14. DELETED

15. You are given the following incomplete Black-Derman-Toy interest rate tree model for the effective annual interest rates:



Calculate the price of a year-4 caplet for the notional amount of $100. The cap rate is 10.5%.

Solution to (15)
First, let us fill in the three missing interest rates in the B-D-T binomial tree. In terms of the notation in Figure 25.4 of McDonald (2013), the missing interest rates are r_d, r_{ddd}, and r_{uud}. We can find these interest rates, because in each period, the interest rates in different states are terms of a *geometric progression*.

$$\frac{0.135}{r_{dd}} = \frac{0.172}{0.135} \Rightarrow r_{dd} = 10.6\%$$
$$\frac{r_{uud}}{0.11} = \frac{0.168}{r_{uud}} \Rightarrow r_{uud} = 13.6\%$$
$$\left(\frac{0.11}{r_{ddd}}\right)^2 = \frac{0.168}{0.11} \Rightarrow r_{ddd} = 8.9\%$$

The payment of a year-4 caplet is made at year 4 (time 4), and we consider its discounted value at year 3 (time 3). At year 3 (time 3), the binomial model has four nodes; at that time, a year-4 caplet has one of four values:

$\frac{16.8-10.5}{1.168} = 5.394$, $\frac{13.6-10.5}{1.136} = 2.729$, $\frac{11-10.5}{1.11} = 0.450$, and 0 because $r_{ddd} = 8.9\%$ which is less than 10.5%.

For the Black-Derman-Toy model, the risk-neutral probability for an up move is ½.
We now calculate the caplet's value in each of the three nodes at time 2:

$$\frac{(5.394+2.729)/2}{1.172} = 3.4654\,,\quad \frac{(2.729+0.450)/2}{1.135} = 1.4004\,,\quad \frac{(0.450+0)/2}{1.106} = 0.2034\,.$$

Then, we calculate the caplet's value in each of the two nodes at time 1:

$$\frac{(3.4654+1.4004)/2}{1.126} = 2.1607\,,\qquad \frac{(1.40044+0.2034)/2}{1.093} = 0.7337\,.$$

Finally, the time-0 price of the year-4 caplet is $\frac{(2.1607+0.7337)/2}{1.09} = 1.3277\,.$

Alternative Solution: The payoff of the year-4 caplet is made at year 4 (at time 4). In a binomial lattice, there are 16 paths from time 0 to time 4.

For the *uuuu* path, the payoff is $(16.8 - 10.5)_+$

For the *uuud* path, the payoff is also $(16.8 - 10.5)_+$

For the *uudu* path, the payoff is $(13.6 - 10.5)_+$

For the *uudd* path, the payoff is also $(13.6 - 10.5)_+$

:

:

We discount these payoffs by the one-period interest rates (annual interest rates) along interest-rate paths, and then calculate their average with respect to the risk-neutral probabilities. In the Black-Derman-Toy model, the risk-neutral probability for each interest-rate path is the same. Thus, the time-0 price of the caplet is

$$\frac{1}{16}\left\{\frac{(16.8-10.5)_+}{1.09\times1.126\times1.172\times1.168} + \frac{(16.8-10.5)_+}{1.09\times1.126\times1.172\times1.168}\right.$$

$$\left. + \frac{(13.6-10.5)_+}{1.09\times1.126\times1.172\times1.136} + \frac{(13.6-10.5)_+}{1.09\times1.126\times1.172\times1.136} + \cdots\cdots\cdots\cdots\right\}$$

$$= \frac{1}{8}\left\{\frac{(16.8-10.5)_+}{1.09\times1.126\times1.172\times1.168}\right.$$

$$+ \frac{(13.6-10.5)_+}{1.09\times1.126\times1.172\times1.136} + \frac{(13.6-10.5)_+}{1.09\times1.126\times1.135\times1.136} + \frac{(13.6-10.5)_+}{1.09\times1.093\times1.135\times1.136}$$

$$+ \frac{(11-10.5)_+}{1.09\times1.126\times1.135\times1.11} + \frac{(11-10.5)_+}{1.09\times1.093\times1.135\times1.11} + \frac{(11-10.5)_+}{1.09\times1.093\times1.106\times1.11}$$

$$\left. + \frac{(9-10.5)_+}{1.09\times1.093\times1.106\times1.09}\right\} = 1.326829.$$

Remark: In this problem, the payoffs are path-independent. The "backward induction" method in the earlier solution is more efficient. However, if the payoffs are path-dependent, then the price will need to be calculated by the "path-by-path" method illustrated in this alternative solution.

16. DELETED

17. You are to estimate a nondividend-paying stock's annualized volatility using its prices in the past nine months.

Month	Stock Price ($/share)
1	80
2	64
3	80
4	64
5	80
6	100
7	80
8	64
9	80

Calculate the historical volatility for this stock over the period.

(A) 83%

(B) 77%

(C) 24%

(D) 22%

(E) 20%

Solution to (17) Answer (A)

This problem is based on Sections 11.3 and 18.5 of McDonald (2013), in particular, Table 18.2 on page 563.

Let $\{r_j\}$ denote the continuously compounded monthly returns. Thus, $r_1 = \ln(64/80)$, $r_2 = \ln(80/64)$, $r_3 = \ln(64/80)$, $r_4 = \ln(80/64)$, $r_5 = \ln(100/80)$, $r_6 = \ln(80/100)$, $r_7 = \ln(64/80)$, and $r_8 = \ln(80/64)$. Note that four of them are ln(1.25) and the other four are –ln(1.25); in particular, their mean is zero.

The (unbiased) sample variance of the non-annualized monthly returns is

$$\frac{1}{n-1}\sum_{j=1}^{n}(r_j-\bar{r})^2 = \frac{1}{7}\sum_{j=1}^{8}(r_j-\bar{r})^2 = \frac{1}{7}\sum_{j=1}^{8}(r_j)^2 = \frac{8}{7}[\ln(1.25)]^2.$$

The annual standard deviation is related to the monthly standard deviation by formula (11.5),

$$\sigma = \frac{\sigma_h}{\sqrt{h}},$$

where $h = 1/12$. Thus, the historical volatility is

$$\sqrt{12}\times\sqrt{\frac{8}{7}}\times\ln(1.25) = 82.6\%.$$

Remarks: Further discussion is given in Section 24.2 of McDonald (2013) (not required for Exam MFE). Suppose that we observe n continuously compounded returns over the time period $[\tau, \tau + T]$. Then,
$h = T/n$, and the historical annual variance of returns is estimated as

$$\frac{1}{h}\frac{1}{n-1}\sum_{j=1}^{n}(r_j-\bar{r})^2 = \frac{1}{T}\frac{n}{n-1}\sum_{j=1}^{n}(r_j-\bar{r})^2 .$$

Now,

$$\bar{r} = \frac{1}{n}\sum_{j=1}^{n}r_j = \frac{1}{n}\ln\frac{S(\tau+T)}{S(\tau)},$$

which is close to zero when n is large. Thus, a simpler estimation formula is

$\frac{1}{h}\frac{1}{n-1}\sum_{j=1}^{n}(r_j)^2$ which is formula (24.2) on page 720, or equivalently, $\frac{1}{T}$

$\frac{n}{n-1}\sum_{j=1}^{n}(r_j)^2$ which is the formula in footnote 9 on page 730. The last formula is related to #10 in this set of sample problems: With probability 1,

$$\lim_{n\to\infty}\sum_{j=1}^{n}[\ln S(jT/n) - \ln S((j-1)T/n)]^2 = \sigma^2 T.$$

18. A market-maker sells 1,000 1-year European gap call options, and delta-hedges the position with shares.

You are given:

(i) Each gap call option is written on 1 share of a nondividend-paying stock.

(ii) The current price of the stock is 100.

(iii) The stock's volatility is 100%.

(iv) Each gap call option has a strike price of 130.

(v) Each gap call option has a payment trigger of 100.

(vi) The risk-free interest rate is 0%.

Under the Black-Scholes framework, determine the initial number of shares in the delta-hedge.

(A) 586

(B) 594

(C) 684

(D) 692

(E) 797

Solution to (18) Answer: (A)

Note that, in this problem, $r = 0$ and $\delta = 0$.

By formula (14.15) in McDonald (2013), the time-0 price of the gap option is

$$C_{\text{gap}} = SN(d_1) - 130N(d_2) = [SN(d_1) - 100N(d_2)] - 30N(d_2) = C - 30N(d_2),$$

where d_1 and d_2 are calculated with $K = 100$ (and $r = \delta = 0$) and $T = 1$, and C denotes the time-0 price of the plain-vanilla call option with exercise price 100.

In the Black-Scholes framework, delta of a derivative security of a stock is the partial derivative of the security price with respect to the stock price. Thus,

$$\begin{aligned}\Delta_{\text{gap}} &= \frac{\partial}{\partial S}C_{\text{gap}} = \frac{\partial}{\partial S}C - 30\frac{\partial}{\partial S}N(d_2) = \Delta_C - 30N'(d_2)\frac{\partial}{\partial S}d_2 \\ &= N(d_1) - 30N'(d_2)\frac{1}{S\sigma\sqrt{T}},\end{aligned}$$

where $N'(x) = \dfrac{1}{\sqrt{2\pi}}e^{-x^2/2}$ is the density function of the standard normal.

Now, with $S = K = 100$, $T = 1$, and $\sigma = 1$,

$$d_1 = [\ln(S/K) + \sigma^2T/2]/(\sigma\sqrt{T}) = (\sigma^2T/2)/(\sigma\sqrt{T}) = ½\sigma\sqrt{T} = ½,$$

and $d_2 = d_1 - \sigma\sqrt{T} = -½$. Hence, at time 0

$$\begin{aligned}\Delta_{\text{gap}} &= N(d_1) - 30N'(d_2)\frac{1}{100} \\ &= N(½) - 0.3N'(-½) \\ &= N(½) - 0.3\frac{1}{\sqrt{2\pi}}e^{-(-½)^2/2} \\ &= 0.69146 - 0.3\frac{e^{-1/8}}{\sqrt{2\pi}} \\ &= 0.58584.\end{aligned}$$

19. Consider a *forward start option* which, 1 year from today, will give its owner a 1-year European call option with a strike price equal to the stock price at that time.

You are given:

(i) The European call option is on a stock that pays no dividends.

(ii) The stock's volatility is 30%.

(iii) The forward price for delivery of 1 share of the stock 1 year from today is 100.

(iv) The continuously compounded risk-free interest rate is 8%.

Under the Black-Scholes framework, determine the price today of the forward start option.

(A) 11.90

(B) 13.10

(C) 14.50

(D) 15.70

(E) 16.80

Solution to (19) Answer: (C)

This problem is based on Exercise 14.21 on page 429 of McDonald (2013).

Let S_1 denote the stock price at the end of one year. Apply the Black-Scholes formula to calculate the price of the at-the-money call one year from today, conditioning on S_1.

$d_1 = [\ln (S_1/S_1) + (r + \sigma^2/2)T]/(\sigma\sqrt{T}\,) = (r + \sigma^2/2)/\sigma = 0.41667$, which turns out to be independent of S_1.

$d_2 = d_1 - \sigma\sqrt{T} = d_1 - \sigma = 0.11667$

The value of the forward start option at time 1 is

$$
\begin{aligned}
C(S_1) &= S_1N(d_1) - S_1e^{-r} N(d_2) \\
&= S_1[N(0.41667) - e^{-0.08} N(0.11667)] \\
&= S_1[0.66154 - e^{-0.08}\times 0.54644] \\
&= 0.157112S_1.
\end{aligned}
$$

(Note that, when viewed from time 0, S_1 is a random variable.)

Thus, the time-0 price of the forward start option must be 0.157112 multiplied by the time-0 price of a security that gives S_1 as payoff at time 1, i.e., multiplied by the prepaid forward price $F_{0,1}^P(S)$. Hence, the time-0 price of the forward start option is

$$0.157112\times F_{0,1}^P(S) = 0.157112\times e^{-0.08}\times F_{0,1}(S) = 0.157112\times e^{-0.08}\times 100 = 14.5033$$

Remark: A key to pricing the forward start option is that d_1 and d_2 turn out to be independent of the stock price. This is the case if the strike price of the call option will be set as a fixed percentage of the stock price at the issue date of the call option.

20. Assume the Black-Scholes framework. Consider a stock, and a European call option and a European put option on the stock. The current stock price, call price, and put price are 45.00, 4.45, and 1.90, respectively.

Investor A purchases two calls and one put. Investor B purchases two calls and writes three puts.

The current elasticity of Investor A's portfolio is 5.0. The current delta of Investor B's portfolio is 3.4.

Calculate the current put-option elasticity.

(A) –0.55
(B) –1.15
(C) –8.64
(D) –13.03
(E) –27.24

Solution to (20) Answer: (D)

Applying the formula

$$\Delta_{\text{portfolio}} = \frac{\partial}{\partial S}\text{portfolio value}$$

to Investor B's portfolio yields

$$3.4 = 2\Delta_C - 3\Delta_P. \tag{1}$$

Applying the elasticity formula

$$\Omega_{\text{portfolio}} = \frac{\partial}{\partial \ln S}\ln[\text{portfolio value}] = \frac{S}{\text{portfolio value}} \times \frac{\partial}{\partial S}\text{portfolio value}$$

to Investor A's portfolio yields

$$5.0 = \frac{S}{2C+P}(2\Delta_C + \Delta_P) = \frac{45}{8.9+1.9}(2\Delta_C + \Delta_P),$$

or

$$1.2 = 2\Delta_C + \Delta_P. \tag{2}$$

Now, $(2) - (1) \quad \Rightarrow \quad -2.2 = 4\Delta_P.$

Hence, time-0 put option elasticity $= \Omega_P = \dfrac{S}{P}\,\Delta_P = \dfrac{45}{1.9} \times -\dfrac{2.2}{4} = -13.03$, which is (D).

Remarks:

(i) If the stock pays no dividends, and if the European call and put options have the same expiration date and strike price, then $\Delta_C - \Delta_P = 1$. In this problem, the put and call do not have the same expiration date and strike price; so this relationship does not hold.

(ii) The statement on page 365 in McDonald (2013) that "[t]he elasticity of a portfolio is the *weighted average* of the elasticities of the portfolio components" may remind students, who are familiar with fixed income mathematics, the concept of *duration*. Formula (3.5.8) on page 101 of *Financial Economics*: *With Applications to Investments, Insurance and Pensions* (edited by H.H. Panjer and published by The Actuarial Foundation in 1998) shows that the so-called *Macaulay duration* is an elasticity.

(iii) In the Black-Scholes framework, the hedge ratio or delta of a portfolio is the partial derivative of the portfolio price with respect to the stock price. In other continuous-time frameworks (which are not in the syllabus of Exam MFE), the hedge ratio may not be given by a partial derivative; for an example, see formula (10.5.7) on page 478 of *Financial Economics*: *With Applications to Investments, Insurance and Pensions*.

21-24. DELETED

25. Consider a chooser option (also known as an as-you-like-it option) on a nondividend-paying stock. At time 1, its holder will choose whether it becomes a European call option or a European put option, each of which will expire at time 3 with a strike price of $100.

The chooser option price is $20 at time $t = 0$.

The stock price is $95 at time $t = 0$. Let $C(T)$ denote the price of a European call option at time $t = 0$ on the stock expiring at time T, $T > 0$, with a strike price of $100.

You are given:

(i) The risk-free interest rate is 0.

(ii) $C(1) = \$4$.

Determine $C(3)$.

(A) $ 9

(B) $11

(C) $13

(D) $15

(E) $17

Solution to (25) Answer: (B)

Let $C(S(t), t, T)$ denote the price at time-t of a European call option on the stock, with exercise date T and exercise price $K = 100$. So,

$$C(T) = C(95, 0, T).$$

Similarly, let $P(S(t), t, T)$ denote the time-t put option price.

At the choice date $t = 1$, the value of the chooser option is

$$\text{Max}[C(S(1), 1, 3), P(S(1),1, 3)],$$

which can expressed as

$$C(S(1), 1, 3) + \text{Max}[0, P(S(1),1, 3) - C(S(1), 1, 3)]. \quad (1)$$

Because the stock pays no dividends and the interest rate is zero,

$$P(S(1),1, 3) - C(S(1), 1, 3) = K - S(1)$$

by put-call parity. Thus, the second term of (1) simplifies as

$$\text{Max}[0, K - S(1)],$$

which is the payoff of a European put option. As the time-1 value of the chooser option is

$$C(S(1), 1, 3) + \text{Max}[0, K - S(1)],$$

its time-0 price *must* be

$$C(S(0), 0, 3) + P(S(0), 0, 1),$$

which, by put-call parity, is

$$C(S(0), 0, 3) + [C(S(0), 0, 1) + K - S(0)]$$
$$= C(3) + [C(1) + 100 - 95] = C(3) + C(1) + 5.$$

Thus,

$$C(3) = 20 - (4 + 5) = 11.$$

Remark: The problem is a modification of Exercise 14.20.b.

26. Consider European and American options on a nondividend-paying stock. You are given:

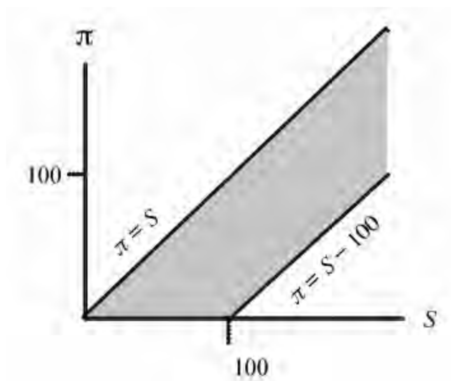
(i) All options have the same strike price of 100.

(ii) All options expire in six months.

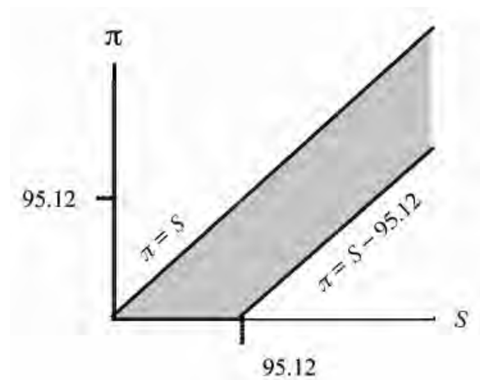
(iii) The continuously compounded risk-free interest rate is 10%.

You are interested in the graph for the price of an option as a function of the current stock price. In each of the following four charts I–IV, the horizontal axis, S, represents the current stock price, and the vertical axis, π, represents the price of an option.

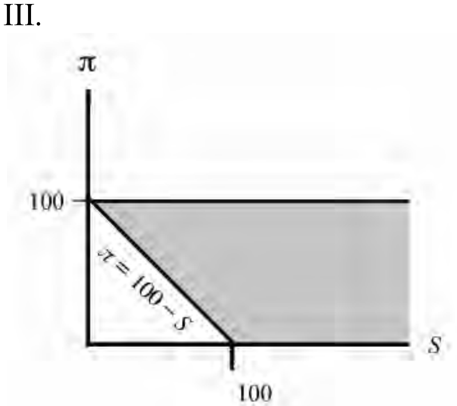
I.



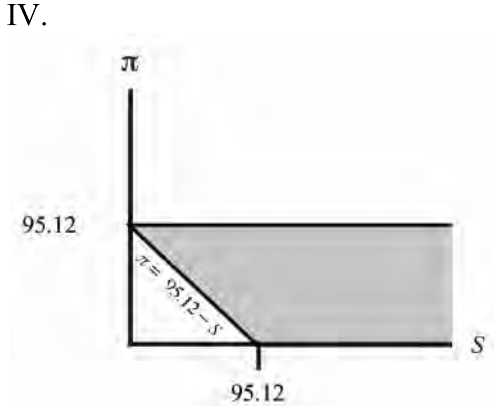
II.



III.



IV.



Match the option with the shaded region in which its graph lies. If there are two or more possibilities, choose the chart with the smallest shaded region.

26. Continued

	European Call	American Call	European Put	American Put
(A)	I	I	III	III
(B)	II	I	IV	III
(C)	II	I	III	III
(D)	II	II	IV	III
(E)	II	II	IV	IV

Solution to (26) Answer: (D)

$T = \frac{1}{2}; \quad \text{PV}_{0,T}(K) = Ke^{-rT} = 100e^{-0.1/2} = 100e^{-0.05} = 95.1229 \approx 95.12.$

By (9.11) on page 277 of McDonald (2013), we have

$$S(0) \geq C_{Am} \geq C_{Eu} \geq \text{Max}[0,\ F_{0,T}^{P}(S) - \text{PV}_{0,T}(K)].$$

Because the stock pays no dividends, the above becomes

$$S(0) \geq C_{Am} = C_{Eu} \geq \text{Max}[0, S(0) - \text{PV}_{0,T}(K)].$$

Thus, the shaded region in II contains C_{Am} and C_{Eu}. (The shaded region in I also does, but it is a larger region.)

By (9.12) on page 277 of McDonald (2013), we have

$$K \geq P_{Am} \geq P_{Eu} \geq \text{Max}[0, \text{PV}_{0,T}(K) - F_{0,T}^{P}(S)]$$
$$= \text{Max}[0, \text{PV}_{0,T}(K) - S(0)]$$

because the stock pays no dividends. However, the region bounded above by $\pi = K$ and bounded below by $\pi = \text{Max}[0, \text{PV}_{0,T}(K) - S]$ is not given by III or IV.

Because an American option can be exercised immediately, we have a tighter lower bound for an American put,

$$P_{Am} \geq \text{Max}[0, K - S(0)].$$

Thus,

$$K \geq P_{Am} \geq \text{Max}[0, K - S(0)],$$

showing that the shaded region in III contains P_{Am}.

For a European put, we can use put-call parity and the inequality $S(0) \geq C_{Eu}$ to get a tighter upper bound,

$$\text{PV}_{0,T}(K) \geq P_{Eu}.$$

Thus,

$$\text{PV}_{0,T}(K) \geq P_{Eu} \geq \text{Max}[0, \text{PV}_{0,T}(K) - S(0)],$$

showing that the shaded region in IV contains P_{Eu}.

Remarks:

(i) It turns out that II and IV can be found on page 156 of Capiński and Zastawniak (2003) *Mathematics for Finance: An Introduction to Financial Engineering*, Springer Undergraduate Mathematics Series.

(ii) The last inequality in (9.9) can be derived as follows. By put-call parity,

$$C_{Eu} = P_{Eu} + F_{0,T}^{P}(S) - e^{-rT}K$$
$$\geq F_{0,T}^{P}(S) - e^{-rT}K \qquad \text{because } P_{Eu} \geq 0.$$

We also have

$$C_{Eu} \geq 0.$$

Thus,

$$C_{Eu} \geq \text{Max}[0,\ F_{0,T}^{P}(S) - e^{-rT}K].$$

(iii) An alternative derivation of the inequality above is to use *Jensen's Inequality* (see, in particular, page 883).

$$C_{Eu} = \mathrm{E}^*\left[e^{-rT}\text{Max}(0, S(T) - K)\right]$$
$$\geq e^{-rT}\text{Max}(0, \mathrm{E}^*[S(T) - K]) \quad \text{because of Jensen's Inequality}$$
$$= \text{Max}(0, \mathrm{E}^*\left[e^{-rT}S(T)\right] - e^{-rT}K)$$
$$= \text{Max}(0, F_{0,T}^{P}(S) - e^{-rT}K).$$

Here, E* signifies risk-neutral expectation.

(iv) That $C_{Eu} = C_{Am}$ for nondividend-paying stocks can be shown by Jensen's Inequality.

27. DELETED

28. Assume the Black-Scholes framework. You are given:

(i) $S(t)$ is the price of a nondividend-paying stock at time t.

(ii) $S(0) = 10$

(iii) The stock's volatility is 20%.

(iv) The continuously compounded risk-free interest rate is 2%.

At time $t = 0$, you write a one-year European option that pays 100 if $[S(1)]^2$ is greater than 100 and pays nothing otherwise.

You delta-hedge your commitment.

Calculate the number of shares of the stock for your hedging program at time $t = 0$.

(A) 20

(B) 30

(C) 40

(D) 50

(E) 60

Solution to (28) Answer: (A)

Note that $[S(1)]^2 > 100$ is equivalent to $S(1) > 10$. Thus, the option is a cash-or-nothing option with strike price 10. The time-0 price of the option is

$$100 \times e^{-rT} N(d_2).$$

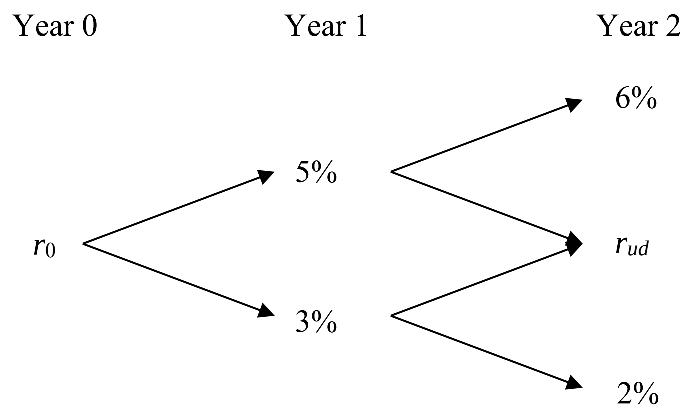
To find the number of shares in the hedging program, we differentiate the price formula with respect to S,

$$\frac{\partial}{\partial S} 100e^{-rT} N(d_2)$$
$$= 100e^{-rT} N'(d_2)\frac{\partial d_2}{\partial S} = 100e^{-rT} N'(d_2)\frac{1}{S\sigma\sqrt{T}}.$$

With $T = 1$, $r = 0.02$, $\delta = 0$, $\sigma = 0.2$, $S = S(0) = 10$, $K = K_2 = 10$, we have $d_2 = 0$ and

$$\begin{aligned}100e^{-rT} N'(d_2)\frac{1}{S\sigma\sqrt{T}} &= 100e^{-0.02} N'(0)\frac{1}{2} \\ &= 100e^{-0.02}\frac{e^{-0^2/2}}{\sqrt{2\pi}}\frac{1}{2} \\ &= \frac{50e^{-0.02}}{\sqrt{2\pi}} \\ &= 19.55.\end{aligned}$$

29. The following is a Black-Derman-Toy binomial tree for effective annual interest rates.



Compute the "volatility in year 1" of the 3-year zero-coupon bond generated by the tree.

(A) 14%

(B) 18%

(C) 22%

(D) 26%

(E) 30%

Solution to (29) Answer: (D)

According to formula (25.45) on page 771 in McDonald (2013), the "volatility in year 1" of an n-year zero-coupon bond in a Black-Derman-Toy model is the number κ such that

$$y(1, n, r_u) = y(1, n, r_d)\, e^{2\kappa},$$

where y, the yield to maturity, is defined by

$$P(1, n, r) = \left(\frac{1}{1+y(1,n,r)}\right)^{n-1}.$$

Here, $n = 3$. To find $P(1, 3, r_u)$ and $P(1, 3, r_d)$, we use the method of *backward induction*.

$$\begin{tikzcd}
 & & P(2, 3, r_{uu}) \\
 & P(1, 3, r_u) \arrow[ur] \arrow[dr] & \\
P(0, 3) \arrow[ur] \arrow[dr] & & P(2, 3, r_{ud}) \\
 & P(1, 3, r_d) \arrow[ur] \arrow[dr] & \\
 & & P(2, 3, r_{dd})
\end{tikzcd}$$

$$P(2, 3, r_{uu}) = \frac{1}{1+r_{uu}} = \frac{1}{1.06},$$

$$P(2, 3, r_{dd}) = \frac{1}{1+r_{dd}} = \frac{1}{1.02},$$

$$P(2, 3, r_{du}) = \frac{1}{1+r_{ud}} = \frac{1}{1+\sqrt{r_{uu}\times r_{dd}}} = \frac{1}{1.03464},$$

$$P(1, 3, r_u) = \frac{1}{1+r_u}[\tfrac{1}{2} P(2, 3, r_{uu}) + \tfrac{1}{2} P(2, 3, r_{ud})] = 0.909483,$$

$$P(1, 3, r_d) = \frac{1}{1+r_d}[\tfrac{1}{2} P(2, 3, r_{ud}) + \tfrac{1}{2} P(2, 3, r_{dd})] = 0.945102.$$

Hence,

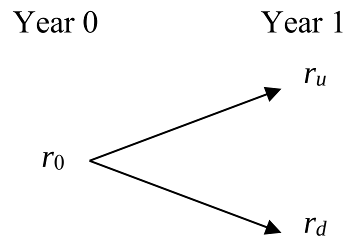
$$e^{2\kappa} = \frac{y(1,3,r_u)}{y(1,3,r_d)} = \frac{[P(1,3,r_u)]^{-1/2}-1}{[P(1,3,r_d)]^{-1/2}-1} = \frac{0.048583}{0.028633},$$

resulting in $\kappa = 0.264348 \approx 26\%$.

30. You are given the following market data for zero-coupon bonds with a maturity payoff of $100.

Maturity (years)	Bond Price ($)	Volatility in Year 1
1	94.34	N/A
2	88.50	10%

A 2-period Black-Derman-Toy interest tree is calibrated using the data from above:



Calculate r_d, the effective annual rate in year 1 in the "down" state.

(A) 5.94%

(B) 6.60%

(C) 7.00%

(D) 7.27%

(E) 7.33%

Solution to (30) Answer: (A)

$$\begin{tikzcd}[row sep=small]
\text{Year 0} & \text{Year 1} \\
 & r_u = r_d e^{2\sigma_1} \\
r_0 \arrow[ur] \arrow[dr] & \\
 & r_d
\end{tikzcd}$$

In a BDT interest rate model, the risk-neutral probability of each "up" move is ½.

Because the "volatility in year 1" of the 2-year zero-coupon bond is 10%, we have

$$\sigma_1 = 10\%.$$

This can be seen from simplifying the right-hand side of (24.51).

We are given $P(0, 1) = 0.9434$ and $P(0, 2) = 0.8850$, and they are related as follows:

$$\begin{aligned}
P(0, 2) &= P(0, 1)[\tfrac{1}{2}P(1, 2, r_u) + \tfrac{1}{2}P(1, 2, r_d)] \\
&= P(0, 1)\left[\frac{1}{2}\frac{1}{1+r_u} + \frac{1}{2}\frac{1}{1+r_d}\right] \\
&= P(0, 1)\left[\frac{1}{2}\frac{1}{1+r_d e^{0.2}} + \frac{1}{2}\frac{1}{1+r_d}\right].
\end{aligned}$$

Thus,

$$\frac{1}{1+r_d e^{0.2}} + \frac{1}{1+r_d} = \frac{2\times 0.8850}{0.9434} = 1.8762,$$

or

$$2 + r_d(1+e^{0.2}) = 1.8762[1 + r_d(1+e^{0.2}) + r_d^{\,2} e^{0.2}],$$

which is equivalent to

$$1.8762 e^{0.2} r_d^{\,2} + 0.8762(1+e^{0.2}) r_d - 0.1238 = 0.$$

The solution set of the quadratic equation is $\{0.0594, -0.9088\}$. Hence,

$$r_d \approx 5.94\%.$$

31. You compute the current delta for a 50-60 bull spread with the following information:

(i) The continuously compounded risk-free rate is 5%.

(ii) The underlying stock pays no dividends.

(iii) The current stock price is $50 per share.

(iv) The stock's volatility is 20%.

(v) The time to expiration is 3 months.

How much does delta change after 1 month, if the stock price does not change?

(A) increases by 0.04

(B) increases by 0.02

(C) does not change, within rounding to 0.01

(D) decreases by 0.02

(E) decreases by 0.04

Solution to (31) Answer: (B)

Assume that the bull spread is constructed by buying a 50-strike call and selling a 60-strike call. (You may also assume that the spread is constructed by buying a 50-strike put and selling a 60-strike put.)

Delta for the bull spread is equal to

(delta for the 50-strike call) – (delta for the 60-strike call).

(You get the same delta value, if put options are used instead of call options.)

Call option delta = $N(d_1)$, where $d_1 = \dfrac{\ln(S/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$

50-strike call:

$$d_1 = \frac{\ln(50/50) + (0.05 + \frac{1}{2} \times 0.2^2)(3/12)}{0.2\sqrt{3/12}} = 0.175, \qquad N(0.175) = 0.56946$$

60-strike call:

$$d_1 = \frac{\ln(50/60) + (0.05 + \frac{1}{2} \times 0.2^2)(3/12)}{0.2\sqrt{3/12}} = -1.64822, \qquad N(-1.64882) = 0.04965$$

Delta of the bull spread = 0.56946 – 0.04965 = 0.51981.

After one month, 50-strike call:

$$d_1 = \frac{\ln(50/50) + (0.05 + \frac{1}{2} \times 0.2^2)(2/12)}{0.2\sqrt{2/12}} = 0.1428869 \qquad N(0.14289) = 0.55681$$

60-strike call:

$$d_1 = \frac{\ln(50/60) + (0.05 + \frac{1}{2} \times 0.2^2)(2/12)}{0.2\sqrt{2/12}} = -2.090087 \qquad N(-2.0901) = 0.01830$$

Delta of the bull spread after one month = 0.55681 – 0.01830 = 0.53851.

The change in delta = 0.53851 − 0.51981 = 0.0187 ≈ 0.02.

32. DELETED

33. You own one share of a nondividend-paying stock. Because you worry that its price may drop over the next year, you decide to employ a *rolling insurance strategy*, which entails obtaining one 3-month European put option on the stock every three months, with the first one being bought immediately.

You are given:

(i) The continuously compounded risk-free interest rate is 8%.

(ii) The stock's volatility is 30%.

(iii) The current stock price is 45.

(iv) The strike price for each option is 90% of the then-current stock price.

Your broker will sell you the four options but will charge you for their total cost now.

Under the Black-Scholes framework, how much do you now pay your broker?

(A) 1.59

(B) 2.24

(C) 2.86

(D) 3.48

(E) 3.61

Solution to (33) Answer: (C)

The problem is a variation of Exercise 14.22, whose solution uses the concept of the forward start option in Exercise 14.21.

Let us first calculate the current price of a 3-month European put with strike price being 90% of the current stock price S.

With $K = 0.9\times S$, $r = 0.08$, $\sigma = 0.3$, and $T = ¼$, we have

$$d_1 = \frac{\ln(S/0.9S) + (r + ½\sigma^2)T}{\sigma\sqrt{T}} = \frac{-\ln(0.9) + (0.08 + ½ \times 0.09) \times ¼}{0.3\sqrt{¼}} = 0.91073677$$

$d_2 = d_1 - \sigma\sqrt{T} = d_1 - 0.3\sqrt{¼} = 0.76074$

$N(-d_1) = N(-0.91074) = 0.18122$

$N(-d_2) = N(-0.76074) = 0.22341$

Put price = $Ke^{-rT}N(-d_2) - SN(-d_1) = 0.9Se^{-0.08 \times 0.25}\times 0.22341 - S\times 0.18122 = 0.015868S$

For the rolling insurance strategy, four put options are needed. Their costs are $0.015868S(0)$ at time 0, $0.015868S(¼)$ at time ¼, $0.015868S(½)$ at time ½, and $0.015868S(¾)$ at time ¾. Their total price at time 0 is the sum of their prepaid forward prices.

Since the stock pays no dividends, we have

$$F_{0,T}^{P}(S(T)) = S(0), \qquad \text{for all } T \geq 0.$$

Hence, the sum of the four prepaid forward prices is

$$0.015868S(0) \times 4 = 0.015868 \times 45 \times 4 = 2.85624 \approx 2.86.$$

34-38. DELETED

39. A discrete-time model is used to model both the price of a nondividend-paying stock and the short-term (risk-free) interest rate. Each period is one year.

At time 0, the stock price is $S_0 = 100$ and the effective annual interest rate is $r_0 = 5\%$.

At time 1, there are only two states of the world, denoted by u and d. The stock prices are $S_u = 110$ and $S_d = 95$. The effective annual interest rates are $r_u = 6\%$ and $r_d = 4\%$.

Let $C(K)$ be the price of a 2-year K-strike European call option on the stock.
Let $P(K)$ be the price of a 2-year K-strike European put option on the stock.

Determine $P(108) – C(108)$.

(A) -2.85

(B) -2.34

(C) -2.11

(D) -1.95

(E) -1.08

Solution to (39) Answer: (B)

We are given that the securities model is a discrete-time model, with each period being one year. Even though there are only two states of the world at time 1, we cannot assume that the model is binomial after time 1. However, the difference, $P(K) - C(K)$, suggests put-call parity.

From the identity

$$x_+ - (-x)_+ = x,$$

we have

$$[K - S(T)]_+ - [S(T) - K]_+ = K - S(T),$$

which yields

$$\begin{aligned} P(K) - C(K) &= F_{0,2}^P(K) - F_{0,2}^P(S) \\ &= \mathrm{PV}_{0,2}(K) - S(0) \\ &= K \times P(0, 2) - S(0). \end{aligned}$$

Thus, the problem is to find $P(0, 2)$, the price of the 2-year zero-coupon bond:

$$\begin{aligned} P(0, 2) &= \frac{1}{1+r_0}\left[p^* \times P(1,2,u) + (1-p^*) \times P(1,2,d)\right] \\ &= \frac{1}{1+r_0}\left[\frac{p^*}{1+r_u} + \frac{1-p^*}{1+r_d}\right]. \end{aligned}$$

To find the risk-neutral probability p^*, we use

$$S_0 = \frac{1}{1+r_0}\left[p^* \times S_u + (1-p^*) \times S_d\right]$$

or

$$100 = \frac{1}{1.05}\left[p^* \times 110 + (1-p^*) \times 95\right].$$

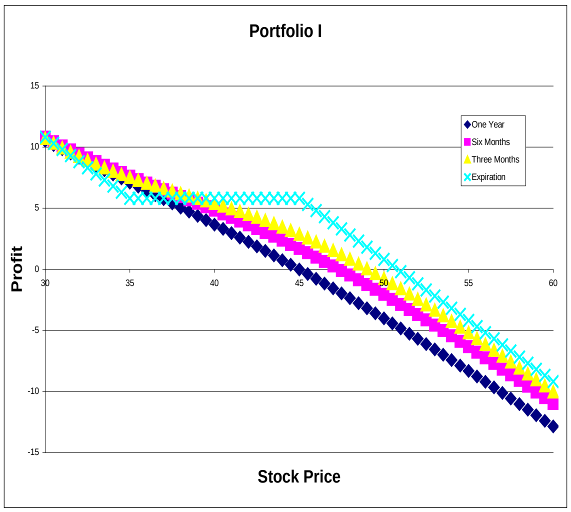
This yields $p^* = \dfrac{105-95}{110-95} = \dfrac{2}{3}$, with which we obtain

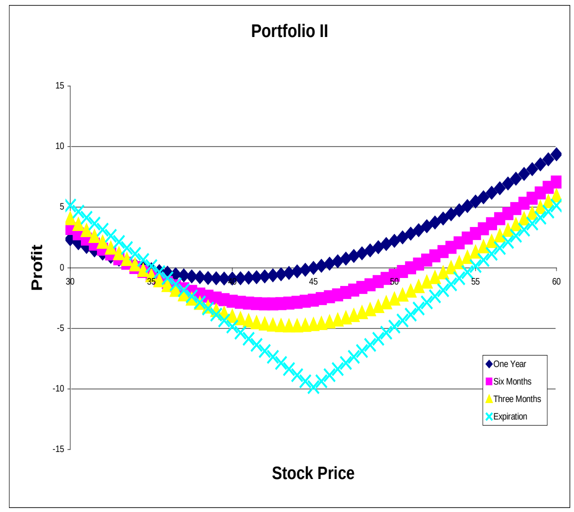
$$P(0, 2) = \frac{1}{1.05}\left[\frac{2/3}{1.06} + \frac{1/3}{1.04}\right] = 0.904232.$$

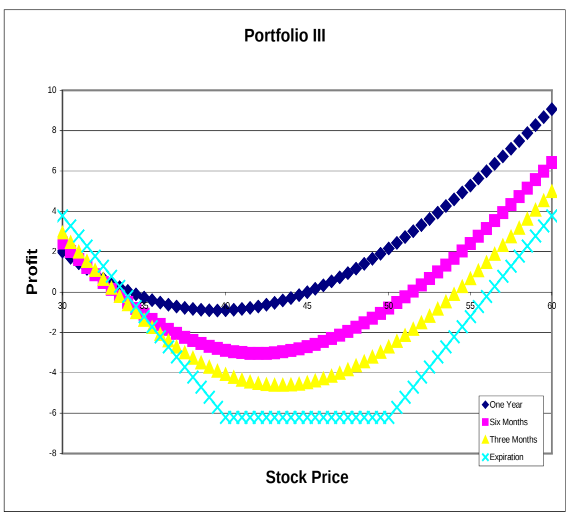
Hence,

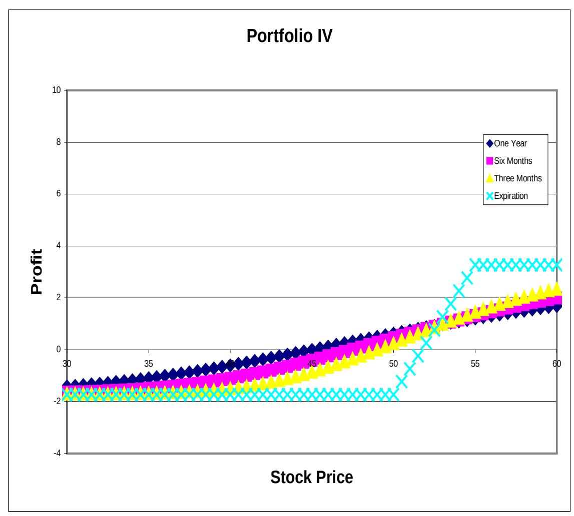
$$P(108) - C(108) = 108 \times 0.904232 - 100 = -2.34294.$$

40. The following four charts are profit diagrams for four option strategies: Bull Spread, Collar, Straddle, and Strangle. Each strategy is constructed with the purchase or sale of two 1-year European options.









Match the charts with the option strategies.

	Bull Spread	Straddle	Strangle	Collar
(A)	I	II	III	IV
(B)	I	III	II	IV
(C)	III	IV	I	II
(D)	IV	II	III	I
(E)	IV	III	II	I

Solution to (40) Answer: (D)

Profit diagrams are discussed Section 12.4 of McDonald (2013). Definitions of the option strategies can be found in the Glossary near the end of the textbook. See also Figure 3.16 on page 85.

The payoff function of a *straddle* is

$$\pi(s) = (K - s)_+ + (s - K)_+ = |s - K| .$$

The payoff function of a *strangle* is

$$\pi(s) = (K_1 - s)_+ + (s - K_2)_+$$

where $K_1 < K_2$.

The payoff function of a *collar* is

$$\pi(s) = (K_1 - s)_+ - (s - K_2)_+$$

where $K_1 < K_2$.

The payoff function of a *bull spread* is

$$\pi(s) = (s - K_1)_+ - (s - K_2)_+$$

where $K_1 < K_2$. Because $x_+ = (-x)_+ + x$, we have

$$\pi(s) = (K_1 - s)_+ - (K_2 - s)_+ + K_2 - K_1 .$$

The payoff function of a *bear spread* is

$$\pi(s) = (s - K_2)_+ - (s - K_1)_+$$

where $K_1 < K_2$.

41. Assume the Black-Scholes framework. Consider a 1-year European contingent claim on a stock.

You are given:

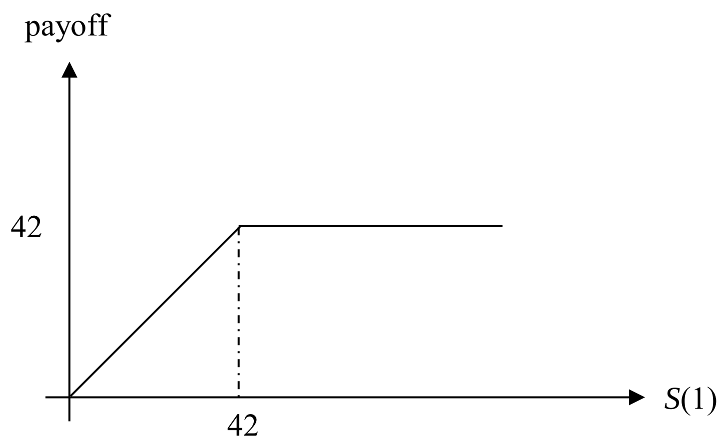
(i) The time-0 stock price is 45.

(ii) The stock's volatility is 25%.

(iii) The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 3%.

(iv) The continuously compounded risk-free interest rate is 7%.

(v) The time-1 payoff of the contingent claim is as follows:



Calculate the time-0 contingent-claim elasticity.

(A) 0.24

(B) 0.29

(C) 0.34

(D) 0.39

(E) 0.44

Solution to (41) Answer: (C)

The payoff function of the contingent claim is

$$\pi(s) = \min(42, s) = 42 + \min(0, s - 42) = 42 - \max(0, 42 - s) = 42 - (42 - s)_+$$

The time-0 price of the contingent claim is

$$\begin{aligned} V(0) &= F_{0,1}^P[\pi(S(1))] \\ &= \mathrm{PV}(42) - F_{0,1}^P[(42 - S(1))_+] \\ &= 42e^{-0.07} - P(45, 42, 0.25, 0.07, 1, 0.03). \end{aligned}$$

We have $d_1 = \dfrac{\ln(45/42) + (0.07 - 0.03 + ½(0.25)^2 \times 1)}{0.25\sqrt{1}} = 0.560971486$

and $d_2 = 0.310971486$. From the Cumulative Normal Distribution Calculator, $N(-d_1) = N(-0.56097) = 0.28741$ and $N(-d_2) = N(-0.31097) = 0.37791$.

Hence, the time-0 put price is

$$P(45, 42, 0.25, 0.07, 1, 0.03) = 42e^{-0.07}(0.37791) - 45e^{-0.03}(0.28741) = 2.247951,$$

which implies $V(0) = 42e^{-0.07} - 2.247951 = 36.91259$.

$$\begin{aligned} \text{Elasticity} &= \frac{\partial \ln V}{\partial \ln S} \\ &= \frac{\partial V}{\partial S} \times \frac{S}{V} \\ &= \Delta_V \times \frac{S}{V} \\ &= -\Delta_{\text{Put}} \times \frac{S}{V}. \end{aligned}$$

$$\begin{aligned} \text{Time-0 elasticity} &= e^{-\delta T} N(-d_1) \times \frac{S(0)}{V(0)} \\ &= e^{-0.03} \times 0.28741 \times \frac{45}{36.91259} \\ &= 0.340025. \end{aligned}$$

Remark: We can also work with $\pi(s) = s - (s - 42)_+$; then

$$V(0) = 45e^{-0.03} - C(45, 42, 0.25, 0.07, 1, 0.03)$$

and

$$\frac{\partial V}{\partial S} = e^{-\delta T} - \Delta_{\text{call}} = e^{-\delta T} - e^{-\delta T} N(d_1) = e^{-\delta T} N(-d_1).$$

42. Prices for 6-month 60-strike European up-and-out call options on a stock S are available. Below is a table of option prices with respect to various H, the level of the barrier. Here, $S(0) = 50$.

H	Price of up-and-out call
60	0
70	0.1294
80	0.7583
90	1.6616
∞	4.0861

Consider a special 6-month 60-strike European "knock-in, partial knock-out" call option that knocks in at $H_1 = 70$, and "partially" knocks out at $H_2 = 80$. The strike price of the option is 60. The following table summarizes the payoff at the exercise date:

H_1 Not Hit	H_1 Hit	
	H_2 Not Hit	H_2 Hit
0	$2 \times \max[S(0.5) - 60, 0]$	$\max[S(0.5) - 60, 0]$

Calculate the price of the option.

(A) 0.6289

(B) 1.3872

(C) 2.1455

(D) 4.5856

(E) It cannot be determined from the information given above.

Solution to (42) Answer: (D)

The "knock-in, knock-out" call can be thought of as a portfolio of
– buying 2 ordinary up-and-in call with strike 60 and barrier H_1,
– writing 1 ordinary up-and-in call with strike 60 and barrier H_2.

Recall also that "up-and-in" call + "up-and-out" call = ordinary call.

Let the price of the ordinary call with strike 60 be p (actually it is 4.0861),
then the price of the UIC ($H_1 = 70$) is $p – 0.1294$
and the price of the UIC ($H_1 = 80$) is $p – 0.7583$.

The price of the "knock-in, knock out" call is $2(p – 0.1294) – (p – 0.7583) = 4.5856$.

Alternative Solution:
Let $M(T) = \max_{0 \le t \le T} S(t)$ be the *running maximum* of the stock price up to time T.
Let I[.] denote the *indicator function*.

For various H, the first table gives the time-0 price of payoff of the form

$$I[H > M(½)] \times [S(½) - 60]_+ .$$

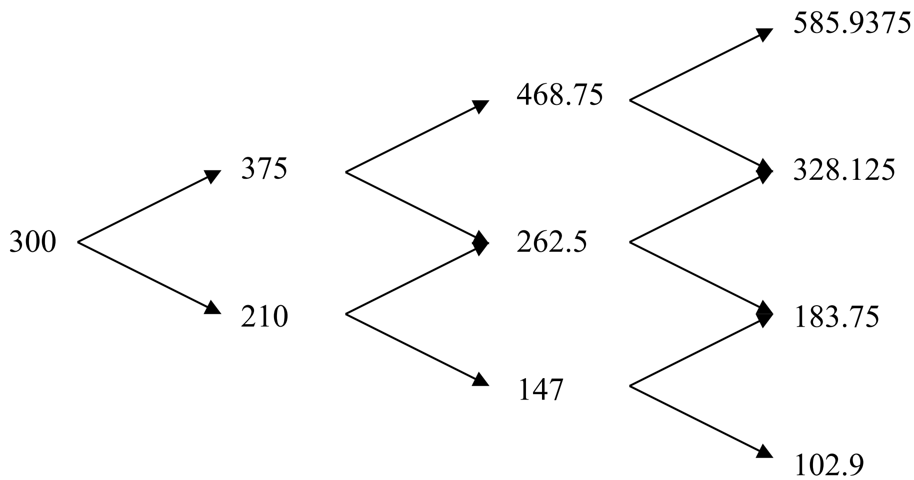
The payoff described by the second table is

$$\begin{aligned}
&I[70 \le M(½)]\{2I[80 > M(½)] + I[80 \le M(½)]\}[S(½) - 60]_+ \\
&= \{1 - I[70 > M(½)]\}\{1 + I[80 > M(½)]\}[S(½) - 60]_+ \\
&= \{1 - I[70 > M(½)] + I[80 > M(½)] - I[70 > M(½)]I[80 > M(½)]\}[S(½) - 60]_+ \\
&= \{1 - 2I[70 > M(½)] + I[80 > M(½)]\}[S(½) - 60]_+ \\
&= \{I[\infty > M(½)] - 2I[70 > M(½)] + I[80 > M(½)]\}[S(½) - 60]_+
\end{aligned}$$

Thus, the time-0 price of this payoff is $4.0861 - 2 \times 0.1294 + 0.7583 = 4.5856$.

43. DELETED

44. Consider the following three-period binomial tree model for a stock that pays dividends continuously at a rate proportional to its price. The length of each period is 1 year, the continuously compounded risk-free interest rate is 10%, and the continuous dividend yield on the stock is 6.5%.



Calculate the price of a 3-year at-the-money American put option on the stock.

(A) 15.86

(B) 27.40

(C) 32.60

(D) 39.73

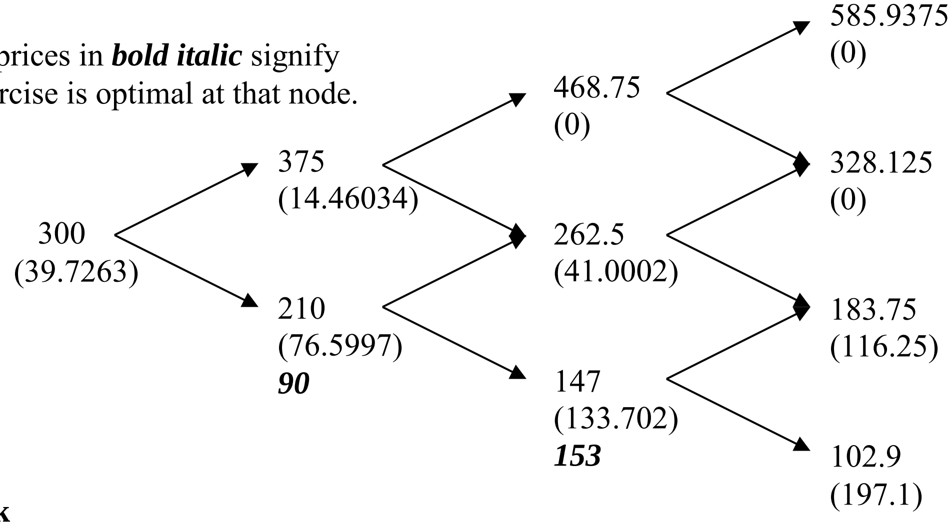
(E) 57.49

Solution to (44) Answer: (D)

By formula (10.5), the risk-neutral probability of an up move is

$$p^* = \frac{e^{(r-\delta)h} - d}{u-d} = \frac{S_0 e^{(r-\delta)h} - S_d}{S_u - S_d} = \frac{300e^{(0.1-0.065)\times 1} - 210}{375-210} = 0.61022.$$

Option prices in ***bold italic*** signify that exercise is optimal at that node.



Remark

If the put option is European, not American, then the simplest method is to use the binomial formula [p. 335, (11.12); p. 574, (19.2)]:

$$e^{-r(3h)}\left[\binom{3}{3}(1-p^*)^3(300-102.9) + \binom{3}{2}p^*(1-p^*)^2(300-183.75) + 0 + 0\right]$$

$= e^{-r(3h)}(1 - p^*)^2[(1 - p^*) \times 197.1 \ + \ 3 \times p^* \times 116.25)]$

$= e^{-r(3h)}(1 - p^*)^2(197.1 \ + \ 151.65p^*)$

$= e^{-0.1 \times 3} \times 0.38978^2 \times 289.63951 \ = 32.5997$

45. DELETED

46. You are to price options on a futures contract. The movements of the futures price are modeled by a binomial tree. You are given:

(i) Each period is 6 months.

(ii) $u/d = 4/3$, where u is one plus the rate of gain on the futures price if it goes up, and d is one plus the rate of loss if it goes down.

(iii) The risk-neutral probability of an up move is 1/3.

(iv) The initial futures price is 80.

(v) The continuously compounded risk-free interest rate is 5%.

Let C_I be the price of a 1-year 85-strike European call option on the futures contract, and C_{II} be the price of an otherwise identical American call option.

Determine $C_{II} - C_I$.

(A) 0

(B) 0.022

(C) 0.044

(D) 0.066

(E) 0.088

Solution to (46) Answer: (E)

By formula (10.21), the risk-neutral probability of an up move is

$$p^* = \frac{1-d}{u-d} = \frac{1/d-1}{u/d-1}.$$

Substituting $p^* = 1/3$ and $u/d = 4/3$, we have

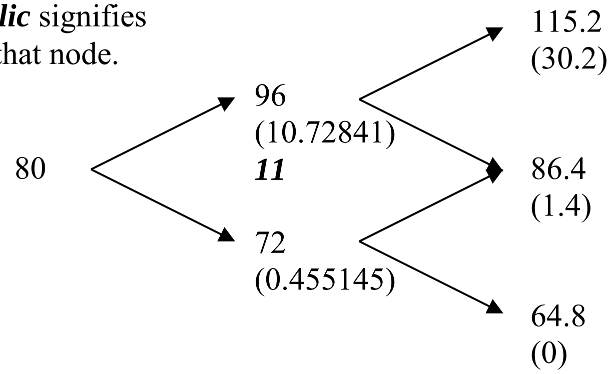
$$\frac{1}{3} = \frac{1/d-1}{4/3-1}.$$

Hence, $d = 0.9$ and $u = (4/3)\times d = 1.2$.

The two-period binomial tree for the futures price and prices of European and American options at $t = 0.5$ and $t = 1$ is given below. The calculation of the European option prices at $t = 0.5$ is given by

$$e^{-0.05\times0.5}[30.2p^* + 1.4(1-p^*)] = 10.72841$$
$$e^{-0.05\times0.5}[1.4p^* + 0\times(1-p^*)] = 0.455145$$

An option price in ***bold italic*** signifies that exercise is optimal at that node.



Thus, $C_{II} - C_I = e^{-0.05\times0.5} \times (11 - 10.72841) \times p^* = 0.088$.

Remarks:

(i) $C_I = e^{-0.05\times0.5}[10.72841p^* + 0.455145(1-p^*)] = 3.78378.$

$C_{II} = e^{-0.05\times0.5}[11p^* + 0.455145(1-p^*)] = 3.87207.$

(ii) A futures price can be treated like a stock with $\delta = r$. With this observation, we can obtain (10.14) from (10.5),

$$p^* = \frac{e^{(r-\delta)h}-d}{u-d} = \frac{e^{(r-r)h}-d}{u-d} = \frac{1-d}{u-d}.$$

Another application is the determination of the price sensitivity of a futures option with respect to a change in the futures price. We learn from page 317 that the price sensitivity of a stock option with respect to a change in the stock price is $e^{-\delta h}\frac{C_u - C_d}{S(u-d)}$. Changing δ to r and S to F yields $e^{-rh}\frac{C_u - C_d}{F(u-d)}$, which is the same as the expression $e^{-rh}\Delta$ given in footnote 7 on page 333.

47. Several months ago, an investor sold 100 units of a one-year European call option on a nondividend-paying stock. She immediately delta-hedged the commitment with shares of the stock, but has not ever re-balanced her portfolio. She now decides to close out all positions.

You are given the following information:

(i) The risk-free interest rate is constant.

(ii)

	Several months ago	Now
Stock price	\$40.00	\$50.00
Call option price	\$ 8.88	\$14.42
Put option price	\$ 1.63	\$ 0.26
Call option delta	0.794	

The put option in the table above is a European option on the same stock and with the same strike price and expiration date as the call option.

Calculate her profit.

(A) \$11

(B) \$24

(C) \$126

(D) \$217

(E) \$240

Solution to (47) Answer: (B)
Let the date several months ago be 0. Let the current date be t.

Delta-hedging at time 0 means that the investor's cash position at time 0 was

$$100[C(0) - \Delta_C(0)S(0)].$$

After closing out all positions at time t, her profit is

$$100\{[C(0) - \Delta_C(0)S(0)]e^{rt} - [C(t) - \Delta_C(0)S(t)]\}.$$

To find the accumulation factor e^{rt}, we can use put-call parity:

$$C(0) - P(0) = S(0) - Ke^{-rT},$$
$$C(t) - P(t) = S(t) - Ke^{-r(T-t)},$$

where T is the option expiration date. Then,

$$e^{rt} = \frac{S(t) - C(t) + P(t)}{S(0) - C(0) + P(0)} = \frac{50 - 14.42 + 0.26}{40 - 8.88 + 1.63} = \frac{35.84}{32.75} = 1.0943511.$$

Thus, her profit is

$$100\{[C(0) - \Delta_C(0)S(0)]e^{rt} - [C(t) - \Delta_C(0)S(t)]\}$$
$$= 100\{[8.88 - 0.794 \times 40] \times 1.09435 - [14.42 - 0.794 \times 50]\}$$
$$= 24.13 \approx 24$$

Alternative Solution: Consider profit as the sum of (i) capital gain and (ii) interest:

(i) capital gain = $100\{[C(0) - C(t)] - \Delta_C(0)[S(0) - S(t)]\}$
(ii) interest = $100[C(0) - \Delta_C(0)S(0)](e^{rt} - 1)$.

Now,

$$\text{capital gain} = 100\{[C(0) - C(t)] - \Delta_C(0)[S(0) - S(t)]\}$$
$$= 100\{[8.88 - 14.42] - 0.794[40 - 50]\}$$
$$= 100\{-5.54 + 7.94\} = 240.00.$$

To determine the amount of interest, we first calculate her cash position at time 0:

$$100[C(0) - \Delta_C(0)S(0)] = 100[8.88 - 40 \times 0.794]$$
$$= 100[8.88 - 31.76] = -2288.00.$$

Hence,

$$\text{interest} = -2288 \times (1.09435 - 1) = -215.87.$$

Thus, the investor's profit is 240.00 – 215.87 = 24.13 ≈ 24.

Third Solution: Use the table format in Section 13.3 of McDonald (2013).

Position	Cost at time 0	Value at time t
Short 100 calls	$-100 \times 8.88 = -888$	$-100 \times 14.42 = -1442$
100Δ shares of stock	$100 \times 0.794 \times 40 = 3176$	$100 \times 0.794 \times 50 = 3970$
Borrowing	$3176 - 888 = 2288$	$2288e^{rt} = 2503.8753$
Overall	0	24.13

Remark: The problem can still be solved if the short-rate is **deterministic** (but not necessarily constant). Then, the accumulation factor e^{rt} is replaced by $\exp[\int_0^t r(s)\mathrm{d}s]$, which can be determined using the put-call parity formulas

$$C(0) - P(0) = S(0) - K\exp[-\int_0^T r(s)\mathrm{d}s],$$

$$C(t) - P(t) = S(t) - K\exp[-\int_t^T r(s)\mathrm{d}s].$$

If interest rates are stochastic, the problem as stated cannot be solved.

48. DELETED

49. You use the usual method in McDonald and the following information to construct a one-period binomial tree for modeling the price movements of a nondividend-paying stock. (The tree is sometimes called a forward tree).

(i) The period is 3 months.

(ii) The initial stock price is $100.

(iii) The stock's volatility is 30%.

(iv) The continuously compounded risk-free interest rate is 4%.

At the beginning of the period, an investor owns an American put option on the stock. The option expires at the end of the period.

Determine the smallest integer-valued strike price for which an investor will exercise the put option at the beginning of the period.

(A) 114

(B) 115

(C) 116

(D) 117

(E) 118

Solution to (49) Answer: (B)

$$u = e^{(r-\delta)h+\sigma\sqrt{h}} = e^{rh+\sigma\sqrt{h}} = e^{(0.04/4)+(0.3/2)} = e^{0.16} = 1.173511$$
$$d = e^{(r-\delta)h-\sigma\sqrt{h}} = e^{rh-\sigma\sqrt{h}} = e^{(0.04/4)-(0.3/2)} = e^{-0.14} = 0.869358$$
S = initial stock price = 100

The problem is to find the smallest integer K satisfying

$$K - S \ > \ e^{-rh}[p^* \times \text{Max}(K - Su, 0) + (1 - p^*) \times \text{Max}(K - Sd, 0)]. \qquad (1)$$

Because the RHS of (1) is nonnegative (the payoff of an option is nonnegative), we have the condition

$$K - S \ > 0. \qquad (2)$$

As $d < 1$, it follows from condition (2) that

$$\text{Max}(K - Sd, 0) \ = \ K - Sd,$$

and inequality (1) becomes

$$K - S \ > \ e^{-rh}[p^* \times \text{Max}(K - Su, 0) + (1 - p^*) \times (K - Sd)]. \qquad (3)$$

If $K \geq Su$, the right-hand side of (3) is

$$e^{-rh}[p^* \times (K - Su) + (1 - p^*) \times (K - Sd)]$$
$$= \ e^{-rh}K - e^{-\delta h}S$$
$$= \ e^{-rh}K - S,$$

because the stock pays no dividends. Thus, if $K \geq Su$, inequality (3) always holds, and the put option is exercised early.

We now investigate whether there is any K, $S < K < Su$, such that inequality (3) holds. If $Su > K$, then $\text{Max}(K - Su, 0) = 0$ and inequality (3) simplifies as

$$K - S \ > \ e^{-rh} \times (1 - p^*) \times (K - Sd),$$

or

$$K \ > \ \frac{1 - e^{-rh}(1 - p^*)d}{1 - e^{-rh}(1 - p^*)} \ S. \qquad (4)$$

The fraction $\dfrac{1 - e^{-rh}(1 - p^*)d}{1 - e^{-rh}(1 - p^*)}$ can be simplified as follows, but this step is not necessary. In McDonald's forward-tree model,

$$1 - p^* \ = \ p^* \times e^{\sigma\sqrt{h}},$$

from which we obtain

$$1 - p^* \ = \ \frac{1}{1 + e^{-\sigma\sqrt{h}}}.$$

Hence, $$\frac{1-e^{-rh}(1-p^*)d}{1-e^{-rh}(1-p^*)} = \frac{1+e^{-\sigma\sqrt{h}}-e^{-rh}d}{1+e^{-\sigma\sqrt{h}}-e^{-rh}}$$
$$= \frac{1+e^{-\sigma\sqrt{h}}-e^{-\sigma\sqrt{h}}}{1+e^{-\sigma\sqrt{h}}-e^{-rh}} \qquad \text{because } \delta = 0$$
$$= \frac{1}{1+e^{-\sigma\sqrt{h}}-e^{-rh}}.$$

Therefore, inequality (4) becomes
$$K > \frac{1}{1+e^{-\sigma\sqrt{h}}-e^{-rh}} S$$
$$= \frac{1}{1+e^{-0.15}-e^{-0.01}} S = 1.148556 \times 100 = 114.8556.$$

Thus, the answer to the problem is $\lceil 114.8556 \rceil = 115$, which is (B).

Alternative Solution:

$$u = e^{(r-\delta)h+\sigma\sqrt{h}} = e^{rh+\sigma\sqrt{h}} = e^{(0.04/4)+(0.3/2)} = e^{0.16} = 1.173511$$
$$d = e^{(r-\delta)h-\sigma\sqrt{h}} = e^{rh-\sigma\sqrt{h}} = e^{(0.04/4)-(0.3/2)} = e^{-0.14} = 0.869358$$
S = initial stock price = 100
$$p^* = \frac{1}{1+e^{\sigma\sqrt{h}}} = \frac{1}{1+e^{0.3/2}} = \frac{1}{1+e^{0.15}} = \frac{1}{1+1.1618} = 0.46257.$$

Then, inequality (1) is
$$K - 100 > e^{-0.01}[0.4626 \times (K - 117.35)_+ + 0.5374 \times (K - 86.94)_+], \qquad (5)$$
and we check three cases: $K \leq 86.94$, $K \geq 117.35$, and $86.94 < K < 117.35$.

For $K \leq 86.94$, inequality (5) cannot hold, because its LHS < 0 and its RHS = 0.
For $K \geq 117.35$, (5) always holds, because its LHS = $K - 100$ while its RHS = $e^{-0.01}K - 100$.
For $86.94 < K < 117.35$, inequality (5) becomes
$$K - 100 > e^{-0.01} \times 0.5374 \times (K - 86.94),$$
or
$$K > \frac{100 - e^{-0.01} \times 0.5374 \times 86.94}{1 - e^{-0.01} \times 0.5374} = 114.85.$$

Third Solution: Use the method of trial and error. For $K = 114, 115, \ldots$, check whether inequality (5) holds.

Remark: An American call option on a nondividend-paying stock is never exercised early. This problem shows that the corresponding statement for American puts is not true.

50. Assume the Black-Scholes framework.

You are given the following information for a stock that pays dividends continuously at a rate proportional to its price.

(i) The current stock price is 0.25.

(ii) The stock's volatility is 0.35.

(iii) The continuously compounded expected rate of stock-price appreciation is 15%.

Calculate the upper limit of the 90% lognormal confidence interval for the price of the stock in 6 months.

(A) 0.393

(B) 0.425

(C) 0.451

(D) 0.486

(E) 0.529

Solution to (50) Answer: (A)

This problem is a modification of #4 in the May 2007 Exam C.

The conditions given are:

(i) $S_0 = 0.25$,

(ii) $\sigma = 0.35$,

(iii) $\alpha - \delta = 0.15$.

We are to seek the number $S_{0.5}^U$ such that $\Pr(S_{0.5} < S_{0.5}^U) = 0.95$.

The random variable $\ln(S_{0.5} / 0.25)$ is normally distributed with

$$\text{mean} = (0.15 - ½ \times 0.35^2) \times 0.5 = 0.044375,$$
$$\text{standard deviation} = 0.35 \times \sqrt{0.5} = 0.24749.$$

Because $N^{-1}(0.95) = 1.64485$, we have

$$0.044375 + 0.24749 N^{-1}(0.95) = 0.451458927.$$

Thus,

$$S_{0.5}^U = 0.25 \times e^{0.45146} = 0.39265.$$

.

Remark The term "confidence interval" as used in Section 18.4 McDonald (2013) seems incorrect, because S_t is a random variable, not an unknown, but constant, parameter. The expression

$$\Pr(S_t^L < S_t < S_t^U) = 1 - p$$

gives the probability that the random variable S_t is between S_t^L and S_t^U, not the "confidence" for S_t to be between S_t^L and S_t^U.

51. Assume the Black-Scholes framework.

The price of a nondividend-paying stock in seven consecutive months is:

Month	Price
1	54
2	56
3	48
4	55
5	60
6	58
7	62

Estimate the continuously compounded expected rate of return on the stock.

(A) Less than 0.28

(B) At least 0.28, but less than 0.29

(C) At least 0.29, but less than 0.30

(D) At least 0.30, but less than 0.31

(E) At least 0.31

Solution to (51) Answer: (E)

This problem is a modification of #34 in the May 2007 Exam C. Note that you are given monthly prices, but you are asked to find an annual rate.

It is assumed that the stock price process is given by

$$\frac{dS(t)}{S(t)} = \alpha\, dt + \sigma\, dZ(t), \qquad t \geq 0.$$

We are to estimate α, using observed values of $S(jh)$, $j = 0, 1, 2, .. , n$, where $h = 1/12$ and $n = 6$. The solution to the stochastic differential equation is

$$S(t) = S(0)\exp[(\alpha - \tfrac{1}{2}\sigma^2)t + \sigma Z(t)].$$

Thus, $\ln[S((j+1)h)/S(jh)]$, $j = 0, 1, 2, \ldots$, are i.i.d. normal random variables with mean $(\alpha - \tfrac{1}{2}\sigma^2)h$ and variance $\sigma^2 h$.

Let $\{r_j\}$ denote the observed continuously compounded monthly returns:

$r_1 = \ln(56/54) = 0.03637$,

$r_2 = \ln(48/56) = -0.15415$,

$r_3 = \ln(55/48) = 0.13613$,

$r_4 = \ln(60/55) = 0.08701$,

$r_5 = \ln(58/60) = -0.03390$,

$r_6 = \ln(62/58) = 0.06669$.

The sample mean is

$$\bar{r} = \frac{1}{n}\sum_{j=1}^{n} r_j = \frac{1}{n}\ln\frac{S(t_{nh})}{S(t_0)} = \frac{1}{6}\ln\frac{62}{54} = 0.023025.$$

The (unbiased) sample variance is

$$\frac{1}{n-1}\sum_{j=1}^{n}(r_j - \bar{r})^2 = \frac{1}{n-1}\left[\sum_{j=1}^{n}(r_j)^2 - n\bar{r}^2\right] = \frac{1}{5}\left[\sum_{j=1}^{6}(r_j)^2 - 6\bar{r}^2\right] = 0.01071.$$

Thus, $\alpha = (\alpha - \tfrac{1}{2}\sigma^2) + \tfrac{1}{2}\sigma^2$ is estimated by

$$(0.023025 + \tfrac{1}{2} \times 0.01071) \times 12 = 0.3405.$$

Remarks:

(i) Let $T = nh$. Then the estimator of $\alpha - ½\sigma^2$ is

$$\frac{\bar{r}}{h} = \frac{1}{nh}\ln\frac{S(T)}{S(0)} = \frac{\ln[S(T)]-\ln[S(0)]}{T-0}.$$

This is a special case of the result that the drift of an arithmetic Brownian motion is estimated by the slope of the straight line joining its first and last observed values. Observed values of the arithmetic Brownian motion in between are not used.

(ii) An (unbiased) estimator of σ^2 is

$$\frac{1}{h}\frac{1}{n-1}\left[\sum_{j=1}^{n}(r_j)^2 - n\bar{r}^2\right] = \frac{1}{T}\left\{\frac{n}{n-1}\sum_{j=1}^{n}(r_j)^2 - \frac{1}{n-1}\left[\ln\frac{S(T)}{S(0)}\right]^2\right\}$$

$$\approx \frac{1}{T}\frac{n}{n-1}\sum_{j=1}^{n}(r_j)^2 \qquad \text{for large } n \text{ (small } h\text{)}$$

$$= \frac{1}{T}\frac{n}{n-1}\sum_{j=1}^{n}\{\ln[S(jT/n)/S((j-1)T/n)]\}^2,$$

which can be found in footnote 9 on page 730 of McDonald (2013). It is equivalent to formula (24.2) on page 720 of McDonald (2013), which is

$$\hat{\sigma}_H^2 = \frac{1}{h}\frac{1}{n-1}\sum_{j=1}^{n}\{\ln[S(jT/n)/S((j-1)T/n)]\}^2.$$

(iii) An important result (McDonald 2013, p. 607, p. 729) is: With probability 1,

$$\lim_{n\to\infty}\sum_{j=1}^{n}\{\ln[S(jT/n)/S((j-1)T/n)]\}^2 = \sigma^2 T,$$

showing that the exact value of σ can be obtained by means of a single sample path of the stock price. Here is an implication of this result. Suppose that an actuary uses a so-called regime-switching model to model the price of a stock (or stock index), with each regime being characterized by a different σ. In such a model, the current regime can be determined by this formula. If the price of the stock can be observed over a time interval, no matter how short the time interval is, then σ is revealed immediately by determining the *quadratic variation* of the logarithm of the stock price.

52. The price of a stock is to be estimated using simulation. It is known that:

(i) The time-t stock price, S_t, follows the lognormal distribution:

$$\ln\left(\frac{S_t}{S_0}\right) \sim \mathcal{N}((\alpha - ½\sigma^2)t, \sigma^2 t)$$

(ii) $S_0 = 50$, $\alpha = 0.15$, and $\sigma = 0.30$.

The following are three uniform (0, 1) random numbers

0.98300 0.03836 0.77935

Use each of these three numbers to simulate a time-2 stock price.

Calculate the mean of the three simulated prices.

(A) Less than 75

(B) At least 75, but less than 85

(C) At least 85, but less than 95

(D) At least 95, but less than 115

(E) At least 115

Solution to (52) Answer: (C)

This problem is a modification of #19 in the May 2007 Exam C.

$U \sim$ Uniform (0, 1)

$\Rightarrow N^{-1}(U) \sim \mathcal{N}(0, 1)$

$\Rightarrow a + bN^{-1}(U) \sim \mathcal{N}(a, b^2)$

The random variable $\ln(S_2 / 50)$ has a normal distribution with mean $(0.15 - ½ \times 0.3^2) \times 2 = 0.21$ and variance $0.3^2 \times 2 = 0.18$, and thus a standard deviation of 0.4243.

Using the Inverse CDF Calculator, we see that the three uniform (0, 1) random numbers correspond to the following three standard normal values: 2.12007, −1.77004, 0.77000. Upon multiplying each by the standard deviation of 0.4243 and adding the mean of 0.21, the resulting normal values are 1.109, −0.541, and 0.537. The simulated stock prices are obtained by exponentiating these numbers and multiplying by 50. This yields 151.57, 29.11, and 85.54. The average of these three numbers is 88.74.

53. Assume the Black-Scholes framework. For a European put option and a European gap call option on a stock, you are given:

(i) The expiry date for both options is T.

(ii) The put option has a strike price of 40.

(iii) The gap call option has strike price 45 and payment trigger 40.

(iv) The time-0 gamma of the put option is 0.07.

(v) The time-0 gamma of the gap call option is 0.08.

Consider a European cash-or-nothing call option that pays 1000 at time T if the stock price at that time is higher than 40.

Find the time-0 gamma of the cash-or-nothing call option.

(A) -5

(B) -2

(C) 2

(D) 5

(E) 8

Solution to (53) Answer: (B)

Let I[.] be the *indicator function*, i.e., I[A] = 1 if the event A is true, and I[A] = 0 if the event A is false. Let K_1 be the strike price and K_2 be the payment trigger of the gap call option. The payoff of the gap call option is

$$[S(T) - K_1] \times \mathrm{I}[S(T) > K_2] \;=\; \underbrace{[S(T) - K_2] \times \mathrm{I}[S(T) > K_2]}_{\substack{\text{payoff of} \\ \text{a } K_2\text{-strike call}}} \;+\; \underbrace{(K_2 - K_1) \times \mathrm{I}[S(T) > K_2]}_{\substack{(K_2 - K_1) \text{ times the payoff of} \\ \text{a cash-or-nothing call} \\ \text{that pays \$1 if } S(T) > K_2}}.$$

Because differentiation is a linear operation, each Greek (except for omega or elasticity) of a portfolio is the sum of the corresponding Greeks for the components of the portfolio (McDonald 2013, page 365). Thus,

$$\text{Gap call gamma} \;=\; \text{Call gamma} \;+\; (K_2 - K_1) \times \text{Cash-or-nothing call gamma}$$

As pointed out on page 358 of McDonald (2013), call gamma equals put gamma. (To see this, differentiate the put-call parity formula twice with respect to S.)

Because $K_2 - K_1 = 40 - 45 = -5$, call gamma = put gamma = 0.07, and gap call gamma = 0.08, we have

$$\text{Cash-or-nothing call gamma} = \frac{0.08 - 0.07}{-5} = -0.002$$

Hence the answer is $1000 \times (-0.002) = -2$.

Remark: Another decomposition of the payoff of the gap call option is the following:

$$[S(T) - K_1] \times \mathrm{I}[S(T) > K_2] \;=\; \underbrace{S(T) \times \mathrm{I}[S(T) > K_2]}_{\substack{\text{payoff of an} \\ \text{asset-or-nothing call}}} \;-\; \underbrace{K_1 \times \mathrm{I}[S(T) > K_2]}_{\substack{K_1 \text{ times the payoff of} \\ \text{a cash-or-nothing call} \\ \text{that pays \$1 if } S(T) > K_2}}.$$

See page 687 of McDonald (2013). Such a decomposition, however, is not useful here.

54. Assume the Black-Scholes framework. Consider two nondividend-paying stocks whose time-t prices are denoted by $S_1(t)$ and $S_2(t)$, respectively.

You are given:

(i) $S_1(0) = 10$ and $S_2(0) = 20$.

(ii) Stock 1's volatility is 0.18.

(iii) Stock 2's volatility is 0.25.

(iv) The correlation between the continuously compounded returns of the two stocks is –0.40.

(v) The continuously compounded risk-free interest rate is 5%.

(vi) A one-year European option with payoff $\max\{\min[2S_1(1), S_2(1)] - 17, 0\}$ has a current (time-0) price of 1.632.

Consider a European option that gives its holder the right to sell either two shares of Stock 1 or one share of Stock 2 at a price of 17 one year from now.

Calculate the current (time-0) price of this option.

(A) 0.67

(B) 1.12

(C) 1.49

(D) 5.18

(E) 7.86

Solution to (54) Answer: (A)

At the option-exercise date, the option holder will sell two shares of Stock 1 or one share of Stock 2, depending on which trade is of lower cost. Thus, the time-1 payoff of the option is

$$\max\{17 - \min[2S_1(1), S_2(1)], 0\},$$

which is the payoff of a 17-strike put on $\min[2S_1(1), S_2(1)]$. Define

$$M(T) = \min[2S_1(T), S_2(T)].$$

Consider put-call parity with respect to $M(T)$:

$$c(K, T) - p(K, T) = F_{0,T}^P(M) - Ke^{-rT}.$$

Here, $K = 17$ and $T = 1$. It is given in (vi) that $c(17, 1) = 1.632$. $F_{0,1}^P(M)$ is the time-0 price of the security with time-1 payoff

$$M(1) = \min[2S_1(1), S_2(1)] = 2S_1(1) - \max[2S_1(1) - S_2(1), 0].$$

Since $\max[2S_1(1) - S_2(1), 0]$ is the payoff of an exchange option, its price can be obtained using (14.16) and (14.17):

$$\sigma = \sqrt{0.18^2 + 0.25^2 - 2(-0.4)(0.18)(0.25)} = 0.361801$$

$$d_1 = \frac{\ln[2S_1(0)/S_2(0)] + ½\sigma^2 T}{\sigma\sqrt{T}} = ½\sigma\sqrt{T} = 0.18090,\ N(d_1) = 0.57178$$

$$d_2 = d_1 - \sigma\sqrt{T} = -½\sigma\sqrt{T} = -0.18090,\ N(d_2) = 0.42822$$

Price of the exchange option = $2S_1(0)N(d_1) - S_2(0)N(d_2) = 20N(d_1) - 20N(d_2) = 2.8712$

Thus,

$$F_{0,1}^P(M) = 2F_{0,1}^P(S_1) - 2.8712 = 2 \times 10 - 2.8712 = 17.1288$$

and

$$p(17, 1) = 1.632 - 17.1288 + 17e^{-0.05} = 0.6741.$$

Remarks: (i) The exchange option above is an "at-the-money" exchange option because $2S_1(0) = S_2(0)$. See also Example 14.3 of McDonald (2013).

(ii) Further discussion on exchange options can be found in Section 23.6, which is not part of the MFE syllabus. Q and S in Section 23.6 correspond to $2S_1$ and S_2 in this problem.

55. Assume the Black-Scholes framework. Consider a 9-month at-the-money European put option on a futures contract. You are given:

(i) The continuously compounded risk-free interest rate is 10%.

(ii) The strike price of the option is 20.

(iii) The price of the put option is 1.625.

If three months later the futures price is 17.7, what is the price of the put option at that time?

(A) 2.09

(B) 2.25

(C) 2.45

(D) 2.66

(E) 2.83

Solution to (55) Answer: (D)

By (12.7), the price of the put option is

$$P = e^{-rT}[KN(-d_2) - FN(-d_1)],$$

where $d_1 = \dfrac{\ln(F/K) + ½\sigma^2 T}{\sigma\sqrt{T}}$, and $d_2 = d_1 - \sigma\sqrt{T}$.

With $F = K$, we have $\ln(F / K) = 0$, $d_1 = \dfrac{½\sigma^2 T}{\sigma\sqrt{T}} = ½\sigma\sqrt{T}$, $d_2 = -½\sigma\sqrt{T}$, and

$$P = Fe^{-rT}[N(½\sigma\sqrt{T}) - N(-½\sigma\sqrt{T})] = Fe^{-rT}[2N(½\sigma\sqrt{T}) - 1] .$$

Putting $P = 1.6$, $r = 0.1$, $T = 0.75$, and $F = 20$, we get

$$\begin{aligned}
&1.625 = 20e^{-0.1\times 0.75}[2N(½\sigma\sqrt{0.75}) - 1]\\
&N(½\sigma\sqrt{0.75}) = 0.54379\\
&½\sigma\sqrt{0.75} = 0.10999\\
&\sigma = 0.254011
\end{aligned}$$

After 3 months, we have $F = 17.7$ and $T = 0.5$; hence

$$d_1 = \frac{\ln(F/K) + ½\sigma^2 T}{\sigma\sqrt{T}} = \frac{\ln(17.7/20) + ½\times 0.254^2 \times 0.5}{0.254\sqrt{0.5}} = -0.59040$$

$N(-d_1) = N(-0.59040) = 0.72254$

$d_2 = d_1 - \sigma\sqrt{T} = -0.59040 - 0.254\sqrt{0.5} = -0.77000$

$N(-d_2) = N(0.77000) = 0.77935$

The put price at that time is

$$\begin{aligned}
P &= e^{-rT}[KN(-d_2) - FN(-d_1)]\\
&= e^{-0.1\times 0.5}[20 \times 0.77935 - 17.7 \times 0.72254]\\
&= 2.66158
\end{aligned}$$

Remarks:

(i) A somewhat related problem is #8 in the May 2007 MFE exam. Also see the box on page 282 and the one on page 560 of McDonald (2013).

(ii) For European call and put options on a futures contract with the same exercise date, the call price and put price are the same if and only if both are at-the-money options. The result follows from put-call parity. See the first equation in Table 9.9 on page 287 of McDonald (2013).

(iii) The point above can be generalized. It follows from the identity

$$[S_1(T) - S_2(T)]_+ \;+\; S_2(T) \;=\; [S_2(T) - S_1(T)]_+ \;+\; S_1(T)$$

that

$$F_{0,T}^P((S_1 - S_2)_+) \;+\; F_{0,T}^P(S_2) \;=\; F_{0,T}^P((S_2 - S_1)_+) \;+\; F_{0,T}^P(S_1).$$

(See also formula 9.8 on page 271.) Note that $F_{0,T}^P((S_1 - S_2)_+)$ and $F_{0,T}^P((S_2 - S_1)_+)$ are time-0 prices of exchange options. The two exchange options have the same price if and only if the two prepaid forward prices, $F_{0,T}^P(S_1)$ and $F_{0,T}^P(S_2)$, are the same.

56. DELETED

57. Michael uses the following method to simulate 8 standard normal random variates:

Step 1: Simulate 8 uniform (0, 1) random numbers $U_1, U_2, \ldots, U_8$.

Step 2: Apply the stratified sampling method to the random numbers so that U_i and U_{i+4} are transformed to random numbers V_i and V_{i+4} that are uniformly distributed over the interval $((i-1)/4, i/4)$, $i = 1, 2, 3, 4$. In each of the four quartiles, a smaller value of U results in a smaller value of V.

Step 3: Compute 8 standard normal random variates by $Z_i = N^{-1}(V_i)$, where N^{-1} is the inverse of the cumulative standard normal distribution function.

Michael draws the following 8 uniform (0, 1) random numbers:

i	1	2	3	4	5	6	7	8
U_i	0.4880	0.7894	0.8628	0.4482	0.3172	0.8944	0.5013	0.3015

Find the difference between the largest and the smallest simulated normal random variates.

(A) 0.35

(B) 0.78

(C) 1.30

(D) 1.77

(E) 2.50

Solution to (57) Answer: (E)

The following transformation in McDonald (2013, page 587),

$$\frac{i-1+u_i}{100}, \qquad i = 1, 2, 3, \ldots, 100,$$

is now changed to

$$\frac{i-1+U_{i \text{ or } i+4}}{4}, \quad i = 1, 2, 3, 4.$$

Since the smallest Z comes from the first quartile, it must come from U_1 or U_5.

Since $U_5 < U_1$, we use U_5 to compute the smallest Z:

$$V_5 = \frac{1-1+0.3172}{4} = 0.0793,$$

$$Z_5 = N^{-1}(0.0793) = -1.41.$$

Since the largest Z comes from the fourth quartile, it must come from U_4 and U_8.

Since $U_4 > U_8$, we use U_4 to compute the largest Z:

$$V_4 = \frac{4-1+0.4482}{4} = 0.86205,$$

$$Z_4 = N^{-1}(0.86205) = 1.08958 \approx 1.09.$$

The difference between the largest and the smallest normal random variates is

$$Z_4 - Z_5 = 1.09 - (-1.41) = 2.50.$$

Remark:

The simulated standard normal random variates are as follows:

i	1	2	3	4	5	6	7	8
U_i	0.4880	0.7894	0.8628	0.4482	0.3172	0.8944	0.5013	0.3015
no stratified sampling	–0.030	0.804	1.093	–0.130	–0.476	1.250	0.003	–0.520
V_i	0.1220	0.4474	0.7157	0.8621	0.0793	0.4736	0.6253	0.8254
Z_i	–1.165	–0.132	0.570	1.090	–1.410	–0.066	0.319	0.936

Observe that there is no U in the first quartile, 4 U's in the second quartile, 1 U in the third quartile, and 3 U's in the fourth quartile. Hence, the V's seem to be more uniform.

For Questions **58** and **59**, you are to assume the Black-Scholes framework.

Let $C(K)$ denote the Black-Scholes price for a 3-month K-strike European call option on a nondividend-paying stock.

Let $\hat{C}(K)$ denote the Monte Carlo price for a 3-month K-strike European call option on the stock, calculated by using 5 random 3-month stock prices simulated under the risk-neutral probability measure.

You are to estimate the price of a 3-month 42-strike European call option on the stock using the formula

$$C^*(42) = \hat{C}(42) + \beta[C(40) - \hat{C}(40)],$$

where the coefficient β is such that the variance of $C^*(42)$ is minimized.

You are given:

(i) The continuously compounded risk-free interest rate is 8%.

(ii) $C(40) = 2.7847$.

(iii) Both Monte Carlo prices, $\hat{C}(40)$ and $\hat{C}(42)$, are calculated using the following 5 random 3-month stock prices:

33.29, 37.30, 40.35, 43.65, 48.90

58. Based on the 5 simulated stock prices, estimate β.

(A) Less than 0.75

(B) At least 0.75, but less than 0.8

(C) At least 0.8, but less than 0.85

(D) At least 0.85, but less than 0.9

(E) At least 0.9

59. Based on the 5 simulated stock prices, compute $C^*(42)$.

(A) Less than 1.7

(B) At least 1.7, but less than 1.9

(C) At least 1.9, but less than 2.2

(D) At least 2.2, but less than 2.6

(E) At least 2.6

Solution to (58) Answer: (B)

$$\text{Var}[C^*(42)] = \text{Var}[\hat{C}(42)] + \beta^2\text{Var}[\hat{C}(40)] - 2\beta\text{Cov}[\hat{C}(42), \hat{C}(40)],$$

which is a quadratic polynomial of β. (See also (19.11) in McDonald.) The minimum of the polynomial is attained at

$$\beta = \text{Cov}[\hat{C}(40), \hat{C}(42)]/\text{Var}[\hat{C}(40)].$$

For a pair of random variables X and Y, we estimate the ratio, Cov[X, Y]/Var[X], using the formula

$$\frac{\sum_{i=1}^{n}(X_i - \bar{X})(Y_i - \bar{Y})}{\sum_{i=1}^{n}(X_i - \bar{X})^2} = \frac{\sum_{i=1}^{n} X_i Y_i - n\bar{X}\bar{Y}}{\sum_{i=1}^{n} X_i^2 - n\bar{X}^2}.$$

We now treat the payoff of the 40-strike option (whose correct price, $C(40)$, is known) as X, and the payoff of the 42-strike option as Y. We do not need to discount the payoffs because the effect of discounting is canceled in the formula above.

Simulated $S(0.25)$	max($S(0.25) - 40$, 0)	max($S(0.25) - 42$, 0)
33.29	0	0
37.30	0	0
40.35	0.35	0
43.65	3.65	1.65
48.90	8.9	6.9

We have $\bar{X} = \frac{0.35 + 3.65 + 8.9}{5} = 2.58$, $\bar{Y} = \frac{1.65 + 6.9}{5} = 1.71$,

$\sum_{i=1}^{n} X_i^2 = 0.35^2 + 3.65^2 + 8.9^2 = 92.655$, and $\sum_{i=1}^{n} X_i Y_i = 3.65 \times 1.65 + 8.9 \times 6.9 = 67.4325$.

So, the estimate for the minimum-variance coefficient β is

$$\frac{67.4325 - 5 \times 2.58 \times 1.71}{92.655 - 5 \times 2.58^2} = 0.764211.$$

Remark: The estimate for the minimum-variance coefficient β can be obtained by using the statistics mode of a scientific calculator very easily. In the following we use TI–30X IIB as an illustration.

Step 1: Press [2nd][DATA] and select "2-VAR".

Step 2: Enter the five data points by the following keystroke:

[ENTER][DATA] 0 ↓ 0 ↓ 0 ↓ 0 ↓ 0.35 ↓ 0 ↓ 3.65 ↓ 1.65 ↓ 8.9 ↓ 6.9 [Enter]

Step 3: Press [STATVAR] and look for the value of "a".

Step 4: Press [2nd][STATVAR] and select "Y" to exit the statistics mode.

You can also find $\overline{X}$, $\overline{Y}$, $\sum_{i=1}^{n} X_i Y_i$, $\sum_{i=1}^{n} X_i^2$ etc in [STATVAR] too.

Below are keystrokes for TI–30XS multiview

Step 1: Enter the five data points by the following keystrokes:

[DATA] 0 ↓ 0 ↓ 0.35 ↓ 3.65 ↓ 8.9 ↓ → 0 ↓ 0 ↓ 0 ↓ 1.65 ↓ 6.9 [Enter]

Step 2: Press [2nd][STAT] and select "2-VAR".

Step 3: Select L1 and L2 for *x* and *y* data. Then select Calc and [ENTER]

Step 4: Look for the value of "a" by scrolling down.

Solution to (59) Answer: (B)

The plain-vanilla Monte Carlo estimates of the two call option prices are:

$$\text{For } K = 40\text{: } e^{-0.08 \times 0.25} \times \frac{0.35 + 3.65 + 8.9}{5} = 2.528913$$

$$\text{For } K = 42\text{: } e^{-0.08 \times 0.25} \times \frac{1.65 + 6.9}{5} = 1.676140$$

The minimum-variance control variate estimate is

$$\begin{aligned} C^*(42) &= \hat{C}(42) + \beta[C(40) - \hat{C}(40)] \\ &= 1.6761 + 0.764211 \times (2.7847 - 2.5289) \\ &= 1.872. \end{aligned}$$

60-74. DELETED

75. You are using Monte Carlo simulation to estimate the price of an option *X*, for which there is no pricing formula. To reduce the variance of the estimate, you use the control variate method with another option *Y*, which has a pricing formula.

You are given:

(i) The naive Monte Carlo estimate of the price of *X* has standard deviation 5.

(ii) The same Monte Carlo trials are used to estimate the price of *Y*.

(iii) The correlation coefficient between the estimated price of *X* and that of *Y* is 0.8.

Calculate the minimum variance of the estimated price of *X*, with *Y* being the control variate.

(A) 1.0

(B) 1.8

(C) 4.0

(D) 9.0

(E) 16.0

Solution to (75) Answer: (D)

Following (19.9), we let $X^* = X_{\text{sim}} + \beta\,(Y_{\text{true}} - Y_{\text{sim}})$. Its variance is

$$\text{Var}(X^*) = \text{Var}(X_{\text{sim}}) + \beta^2\,\text{Var}(Y_{\text{sim}}) - 2\beta\,\text{Cov}(X_{\text{sim}}, Y_{\text{sim}}),$$

which is (19.10).

To find the optimal β, differentiate the RHS with respect to β and equate the derivative to 0. The solution of the resulting equation is the optimal β, which is

$$\frac{\text{Cov}(X_{\text{sim}}, Y_{\text{sim}})}{\text{Var}(Y_{\text{sim}})},$$

a result that can be found on page 585 in McDonald (2013).

Thus, that the minimum of Var(X^*) is

$$\text{Var}(X_{\text{sim}}) + \left[\frac{\text{Cov}(X_{\text{sim}}, Y_{\text{sim}})}{\text{Var}(Y_{\text{sim}})}\right]^2 \text{Var}(Y_{\text{sim}}) - 2\frac{\text{Cov}(X_{\text{sim}}, Y_{\text{sim}})}{\text{Var}(Y_{\text{sim}})}\text{Cov}(X_{\text{sim}}, Y_{\text{sim}})$$

$$= \text{Var}(X_{\text{sim}}) - \frac{[\text{Cov}(X_{\text{sim}}, Y_{\text{sim}})]^2}{\text{Var}(Y_{\text{sim}})}$$

$$= \text{Var}(X_{\text{sim}}) \left\{1 - \frac{[\text{Cov}(X_{\text{sim}}, Y_{\text{sim}})]^2}{\text{Var}(X_{\text{sim}})\text{Var}(Y_{\text{sim}})}\right\}$$

$$= \text{Var}(X_{\text{sim}})\{1 - [\text{Corr}(X_{\text{sim}}, Y_{\text{sim}})]^2\}$$

$$= 5^2(1 - 0.8^2)$$

$$= 9.$$

Remarks: (i) For students who have learned the concept of *inner product* (*scalar product* or *dot product*), here is a way to view the problem. Given two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, we want to minimize the length $\|\boldsymbol{x} - \beta\boldsymbol{y}\|$ by varying β. To find the optimal β, we differentiate $\|\boldsymbol{x} - \beta\boldsymbol{y}\|^2$ with respect to β and equate the derivative with 0. The optimal β is $\langle\boldsymbol{x}, \boldsymbol{y}\rangle/\|\boldsymbol{y}\|^2$. Hence,

$$\underset{\beta}{\text{Minimum}}\,\|\boldsymbol{x} - \beta\boldsymbol{y}\|^2 = \|\boldsymbol{x} - \frac{\langle\boldsymbol{x}, \boldsymbol{y}\rangle}{\|\boldsymbol{y}\|^2}\boldsymbol{y}\|^2 = \|\boldsymbol{x}\|^2\left(1 - \frac{\langle\boldsymbol{x}, \boldsymbol{y}\rangle^2}{\|\boldsymbol{x}\|^2\|\boldsymbol{y}\|^2}\right).$$

(ii) The quantity $\dfrac{\langle\boldsymbol{x}, \boldsymbol{y}\rangle}{\|\boldsymbol{x}\|\cdot\|\boldsymbol{y}\|}$ is the cosine of the angle between the vectors $\boldsymbol{x}$ and $\boldsymbol{y}$.

(iii) A related formula is (4.4) in McDonald (2013).

76. You are given the following information about a Black-Derman-Toy binomial tree modeling the movements of effective annual interest rates:

(i) The length of each period is one year.

(ii) In the first year, $r_0 = 9\%$.

(iii) In second year, $r_u = 12.6\%$ and $r_d = 9.3\%$

(iv) In third year, $r_{uu} = 17.2\%$ and $r_{dd} = 10.6\%$. The value of r_{ud} is not provided.

Calculate the price of a 3-year interest-rate cap for notional amount 10,000 and cap rate 11.5%.

(A) 202

(B) 207

(C) 212

(D) 217

(E) 222

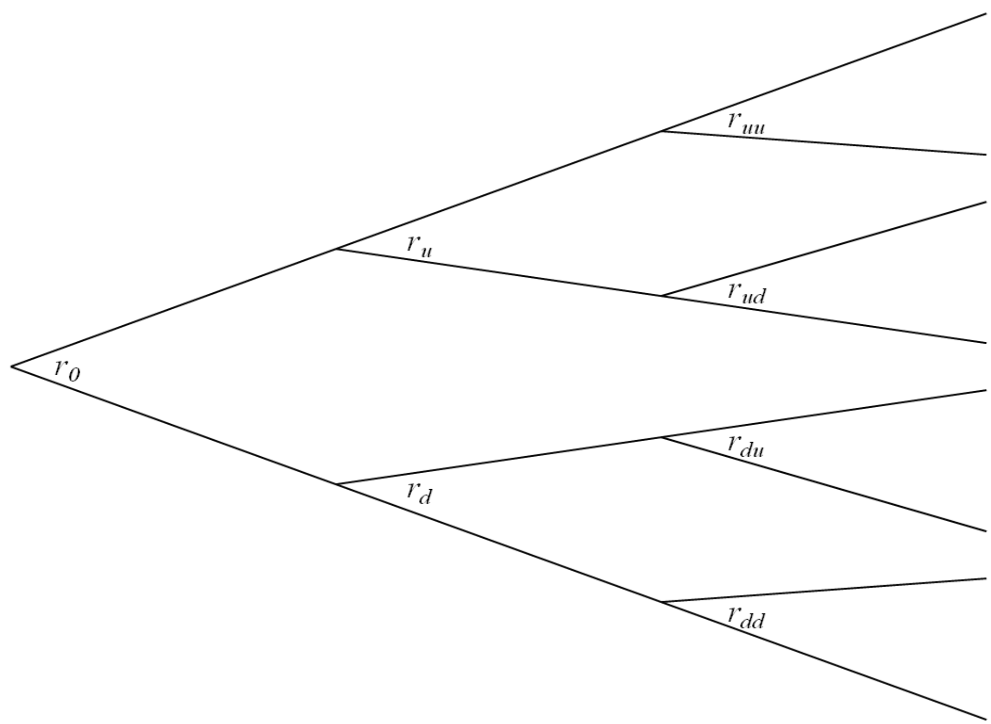
Solution to (76) Answer: (D)

A related problem is #15 in this set of sample questions and solutions.

Caps are usually defined so that the initial rate, r_0, even if it is greater than the cap rate, does not lead to a payoff, i.e., the year-1 caplet is disregarded. In any case, the year-1 caplet in this problem has zero value because r_0 is lower than the cap rate.

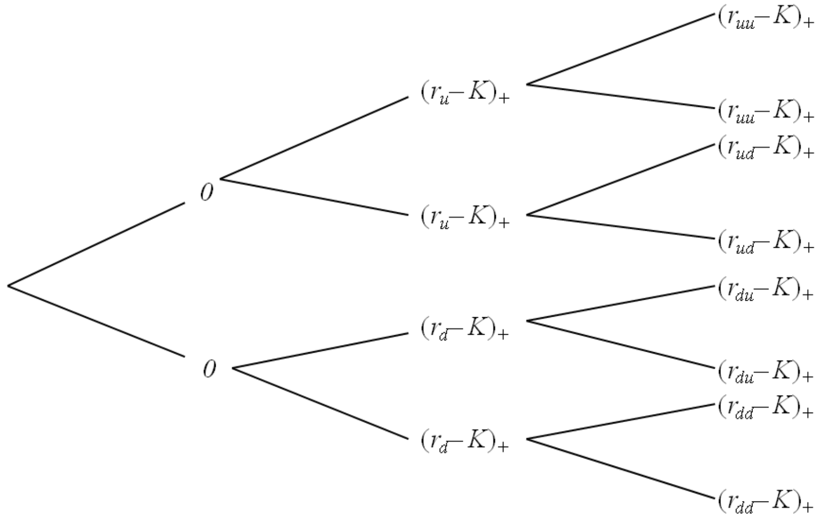
Since a 3-year cap is the sum of a year-2 caplet and a year-3 caplet, one way to price a 3-year cap is to price each of the two caplets and then add up the two prices. However, because the payoffs or cashflows of a cap are not path-dependent, the method of *backward induction* can be applied, which is what we do next.

It seems more instructive if we do not assume that the binomial tree is recombining, i.e., we do not assume $r_{ud} = r_{du}$. Thus we have the following three-period (three-year) interest rate tree.

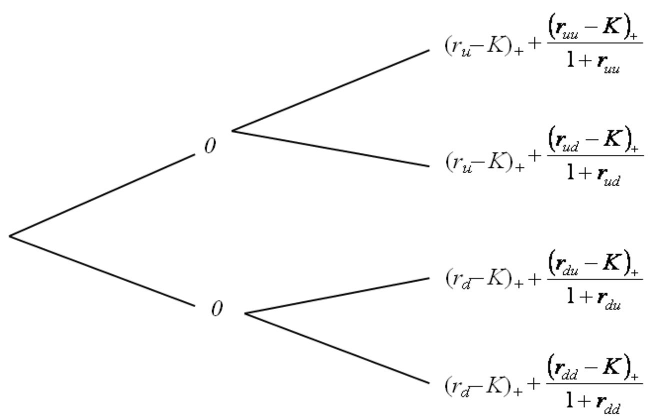


We also do not assume the risk-neutral probabilities to be ½ and ½. We use p^* to denote the risk-neutral probability of an up move, and q^* the probability of a down move.

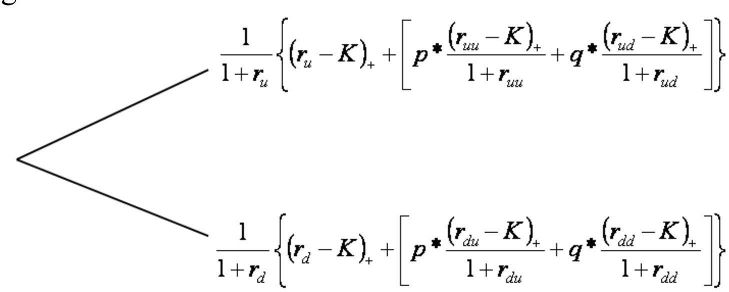
In the next diagram, we show the payoffs or cashflows of a 3-year interest-rate cap for notional amount 1 and cap rate K. Here, $(r - K)_+$ means $\max(0, r - K)$.



Discounting and averaging the cashflows at time 3 back to time 2:



Moving back from time 2 back to time 1:



Finally, we have the time-0 price of the 3-year interest-rate cap for notional amount 1 and cap rate K:

$$\frac{1}{1+r_0}\left\{p^*\frac{1}{1+r_u}\left[(r_u-K)_+ + p^*\frac{(r_{uu}-K)_+}{1+r_{uu}} + q^*\frac{(r_{ud}-K)_+}{1+r_{ud}}\right]\right.$$
$$\left. + q^*\frac{1}{1+r_d}\left[(r_d-K)_+ + p^*\frac{(r_{du}-K)_+}{1+r_{du}} + q^*\frac{(r_{dd}-K)_+}{1+r_{dd}}\right]\right\}. \qquad (1)$$

As we mentioned earlier, the price of a cap can also be calculated as the sum of caplet prices. The time-0 price of a year-2 caplet is

$$\frac{1}{1+r_0}\left[p^*\frac{(r_u-K)_+}{1+r_u} + q^*\frac{(r_d-K)_+}{1+r_d}\right].$$

The time-0 price of a year-3 caplet is

$$\frac{1}{1+r_0}\left\{p^*\frac{1}{1+r_u}\left[p^*\frac{(r_{uu}-K)_+}{1+r_{uu}} + q^*\frac{(r_{ud}-K)_+}{1+r_{ud}}\right] + q^*\frac{1}{1+r_d}\left[p^*\frac{(r_{du}-K)_+}{1+r_{du}} + q^*\frac{(r_{dd}-K)_+}{1+r_{dd}}\right]\right\}. \quad (2)$$

It is easy to check that the sum of these two caplet pricing formulas is the same as expression (1).

Rewriting expression (2) as

$$(p^*)^2\frac{(r_{uu}-K)_+}{(1+r_0)(1+r_u)(1+r_{uu})} + p^*q^*\frac{(r_{ud}-K)_+}{(1+r_0)(1+r_u)(1+r_{ud})}$$
$$+q^*p^*\frac{(r_{du}-K)_+}{(1+r_0)(1+r_d)(1+r_{du})} + (q^*)^2\frac{(r_{du}-K)_+}{(1+r_0)(1+r_d)(1+r_{dd})}$$

shows the path-by-path nature of the year-3 caplet price.

A Black-Derman-Toy interest rate tree is a recombining tree (hence $r_{ud} = r_{du}$) with $p^* = q^* = ½$. Expression (1) simplifies as

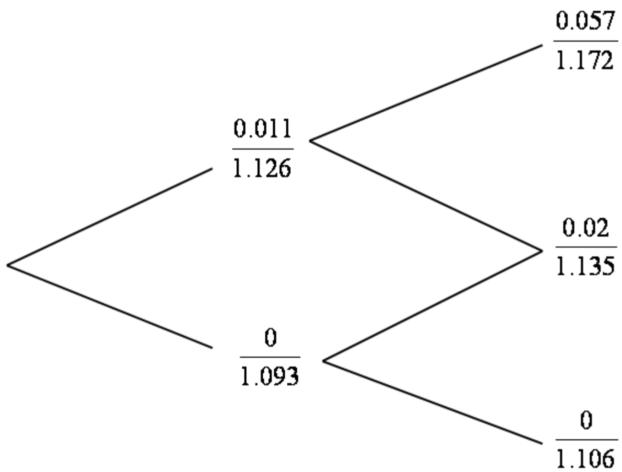
$$\frac{1}{2}\frac{1}{1+r_0}\left\{\frac{1}{1+r_u}\left[(r_u-K)_+ + \frac{1}{2}\left(\frac{(r_{uu}-K)_+}{1+r_{uu}} + \frac{(r_{ud}-K)_+}{1+r_{ud}}\right)\right]\right.$$
$$\left. + \frac{1}{1+r_d}\left[(r_d-K)_+ + \frac{1}{2}\left(\frac{(r_{ud}-K)_+}{1+r_{ud}} + \frac{(r_{dd}-K)_+}{1+r_{dd}}\right)\right]\right\}. \qquad (3)$$

In this problem, the value of r_{ud} is not given. In each period of a B-D-T model, the interest rates in different states are terms of a geometric progression. Thus, we have

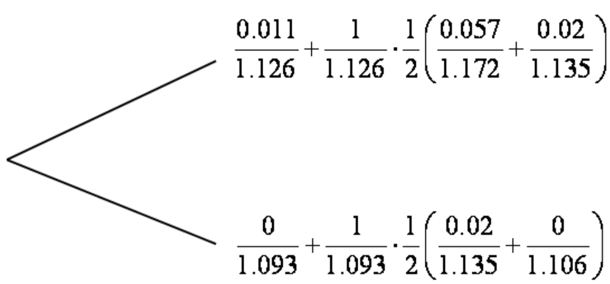
$$0.172/r_{ud} = r_{ud}/0.106,$$

from which we obtain $r_{ud} = 0.135$. With this value, we can price the cap using (3).

Instead of using (3), we now solve the problem directly. As in Figure 25.6 on page 773 of McDonald (2013), we first discount each cap payment to the beginning of the payment year. The following tree is for notational amount of 1 (and $K = 0.115$).



Discounting and averaging the cashflows at time 2 back to time 1:



Thus the time-0 price of the cap is

$$\frac{1}{1.09}\cdot\frac{1}{2}\left\{\frac{1}{1.126}\left[0.011+\frac{1}{2}\left(\frac{0.057}{1.172}+\frac{0.02}{1.135}\right)\right] + \frac{1}{1.093}\left[0+\frac{1}{2}\left(\frac{0.02}{1.135}+\frac{0}{1.106}\right)\right]\right\}$$

$$=\frac{1}{1.09}\cdot\frac{1}{2}\left\{\frac{1}{1.126}\times 0.044128 + \frac{1}{1.093}\times 0.008811\right\}$$

$$= 0.02167474.$$

Answer (D) is correct, because the notation amount is 10,000.

Remark: The prices of the two caplets for notional amount 10,000 and $K = 0.115$ are 44.81 and 171.94. The sum of these two prices is 216.75.