
SOCIETY OF ACTUARIES
Quantitative Finance and Investment Core

Exam QFICORE

MORNING SESSION

Date: Wednesday, April 25, 2018

Time: 8:30 a.m. – 11:45 a.m.

INSTRUCTIONS TO CANDIDATES

General Instructions

1. This examination has a total of 100 points. It consists of a morning session (worth 60 points) and an afternoon session (worth 40 points).
 - a) The morning session consists of 9 questions numbered 1 through 9.
 - b) The afternoon session consists of 7 questions numbered 10 through 16.

The points for each question are indicated at the beginning of the question.

2. Failure to stop writing after time is called will result in the disqualification of your answers or further disciplinary action.
3. While every attempt is made to avoid defective questions, sometimes they do occur. If you believe a question is defective, the supervisor or proctor cannot give you any guidance beyond the instructions on the exam booklet.

Written-Answer Instructions

1. Write your candidate number at the top of each sheet. Your name must not appear.
2. Write on only one side of a sheet. Start each question on a fresh sheet. On each sheet, write the number of the question that you are answering. Do not answer more than one question on a single sheet.
3. The answer should be confined to the question as set.
4. When you are asked to calculate, show all your work including any applicable formulas. When you are asked to recommend, provide proper justification supporting your recommendation.
5. When you finish, insert all your written-answer sheets into the Essay Answer Envelope. Be sure to hand in all your answer sheets because they cannot be accepted later. Seal the envelope and write your candidate number in the space provided on the outside of the envelope. Check the appropriate box to indicate morning or afternoon session for Exam QFICORE.
6. Be sure your written-answer envelope is signed because if it is not, your examination will not be graded.

Tournez le cahier d'examen pour la version française.

****BEGINNING OF EXAMINATION****
Morning Session

1. (6 points) Let r be the domestic risk-free rate and f the foreign risk-free rate. Let S_t be the price of 1 unit of foreign currency in terms of the domestic currency. Assume S_t follows the stochastic differential equation (SDE)

$$dS_t = (r - f)S_t dt + \sigma S_t dW_t$$

where W_t is a standard Wiener process under the probability measure \mathbb{P} , and $r, f, \sigma > 0$ are constants.

- (a) (1 point) Calculate $\text{Prob}(W_1 + W_2 > 1)$.
- (b) (1 point) Solve the SDE for S_t .

Let $Z_t = \frac{1}{S_t}$.

- (c) (2 points) Show that
- (i) $dZ_t = Z_t [(f - r + \sigma^2)dt - \sigma dW_t]$.
- (ii) $Z_t e^{(r-f)t}$ is not a martingale under the measure \mathbb{P} .
- (d) (2 points) Construct a measure $\tilde{\mathbb{P}}$ such that $Z_t e^{(r-f)t}$ is a martingale under $\tilde{\mathbb{P}}$.

2. (7 points) Assume the Black-Scholes framework.

(a) (1 point) Describe the assumptions of the Black-Scholes model.

Let $S(t)$ be the time- t price of a stock. You are given:

- The stock does not pay dividends.
- The volatility of the stock is σ .
- The continuously compounded risk-free interest rate is r .

A power contract on the stock matures at time T . Its time- t price for all $t \leq T$ is given by $F(t) = S(t)^{\left(\frac{c}{\sigma^2}\right)}$, where c is a positive constant.

(b) (1.5 points) Determine C .

Let $V(t, F)$ be the time- t value of a European option written on the power contract.

(c) (3 points) Show that $V(t, F)$ satisfies the following partial differential equation.

$$\frac{\partial V(t, F)}{\partial t} + rF \frac{\partial V(t, F)}{\partial F} + \frac{2r^2 F^2}{\sigma^2} \frac{\partial^2 V(t, F)}{\partial F^2} - rV(t, F) = 0$$

(d) (1.5 points) Show, based on part (c), that the time- t value of a K -strike European put written on the power contract is

$$V(t, F) = Ke^{-r(T-t)}N(-d_2) - FN(-d_1),$$

$$\text{where } d_1 = \frac{\ln\left(\frac{F}{K}\right) + \left(r + \frac{2r^2}{\sigma^2}\right)(T-t)}{\frac{2r}{\sigma}\sqrt{T-t}}, \text{ and } d_2 = d_1 - \frac{2r}{\sigma}\sqrt{T-t}.$$

(Hint: Use the fact that the time- t value of a put option on the stock is

$$P(t, S) = Ke^{-r(T-t)}N(-d_2^*) - SN(-d_1^*),$$

$$\text{where } d_1^* = \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}, \text{ and } d_2^* = d_1^* - \sigma\sqrt{T-t}.)$$

3. (8 points) Assume that the stock price S_t follows the process:

$$dS_t = rS_t dt + \sigma S_t dW_t$$

where r is the constant risk-free rate, $\sigma > 0$ is the constant stock price volatility and $\{W_t : 0 \leq t \leq T\}$ is a standard Wiener process under the risk-neutral measure \mathbb{Q} .

- (a) (1 point) Derive the SDE satisfied by the process $X_t = S_t^2$.

Consider the time- t price $f(S_t, t)$, (f_t for short), of a contract which has the terminal payoff at expiry T :

$$f_T = S_T^2$$

- (b) (3 points) Derive the following:
- The expression for the time- t value f_t in terms of S_t ,
 - The delta of the contract.

Denote by $c_t(K, T)$ the price of a call option at time t on the stock with expiry T and strike K . Assume that call options with all strikes $K \in [0, \infty)$ trade in the market. The terminal payoff f_T can then be replicated by a portfolio of these call options with $g(K)$ units of the call with strike K . That is,

$$f_T = \int_0^\infty g(K) \max(S_T - K, 0) dK$$

- (c) (3 points) Derive the following:
- $g(K)$ for all $K \in [0, \infty)$,
 - The expression for f_t in terms of $c_t(K, T)$, assuming that no arbitrage exists.
- (d) (1 point) Compare and contrast the hedging strategies that can be developed using the results obtained in (b) and (c).

4. (7 points)

- (a) (0.5 points) State the conditions for a process to be a martingale.
- (b) (3 points) Show that each of the following processes is a martingale by using Ito's formula to verify that it satisfies all the conditions in part (a).
- (i) $W_t^3 - 3tW_t$,
- (ii) $e^{\lambda W_t - 0.5\lambda^2 t}$.

Let X be a random variable with $E[|X|] < \infty$. Define M_t as $M_t = E[X | I_t]$ where I_t is a filtration.

- (c) (1 point) Show that M_t is a martingale.
- (d) (2.5 points) Find the process $g(t, W_t)$ for each of the following martingales such that $M_t = M_0 + \int_0^t g(u, W_u) dW_u$ for $t \leq T$.
- (i) $M_t = E[W_T^3 | I_t]$,
- (ii) $M_t = E[e^{\lambda W_T} | I_t]$.

5. (6 points) Let:

- t = time variable
- S = price of a stock at time t
- V = price of an option on the stock at time t
- r = risk-free interest rate
- σ = implied volatility of the stock price

(a) (0.5 points) Identify each of following partial derivatives with the named variables:

Partial derivatives: $\frac{\partial V}{\partial t}$, $\frac{\partial V}{\partial S}$, $\frac{\partial^2 V}{\partial S^2}$, $\frac{\partial V}{\partial r}$,

Named variables: Delta, Gamma, Rho, Speed, Theta, Vega

Assume that the option price follows the Black-Scholes equation. Suppose you bought one option and delta-hedged your position with the underlying stock. Your portfolio value is defined as the sum of the value of the option and the value of the position in the underlying stock. At time t , you are given: $S = 100$, $V = 11.48$, $r = 0.01$, $\sigma = 0.1$, Delta = 0.89, Gamma = 0.02.

- (b) (1 point) Determine your portfolio value and Theta at time t .
- (c) (2 points) Outline characteristics of delta hedging and suggest another hedging strategy that could mitigate the weaknesses of delta hedging.

Also at time t , you fitted the following quadratic function using the available market data at that time to reflect the relationship between the option's implied volatility σ and its strike price K :

$$\sigma = 0.003\% * K^2 - 0.64\% * K + 49.5\%$$

The stock price S changes from 100 at time t to 120 at time $t+1$.

- (d) (1 point) Calculate the at-the-money option's implied volatility at time $t+1$ based on the assumption that implied volatilities exhibit
- (i) *Sticky strike* behavior,
 - (ii) *Sticky delta* behavior.

5. Continued

- (e) (0.5 points) Identify one market in which sticky strike behavior is commonly observed, and one market in which sticky delta behavior is commonly observed.

The CFO of your company emailed you his vision for the company's hedging program:

"We're not really looking to make money with the hedging program – in fact, we know that we will lose money on occasion. Ideally, the hedge program will have minimal fluctuation in daily profit and loss."

- (f) (1 point) Recommend and justify the volatility measure that should be used for the hedging program in order to be consistent with the CFO's vision.

6. (6 points) You are given the following information about a European put option on a non-dividend paying stock as of today:

- Stock price = \$100
- Put option's strike price = \$100
- Put option's remaining time to expiration = 1 year
- Continuously compound risk-free interest rate = 2% for all maturities
- Market implied volatility of the stock = 20%
- All assumptions underlying the Black-Scholes option pricing model hold

You can trade the put option today only, but you can trade the stock on any day the market is open. All your trades are to be conducted at the market price.

(a) (0.5 points) Calculate the market price of the put option.

You firmly believe that the put option is actually worth \$5.94 today.

- (b) (1.5 points) Propose a trading strategy today involving the put option and/or its underlying stock such that the strategy will generate daily profit for you until the option expires.
- (c) (2.5 points) Derive the present value of the total profit of your strategy. Define all terms and symbols used in your derivation.
- (d) (0.5 points) Determine the total cash amount generated (or required) for risk-free investment (or borrowing) today as a result of executing your strategy in parts (b) and (c).
- (e) (1 point) Outline the advantages and disadvantages of your strategy in parts (b) and (c).

7. (9 points) A short rate process is defined by:

$$dr_t = (\mu_t - \theta r_t) dt + \sigma_t dW_t$$

where μ_t and σ_t are both deterministic and time-dependent, θ is a positive constant, and W_t is a standard Wiener process.

- (a) (1.5 points) Determine the distribution of r_t including its expected value and variance.

You are given that
$$\int_0^T \int_0^t e^{-\theta(t-s)} \sigma_s dW_s dt = \int_0^T \sigma_t \left(\frac{1 - e^{-\theta(T-t)}}{\theta} \right) dW_t.$$

- (b) (3 points) Derive the time-0 price $P(0, T)$ of a zero-coupon bond maturing at time T .

Now express r_t as $r_t = \alpha_t + x_t$, where α_t is a deterministic function and x_t follows the SDE

$$dx_t = -\theta x_t dt + \sigma_t dW_t$$

with $x_0 = 0$, and assume θ and σ_t are known.

- (c) (1.5 points) Express α_t in terms of μ_t .

Also given:

$$\mu_t = \frac{\partial f(0, t)}{\partial t} + \theta f(0, t) + \int_0^t \sigma_s^2 e^{-2\theta(t-s)} ds$$

where $f(0, t)$ describes the current observed forward term structure.

- (d) (2 points) Show that

$$\alpha_t = r_0 e^{-\theta t} + f(0, t) - f(0, 0) e^{-\theta t} + \int_0^t \frac{\sigma_v^2}{\theta} e^{-\theta(t-v)} (1 - e^{-\theta(t-v)}) dv$$

- (e) (1 point) Describe one use/application and one disadvantage of yield curve fitting for a one-factor model.

8. (6 points) The risk-free spot interest rate $r(t)$ at time t is modeled by the following stochastic process

$$dr(t) = \eta(t) dt + \sqrt{\beta} dW_t$$

where $\beta > 0$ is a constant, $\eta(t)$ is a deterministic function, and W_t is a standard Wiener process under risk-neutral measure.

The corresponding price $Z(r(t); t, T)$ at time t of a zero-coupon bond with \$1 principal maturing at time T is given by

$$Z(r(t); t, T) = e^{A(t, T) - B(t, T)r(t)}$$

where

$$A(t, T) = -\int_t^T \eta(s)(T-s) ds + \frac{1}{6} \beta (T-t)^3$$

$$B(t, T) = T - t$$

- (a) (1 point) Compare and contrast the HJM approach versus the classical approach with regard to pricing interest-sensitive securities.

At initial time t^* , denote by r^* the observed spot interest rate, and $M(t^*, t)$ the observed market price of a zero-coupon bond with \$1 principal maturing at time t .

- (b) (2.5 points) Derive an expression $\eta^*(t)$ for $\eta(t)$ calibrated to the market data such that $r(t^*) = r^*$ and $Z(r^*; t^*, t) = M(t^*, t)$.
- (c) (1 point) Recommend an approach that should be used to fit the above spot interest rate model to the initial term structure.

Let $F(t, T)$ be the instantaneous forward rate at time t with maturity at time T .

- (d) (1 point) Derive the stochastic differential equation governing $F(t, T)$.
- (e) (0.5 points) Assess the relationship between the forward rate dynamics and the initial calibration of the above spot interest rate model.

9. (5 points) You are given the following ABC company's stock price data:

Day (t)	1	2	3	4	...
ABC Stock Price (p_t)	\$10.1	\$11.2	\$10.0	\$10.5	...

You estimated a GARCH(1,1) model for ABC stock's volatility, using the above data and the maximum likelihood method, as below:

$$u_t = \frac{p_t}{p_{t-1}} - 1, \quad t = 2, 3, \dots$$

$$\sigma_3^2 = u_2^2,$$

and

$$\sigma_t^2 = 0.00002 + 0.1 * u_{t-1}^2 + 0.8 * \sigma_{t-1}^2, \quad t = 4, 5, \dots$$

- (a) (2.5 points) Calculate the maximum value of the log likelihood function, ignoring the constant multiplicative factor, of your model using the given ABC stock prices.

Assume that there are 252 trading days per year.

- (b) (0.5 points) Determine the long-run average annual volatility implied by your GARCH(1,1) model.

The daily volatility forecast for Day N ($N > 4$), made on Day 4, is 5.9%.

- (c) (1 point) Determine the number N .
- (d) (1 point) Describe strengths and weaknesses of GARCH models.

****END OF EXAMINATION****
Morning Session

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