Tables for Exam STAM

The reading material for Exam STAM includes a variety of textbooks. Each text has a set of probability distributions that are used in its readings. For those distributions used in more than one text, the choices of parameterization may not be the same in all of the books. This may be of educational value while you study, but could add a layer of uncertainty in the examination. For this latter reason, we have adopted one set of parameterizations to be used in examinations. This set will be based on Appendices A & B of *Loss Models: From Data to Decisions* by Klugman, Panjer and Willmot. A slightly revised version of these appendices is included in this note. A copy of this note will also be distributed to each candidate at the examination.

Each text also has its own system of dedicated notation and terminology. Sometimes these may conflict. If alternative meanings could apply in an examination question, the symbols will be defined.

For Exam STAM, in addition to the abridged table from *Loss Models*, sets of values from the standard normal and chi-square distributions will be available for use in examinations. These are also included in this note.

When using the normal distribution, choose the nearest z-value to find the probability, or if the probability is given, choose the nearest z-value. No interpolation should be used.

> Example: If the given z-value is 0.759, and you need to find $\Pr(Z < 0.759)$ from the normal distribution table, then choose the probability for z-value = 0.76: $\Pr(Z < 0.76) = 0.7764$.

When using the normal approximation to a discrete distribution, use the continuity correction.

The density function for the standard normal distribution is $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$.

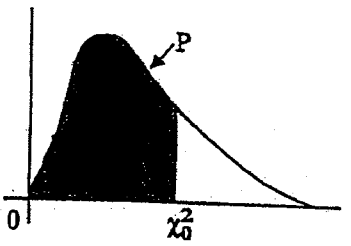
NORMAL DISTRIBUTION TABLE

Entries represent the area under the standardized normal distribution from $-\infty$ to z, Pr(Z<z)
The value of z to the first decimal is given in the left column. The second decimal place is given in the top row.

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Values of z for selected values of Pr(Z<z)							
z	0.842	1.036	1.282	1.645	1.960	2.326	2.576
Pr(Z<z)	0.800	0.850	0.900	0.950	0.975	0.990	0.995

χ^2 Distribution



The table below gives the value x_0^2 for which $P[x^2 < x_0^2] =$ P for a given number of degrees of freedom and a given value of P.

Degrees of Freedom	Values of P									
	0.005	**0.010**	**0.025**	**0.050**	**0.100**	**0.900**	**0.950**	**0.975**	**0.990**	**0.995**
1	---	---	0.001	0.004	0.016	2.706	3.841	5.024	6.635	7.879
2	0.01	0.020	0.051	0.103	0.211	4.605	5.991	7.378	9.210	10.597
3	0.072	0.115	0.216	0.352	0.584	6.251	7.815	9.348	11.345	12.838
4	0.207	0.297	0.484	0.711	1.064	7.779	9.488	11.143	13.277	14.860
5	0.412	0.554	0.831	1.145	1.610	9.236	11.070	12.833	15.086	16.750
6	0.676	0.872	1.237	1.635	2.204	10.645	12.592	14.449	16.812	18.548
7	0.989	1.239	1.690	2.167	2.833	12.017	14.067	16.013	18.475	20.278
8	1.344	1.646	2.180	2.733	3.490	13.362	15.507	17.535	20.090	21.955
9	1.735	2.088	2.700	3.325	4.168	14.684	16.919	19.023	21.666	23.589
10	2.156	2.558	3.247	3.940	4.865	15.987	18.307	20.483	23.209	25.188
11	2.603	3.053	3.816	4.575	5.578	17.275	19.675	21.920	24.725	26.757
12	3.074	3.571	4.404	5.226	6.304	18.549	21.026	23.337	26.217	28.300
13	3.565	4.107	5.009	5.892	7.042	19.812	22.362	24.736	27.688	29.819
14	4.075	4.660	5.629	6.571	7.790	21.064	23.685	26.119	29.141	31.319
15	4.601	5.229	6.262	7.261	8.547	22.307	24.996	27.488	30.578	32.801
16	5.142	5.812	6.908	7.962	9.312	23.542	26.296	28.845	32.000	34.267
17	5.697	6.408	7.564	8.672	10.085	24.769	27.587	30.191	33.409	35.718
18	6.265	7.015	8.231	9.390	10.865	25.989	28.869	31.526	34.805	37.156
19	6.844	7.633	8.907	10.117	11.651	27.204	30.144	32.852	36.191	38.582
20	7.434	8.260	9.591	10.851	12.443	28.412	31.410	34.170	37.566	39.997

Excerpts from the Appendices to *Loss Models: From Data to Decisions, 5th edition*

September 12, 2018

Appendix A

An Inventory of Continuous Distributions

A.1 Introduction

The incomplete gamma function is given by

$$\Gamma(\alpha; x) = \frac{1}{\Gamma(\alpha)} \int_0^x t^{\alpha-1} e^{-t}\, dt, \quad \alpha > 0,\ x > 0,$$

with

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\, dt, \quad \alpha > 0.$$

Also, define

$$G(\alpha; x) = \int_x^\infty t^{\alpha-1} e^{-t}\, dt, \quad x > 0.$$

At times we will need this integral for nonpositive values of α. Integration by parts produces the relationship

$$G(\alpha; x) = -\frac{x^\alpha e^{-x}}{\alpha} + \frac{1}{\alpha} G(\alpha + 1; x).$$

This process can be repeated until the first argument of G is $\alpha + k$, a positive number. Then it can be evaluated from

$$G(\alpha + k; x) = \Gamma(\alpha + k)[1 - \Gamma(\alpha + k; x)].$$

The incomplete beta function is given by

$$\beta(a, b; x) = \frac{\Gamma(a + b)}{\Gamma(a)\Gamma(b)} \int_0^x t^{a-1}(1 - t)^{b-1}\, dt, \quad a > 0,\ b > 0,\ 0 < x < 1.$$

A.2 Transformed Beta Family

A.2.2 Three-Parameter Distributions

A.2.2.1 Generalized Pareto—α, θ, τ

$$
\begin{aligned}
f(x) &= \frac{\Gamma(\alpha+\tau)}{\Gamma(\alpha)\Gamma(\tau)} \frac{\theta^\alpha x^{\tau-1}}{(x+\theta)^{\alpha+\tau}}, \qquad F(x) = \beta(\tau, \alpha; u), \quad u = \frac{x}{x+\theta}, \\
\mathrm{E}[X^k] &= \frac{\theta^k \Gamma(\tau+k)\Gamma(\alpha-k)}{\Gamma(\alpha)\Gamma(\tau)}, \quad -\tau < k < \alpha, \\
\mathrm{E}[X^k] &= \frac{\theta^k \tau(\tau+1)\cdots(\tau+k-1)}{(\alpha-1)\cdots(\alpha-k)} \quad \text{if } k \text{ is a positive integer}, \\
\mathrm{E}[(X \wedge x)^k] &= \frac{\theta^k \Gamma(\tau+k)\Gamma(\alpha-k)}{\Gamma(\alpha)\Gamma(\tau)} \beta(\tau+k, \alpha-k; u) + x^k[1-F(x)], \quad k > -\tau, \\
\text{Mode} &= \theta \frac{\tau-1}{\alpha+1}, \quad \tau > 1, \text{ else } 0.
\end{aligned}
$$

A.2.2.2 Burr—α, θ, γ

(Burr Type XII, Singh–Maddala)

$$
\begin{aligned}
f(x) &= \frac{\alpha\gamma(x/\theta)^\gamma}{x[1+(x/\theta)^\gamma]^{\alpha+1}}, \qquad F(x) = 1-u^\alpha, \quad u = \frac{1}{1+(x/\theta)^\gamma}, \\
\mathrm{VaR}_p(X) &= \theta[(1-p)^{-1/\alpha} - 1]^{1/\gamma}, \\
\mathrm{E}[X^k] &= \frac{\theta^k \Gamma(1+k/\gamma)\Gamma(\alpha-k/\gamma)}{\Gamma(\alpha)}, \quad -\gamma < k < \alpha\gamma, \\
\mathrm{E}[(X \wedge x)^k] &= \frac{\theta^k \Gamma(1+k/\gamma)\Gamma(\alpha-k/\gamma)}{\Gamma(\alpha)} \beta(1+k/\gamma, \alpha-k/\gamma; 1-u) + x^k u^\alpha, \quad k > -\gamma, \\
\text{Mode} &= \theta \left(\frac{\gamma-1}{\alpha\gamma+1} \right)^{1/\gamma}, \quad \gamma > 1, \text{ else } 0.
\end{aligned}
$$

A.2.2.3 Inverse Burr—τ, θ, γ

$$
\begin{aligned}
f(x) &= \frac{\tau\gamma(x/\theta)^{\gamma\tau}}{x[1+(x/\theta)^{\gamma}]^{\tau+1}}, \qquad F(x) = u^{\tau}, \quad u = \frac{(x/\theta)^{\gamma}}{1+(x/\theta)^{\gamma}}, \\
\mathrm{VaR}_p(X) &= \theta(p^{-1/\tau}-1)^{-1/\gamma}, \\
\mathrm{E}[X^k] &= \frac{\theta^k\Gamma(\tau+k/\gamma)\Gamma(1-k/\gamma)}{\Gamma(\tau)}, \quad -\tau\gamma < k < \gamma, \\
\mathrm{E}[(X\wedge x)^k] &= \frac{\theta^k\Gamma(\tau+k/\gamma)\Gamma(1-k/\gamma)}{\Gamma(\tau)}\beta(\tau+k/\gamma, 1-k/\gamma; u) + x^k[1-u^{\tau}], \quad k > -\tau\gamma, \\
\text{Mode} &= \theta\left(\frac{\tau\gamma-1}{\gamma+1}\right)^{1/\gamma}, \quad \tau\gamma > 1, \text{ else } 0.
\end{aligned}
$$

A.2.3 Two-Parameter Distributions

A.2.3.1 Pareto—α, θ

(Pareto Type II, Lomax)

$$
\begin{aligned}
f(x) &= \frac{\alpha\theta^{\alpha}}{(x+\theta)^{\alpha+1}}, \qquad F(x) = 1-\left(\frac{\theta}{x+\theta}\right)^{\alpha}, \\
\mathrm{VaR}_p(X) &= \theta[(1-p)^{-1/\alpha}-1], \\
\mathrm{E}[X^k] &= \frac{\theta^k\Gamma(k+1)\Gamma(\alpha-k)}{\Gamma(\alpha)}, \quad -1 < k < \alpha, \\
\mathrm{E}[X^k] &= \frac{\theta^k k!}{(\alpha-1)\cdots(\alpha-k)}, \quad \text{if } k \text{ is a positive integer}, \\
\mathrm{E}[X\wedge x] &= \frac{\theta}{\alpha-1}\left[1-\left(\frac{\theta}{x+\theta}\right)^{\alpha-1}\right], \quad \alpha \neq 1, \\
\mathrm{E}[X\wedge x] &= -\theta\ln\left(\frac{\theta}{x+\theta}\right), \quad \alpha = 1, \\
\mathrm{TVaR}_p(X) &= \mathrm{VaR}_p(X) + \frac{\theta(1-p)^{-1/\alpha}}{\alpha-1}, \quad \alpha > 1, \\
\mathrm{E}[(X\wedge x)^k] &= \frac{\theta^k\Gamma(k+1)\Gamma(\alpha-k)}{\Gamma(\alpha)}\beta[k+1, \alpha-k; x/(x+\theta)] \\
&\quad + x^k\left(\frac{\theta}{x+\theta}\right)^{\alpha}, \quad k > -1, \, k \neq \alpha, \\
\mathrm{E}[(X\wedge x)^{\alpha}] &= \theta^{\alpha}\left(\frac{x}{x+\theta}\right)^{\alpha}\left[1+\alpha\sum_{n=0}^{\infty}\frac{[x/(x+\theta)]^{n+1}}{\alpha+n+1}\right], \\
\text{Mode} &= 0.
\end{aligned}
$$

A.2.3.2 Inverse Pareto—τ, θ

$$
\begin{aligned}
f(x) &= \frac{\tau\theta x^{\tau-1}}{(x+\theta)^{\tau+1}}, \qquad F(x) = \left(\frac{x}{x+\theta}\right)^{\tau}, \\
\mathrm{VaR}_p(X) &= \theta[p^{-1/\tau} - 1]^{-1}, \\
\mathrm{E}[X^k] &= \frac{\theta^k \Gamma(\tau+k)\Gamma(1-k)}{\Gamma(\tau)}, \quad -\tau < k < 1, \\
\mathrm{E}[X^k] &= \frac{\theta^k (-k)!}{(\tau-1)\cdots(\tau+k)} \quad \text{if } k \text{ is a negative integer}, \\
\mathrm{E}[(X \wedge x)^k] &= \theta^k \tau \int_0^{x/(x+\theta)} y^{\tau+k-1}(1-y)^{-k} dy + x^k \left[1 - \left(\frac{x}{x+\theta}\right)^{\tau}\right], \quad k > -\tau, \\
\text{Mode} &= \theta\frac{\tau-1}{2}, \quad \tau > 1, \text{ else } 0.
\end{aligned}
$$

A.2.3.3 Loglogistic—γ, θ

(Fisk)

$$
\begin{aligned}
f(x) &= \frac{\gamma(x/\theta)^{\gamma}}{x[1+(x/\theta)^{\gamma}]^2}, \qquad F(x) = u, \quad u = \frac{(x/\theta)^{\gamma}}{1+(x/\theta)^{\gamma}}, \\
\mathrm{VaR}_p(X) &= \theta(p^{-1} - 1)^{-1/\gamma}, \\
\mathrm{E}[X^k] &= \theta^k \Gamma(1+k/\gamma)\Gamma(1-k/\gamma), \quad -\gamma < k < \gamma, \\
\mathrm{E}[(X \wedge x)^k] &= \theta^k \Gamma(1+k/\gamma)\Gamma(1-k/\gamma)\beta(1+k/\gamma, 1-k/\gamma; u) + x^k(1-u), \quad k > -\gamma, \\
\text{Mode} &= \theta\left(\frac{\gamma-1}{\gamma+1}\right)^{1/\gamma}, \quad \gamma > 1, \text{ else } 0.
\end{aligned}
$$

A.2.3.4 Paralogistic—α, θ

This is a Burr distribution with $\gamma = \alpha$.

$$
\begin{aligned}
f(x) &= \frac{\alpha^2(x/\theta)^{\alpha}}{x[1+(x/\theta)^{\alpha}]^{\alpha+1}}, \qquad F(x) = 1 - u^{\alpha}, \quad u = \frac{1}{1+(x/\theta)^{\alpha}}, \\
\mathrm{VaR}_p(X) &= \theta[(1-p)^{-1/\alpha} - 1]^{1/\alpha}, \\
\mathrm{E}[X^k] &= \frac{\theta^k \Gamma(1+k/\alpha)\Gamma(\alpha-k/\alpha)}{\Gamma(\alpha)}, \quad -\alpha < k < \alpha^2, \\
\mathrm{E}[(X \wedge x)^k] &= \frac{\theta^k \Gamma(1+k/\alpha)\Gamma(\alpha-k/\alpha)}{\Gamma(\alpha)}\beta(1+k/\alpha, \alpha-k/\alpha; 1-u) + x^k u^{\alpha}, \quad k > -\alpha, \\
\text{Mode} &= \theta\left(\frac{\alpha-1}{\alpha^2+1}\right)^{1/\alpha}, \quad \alpha > 1, \text{ else } 0.
\end{aligned}
$$

A.2.3.5 Inverse Paralogistic—τ, θ

This is an inverse Burr distribution with $\gamma = \tau$.

$$\begin{aligned}
f(x) &= \frac{\tau^2 (x/\theta)^{\tau^2}}{x[1+(x/\theta)^\tau]^{\tau+1}}, \qquad F(x) = u^\tau, \quad u = \frac{(x/\theta)^\tau}{1+(x/\theta)^\tau}, \\
\mathrm{VaR}_p(X) &= \theta(p^{-1/\tau} - 1)^{-1/\tau}, \\
\mathrm{E}[X^k] &= \frac{\theta^k \Gamma(\tau + k/\tau)\Gamma(1 - k/\tau)}{\Gamma(\tau)}, \quad -\tau^2 < k < \tau, \\
\mathrm{E}[(X \wedge x)^k] &= \frac{\theta^k \Gamma(\tau + k/\tau)\Gamma(1 - k/\tau)}{\Gamma(\tau)} \beta(\tau + k/\tau, 1 - k/\tau; u) + x^k[1 - u^\tau], \quad k > -\tau^2, \\
\text{Mode} &= \theta\,(\tau - 1)^{1/\tau}, \quad \tau > 1, \text{ else } 0.
\end{aligned}$$

A.3 Transformed Gamma Family

A.3.2 Two-Parameter Distributions

A.3.2.1 Gamma—α, θ

(When $\alpha = n/2$ and $\theta = 2$, it is a chi-square distribution with n degrees of freedom.)

$$\begin{aligned}
f(x) &= \frac{(x/\theta)^\alpha e^{-x/\theta}}{x\Gamma(\alpha)}, \qquad F(x) = \Gamma(\alpha; x/\theta), \\
\mathrm{E}[X^k] &= \frac{\theta^k \Gamma(\alpha + k)}{\Gamma(\alpha)}, \quad k > -\alpha, \\
\mathrm{E}[X^k] &= \theta^k (\alpha + k - 1) \cdots \alpha \quad \text{if } k \text{ is a positive integer}, \\
\mathrm{E}[(X \wedge x)^k] &= \frac{\theta^k \Gamma(\alpha + k)}{\Gamma(\alpha)} \Gamma(\alpha + k; x/\theta) + x^k[1 - \Gamma(\alpha; x/\theta)], \quad k > -\alpha, \\
\mathrm{E}[(X \wedge x)^k] &= \alpha(\alpha + 1) \cdots (\alpha + k - 1)\theta^k \Gamma(\alpha + k; x/\theta) \\
&\quad + x^k[1 - \Gamma(\alpha; x/\theta)] \quad \text{if } k \text{ is a positive integer}, \\
M(t) &= (1 - \theta t)^{-\alpha}, \quad t < 1/\theta, \\
\text{Mode} &= \theta(\alpha - 1), \quad \alpha > 1, \text{ else } 0.
\end{aligned}$$

A.3.2.2 Inverse Gamma—α, θ

(Vinci)

$$\begin{aligned}
f(x) &= \frac{(\theta/x)^\alpha e^{-\theta/x}}{x\Gamma(\alpha)}, \qquad F(x) = 1 - \Gamma(\alpha; \theta/x), \\
\mathrm{E}[X^k] &= \frac{\theta^k \Gamma(\alpha - k)}{\Gamma(\alpha)}, \quad k < \alpha, \\
\mathrm{E}[X^k] &= \frac{\theta^k}{(\alpha-1)\cdots(\alpha-k)} \quad \text{if } k \text{ is a positive integer}, \\
\mathrm{E}[(X \wedge x)^k] &= \frac{\theta^k \Gamma(\alpha - k)}{\Gamma(\alpha)}[1 - \Gamma(\alpha - k; \theta/x)] + x^k \Gamma(\alpha; \theta/x) \\
&= \frac{\theta^k G(\alpha - k; \theta/x)}{\Gamma(\alpha)} + x^k \Gamma(\alpha; \theta/x), \text{ all } k, \\
\text{Mode} &= \theta/(\alpha + 1).
\end{aligned}$$

A.3.2.3 Weibull—θ, τ

$$\begin{aligned}
f(x) &= \frac{\tau(x/\theta)^\tau e^{-(x/\theta)^\tau}}{x}, \qquad F(x) = 1 - e^{-(x/\theta)^\tau}, \\
\mathrm{VaR}_p(X) &= \theta[-\ln(1-p)]^{1/\tau}, \\
\mathrm{E}[X^k] &= \theta^k \Gamma(1 + k/\tau), \quad k > -\tau, \\
\mathrm{E}[(X \wedge x)^k] &= \theta^k \Gamma(1 + k/\tau)\Gamma[1 + k/\tau; (x/\theta)^\tau] + x^k e^{-(x/\theta)^\tau}, \quad k > -\tau, \\
\text{Mode} &= \theta\left(\frac{\tau - 1}{\tau}\right)^{1/\tau}, \quad \tau > 1, \text{ else } 0.
\end{aligned}$$

A.3.2.4 Inverse Weibull—θ, τ

(log-Gompertz)

$$\begin{aligned}
f(x) &= \frac{\tau(\theta/x)^\tau e^{-(\theta/x)^\tau}}{x}, \qquad F(x) = e^{-(\theta/x)^\tau}, \\
\mathrm{VaR}_p(X) &= \theta(-\ln p)^{-1/\tau}, \\
\mathrm{E}[X^k] &= \theta^k \Gamma(1 - k/\tau), \quad k < \tau, \\
\mathrm{E}[(X \wedge x)^k] &= \theta^k \Gamma(1 - k/\tau)\{1 - \Gamma[1 - k/\tau; (\theta/x)^\tau]\} + x^k \left[1 - e^{-(\theta/x)^\tau}\right], \\
&= \theta^k G[1 - k/\tau; (\theta/x)^\tau] + x^k \left[1 - e^{-(\theta/x)^\tau}\right], \quad \text{all } k, \\
\text{Mode} &= \theta\left(\frac{\tau}{\tau + 1}\right)^{1/\tau}.
\end{aligned}$$

A.3.3 One-Parameter Distributions

A.3.3.1 Exponential—θ

$$\begin{aligned}
f(x) &= \frac{e^{-x/\theta}}{\theta}, \qquad F(x) = 1 - e^{-x/\theta}, \\
\mathrm{VaR}_p(X) &= -\theta \ln(1-p), \\
\mathrm{E}[X^k] &= \theta^k \Gamma(k+1), \quad k > -1, \\
\mathrm{E}[X^k] &= \theta^k k! \quad \text{if } k \text{ is a positive integer}, \\
\mathrm{E}[X \wedge x] &= \theta(1 - e^{-x/\theta}), \\
\mathrm{TVaR}_p(X) &= -\theta \ln(1-p) + \theta, \\
\mathrm{E}[(X \wedge x)^k] &= \theta^k \Gamma(k+1)\Gamma(k+1; x/\theta) + x^k e^{-x/\theta}, \quad k > -1, \\
\mathrm{E}[(X \wedge x)^k] &= \theta^k k! \Gamma(k+1; x/\theta) + x^k e^{-x/\theta} \quad \text{if } k > -1 \text{ is an integer}, \\
M(z) &= (1 - \theta z)^{-1}, \quad z < 1/\theta, \\
\text{Mode} &= 0.
\end{aligned}$$

A.3.3.2 Inverse Exponential—θ

$$\begin{aligned}
f(x) &= \frac{\theta e^{-\theta/x}}{x^2}, \qquad F(x) = e^{-\theta/x}, \\
\mathrm{VaR}_p(X) &= \theta(-\ln p)^{-1}, \\
\mathrm{E}[X^k] &= \theta^k \Gamma(1-k), \quad k < 1, \\
\mathrm{E}[(X \wedge x)^k] &= \theta^k G(1-k; \theta/x) + x^k(1 - e^{-\theta/x}), \quad \text{all } k, \\
\text{Mode} &= \theta/2.
\end{aligned}$$

A.5 Other Distributions

A.5.1.1 Lognormal—μ,σ

(μ can be negative)

$$\begin{aligned}
f(x) &= \frac{1}{x\sigma\sqrt{2\pi}} \exp(-z^2/2) = \phi(z)/(\sigma x), \quad z = \frac{\ln x - \mu}{\sigma}, \\
F(x) &= \Phi(z), \\
\mathrm{E}[X^k] &= \exp\left(k\mu + \tfrac{1}{2}k^2\sigma^2\right), \\
\mathrm{E}[(X \wedge x)^k] &= \exp\left(k\mu + \tfrac{1}{2}k^2\sigma^2\right) \Phi\left(\frac{\ln x - \mu - k\sigma^2}{\sigma}\right) + x^k[1 - F(x)], \\
\text{Mode} &= \exp(\mu - \sigma^2).
\end{aligned}$$

A.5.1.2 Inverse Gaussian—μ, θ

$$
\begin{aligned}
f(x) &= \left(\frac{\theta}{2\pi x^3}\right)^{1/2} \exp\left(-\frac{\theta z^2}{2x}\right), \quad z = \frac{x-\mu}{\mu}, \\
F(x) &= \Phi\left[z\left(\frac{\theta}{x}\right)^{1/2}\right] + \exp\left(\frac{2\theta}{\mu}\right)\Phi\left[-y\left(\frac{\theta}{x}\right)^{1/2}\right], \quad y = \frac{x+\mu}{\mu}, \\
\mathrm{E}[X] &= \mu, \quad \mathrm{Var}[X] = \mu^3/\theta, \\
\mathrm{E}[X^k] &= \sum_{n=0}^{k-1} \frac{(k+n-1)!}{(k-n-1)!n!}\frac{\mu^{n+k}}{(2\theta)^n}, \quad k = 1, 2, \ldots, \\
\mathrm{E}[X \wedge x] &= x - \mu z \Phi\left[z\left(\frac{\theta}{x}\right)^{1/2}\right] - \mu y \exp(2\theta/\mu)\Phi\left[-y\left(\frac{\theta}{x}\right)^{1/2}\right], \\
M(z) &= \exp\left[\frac{\theta}{\mu}\left(1 - \sqrt{1 - \frac{2\mu^2}{\theta}z}\right)\right], \quad z < \frac{\theta}{2\mu^2}.
\end{aligned}
$$

A.5.1.3 Log-t—r, μ, σ

(μ can be negative) Let Y have a t distribution with r degrees of freedom. Then $X = \exp(\sigma Y + \mu)$ has the log-t distribution. Positive moments do not exist for this distribution. Just as the t distribution has a heavier tail than the normal distribution, this distribution has a heavier tail than the lognormal distribution.

$$
\begin{aligned}
f(x) &= \frac{\Gamma\left(\frac{r+1}{2}\right)}{x\sigma\sqrt{\pi r}\,\Gamma\left(\frac{r}{2}\right)\left[1 + \frac{1}{r}\left(\frac{\ln x - \mu}{\sigma}\right)^2\right]^{(r+1)/2}}, \\
F(x) &= F_r\left(\frac{\ln x - \mu}{\sigma}\right) \text{ with } F_r(t) \text{ the cdf of a } t \text{ distribution with } r \text{ df}, \\
F(x) &= \begin{cases} \frac{1}{2}\beta\left[\frac{r}{2}, \frac{1}{2}; \frac{r}{r + \left(\frac{\ln x - \mu}{\sigma}\right)^2}\right], & 0 < x \le e^{\mu}, \\ 1 - \frac{1}{2}\beta\left[\frac{r}{2}, \frac{1}{2}; \frac{r}{r + \left(\frac{\ln x - \mu}{\sigma}\right)^2}\right], & x \ge e^{\mu}. \end{cases}
\end{aligned}
$$

A.5.1.4 Single-Parameter Pareto—α, θ

$$\begin{aligned}
f(x) &= \frac{\alpha\theta^\alpha}{x^{\alpha+1}}, \quad x > \theta, \qquad F(x) = 1 - \left(\frac{\theta}{x}\right)^\alpha, \quad x > \theta, \\
\mathrm{VaR}_p(X) &= \theta(1-p)^{-1/\alpha}, \\
\mathrm{E}[X^k] &= \frac{\alpha\theta^k}{\alpha - k}, \quad k < \alpha, \\
\mathrm{E}[(X \wedge x)^k] &= \frac{\alpha\theta^k}{\alpha - k} - \frac{k\theta^\alpha}{(\alpha - k)x^{\alpha-k}}, \ x \geq \theta, \ k \neq \alpha, \\
\mathrm{E}[(X \wedge x)^\alpha] &= \theta^\alpha[1 + \alpha \ln(x/\theta)], \\
\mathrm{TVaR}_p(X) &= \frac{\alpha\theta(1-p)^{-1/\alpha}}{\alpha - 1}, \quad \alpha > 1, \\
\mathrm{Mode} &= \theta.
\end{aligned}$$

Note: Although there appear to be two parameters, only α is a true parameter. The value of θ must be set in advance.

A.6 Distributions with Finite Support

For these two distributions, the scale parameter θ is assumed known.

A.6.1.1 Generalized Beta—a, b, θ, τ

$$\begin{aligned}
f(x) &= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} u^a (1-u)^{b-1} \frac{\tau}{x}, \quad 0 < x < \theta, \quad u = (x/\theta)^\tau, \\
F(x) &= \beta(a, b; u), \\
\mathrm{E}[X^k] &= \frac{\theta^k \Gamma(a+b)\Gamma(a+k/\tau)}{\Gamma(a)\Gamma(a+b+k/\tau)}, \quad k > -a\tau, \\
\mathrm{E}[(X \wedge x)^k] &= \frac{\theta^k \Gamma(a+b)\Gamma(a+k/\tau)}{\Gamma(a)\Gamma(a+b+k/\tau)} \beta(a+k/\tau, b; u) + x^k[1 - \beta(a, b; u)].
\end{aligned}$$

A.6.1.2 Beta—a, b, θ

The case $\theta = 1$ has no special name but is the commonly used version of this distribution.

$$
\begin{aligned}
f(x) &= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} u^a (1-u)^{b-1} \frac{1}{x}, \quad 0 < x < \theta, \quad u = x/\theta, \\
F(x) &= \beta(a, b; u), \\
\mathrm{E}[X^k] &= \frac{\theta^k \Gamma(a+b)\Gamma(a+k)}{\Gamma(a)\Gamma(a+b+k)}, \quad k > -a, \\
\mathrm{E}[X^k] &= \frac{\theta^k a(a+1)\cdots(a+k-1)}{(a+b)(a+b+1)\cdots(a+b+k-1)} \quad \text{if } k \text{ is a positive integer,} \\
\mathrm{E}[(X \wedge x)^k] &= \frac{\theta^k a(a+1)\cdots(a+k-1)}{(a+b)(a+b+1)\cdots(a+b+k-1)} \beta(a+k, b; u) \\
&\quad + x^k [1 - \beta(a, b; u)].
\end{aligned}
$$

Appendix B

An Inventory of Discrete Distributions

B.1 Introduction

The 16 models presented in this appendix fall into three classes. The divisions are based on the used to compoute the probabilities. For some of the more familiar distributions these formulas will look different from the ones you may have learned, but they produce the same probabilities. After each name, the parameters are given. All parameters are positive unless otherwise indicated. In all cases, p_k is the probability of observing k losses. inxx For finding moments, the most convenient form is to give the factorial moments. The jth factorial moment is $\mu_{(j)} = \mathrm{E}[N(N-1)\cdots(N-j+1)]$. We have $\mathrm{E}[N] = \mu_{(1)}$ and $\mathrm{Var}(N) = \mu_{(2)} + \mu_{(1)} - \mu_{(1)}^2$.

The estimators presented are not intended to be useful estimators but, rather, provide starting values for maximizing the likelihood (or other) function. For determining starting values, the following quantities are used (where n_k is the observed frequency at k [if, for the last entry, n_k represents the number of observations at k or more, assume it was at exactly k] and n is the sample size):

$$\hat{\mu} = \frac{1}{n}\sum_{k=1}^{\infty} k n_k, \quad \hat{\sigma}^2 = \frac{1}{n}\sum_{k=1}^{\infty} k^2 n_k - \hat{\mu}^2.$$

When the method of moments is used to determine the starting value, a circumflex (e.g., $\hat{\lambda}$) is used. For any other method, a tilde (e.g., $\tilde{\lambda}$) is used. When the starting value formulas do not provide admissible parameter values, a truly crude guess is to set the product of all λ and β parameters equal to the sample mean and set all other parameters equal to 1. If there are two λ or β parameters, an easy choice is to set each to the square root of the sample mean.

The last item presented is the probability generating function,

$$P(z) = \mathrm{E}[z^N].$$

B.2 The $(a, b, 0)$ Class

The distributions in this class have support on $0, 1, \ldots$. For this class, a particular distribution is specified by setting p_0 and then using $p_k = (a + b/k)p_{k-1}$. Specific members are created by setting p_0, a, and b. For any member, $\mu_{(1)} = (a + b)/(1 - a)$, and for higher j, $\mu_{(j)} = (aj + b)\mu_{(j-1)}/(1 - a)$. The variance is $(a + b)/(1 - a)^2$.

B.2.1.1 Poisson—λ

$$\begin{aligned} p_0 &= e^{-\lambda}, \quad a = 0, \quad b = \lambda, \qquad p_k = \frac{e^{-\lambda}\lambda^k}{k!}, \\ \mathrm{E}[N] &= \lambda, \quad \mathrm{Var}[N] = \lambda, \\ \hat{\lambda} &= \hat{\mu}, \\ P(z) &= e^{\lambda(z-1)}. \end{aligned}$$

B.2.1.2 Geometric—β

$$\begin{aligned} p_0 &= \frac{1}{1+\beta}, \quad a = \frac{\beta}{1+\beta}, \quad b = 0, \qquad p_k = \frac{\beta^k}{(1+\beta)^{k+1}}, \\ \mathrm{E}[N] &= \beta, \quad \mathrm{Var}[N] = \beta(1+\beta), \\ \hat{\beta} &= \hat{\mu}, \\ P(z) &= [1 - \beta(z-1)]^{-1}, \quad -(1 + 1/\beta) < z < 1 + 1/\beta. \end{aligned}$$

This is a special case of the negative binomial with $r = 1$.

B.2.1.3 Binomial—q, m

($0 < q < 1$, m an integer)

$$\begin{aligned} p_0 &= (1-q)^m, \quad a = -\frac{q}{1-q}, \quad b = \frac{(m+1)q}{1-q}, \\ p_k &= \binom{m}{k} q^k (1-q)^{m-k}, \quad k = 0, 1, \ldots, m, \\ \mathrm{E}[N] &= mq, \quad \mathrm{Var}[N] = mq(1-q), \\ \hat{q} &= \hat{\mu}/m, \\ P(z) &= [1 + q(z-1)]^m. \end{aligned}$$

B.2.1.4 Negative Binomial—β, r

$$\begin{aligned}
p_0 &= (1+\beta)^{-r}, \quad a = \frac{\beta}{1+\beta}, \quad b = \frac{(r-1)\beta}{1+\beta}, \\
p_k &= \frac{r(r+1)\cdots(r+k-1)\beta^k}{k!(1+\beta)^{r+k}}, \\
\mathrm{E}[N] &= r\beta, \quad \mathrm{Var}[N] = r\beta(1+\beta), \\
\hat{\beta} &= \frac{\hat{\sigma}^2}{\hat{\mu}} - 1, \quad \hat{r} = \frac{\hat{\mu}^2}{\hat{\sigma}^2 - \hat{\mu}}, \\
P(z) &= [1-\beta(z-1)]^{-r}, \quad -(1+1/\beta) < z < 1 + 1/\beta.
\end{aligned}$$

B.3 The $(a, b, 1)$ Class

To distinguish this class from the $(a, b, 0)$ class, the probabilities are denoted $\Pr(N = k) = p_k^M$ or $\Pr(N = k) = p_k^T$ depending on which subclass is being represented. For this class, p_0^M is arbitrary (i.e., it is a parameter), and then p_1^M or p_1^T is a specified function of the parameters a and b. Subsequent probabilities are obtained recursively as in the $(a, b, 0)$ class: $p_k^M = (a + b/k)p_{k-1}^M$, $k = 2, 3, \ldots$, with the same recursion for p_k^T. There are two subclasses of this class. When discussing their members, we often refer to the "corresponding" member of the $(a, b, 0)$ class. This refers to the member of that class with the same values for a and b. The notation p_k will continue to be used for probabilities for the corresponding $(a, b, 0)$ distribution.

B.3.1 The Zero-Truncated Subclass

The members of this class have $p_0^T = 0$, and therefore it need not be estimated. These distributions should only be used when a value of zero is impossible. The first factorial moment is $\mu_{(1)} = (a+b)/[(1-a)(1-p_0)]$, where p_0 is the value for the corresponding member of the $(a, b, 0)$ class. For the logarithmic distribution (which has no corresponding member), $\mu_{(1)} = \beta/\ln(1+\beta)$. Higher factorial moments are obtained recursively with the same formula as with the $(a, b, 0)$ class. The variance is $(a+b)[1-(a+b+1)p_0]/[(1-a)(1-p_0)]^2$. For those members of the subclass that have corresponding $(a, b, 0)$ distributions, $p_k^T = p_k/(1-p_0)$.

B.3.1.1 Zero-Truncated Poisson—λ

$$\begin{aligned}
p_1^T &= \frac{\lambda}{e^\lambda - 1}, \quad a = 0, \quad b = \lambda, \\
p_k^T &= \frac{\lambda^k}{k!(e^\lambda - 1)}, \\
\mathrm{E}[N] &= \lambda/(1 - e^{-\lambda}), \quad \mathrm{Var}[N] = \lambda[1 - (\lambda + 1)e^{-\lambda}]/(1 - e^{-\lambda})^2, \\
\tilde{\lambda} &= \ln(n\hat{\mu}/n_1), \\
P(z) &= \frac{e^{\lambda z} - 1}{e^\lambda - 1}.
\end{aligned}$$

B.3.1.2 Zero-Truncated Geometric—β

$$\begin{aligned}
p_1^T &= \frac{1}{1+\beta}, \quad a = \frac{\beta}{1+\beta}, \quad b = 0, \\
p_k^T &= \frac{\beta^{k-1}}{(1+\beta)^k}, \\
\mathrm{E}[N] &= 1 + \beta, \quad \mathrm{Var}[N] = \beta(1+\beta), \\
\hat{\beta} &= \hat{\mu} - 1, \\
P(z) &= \frac{[1 - \beta(z-1)]^{-1} - (1+\beta)^{-1}}{1 - (1+\beta)^{-1}}, \quad -(1 + 1/\beta) < z < 1 + 1/\beta.
\end{aligned}$$

This is a special case of the zero-truncated negative binomial with $r = 1$.

B.3.1.3 Logarithmic—β

$$\begin{aligned}
p_1^T &= \frac{\beta}{(1+\beta)\ln(1+\beta)}, \quad a = \frac{\beta}{1+\beta}, \quad b = -\frac{\beta}{1+\beta}, \\
p_k^T &= \frac{\beta^k}{k(1+\beta)^k \ln(1+\beta)}, \\
\mathrm{E}[N] &= \beta/\ln(1+\beta), \quad \mathrm{Var}[N] = \frac{\beta[1 + \beta - \beta/\ln(1+\beta)]}{\ln(1+\beta)}, \\
\tilde{\beta} &= \frac{n\hat{\mu}}{n_1} - 1 \quad \text{or} \quad \frac{2(\hat{\mu} - 1)}{\hat{\mu}}, \\
P(z) &= 1 - \frac{\ln[1 - \beta(z-1)]}{\ln(1+\beta)}, \quad -(1 + 1/\beta) < z < 1 + 1/\beta.
\end{aligned}$$

This is a limiting case of the zero-truncated negative binomial as $r \to 0$.

B.3.1.4 Zero-Truncated Binomial—q, m,

($0 < q < 1$, m an integer)

$$
\begin{aligned}
p_1^T &= \frac{m(1-q)^{m-1}q}{1-(1-q)^m}, \quad a = -\frac{q}{1-q}, \quad b = \frac{(m+1)q}{1-q}, \\
p_k^T &= \frac{\binom{m}{k}q^k(1-q)^{m-k}}{1-(1-q)^m}, \quad k = 1, 2, \ldots, m, \\
\mathrm{E}[N] &= \frac{mq}{1-(1-q)^m}, \\
\mathrm{Var}[N] &= \frac{mq[(1-q)-(1-q+mq)(1-q)^m]}{[1-(1-q)^m]^2}, \\
\tilde{q} &= \frac{\hat{\mu}}{m}, \\
P(z) &= \frac{[1+q(z-1)]^m-(1-q)^m}{1-(1-q)^m},.
\end{aligned}
$$

B.3.1.5 Zero-Truncated Negative Binomial—β, r ($r > -1, r \neq 0$)

$$
\begin{aligned}
p_1^T &= \frac{r\beta}{(1+\beta)^{r+1}-(1+\beta)}, \quad a = \frac{\beta}{1+\beta}, \quad b = \frac{(r-1)\beta}{1+\beta}, \\
p_k^T &= \frac{r(r+1)\cdots(r+k-1)}{k![(1+\beta)^r-1]}\left(\frac{\beta}{1+\beta}\right)^k, \\
\mathrm{E}[N] &= \frac{r\beta}{1-(1+\beta)^{-r}}, \\
\mathrm{Var}[N] &= \frac{r\beta[(1+\beta)-(1+\beta+r\beta)(1+\beta)^{-r}]}{[1-(1+\beta)^{-r}]^2}, \\
\tilde{\beta} &= \frac{\hat{\sigma}^2}{\hat{\mu}}-1, \quad \tilde{r} = \frac{\hat{\mu}^2}{\hat{\sigma}^2-\hat{\mu}}, \\
P(z) &= \frac{[1-\beta(z-1)]^{-r}-(1+\beta)^{-r}}{1-(1+\beta)^{-r}}, \quad -(1+1/\beta) < z < 1+1/\beta.
\end{aligned}
$$

This distribution is sometimes called the extended truncated negative binomial distribution because the parameter r can extend below zero.

B.3.2 The Zero-Modified Subclass

A zero-modified distribution is created by starting with a truncated distribution and then placing an arbitrary amount of probability at zero. This probability, p_0^M, is a parameter. The remaining probabilities are adjusted accordingly. Values of p_k^M can be determined from the corresponding zero-truncated distribution as $p_k^M = (1 - p_0^M)p_k^T$ or from the corresponding $(a, b, 0)$ distribution as $p_k^M = (1 - p_0^M)p_k/(1 - p_0)$. The same recursion used for the zero-truncated subclass applies.

The mean is $1 - p_0^M$ times the mean for the corresponding zero-truncated distribution. The variance is $1 - p_0^M$ times the zero-truncated variance plus $p_0^M(1 - p_0^M)$ times the square of the zero-truncated mean. The probability generating function is $P^M(z) = p_0^M + (1 - p_0^M)P(z)$, where $P(z)$ is the probability generating function for the corresponding zero-truncated distribution.

The maximum likelihood estimator of p_0^M is always the sample relative frequency at zero.