ACTUARIAL RESEARCH CLEARING HOUSE
1980 VOL. 1

AN APPROXIMATE SOLUTION FOR THE UNKNOWN RATE OF INTEREST

For An Annuity Certain

Dr. Murray Silver
Department of Insurance and Risk
Temple University
Philadelphia, Pa. 19122

ABSTRACT

If $a_{\overline{n}|} = k$, find the unknown rate of interest. This paper derives the approximate solutions $i = \dfrac{1 - (\frac{k}{n})^2}{k}$ and $i = \dfrac{\left(1 + \frac{1 - (\frac{k}{n})^2}{k}\right)^{n+1} - \left(\frac{1 - (\frac{k}{n})^2}{k}\right)(n+1) - 1}{k\left(1 + \frac{1 - (\frac{k}{n})^2}{n}\right)^{n+1} - n}$.

INTRODUCTION:

On page 60 of [1], Kellison considers the problem of finding the unknown rate of interest determined by the equation $a_{\overline{n}|} = k$. Three different solutions are offered: Interpolation in the tables

Treating $a_{\overline{n}|} = v + v^2 + \ldots v^n = k$ as an n^{th} degree polynomial whose roots are to be found

Iteration by means of $i = \dfrac{1 - (1+i)^{-n}}{k}$

The purpose of this paper is to derive approximate analytical formulae and discuss their accuracy.

Derivation

$a_{\overline{n}|} = v + v^2 + \ldots v^n = k$. If we multiply by $(1 - v)$ and rearrange, we obtain

$$f(v) = v^{n+1} - (1+k)\, v + k \qquad (1)$$

$$f'(v) = (n+1)\, v^n - (1+k) \qquad (2)$$

$$f''(v) = (n+1)\, n\, v^{n-1} > 0 \text{ for } v > 0 \qquad (3)$$

From elementary calculus, the graph of this function on the interval (0,1) is easily obtained. The following facts about the graph on (0,1) are stated without proof: the graph is concave upwards.

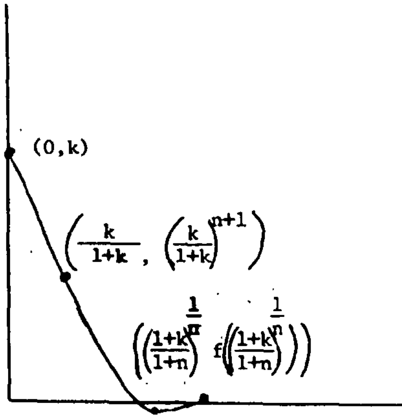
The graph passes through the points (o,k), $\left(\frac{k}{1+k}, \left(\frac{k}{1+k}\right)^{n+1}\right)$ and $(1,0)$.

The graph possesses exactly one minimum at $\left(\frac{1+k}{1+n}\right)^{\frac{1}{n}}$

Between 0 and 1, the graph possesses exactly one root r.

$f(\nu) > 0$ for $0 < \nu < r$ and $f(\nu) < 0$ for $r < \nu < 1$. $\frac{k}{1+k} < r < \left(\frac{1+k}{1+n}\right)^{\frac{1}{n}}$

The graph is shown in the figure.



FIGURE

We now derive our first estimate for r.

$$\frac{k}{n} = \frac{r + r^2 + r^3 + \ldots r^n}{n} > \sqrt[n]{r^{\frac{n(n+1)}{2}}}$$

by the classical inequality on the arithmetic and geometric means.

Hence, $\left(\frac{k}{n}\right)^{\frac{2}{n+1}} > r.$ (4)

Since $i > o$, $k < n$ and

$k + k\,n < n + k\,n$ or $\frac{k}{n} < \frac{1+k}{1+n}$. Whence,

$\left(\frac{k}{n}\right)^{2n} < \left(\frac{1+k}{1+n}\right)^{n+1}$ or $\left(\frac{k}{n}\right)^{\frac{2}{n+1}} < \left(\frac{1+k}{1+n}\right)^{\frac{1}{n}}$. This inequality together with

(4) gives: $r < \left(\frac{k}{n}\right)^{\frac{2}{n+1}} < \left(\frac{1+k}{1+n}\right)^{\frac{1}{n}}$. (5)

Since $\left(\frac{k}{n}\right)^{\frac{2}{n+1}} > r$ by (4), we shall choose the slightly smaller quantity $\left(\frac{k}{n}\right)^{\frac{2}{n}}$ as our first approximation for r. The values of these two quantities are tabulated in table I. These values are most easily checked by converting the approximate values of ν to i. The exact solution is given in column 1 and the approximate values of i are shown in columns 3 and 5. Table I shows that $\left(\frac{k}{n}\right)^{2n}$ gives slightly more accurate values in the range of interest of most financial problems. More important, the subsequent expressions become algebrically simpler with this choice. However, for $n < 9$, this is no longer true, because $\left(\frac{k}{n}\right)^{\frac{2}{n+1}}$ cannot be replaced by $\left(\frac{k}{n}\right)^{\frac{2}{n}}$ without an excessive loss of accuracy. We therefore assume $n > 9$.

In order to improve this estimate for the root, we shall iterate using the formula $i = \frac{1 - \nu^n}{k}$ (6) (see [1], page 61). Let $\left(\frac{k}{n}\right)^{\frac{2}{n}}$ be the starting value; one iteration produces $i = \dfrac{1 - \left(\frac{k}{n}\right)^2}{k}$ (7)

These values are tabulated in table II, column 2 and agree reasonably well with the exact values of column 1. Except for the first two values, the relative error is less than .05 for all tabulated values. If all calculated values are rounded to two decimal places, formula (7) is accurate to two decimal places in all tabulated values except for $n = 20$ and $i > .13$. This formula is easily calculated by hand and is sufficient for many purposes. If a high degree of accuracy is required, formula (7) is an excellent starting value for any of the standard iterative techniques.

The obvious extension of formula (7) is to perform a second iteration. This yields:

$$i = \frac{1 - \left(1 + \left(\frac{1 - \left(\frac{k}{n}\right)^2}{k}\right)\right)^{-n}}{k} \qquad (8)$$

Formula (8) is tabulated in column 3 of table II. When rounded off, this estimate is accurate to two decimal places in every tabulated value. While this slight improvement is disappointing, it is easily anticipated. Following Kellison [2] page 254, the rate of convergence is determined by

$\frac{d}{di}\left(\frac{1 - \nu^n}{k}\right) = \frac{n}{k}\ \nu^{n+1} \doteq \frac{k}{n}$ where $\nu \doteq \left(\frac{k}{n}\right)^{\frac{2}{n+1}}$. Since $\frac{k}{n}$ approaches 1 for small i, the rate of convergence must, in general, be slow.

We shall now use formula (7) as the starting value for the Newton-Raphson Technique. If we apply this method to equation (1), using in addition equation (2):

$$\nu_2 = \nu_1 - \frac{\nu_1^{n+1} - (1+k)\nu_1 + k}{(n+1)\nu_1^n - (1+k)} = \frac{n\,\nu_1^{n+1} - k}{(n+1)\nu_1^n - (1+k)}$$

ν can be eliminated in favor of i:

$$i = \frac{(1+i_1)^{n+1} - i_1\,(n+1) - 1}{k\,(1+i_1)^{n+1} - n} \qquad (9)$$

Finally, insert formula (7) into equation (9):

$$i = \frac{\left(1 + \frac{1 - \left(\frac{k}{n}\right)^2}{k}\right)^{n+1} - (n+1)\left(\frac{1 - \left(\frac{k}{n}\right)^2}{k}\right) - 1}{k\left(1 + \frac{1 - \left(\frac{k}{n}\right)^2}{k}\right)^{n+1} - n} \qquad (10)$$

The values given by formula (10) are tabulated in column 4 of table II. It should be noted that all values are slightly larger than the true value. That this must be the case, is now demonstrated. By (3), the graph of (1) on (0,1) is a convex curve; thus, the graph always lies to one side of any tangent line. By (5) and the fact that (7) lies even closer to the root,

$$\frac{1 - \left(\frac{k}{n}\right)^2}{k} < \left(\frac{1+k}{1+n}\right)^{\frac{1}{n}}$$. Therefore, the tangent line of the Newton-Raphson method always lies under the curve and must cross the x axis to the left of the true root. Hence, the estimate for ν is always too small and the estimate for i too large. In recognition of this fact, we must not round off

the results of (10); we must adjust downwards. This is most simply achieved by truncation. We shall therefore, agree to keep the first four decimal places and truncate everything thereafter. If we apply this scheme to column 4 of table II, all tabulated results are accurate to four decimal places. In summary, formula (10) represents an approximate analytical solution to the problem of this paper for $n \geqslant 9$, if the first four digits are taken without rounding off. If needed, further accuracy is quickly obtained by using equation (9) (Newton-Raphson) iteratively.

REFERENCES

[1] Kellison, S. G. (1970) The Theory of Interest, Richard D. Irwin

[2] Kellison, S. G. (1975) Fundamentals of Numerical Analysis, Richard D. Irwin

TABLE I

n = 10 i	$\left(\frac{k}{n}\right)^{\frac{2}{n}} \doteq r$	$\left(\frac{k}{r}\right)^{\frac{-2}{n}} - 1 \doteq i$	$\left(\frac{k}{n}\right)^{\frac{2}{n+1}} \doteq r$	$\left(\frac{k}{n}\right)^{\frac{-2}{n+1}} - 1 \doteq i$
.01	.989	.011	.990	.010
.03	.969	.032	.972	.029
.07	.932	.073	.938	.066
.09	.915	.093	.923	.084
.11	.900	.112	.908	.101
.15	.871	.148	.882	.134
.21	.858	.165	.870	.149
n = 20				
.01	.990	.010	.990	.001
.03	.971	.030	.972	.029
.07	.938	.066	.941	.062
.09	.925	.082	.928	.078
.11	.912	.096	.916	.092
.15	.890	.123	.895	.117
.21	.864	.157	.870	.149
n = 30				
.01	.990	.010	.990	.010
.03	.972	.029	.973	.028
.07	.943	.061	.945	.059
.09	.931	.074	.933	.072
.11	.921	.086	.923	.083
.15	.904	.107	.907	.103
.21	.884	.131	.888	.126
n = 50 i				
.01	.990	.001	.990	.001
.03	.974	.027	.974	.026
.07	.950	.053	.951	.052
.09	.941	.063	.942	.061
.11	.934	.071	.935	.069
.15	.923	.084	.924	.082
.21	.910	.099	.912	.097

TABLE II

i	$\frac{1-\left(\frac{k}{n}\right)^2}{k}$	$\frac{1-\left(1+\left(1-\left(\frac{k}{n}\right)^2\right)\right)^{-n}}{k}$	$\frac{\left(\frac{1-\left(\frac{k}{n}\right)^2}{k}\right)^{n+1}-\left(\frac{1-\left(\frac{k}{n}\right)^2}{k}\right)-1}{k\left(1+\left(\frac{1-\left(\frac{k}{n}\right)^2}{k}\right)\right)^{n+1}-n}$
n = 10			
.01	.01087	.01082	.010062
.03	.03193	.03162	.030097
.05	.05229	.05171	.050078
.07	.07214	.07143	.070046
.09	.09164	.09098	.090020
.11	.11091	.11049	.110005
.13	.13003	.13001	.130000
.15	.14906	.14960	.150003
.17	.16808	.16926	.170011
.19	.18708	.18900	.190021
.21	.20612	.20881	.210031
n = 20			
.01	.01030	.01027	.010008
.03	.03002	.03002	.030000
.05	.04909	.04947	.050012
.07	.06791	.06903	.070038
.09	.08673	.08879	.090063
.11	.10567	.10873	.110076
.13	.12479	.12880	.130078
.15	.14411	.14895	.150071
.17	.16362	.16911	.170061
.19	.18329	.18928	.190049
.21	.20310	.20943	.210039
n = 30			
.01	.01007	.01006	.010000
.03	.02924	.02953	.030014
.05	.04798	.04910	.050050
.07	.06680	.06900	.070070
.09	.08592	.08913	.090067
.11	.10537	.10933	.110053
.13	.12508	.12952	.130037
.15	.14501	.14968	.150024
.17	.16507	.16979	.170015
.19	.18522	.18987	.190009
.21	.20542	.20992	.210005

TABLE II CONTINUED

n = 40			
.01	.00993	.00995	.010000
.03	.02882	.02938	.030032
.05	.04755	.04919	.050060
.07	.06668	.06934	.070055
.09	.08624	.08956	.090037
.11	.10612	.10974	.110020
.13	.12622	.12986	.130010
.15	.14641	.14993	.150005
.17	.16665	.16996	.170002
.19	.18689	.18998	.190001
.21	.20713	.20999	.210001
n = 60			
.01	.00976	.00982	.010005
.03	.02845	.02942	.030045
.05	.04758	.04958	.050038
.07	.06733	.06980	.070017
.09	.08745	.08992	.090006
.11	.10769	.10997	.110002
.13	.12795	.12999	.130001
.15	.14818	.15000	.150000
.17	.16838	.17000	.170000
.19	.18854	.19000	.190000
.21	.20868	.21000	.210000