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How Long is Long in Longevity?

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How long is long in longevity?

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Abstract

Longevity refers to the ability to last for a long time. A fundamental question is: how long is a long time? Over the history, the onset of longevity has been defined in terms of fixed chronological ages, (i.e. 60 years, 70 years, etc.). These thresholds ages are arbitrarily defined, and are not informative about how old individuals are.

The aim of this article is to explore the onset of longevity. First, I revisit thresholds of old age, their mathematical representation, and their demographic interpretation. Second, I propose a framework that allow us to sketch a formal definition of when a long life begins. Rather than looking at chronological ages, I describe start of longevity in terms of survivorship-ages (i.e., the age where s proportion of the population is still alive). This framework allows to link the commencement of longevity with theoretical insights from evolutionary biology and reliability theory.

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1 Introduction

Longevity has arbitrarily been defined to start between ages 60 and 70 throughout human history. This can be traced back to ancient Greece where men were requested to complete military service until age 60. In medieval Europe, age 70 was the age of tax exemption (Johnson et al., 1998). After these ages, people were considered “old enough” and uncapable to perform activities in benefit to the community. Augustin de Hippo categorized the lifetime of an individual into seven ages (or decades), where old age began at the seventh (Thane, 2020). The idea that old age started at the seventh decade has also prevailed in literature. For example, William Shakespeare describes the Seven Ages of Man in his play “As You Like it”, where the seventh decade is depicted as the loss of functional activities, decrepitude, and solitude (Shakespeare, 1767). Shakespeare writes about old age as a reference to the end of life: ”Last scene of all; that ends this strange, eventful history; is second childishness and mere oblivion; sans teeth, sans eyes, sans taste, sans everything”. Shakespeare’s view on old age is in stark contrast of the depiction of a 70-year-old living in modern industrialized populations, where many individuals survive to the seventh decade (and even older ages) in good health (Alvarez et al., 2021b; Hitt et al., 1999; Medford and Alvarez, 2021).

Current definitions of the onset of longevity are associated with retirement ages. In 1889, when designing the first pension system in Germany, the chancellor Otto von Bismarck established pensions to be paid not at a fixed age but when a worker was certified unfit to work. Bismarck chose, what he believed was in the interest of the state, to acknowledge diversity in aging rather than to impose chronology (Thane, 2020). Later in 1891, Denmark also introduced state-funded pension at age 65 for the very poor and “respectable” people (i.e., those with no record of drunkenness or other bad behavior). New Zealand and Australia followed Denmark’s initiative by fixing retirement at age 65, whereas Britain did so at age 70 (Thane, 2020). Since then, the idea that old age starts at retirement has prevailed and ages between 60 and 70 are the predilect thresholds of old age.

In recent years, some European countries such as Finland, Denmark and Italy have addressed the fact that people live longer than before and that fixed chronological ages do not provide any informative measure of when a person should retire from working. These countries have adopted policies that change pension age and link it to developments in life expectancy (Alvarez et al., 2021a). Under this perspective, pension age is not fixed, but it will increase over time as life expectancy increases, reflecting longevity increases.

Indeed, life expectancy is the single most known indicator of longevity (Luy et al., 2020). It is defined as the average lifespan in a population. It has been shown that the maximum life expectancy in a population is increasing at a rate of 2 months per decade in the most longevous populations (Oeppen and Vaupel, 2002). There are no doubts that life expectancy is a useful measure of longevity. However, its main disadvantage is that it only depicts the mean lifespan of a population, and neglects information about the distribution of lifespans. Additional summary population indicators have been proposed to account for this drawback of life expectancy (i.e. the modal age, percentiles of the death distribution etc.). There are two questions that arise from this. First, does longevity begin when the mean lifespan has died (i.e., at life expectancy)? Second, what other indicators provide insights about the onset of longevity?

In this paper I review demographic indicators that are natural candidates to define the onset of longevity. Then I propose a new perspective to assess the onset of longevity. Rather than looking at chronological ages, I define the onset longevity in terms of s-ages (i.e. the age at which a proportion s of the population is still alive). I derive the mathematical properties of s-ages and compare them to meaningful thresholds of longevity. Finally, I relate the proposed threshold to definitions stemming from different disciplines.

2 The onset of longevity: in the quest for a formal definition

As mentioned in the introduction, life expectancy is the most well-known indicator of longevity. It is formally defined as

$$e(x) = \frac{1}{s(x)} \int_x^{\omega} s(a)da, \tag{1}$$

where $s(x)$ is the survival function at age x. Life expectancy is expressed in years and indicates how long the average lifespan is in a population. It is customary measured at birth (i.e. $x = 0$). Thus, if longevity is defined as living beyond the average, then life expectancy is the natural threshold of longevity (Canudas-Romo, 2010).

Another common demographic indicator is the modal age at death, otherwise defined as the most common age at death (Canudas-Romo, 2008; Diaconu et al., 2022; Horiuchi et al., 2013). Let $f(x)$ be the probability density function of the death distribution, then the mode is defined as the age M, so that satisfies the condition $\frac{df(x)}{dx}|_{x=M} = 0$. Canudas-Romo (2008) showed that derivative of $f(x)$ equals

$$\frac{df(x)}{dx} = f(x)[\frac{d\ln\mu(x)}{dx} - \mu(x)], \tag{2}$$

Where $mu(x)$ is the force of mortality at age x. Then $\frac{df(x)}{dx}$ equals zero when either $f(x)$ is zero or when $\frac{d\ln\mu(x)}{dx} - \mu(x)$ equals zero. In the first case, $f(x) = 0$ indicates no deaths. The second case implies that at the modal age at death M, the force of mortality equals its relative derivative with respect to age x,

$$\mu(x) = \frac{d\ln\mu(x)}{dx}|_{x=M} \tag{3}$$

The modal age at death has a meaningful demographic interpretation and it is used to describe the dynamics of the postponement of deaths toward higher ages. However, this indicator has a fundamental disadvantage. In populations with high infant mortality (i.e. some Latin American population, see Alvarez et al. (2019)), there might be "two modes" in the distribution of lifespans; one located at birth and the other one located in the age range 70-90 in modern populations. Mathematically, Equation (2) has two local maximas and defining the onset of longevity in terms of the mode is problematic. A common "solution" to this issue is to truncate the death distribution at age 5 or 10 and then calculate M. However, this approach is arbitrary, and might yield to biased results.

Another meaningful demographic indicator is the entropy of the survival function (Aburto et al., 2019). This indicator can be understood in two complementary ways. First, it can be interpreted as the elasticity of life expectancy. In this case, the entropy provides insights such as "If mortality is reduced at all ages by $x\%$, then life expectancy will reduce by $y\%$". Second, the entropy can also be seen as a measure of lifespan variability, so that the higher the values for the entropy, the higher the variability (or less equal) the lifespans are in a population. Formally the entropy is defined as

$$h(x) = \frac{\int_x^{\omega} s(a)\ln s(a)da}{s(x)}. \tag{4}$$

Depending on where mortality reductions take place, the entropy tends to increase or decrease (Aburto et al., 2019). Thus, if mortality reductions over time occur at all ages, there exists a unique threshold, age t, that separates positive from negative contributions to the entropy h(x), resulting from those mortality reductions. Let $H(x)$ be the cumulative hazard at age x, so that $H(x) = \int_0^x \mu(a)da$, then the threshold age t is reached when the following condition is fulfilled:

$$h(t) = 1 + h(0) - H(t). \tag{5}$$

Given that all functions in Equation (5) are positive, then it is guaranteed that the threshold age t is unique. Threshold age t is a meaningful indicator that separates "young deaths" and "old deaths" according to where improvements in mortality take place. Therefore, threshold age t can be conceived as the onset of longevity.

3 Survivorship ages

All indicators described above, are defined in terms of chronological ages. Alvarez and Vaupel (2023) introduced a framework to study mortality in terms of a different scale: survivorship ages. Survivorship ages (or s-ages) are defined as the age $x(s)$, where a proportion s of the population is still alive so that $s \in [0,1]$. This framework is based on a fundamental fact: every individual has a chronological age and given that individuals live in populations and death rates fall relentlessly with age, every individual also has a unique survivorship age. Thus s-ages can be thought as the inverse of the survival function so that $x(s) = s^{-1}(x)$. This perspective links population dynamics to the individual lifespan as the s-age of the individual depends on the "duration" of the other individuals who belong to the same population.

Given that $s(x)$ is continous and differentiable over time, the inverse function theorem guarantees that $x(s)$ is continous and differentiable over time. Therefore, Alvarez and Vaupel (2023) define $\psi(s) = -\dfrac{dx(s)}{ds}$ as probability density function of $x(s)$ and show that functions $\psi(s)$ and $f(x)$ are also reciprocal so that

$$\frac{dx(s)}{ds} = [\frac{ds(x)}{dx}]^{-1}, \tag{6}$$

and

$$\psi(s) = [f(x)]^{-1} \tag{7}$$

Furthermore, Alvarez and Vaupel (2023) show that the force of mortality can be expressed as

$$\mu(s) = [s\psi(s)]^{-1} \tag{8}$$

Similarly, life expectancy can be re-expressed in terms of s-ages:

$$e(s) = \frac{1}{s}\int_0^s x(a)da \tag{9}$$

As shown in this section, the study of longevity in terms of survivorship-ages opens the door to new avenues of research including the definition of the onset s-age of longevity.

4 The onset of longevity in terms of s-ages

Contrary to chronological ages, s-ages provide an informative description of the dynamics of the population. As shown in Equation (8), they link the risk of dying to the actual source of change: death and survival. The question is: what s-ages are candidates to describe the onset of longevity?

For this purpose, I refer again to the cumulative hazard function $H(x) = \int_0^x \mu(a)da$. This function has a meaningful interpretation in different applications. In reliability theory it is defined as the "hazard potential" (Singpurwalla, 2006), where "potential" refers to a feature parallel to that of potential energy in physics. In reliability applications, $H(x)$ describes the accumulated damage that a system is exposed until failure. Analogously, I define here the $H(x)$ as the amount of mortality that the population is exposed at age x.

Let us now focus on the amount of mortality that would be necessary to kill the individual with the average lifespan. For this purpose, I define $\bar{H}$ as the weighted average of the hazard potential, where the weights are given by the density function $f(x) = \mu(x)s(x)$, so that:

$$\bar{H} = \frac{\int_0^\omega H(a)f(a)da}{\int_0^\omega f(a)da} = \frac{\int_0^\omega \mu(a)\int_0^a f(u)duda}{\int_0^\omega f(a)da} = \frac{\int_0^\omega \mu(a)s(a)da}{\int_0^\omega f(a)da} = 1 \tag{10}$$

The fact that $\bar{H} = 1$ is a very insightful result. It indicates that when the cumulative hazard reaches the unity (i.e. $H(x) = \bar{H} = 1$), the population has been exposed to "sufficient" mortality to kill the individual with the average lifespan. This is a meaningful result to formally define the onset of longevity.

In terms of s, the cumulative hazard of s is reduced to $H(s) = -\ln s$. Thus, the threshold $\bar{H} = \ln s = 1$ is reached when $s \approx 0.37$. This indicates that the longevity begins when survival levels are about 0.37. In other words, only 37% of the population is longevous and this threshold is reached at the s-age $x(0.37)$.

5 Biological perspectives on the onset of longevity

So far, I have described the onset of longevity in terms of the properties and common interpretation of demographic indices. This can also be done in terms of postulates from other disciplines. For example, a prevailing view in evolutionary biology is well described by Carey and Gruenfelder (1997). They argue that in the Darwinian sense of evolution "longevity starts when there is no longer age-specific pressure from natural selection. However, health and vitality trajectories are determined by evolutionary forces at young ages."

Vaupel (2003) follows up on Carey and Gruenfelder (1997) insight and uses a baseball analogy to describe the onset of longevity: "The speed and trajectory of a ball are governed by the pitcher's strength and skill up to the moment when the ball leaves the pitchers hand. Thereafter, the ball's course is determined by the force of gravity acting on the momentum of the ball. Similarly, the remaining trajectory of life is determined by the force of failure acting on the momentum produced by Darwinian forces operating earlier in life". Vaupel (2003) coins the term "post-Darwinian longevity" in reference to the "force of failure" does not hinge on classic Darwinian description of the interaction between fitness and natural selection. By evoking the insights of Carey and Gruenfelder (1997) and Vaupel (2003) and relating them to the definition of hazard potential described above, I postulate that $\bar{H} = 1$ is the threshold where the force of failure becomes the main force that governs the length of life. In consequence, $x(0.37)$ can be seen as the s-age that defines the onset of post-Darwinian longevity.

6 Empirical illustration of the onset of longevity

In this section I use empirical data to illustrate the indices of longevity introduced in previous sections. I use sex-specific mortality data from the (Human Mortality Database, 2020) from the Denmark, France and the United States. I start the analysis at birth and cover the period 1950-2020. I use P-Splines to produce continuous and smoothed mortality surfaces. In particular, continuous surfaces are required in the calculation of s-ages (Alvarez and Vaupel, 2023).

Figure 1 shows the survival function $s(x)$ for females in the United States for the years 1950, 1970, 1990 and 2017. This function indicates the proportion of survivors at any chronological age x. The circles indicate the location of s-ages $x(s)$. Function $x(s)$ indicates the age at any survival level $s \in [0,1]$. Given that both functions are reciprocal, $x(s) = s^{-1}(x)$, it is straightforward to obtain one in terms of the other one. We can observe that the location of s-ages has shifted over time. This pattern indicates the steady postponement of survival (Alvarez and Vaupel, 2023; Zuo et al., 2018)

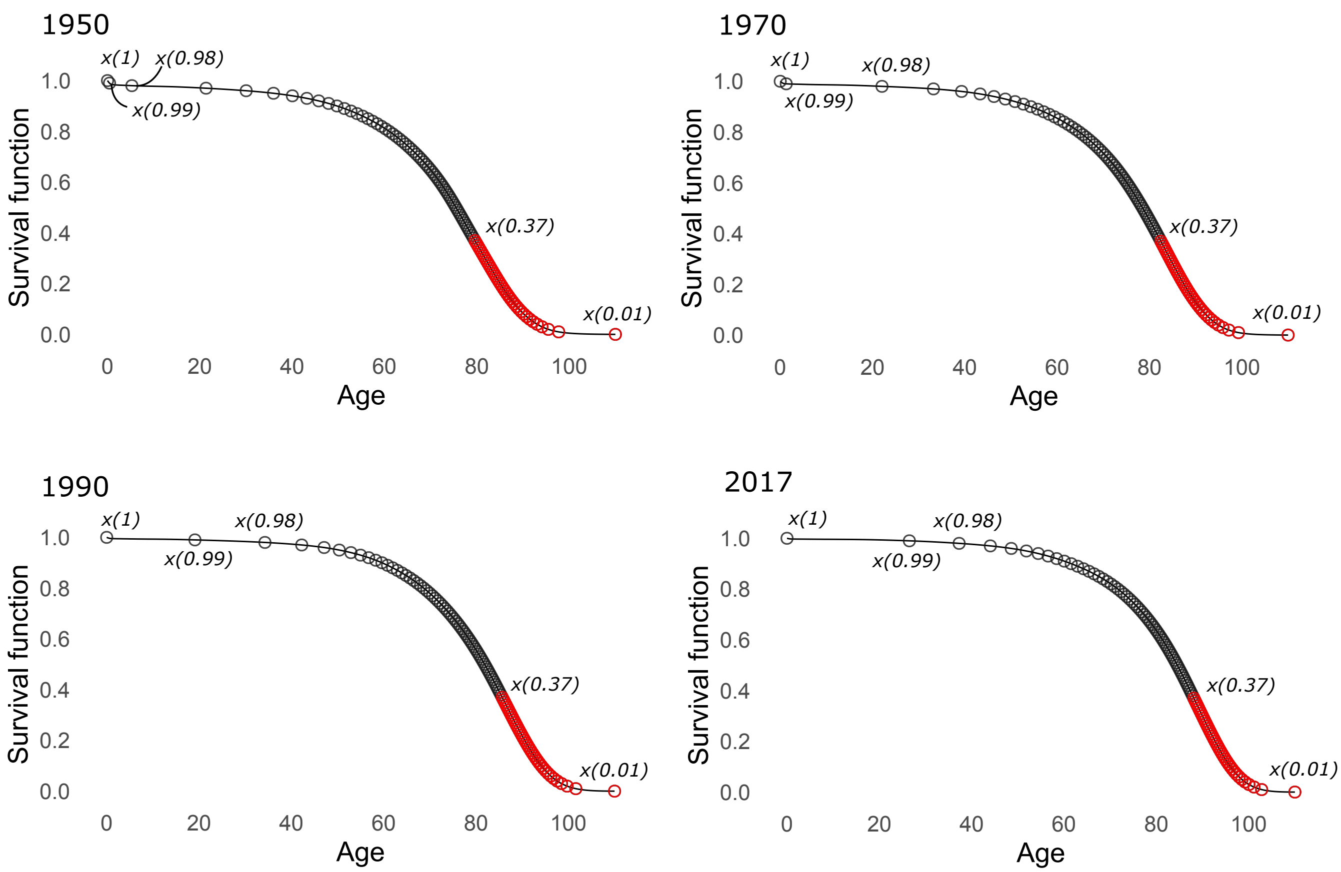

Figure 1: Survivorship function for females in the United States, 1950, 1970, 1990 and 2017. Circles indicate the location of survivorship-ages. Red circles indicate those s-ages that pertain to the tail of longevity as defined by the onset s-age $x(0.37)$.

Red circles in Figure 1 indicate those s-ages that are above the threshold of longevity $\bar{H}$ defined in the previous section. As mentioned before, this threshold can be interpreted as the value of the hazard potential where the population has been exposed to enough mortality to kill the average individual. In terms of s-ages this refers to the s-age $x(0.37)$. Thus, red circles indicate s-ages where post-Darwinian longevity prevails. Under this perspective, individuals making it to any of the red circles have survived "long enough" in comparison to their peers. Note that the threshold s-age $x(0.37)$ indicates that there is always 37% of the survivors in any population that become longevous. However, s-ages below $x(0.37)$ have also shifted over time, which is an indication that the tail of longevity is extending.

How does trends over time in s-age $x(0.37)$ compare with other thresholds of longevity? In Figure 2, I show trends over time in life expectancy, modal age at death, threshold to the entropy and the s-age $x(0.37)$ for females in Denmark, France and the United States. These indicators represent a characteristic of the population and capture well the postponement in survival. Figure 2 shows many regularities in these indicators. First, they all move toward older ages over time. Second, they are located above age 70, entailing that, regardless of the used indicator, longevity begins well after retirement ages. Third, the development of these thresholds is remarkably similar across the populations studied here. I analyze these similarities in detail.

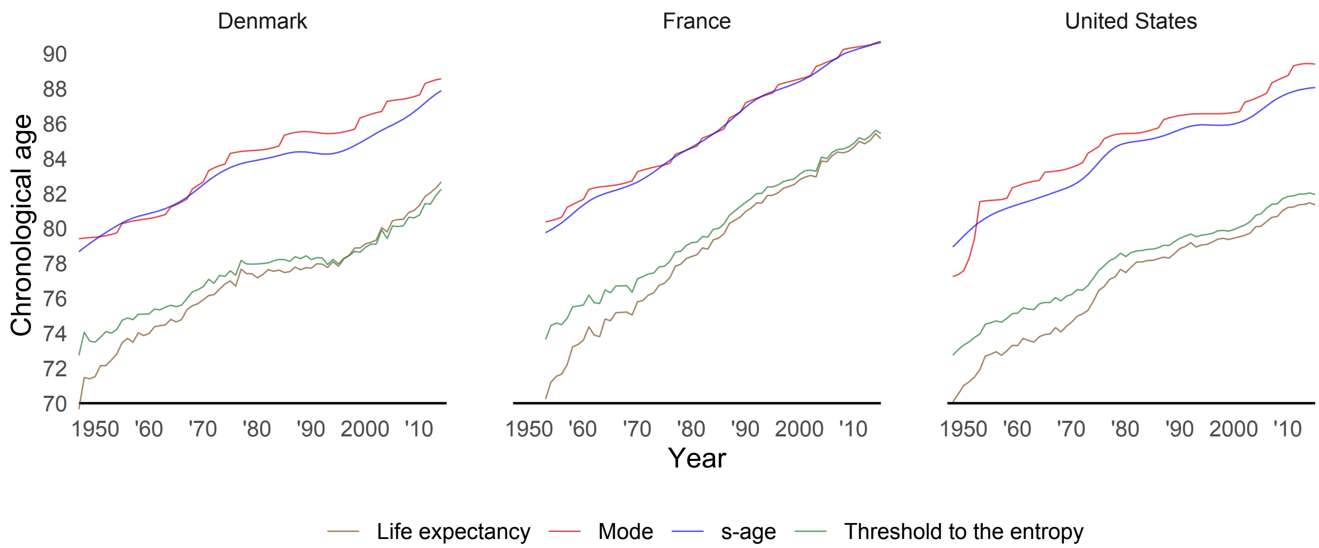

Figure 2: Trends in measures that describe the onset of longevity. Females in Denmark, France, and United States, 1950-2020.

Trends in life expectancy and the threshold to the entropy depicted in Figure 2 are close to each other and have a similar development over time, which is in line with Aburto et al. (2019) results. The entropy is a characteristic of life expectancy so that it is an indicator of the sensitivity to life expectancy to changes in mortality rates. Thus, the threshold age to the entropy entails that mortality reductions at ages above life expectancy have a more pronounced effect in increases in life expectancy than improvements occurring at younger ages. Life expectancy and the threshold to the entropy both serve as indicators of the onset of longevity, when longevity is defined in terms of those that survive beyond the average lifespan in the population.

Figure 2 also shows that the s-age $x(0.37)$ and the mode are close to each other for the three populations analyzed here. The similarity between M and $x(0.37)$ indicates that the mode is achieved to s-ages close to $x(0.37)$. In terms of the hazard this indicates that both indicators are achieved when the hazard potential equals $\bar{H} = 1$.

Indeed, Canudas-Romo (2008) uses the Gompertz mortality change model (Bongaarts and Feeney, 2002, 2003) to unravel an interesting characteristic of the modal age at death. This model is a generalization of the standard Gompertz function and the mortality hazard is denoted as:

$$\mu(x,t) = \mu(0,t)e^{bx}, \tag{11}$$

where $\mu(0,t)$ is the initial value of mortality at time t. This means that the initial value of the hazard changes over time, in contrast to the regular Gompertz model where this value is fixed to $\mu(0,t) = a$. Under this model, the mode at time t is $M(t) = \frac{\ln[b]-\ln[\mu(0,t)]}{b}$ and the survival function at the mode is expressed as $s(M,t) = e^{\frac{\mu(0,t)}{b}-1}$. Thus, when $\mu(0,t)$ declines over time, the survival function at the mode M, approaches $s = 0.37$:

$$\lim_{\mu(0,t)\to 0} s(M,t) = e^{-1} \approx 0.37, \tag{12}$$

and in consequence the mode $M(t)$ approaches the s-age $x(0.37)$. This finding sheds light on the relationship between the most common age at death and the s-age where the population has been exposed to enough mortality to kill the average.

7 Concluding remarks

When does *old age* begin?. In this article I reflect on this basic question. First, I review common demographic indicators that are natural candidates to answer this issue. Next, I introduce s-ages as meaningful quantities to outline the onset of longevity, and link them to perspectives stemming from different disciplines (i.e. reliability and evolutionary biology).

It is striking that throughout most of the human history, old age has been considered to start at a specific chronological age, which has been fixed at ages between 60 and 70 depending on when (or at what age) a person is officially perceived unable to perform certain activity (e.g. pay taxes, retire from work, etc.). These definitions are arbitrarily set and have prevailed over centuries such that, even in modern times, these fixed-thresholds of old age are closely linked to retirement ages.

I show in this study that fixed chronological ages have little or nothing to do with longevity. In particular because any of the indicators analyzed here are located at ages way above age 70 and all of them increase over time. These patterns reflect the fact that more and more individuals survive to ages 60 and 70 in good health and vitality, and at the same time, have good prospects of surviving to even higher advanced ages (Alvarez et al., 2021b; Medford and Alvarez, 2021). This simple analysis clearly indicates a regularity: the commencement of longevity is dynamic rather than fixed.

The fundamental characteristic of longevity is the comparison of durations across individuals so that one must compare its duration (or lifespan) with the duration of their peers to determine that someone (or something) has lasted *long enough*. Thus, the concept of longevity only makes sense in a population as more than one individual is necessary to make such comparisons. Each of the indicators reviewed in this article (i.e. life expectancy, mode, threshold to the entropy, $x(.37)$) depict a specific characteristic of the population of interest, and in consequence, are informative of how long individuals have lasted in that specific population and can be used to define old age.

It is also important to highlight that there is no unique way to define longevity and that the indicators reviewed here are some of the multiple choices that one can use to this aim. Regardless of the indicator used, it is more meaningful to refer to a specific characteristic of the population, rather than refer to a fixed chronological age.

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