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## Semi Monte Carlo - a New Variance Reduction Method

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# Semi Monte Carlo – a New Variance Reduction Method

# Agenda

1. Problem Formulation
2. The Idea of Semi Monte Carlo
3. Experiments
4. Conclusion
5. Q&A

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Problem Formulation

# Problem Setting

- » Evaluate a mean of some function over a given measure when analytic calculations are impractical.

$$E_P\{f(\omega)\} \equiv \int_{(\Omega)} f(\omega)p(\omega)d\omega$$

- » Monte Carlo method is often used with some modifications
  - Control variates
  - Low discrepancy sequences
  - ...
- » Another well-known method to do it – use numeric integration (a lot of schemes)

# Methods Comparison

## Pro and Con

	Monte Carlo	Numeric integration
Pro	<ul style="list-style-type: none"><li>• <b>Does not depend on dimension</b></li><li>• Does not depend on smoothness</li><li>• No bias</li><li>• Simple error estimate based on data obtained</li><li>• Some variance reduction methods available</li></ul>	<ul style="list-style-type: none"><li>• <b>Fast convergence</b></li><li>• Control over convergence</li><li>• Absolute error estimate</li><li>• Variety of schemes to choose from</li></ul>
Contra	<ul style="list-style-type: none"><li>• <b>Slow convergence</b></li><li>• Weak control over convergence</li><li>• Stochastic error estimate – variance of result</li></ul>	<ul style="list-style-type: none"><li>• <b>Is effective for small dimensions only</b></li><li>• Relies on an a-priory smoothness of the integrand (estimates for the function class)</li><li>• May provide a bias, hard to estimate</li></ul>

# 2

## The Idea of Semi Monte Carlo

# The Idea of Semi Monte Carlo

Combine Numeric Integration and Monte Carlo Simulation (SMC)

- » Combine numeric integration and Monte-Carlo simulation to use best of both
- » Why combine?
  - Monte Carlo gives variance but no bias
  - Numeric integration scheme (NIS) gives bias but no variance
  - Trade one for another to minimize Mean Square Error (MSE)
- » How to combine?
  - Use NIS for few critical dimensions and MC for many less variative
  - Based on some a-priory knowledge of the distribution

# The Main Idea of Semi Monte Carlo

## Technical Definition

» We are calculating  $E_P\{f(\omega)\} \equiv \int_{(\Omega)} f(\omega)p(\omega)d\omega$

– Split the probability density:  $(\omega) \equiv (x, y)$  and  $p(\omega) \equiv p(x, y) = p_X(x) \cdot p_Y(y|x)$ ;

– Then

$$E_P\{f(\omega)\} = E_x \left\{ E_y \{f(x, y)|x\} \right\} = \int_{(X)} \left\{ \int_{(Y|x)} f(x, y)p_Y(y|x)dy \right\} p_X(x)dx$$

» Apply NIS to the outer integral and MC to the inner integral

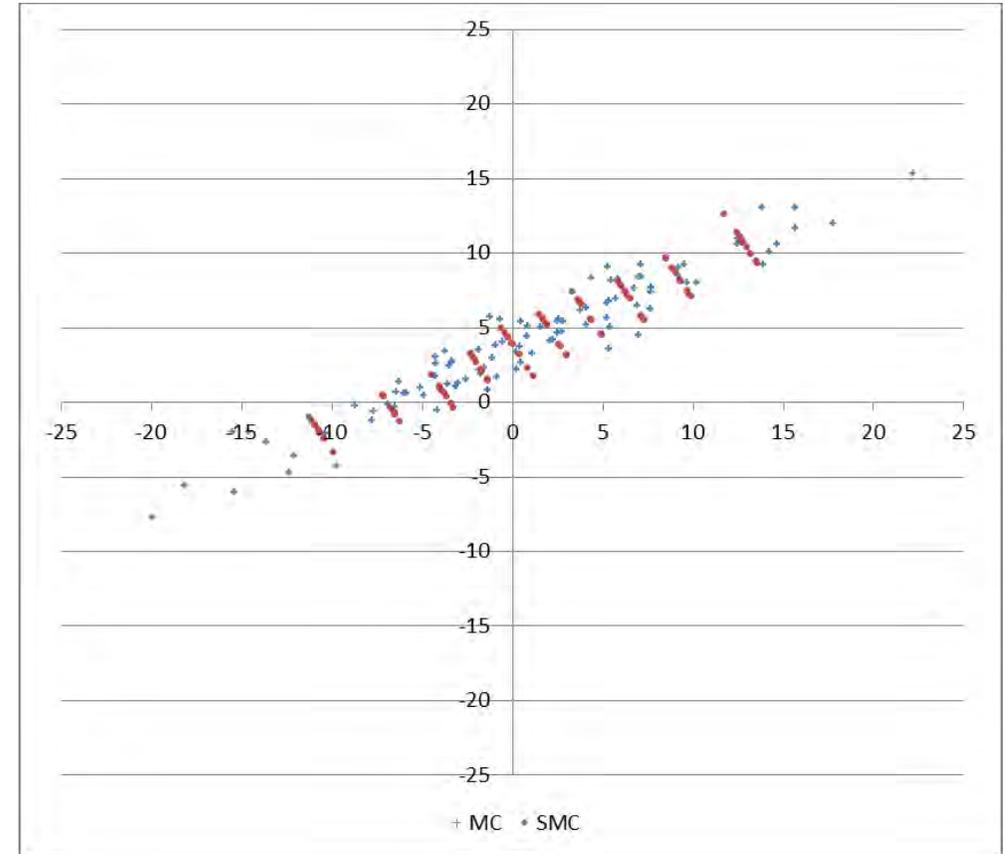
$$E_P\{f(\omega)\} \approx \sum_j \left\{ \int_{(Y|x)} f(x_j, y)p_Y(y|x_j)dy \right\} w_j \approx \sum_j \left( w_j \frac{1}{N_j} \sum_i f(x_j, y_i) \right)$$

– Here  $x_j$  are NIS nodes and  $y_i$  are independent random variables with distributions  $p_Y(y|x_j)dy$ ,  $N_j$  is the number of MC simulations for the node  $x_j$

# The Main Idea

## Graphic Illustration

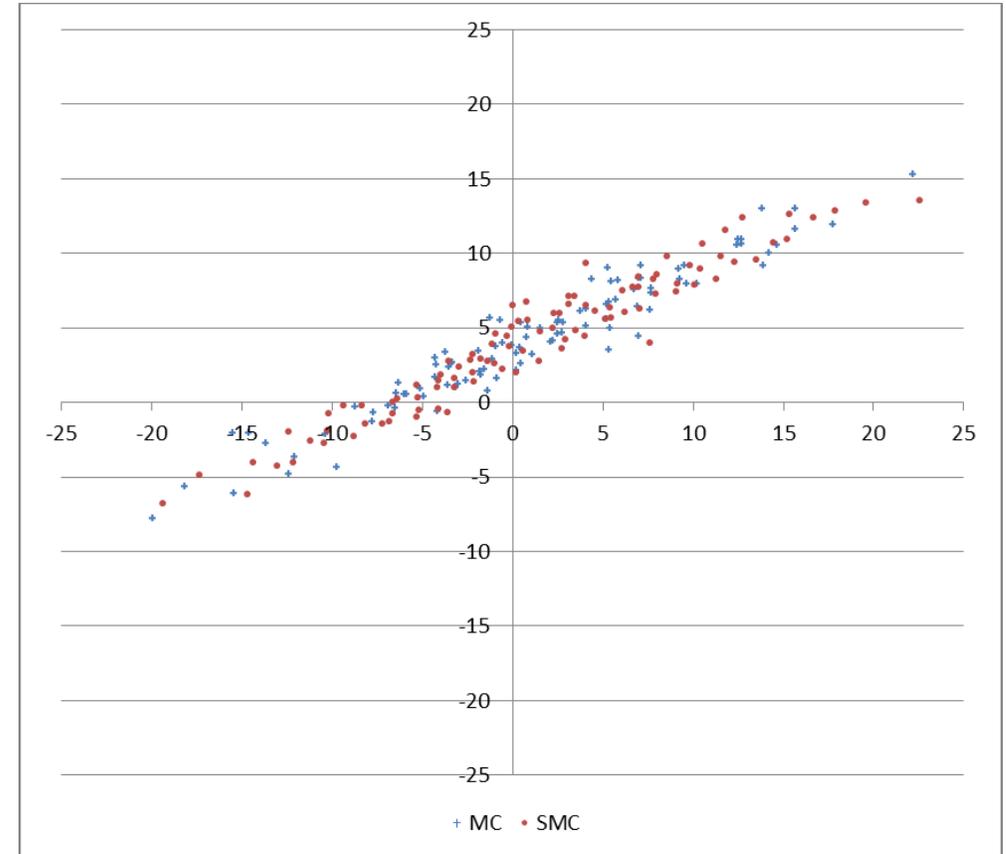
- » Consider 2-dimensional Normal distribution with correlation matrix far from unit
  - Choose principal component(s)
  - Use numeric integration to find nodes for principal component(s)
  - Use MC conditional on this component to generate the rest of coordinates
- » Blue points – usual MC randomly picked
- » Red points – long coordinate picked for numeric integration, short – by MC
- » Far from optimal choice



# The Main Idea

## Graphic Illustration 2

- » Changing the number of nodes in integration scheme; total number of points is preserved
- » Blue points – usual MC randomly picked
- » Red points – long coordinate picked for numeric integration, short – by MC
- » Better choice - hard to distinguish visually



# The Main Idea

## Parameters Choice

- » The main problem: balance bias from NIS and variance from MC
- » Splitting into  $x$  and  $y$ 
  - Use principal components as outer integral for NIS
  - Find the sharp decline of variance of components to split them
- » Choice of the NIS
  - Based on the probability density  $p_X(x)$
  - Take into account symmetry of the distribution
  - Can you use different weights  $w_j$ ?
- » Balance of nodes
  - Usually one MC simulation per NIS node (see illustration)

# The Main Idea

## Analogies

- » There are many ways to think of SMC:
  - Principal components analysis
  - Stratified sampling
  - Brownian bridge extension
  - Low discrepancy sequence
  - ...
- » All of these add to intuition and help to find proper parameters

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Experiments

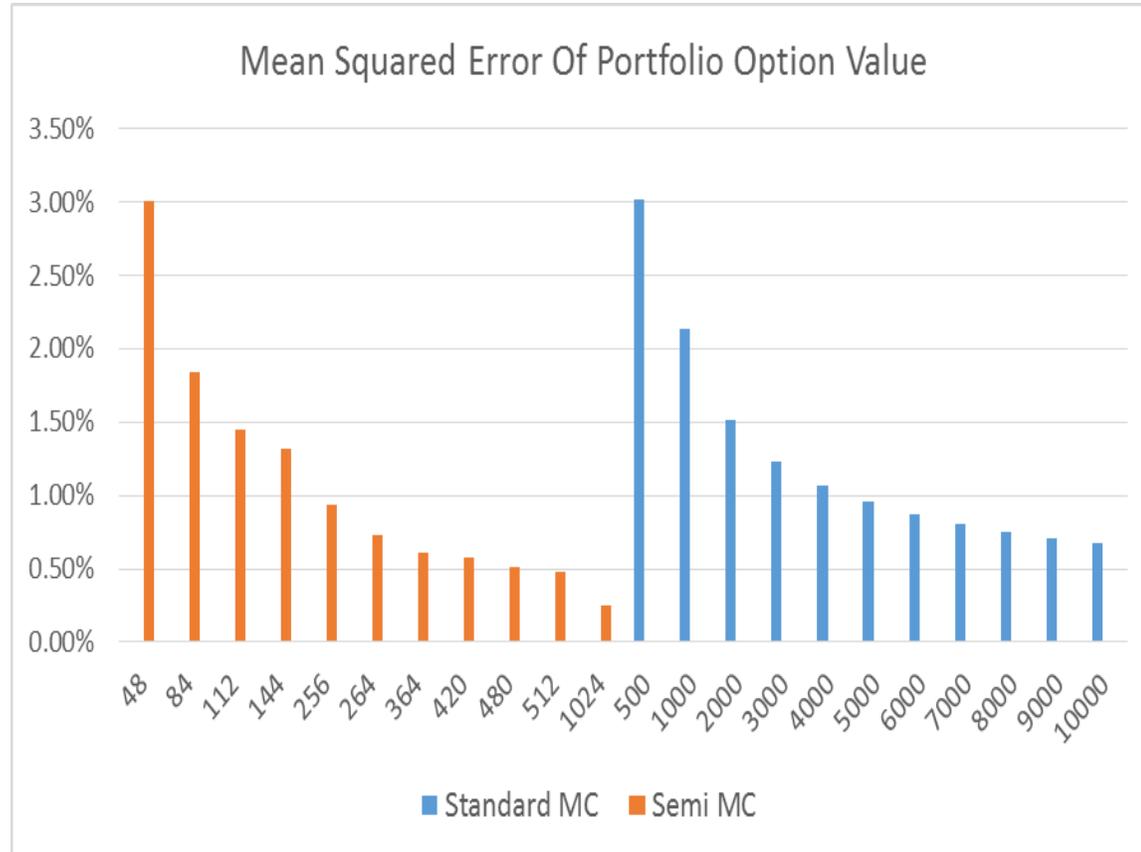
# Experiments

## Hull White and Lognormal Risk - Neutral Model for GMWB

- » Finding the total option values and associated Greeks calculated for the GMWB within a realistic block of approximately 6,000 annuity policies
- » Hull White and Lognormal Risk - Neutral Model is convenient to find the variables split since the distribution is normal
- » Benchmark
  - 70,000 scenarios HWLN Monte Carlo
  - Error estimated by variance (= MSE for Monte Carlo)
- » Compression ratio is defined as 
$$\frac{\text{Number of scenarios with MC producing the same MSE}}{\text{Number of scenarios with SMC}}$$

# Experiments

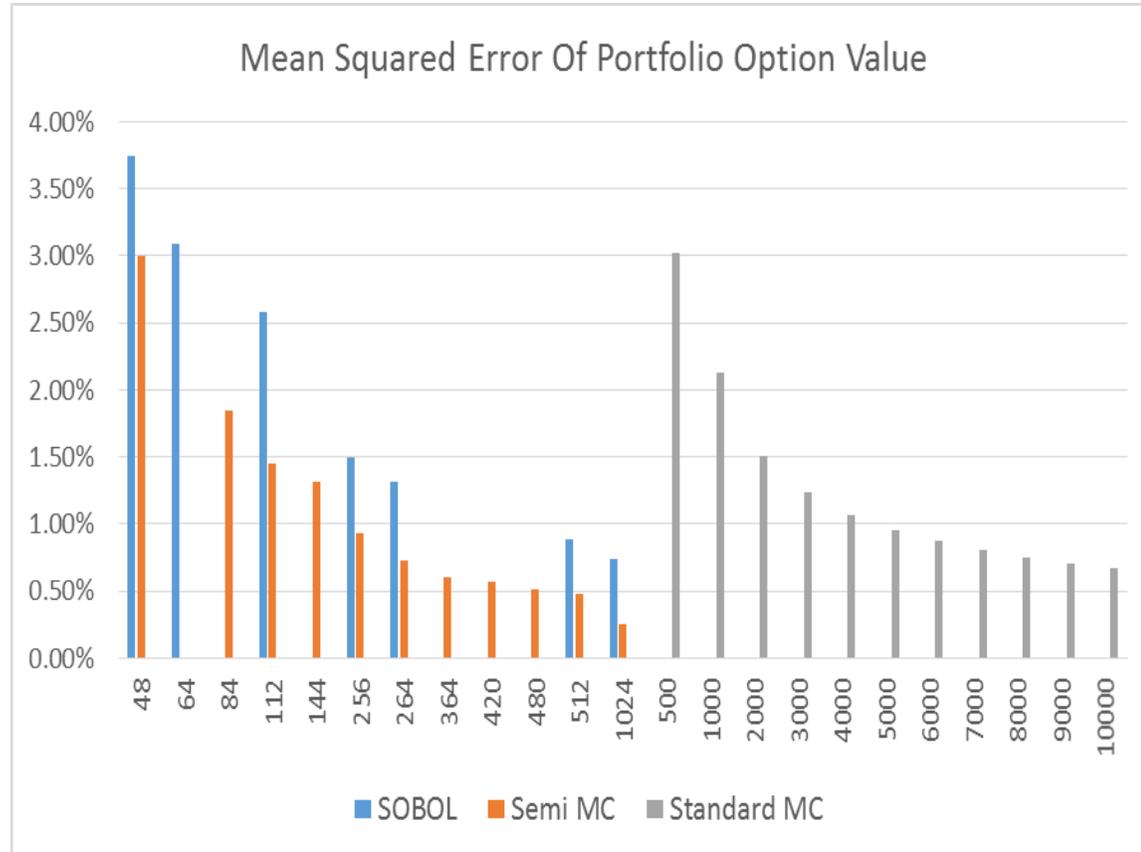
## RSME Comparison: SMC and MC



RSME	Number of Scenarios Standard MC	Number of Scenarios Semi MC	Compression Ratio
1.00%	5,000	256	20:1
1.50%	2,000	112	18:1
2.00%	1,000	84	12:1

# Experiments

## RSME Comparison: SMC and Sobol Sequence



RSME	Number of Scenarios Standard MC	Number of Scenarios Sobol	Compression Ratio	Number of Scenarios Semi MC	Compression Ratio
1.00%	5,000	512	10:1	256	20:1
1.50%	2,000	256	8:1	112	18:1
3.00%	500	64	8:1	48	10:1

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Final Remarks

# Final Remarks

## Conclusion

### » Advantages

- Preserves distribution, not average only
- Very flexible – allows for adjustments
- Improves performance when combined with other methods
  - › Policies clustering
  - › Per policy generate
  - › Low discrepancy sequences

### » Problems to be resolved

- Parameters optimization:
  - › Splitting for non-linear models (Heston, Libor,...)?
  - › Optimal dimension choice
  - › Optimal NIS choice
- Hard to evaluate an error without finding the benchmark
- Harder to explain

# Formula for MSE

Just a reference for Math fans

» Mean Squared Error for SMC looks like (2-nd row is for optimal choice of  $N_j$ )

$$\begin{aligned}MSE_{SMC} &= \left[ \bar{f} - \sum_j w_j E_y \{ f(x_j, y) | x_j \} \right]^2 + \sum_j \frac{w_j^2}{N_j} \text{Var}_Y \{ f(x_j, y) | x_j \} \\ &\approx \left[ \bar{f} - \sum_j w_j E_y \{ f(x_j, y) | x_j \} \right]^2 + \frac{1}{N} \left[ \int_{(\Omega)} \text{std}_y \{ f(x, y) | x \} p_X(x) dx \right]^2 \\ &= C_1 \cdot M^{-2p} + C_2 \cdot N^{-1}\end{aligned}$$

Here  $\bar{f}$  is a true value,  $w_j$  are NIS weights,  $N_j$  – number of simulations for  $x_j$ ,  $p$  depends on NIS,  $M$  is number of NIS nodes,  $N = \sum_{j=1}^M N_j$

» It helps to balance split and  $N_j$ .

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Thank You!

Q&A

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