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Semi Monte Carlo - a New Variance Reduction Method

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Semi Monte Carlo – a New Variance Reduction Method

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Agenda

1. Problem Formulation
2. The Idea of Semi Monte Carlo
3. Experiments
4. Conclusion
5. Q&A

1 Problem Formulation

Problem Setting

- Evaluate a mean of some function over a given measure when analytic calculations are impractical.

$$E_P\{f(\omega)\} \equiv \int_{(\Omega)} f(\omega)p(\omega)d\omega$$

- Monte Carlo method is often used with some modifications
  - Control variates
  - Low discrepancy sequences
  - …
- Another well-known method to do it – use numeric integration (a lot of schemes)

Methods Comparison

Pro and Con

	Monte Carlo	Numeric integration
Pro	• **Does not depend on dimension** • Does not depend on smoothness • No bias • Simple error estimate based on data obtained • Some variance reduction methods available	• **Fast convergence** • Control over convergence • Absolute error estimate • Variety of schemes to choose from
Contra	• **Slow convergence** • Weak control over convergence • Stochastic error estimate – variance of result	• **Is effective for small dimensions only** • Relies on an a-priory smoothness of the integrand (estimates for the function class) • May provide a bias, hard to estimate

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The Idea of Semi Monte Carlo

The Idea of Semi Monte Carlo

Combine Numeric Integration and Monte Carlo Simulation (SMC)

- Combine numeric integration and Monte-Carlo simulation to use best of both
- Why combine?
  - Monte Carlo gives variance but no bias
  - Numeric integration scheme (NIS) gives bias but no variance
  - Trade one for another to minimize Mean Square Error (MSE)
- How to combine?
  - Use NIS for few critical dimensions and MC for many less variative
  - Based on some a-priory knowledge of the distribution

The Main Idea of Semi Monte Carlo

Technical Definition

- We are calculating $E_P\{f(\omega)\} \equiv \int_{(\Omega)} f(\omega)p(\omega)d\omega$
  - Split the probability density: $(\omega) \equiv (x,y)$ and $p(\omega) \equiv p(x,y) = p_X(x) \cdot p_Y(y|x)$;
  - Then

$$E_P\{f(\omega)\} = E_x\left\{E_y\{f(x,y)|x\}\right\} = \int_{(X)} \left\{ \int_{(Y|x)} f(x,y)p_Y(y|x)dy \right\} p_X(x)dx$$

- Apply NIS to the outer integral and MC to the inner integral

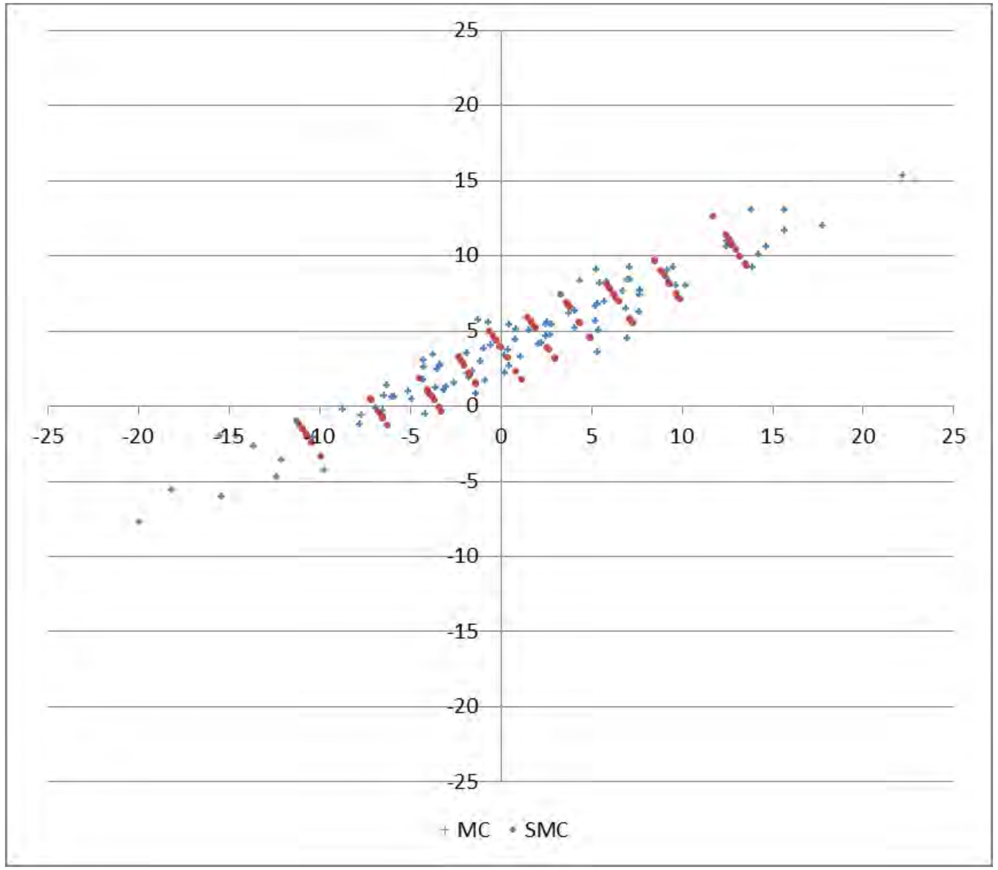
$$E_P\{f(\omega)\} \approx \sum_j \left\{ \int_{(Y|x)} f\left(x_j,y\right)p_Y\left(y|x_j\right)dy \right\} w_j \approx \sum_j \left(w_j \frac{1}{N_j} \sum_i f\left(x_j,y_i\right) \right)$$

  - Here x_j are NIS nodes and y_i are independent random variables with distributions $p_Y\left(y\middle|x_j\right)dy$, N_j is the number of MC simulations for the node x_j

The Main Idea

Graphic Illustration

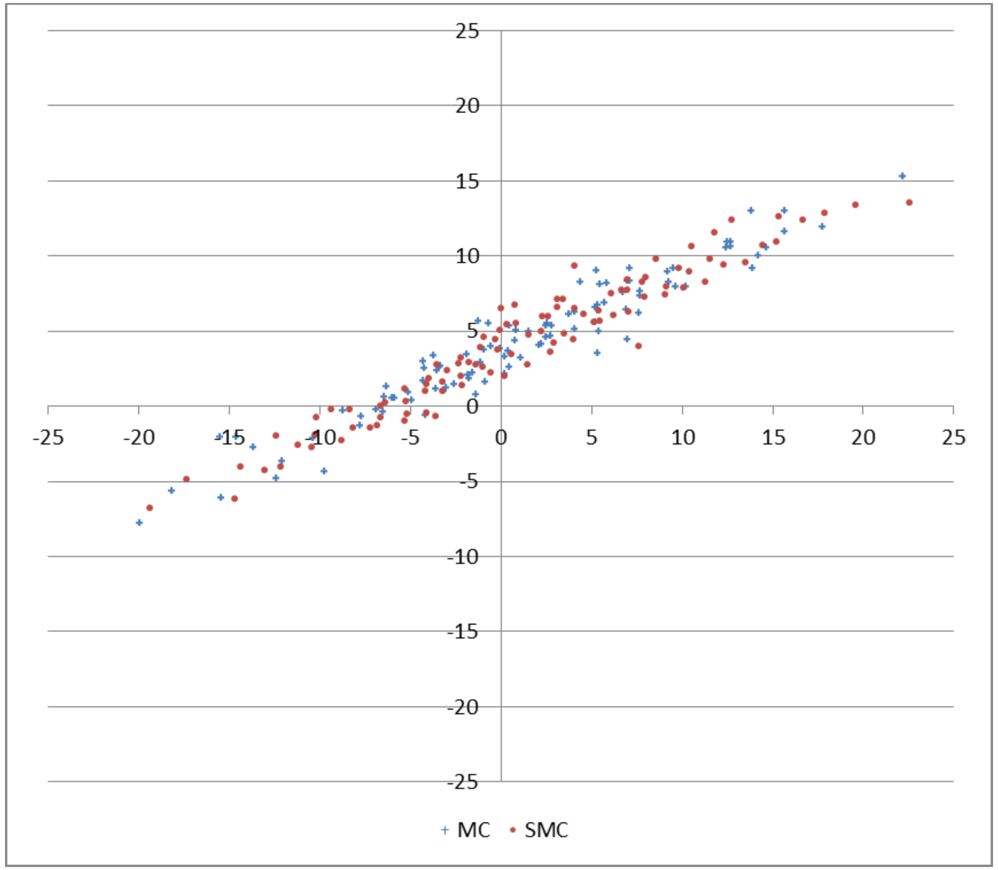
- Consider 2-dimensional Normal distribution with correlation matrix far from unit
  - Choose principal component(s)
  - Use numeric integration to find nodes for principal component(s)
  - Use MC conditional on this component to generate the rest of coordinates
- Blue points – usual MC randomly picked
- Red points – long coordinate picked for numeric integration, short – by MC
- Far from optimal choice



The Main Idea

Graphic Illustration 2

- Changing the number of nodes in integration scheme; total number of points is preserved
- Blue points – usual MC randomly picked
- Red points – long coordinate picked for numeric integration, short – by MC
- Better choice - hard to distinguish visually



The Main Idea

Parameters Choice

- The main problem: balance bias from NIS and variance from MC
- Splitting into x and y
  - Use principal components as outer integral for NIS
  - Find the sharp decline of variance of components to split them
- Choice of the NIS
  - Based on the probability density $p_X(x)$
  - Take into account symmetry of the distribution
  - Can you use different weights w_j?
- Balance of nodes
  - Usually one MC simulation per NIS node (see illustration)

The Main Idea

Analogies

- There are many ways to think of SMC:
  - Principal components analysis
  - Stratified sampling
  - Brownian bridge extension
  - Low discrepancy sequence
  - …
- All of these add to intuition and help to find proper parameters

3 Experiments

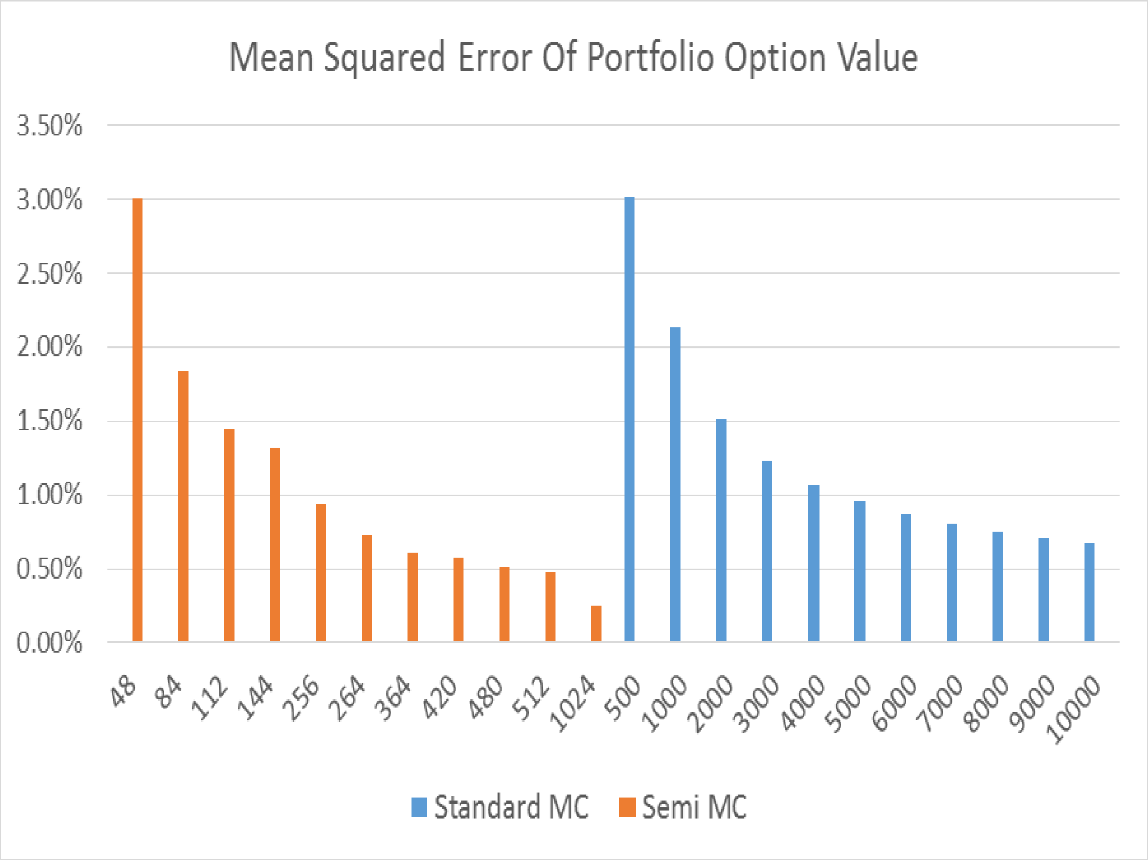
Experiments

Hull White and Lognormal Risk - Neutral Model for GMWB

- Finding the total option values and associated Greeks calculated for the GMWB within a realistic block of approximately 6,000 annuity policies
- Hull White and Lognormal Risk - Neutral Model is convenient to find the variables split since the distribution is normal
- Benchmark
  - 70,000 scenarios HWLN Monte Carlo
  - Error estimated by variance (= MSE for Monte Carlo)
- Compression ratio is defined as $\frac{Number\ of\ scenarios\ with\ MC\ producing\ the\ same\ MSE}{Number\ of\ scenarios\ with\ SMC}$

Experiments

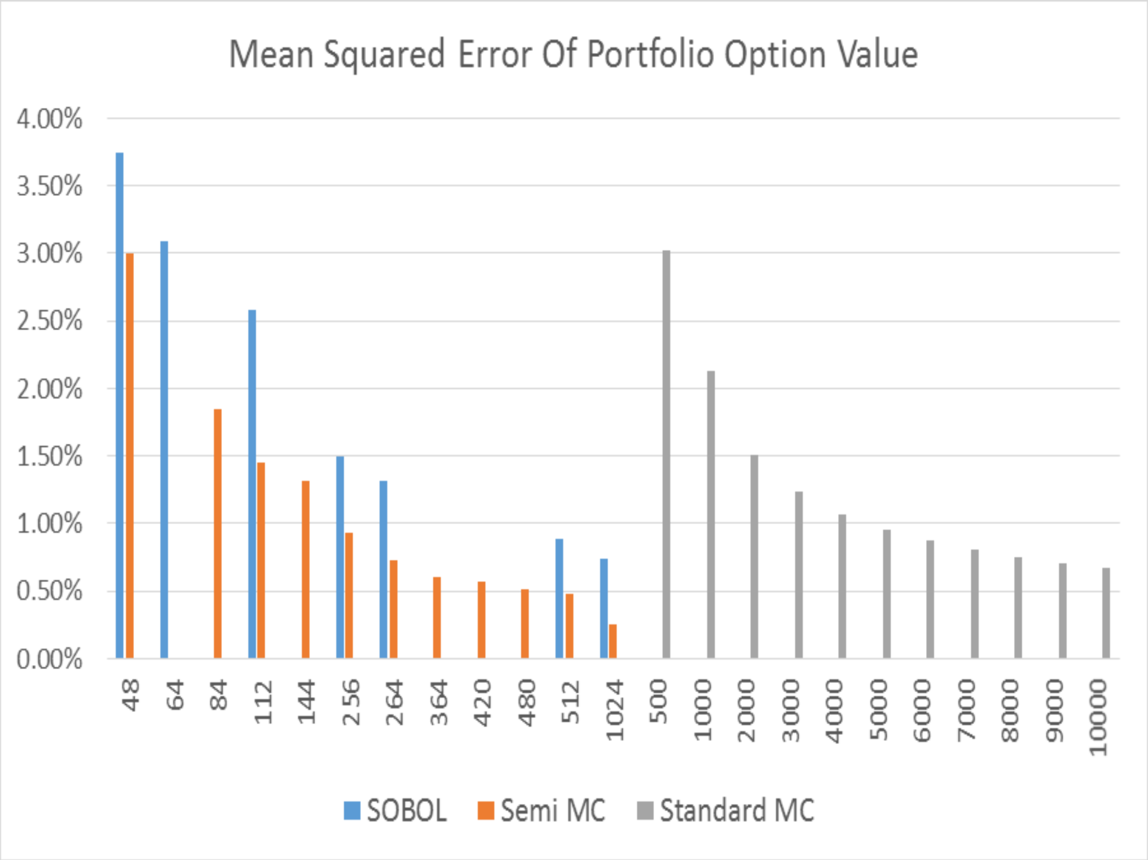
RSME Comparison: SMC and MC



RSME	Number of Scenarios Standard MC	Number of Scenarios Semi MC	Compres-sion Ratio
1.00%	5,000	256	20:1
1.50%	2,000	112	18:1
2.00%	1,000	84	12:1

Experiments

RSME Comparison: SMC and Sobol Sequence



RSME	Number of Scenarios Standard MC	Number of Scenarios Sobol	Com-pression Ratio	Number of Scenarios Semi MC	Com-pression Ratio
1.00%	5,000	512	10:1	256	20:1
1.50%	2,000	256	8:1	112	18:1
3.00%	500	64	8:1	48	10:1

4 Final Remarks

Final Remarks

Conclusion

- Advantages
  - Preserves distribution, not average only
  - Very flexible – allows for adjustments
  - Improves performance when combined with other methods
    - Policies clustering
    - Per policy generate
    - Low discrepancy sequences

- Problems to be resolved
  - Parameters optimization:
    - Splitting for non-linear models (Heston, Libor,…)?
    - Optimal dimension choice
    - Optimal NIS choice
  - Hard to evaluate an error without finding the benchmark
  - Harder to explain

Formula for MSE

Just a reference for Math fans

- Mean Squared Error for SMC looks like (2-nd row is for optimal choice of N_j)

$$MSE_{SMC} = \left[\bar{f} - \sum_j w_j E_y\{f(x_j, y)|x_j\}\right]^2 + \sum_j \frac{w_j^2}{N_j} Var_Y\{f(x_j, y)|x_j\}$$

$$\gtrsim \left[\bar{f} - \sum_j w_j E_y\{f(x_j, y)|x_j\}\right]^2 + \frac{1}{N}\left[\int_{(\Omega)} std_y\{f(x, y)|x\} p_X(x) dx\right]^2$$

$$= C_1 \cdot M^{-2p} + C_2 \cdot N^{-1}$$

Here $\bar{f}$ is a true value, w_j are NIS weights, N_j– number of simulations for x_j, p depends on NIS, M is number of NIS nodes, $N = \sum_{j=1}^{M} N_j$

- It helps to balance split and N_j.

5 | Thank You! Q&A

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