State-of-the-Art Hybrid Modeling for Fixed Indexed Annuities and Variable Annuities

Presenter(s):

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State-of-the-art Hybrid Modeling for Fixed Indexed Annuities and Variable Annuities

Russell Goyder
Equity-Based Insurance Guarantees Conference
Chicago, November 17th 2014
Session 3A: 1530 – 1700 hours
Outline

1. Brief review of annuity modeling
2. Hybrid modeling challenges
3. Copula hybrid models
4. Contract representation
5. Analytic risk
6. Valuation compiler
7. Examples
8. Conclusion
Brief Review of Annuity Modeling
The first annuities

1759
First annuity in America offered by a Pennsylvania company to Presbyterian ministers and their families.

1912
The Pennsylvania Company for Insurance on Lives and Granting Annuities offers annuities to the general public.

1952
First variable annuity is issued by TIAA-CREF.

1959
The Supreme Court holds that variable annuities are subject to federal securities regulation.

1982
The Tax Equity and Fiscal Responsibility Act of 1982 allows annuities to keep their valuable tax-deferred status.

Sources: IRI Insight Issue 5 Volume 4 and www.limra.com
The Tax Reform Act of 1984 eliminates the double taxation of realized capital gains of separate accounts.

1984

1995

Annuity industry sales top $100 billion for the year.

1995

1996

GMIB introduced.

1996

1997

Total annuity assets top $1 trillion.

1997

1999

Variable annuity sales top $100 billion a year.

1999

Sources: IRI Insight Issue 5 Volume 4 and www.limra.com
Modern landscape

- 2002: GMAB introduced
- 2004: GMWB introduced
- 2005: Indexed annuity sales top $25 billion for the year
- 2006: The Pension Protection Act of 2006 overhauls the federal pension plan
- 2012: Living benefits make up 84% of all North American sales

Sources: IRI Insight Issue 5 Volume 4 and www.limra.com

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Accumulation phase

Guaranteed benefits are functions of several possible end values.

(PR = Participation Rate)
GMxB features

Accumulation benefit: rollover options

Death benefit

Accumulation benefit

Income Benefit: rollover options

Income benefit

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GMxB modeling literature

1990’s

- Ravindran + Edelist (1996)

2000
- Milevsky + Posner
- Windcliff et al

2002
- Boyle + Hardy
- Ballotta + Haberman

2004
- Milevsky + Salisbury
- van Haastrecht et al

2006
- Coleman et al
- Chen + Forsyth
- Dai et al
- Chen et al
- Bauer et al
- Ulm

2008
- Marshall et al
- van Haastrecht et al

2010
- Benhamou et al
- Blamont + Sagoo
- Kling et al

2012
- Feng + Volkmer
- Krayzler et al

2014
- Huang et al
- Benhamou et al
- Blamont + Sagoo
- Krayzler et al
Model and contract breakdown

- Sparse coverage of the model-contract matrix
- Very limited modeling of equity and rates smiles

<table>
<thead>
<tr>
<th>Mortality</th>
<th>Interest rates</th>
<th>Fund value</th>
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<tbody>
<tr>
<td>Basic</td>
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<td>Lognormal</td>
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<td>Stochastic</td>
<td>1f HJM</td>
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<tr>
<td>Basic</td>
<td>Hull White 2f</td>
<td>Schoebel-Zhu</td>
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<td>CIR</td>
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Article Count

- Sparse coverage of the model-contract matrix
- Very limited modeling of equity and rates smiles
Themes in the literature

Actuarial
- Biometric emphasis
- Stochastic mortality
- Lognormal fund dynamics
- Deterministic interest rates
- Good coverage of contract features
- Monte Carlo / practical pricing methods

Derivatives
- Financial emphasis
- Simple mortality
- Smile models for fund
- Interest rate models
- Focus on GMAB and GMIB
- Focus on (quasi) analytic solutions
Literature summary

- Many complex contract features
- Contract coverage is patchy
- Mortality risks treated independently

- Hybrid models for equity-rates exposure
- Each hybrid model is a research project
- Smile models are pretty rare
Hybrid Modeling Challenges
Why is hybrid modeling so hard?

- Stochastic rates destroy equity calibrations
- Careful analysis for consistent measure
- Simulation headaches
- Non-intuitive correlation
To illustrate:

• Simple example
  – Yet sufficient to illustrate the above challenges
  – Lognormal equity, Hull White interest rates

\[
\frac{dS(t)}{S(t)} = r(t)dt + \sigma_S(t)dW_S(t)
\]

\[
dr(t) = [\theta(t) - \alpha r(t)] dt + \sigma_r(t)dW_r(t)
\]

\[
\rho dt = \langle dW_S(t), dW_r(t) \rangle
\]
Equity-rates example: calibration

Canonical Treatment

Forward:

\[ F_T(t) = \frac{S(t)}{P(t, T)} \]

is a martingale:

\[ \frac{dF_T(t)}{F_T(t)} = \sigma_{F_T}(t)dW_{F_T}(t) \]

where

\[ \sigma_{F_T}(t) = \sqrt{\sigma_S^2(t) + 2\rho \sigma_T(t) \sigma_S(t) B(t, T) + \sigma_r^2(t) B^2(t, T)} \]

and

\[ B(t, T) = \frac{1 - \exp \left( -\alpha (T - t) \right) }{\alpha} \]
Equity-rates example: calibration

Calibrate rates model parameters $a$ and $\sigma_r(t)$

Price options via

$$S(T) = F_T(T) = F_T(0) \exp \left( \Sigma_T(t) W_{FT}(t) - \frac{1}{2} \Sigma_T^2(t) t \right) \bigg|_{t=T}$$

where

$$\Sigma_T(t) = \sqrt{\frac{1}{t} \int_0^t \sigma^2_{FT}(u) du}$$

$$= \sqrt{\frac{1}{t} \int_0^t \sigma^2_S(u) + 2\rho \sigma_r(u) \sigma_S(u) B(u, T) + \sigma_r^2(u) B^2(u, T) du}$$

For a given value of $\rho$, solve for equity vol $\sigma_S(t)$
Equity-rates example: valuation

Bring into common measure, say $T$-forward

\[
\frac{dS(t)}{S(t)} = \left[ r(t) - \rho \sigma_S(t) \sigma_r(t) B(t, T) \right] dt + \sigma_S(t) dW^T_S(t)
\]

\[
dr(t) = \left[ \theta(t) - \sigma_r(t) B(t, T) - \alpha r(t) \right] dt + \sigma_r(t) dW^T_r(t)
\]

Given $x_i \sim \Phi(0,1)$ and $i^{th}$ short rate $r_i$,

\[
S_{i+1} = S_i \left( 1 + \sigma_S(t_i) x_i \sqrt{\Delta t_i} + (r_i - \rho \sigma_S(t_i) \sigma_r(t_i) B(t_i, T)) \right)
\]
Equity-rates example: valuation

Short rate

\[ r_{i+1} = r_{i-1}e^{-a\Delta t_i} - M(t_{i-1}, t_i, T) + z_i \sqrt{V(t_{i-1}, t_i)} + \alpha(t_i) \]

where

\[ \alpha(t) = f(0, t) + \frac{\sigma_r^2(t)}{2a^2} (1 - e^{-at})^2 \]

\[ M(s, t, T) = \frac{\sigma_r^2(t)}{2a^2} \left(1 - e^{-a(T-s)} - e^{-a(T-t)} + e^{-a(T+t-2s)}\right) \]

\[ V(s, T) = \frac{\sigma_r^2(t)}{2a^2} \left(1 - e^{-2a(T-s)}\right) \]

and

\[ z_i = \rho x_i + \sqrt{1 - \rho^2} y_i \quad x_i, y_i \sim N(0, 1) \]
Equity-rates example: correlation

- Calibrate correlation if have quotes
- Rare, so often combination of intuition and historical analysis

Exercise caution - given two functions $f, g : \mathbb{R} \rightarrow \mathbb{R}$

$$\mathbb{E}^T [f(r(t)) g(S(t))] = \mathbb{E}^T [f(r(t))] \mathbb{E}^T [g(S(t))]$$

Not when $\rho = 0$, but when

$$\rho = \frac{\sigma_r(t) B(t, T)}{\sigma_S(t)}$$
Summary

• Hybrid models:
  - Require bespoke calibration techniques
  - Need significant quantitative research
  - Need bespoke software implementation
  - Contain confusing correlation parameters

Hybrid models are rare and low-fi
Our goals

Configure (not code) arbitrary hybrid models

Handle any number of underlyings

Multi-factor marginals

Correlations that make sense

Fund value (SLV)

RATES (LMM)

Withdrawal

Mortality

Copula

Full coverage of “model – contract” matrix

Parallel computation and in-memory caching

Analytic Risk
Copula Hybrid Models
Model the forward directly?

\[ \frac{dS(t)}{S(t)} = r(t)dt + \sigma_S(t)dW_S(t) \]

\[ dr(t) = [\theta(t) - ar(t)] dt + \sigma_r(t)dW_r(t) \]

\[ \rho dt = \langle dW_S(t), dW_r(t) \rangle \]

\[ \frac{dF_T(t)}{F_T(t)} = \sigma_F(t)dW_{F_T}(t) \]

\[ dr(t) = [\theta(t) - ar(t)] dt + \sigma_r(t)dW_r(t) \]

\[ \rho' dt = \langle dW_{F_T}(t), dW_r(t) \rangle \]
Model the forward directly?

\[
\frac{dS(t)}{S(t)} = r(t)dt + \sigma_S(t)dW_S(t)
\]

\[
dr(t) = [\theta(t) - ar(t)] dt + \sigma_r(t)dW_r(t)
\]

\[
\rho dt = \langle dW_S(t), dW_r(t) \rangle
\]

\[
\frac{dF_T(t)}{F_T(t)} = \sigma_{F_T}(t)dW_{F_T}(t)
\]

\[
d\tau(t) = [\theta(t) - ar(t)] dt + \sigma_r(t)dW_r(t)
\]

\[
\rho' dt = \langle dW_{F_T}(t), dW_r(t) \rangle
\]
Decoupled equations

- Reuse existing calibrations
- Reuse existing marginal distribution generators
- Intuitive correlation
  - $\rho' = 0$ means independent observables

What about a consistent measure?
Problem

Anchored to time $T$. Recover spot price at $T$:

$$S(T) = F_T(T) = F_T(0) \exp \left( \Sigma_T(t) W_{FT}(t) - \frac{1}{2} \Sigma^2_T(t) t \right) \bigg|_{t=T}$$

But for any other time $t$, introduce a drift $\chi_T(t)$

$$S(t) = F_t(0) \exp \left( \Sigma_t(t) W_{FT}(t) - \frac{1}{2} \Sigma^2_t(t) t + \chi_T(t) \right)$$

where

$$\chi_T(t) = -\rho' \Sigma_t(t) \sqrt{t} B(t, T) \sqrt{\int_0^t e^{-2\alpha(t-\tau)} \sigma^2_r(\tau) d\tau}$$

- Drift correction tractable in this example
- In general it is not
**Numeraire corrections**

Define factor \( \psi_T(t) \):

\[
S(t) = F_t(0) \exp \left( \Sigma_t(t) W_{FT}(t) - \frac{1}{2} \Sigma_t^2(t) t \right) \exp (\chi_T(t))
\]

\[\equiv S'(t) \psi_T(t)\]

where \( \psi_T(t) = \exp(\chi_T(t)) \). No arbitrage if

\[
\mathbb{E}^{T} \left[ \mathcal{N}^{T}(t) S(t) \right] = P(0,t) \mathbb{E}^{t} \left[ S(t) \right] \equiv P(0,t) F_t(0)
\]

where

\[
\mathcal{N}^{T}(t) = \frac{P(0,T')}{P(t,T)}
\]
Numeraire corrections

Substitute for $S(t)$: $\psi_T(t) = \frac{P(0, t)F_t(0)}{\mathbb{E}^T [\mathcal{N}^T(t)S'(t)\big]}$

1. Denominator is “run hybrid simulation ignoring drift issue”
2. Simple payoff converges very quickly
3. Need one correction factor per stock observation
4. Calculate as a preprocessing step
5. Scale each path by $\psi_T(t)$
6. Payoff independent
7. Model independent
8. Combine with compiler techniques

Multiple currencies: prices

- Asset (foreign) currency $A$, numeraire (domestic) currency $B$
- $B$ worth of one $A$ is $X(t)$

$$X(t) = X'(t) \frac{P_B(0, t) \overline{X}(t)}{E_{B,T}^{B,T} \left[ N_B^T(t) X'(t) \right]}$$

$$S_A(t) = S'_A(t) \frac{P_B(0, t) \overline{X}(t) F_A(t)}{E_{B,T}^{B,T} \left[ N_B^T(t) X(t) S'_A(t) \right]}$$

Lognormal: \[ \rightarrow S_A(t) = S'_A(t) \exp \left( -\rho \sigma_X \sigma_S t \right) \]

- Note use of $X(t)$ in correction for equity
- For $n$ currencies, choose one as numeraire, apply above to rest
Multiple currencies: rates

For rate from $t \rightarrow u$ fixed at $s$ in currency $A$, numeraire $P_B(t, T)$

$$\mathbb{E}^{B,T} \left[ \mathcal{N}_B^T(u)X(u)L_A(s) \mid \mathcal{F}_s \right] = X(0)P_A(0, u)\bar{L}_A(s)$$

SO

$$L_A(s) = L'_A(s) \frac{\bar{X}(u)P_B(0, u)\bar{L}_A(s)}{\mathbb{E}^{B,T} \left[ \mathcal{N}_B^T(u)X(u)L_A(s) \mid \mathcal{F}_s \right]}$$
Any model?

• Credit (default intensity), Inflation, Multi-curve
  – By analogy with the above
• Jumps – implicitly through pdf
• Multi-factor models / discretized SDEs
  – Clarity on marginal vs copula
• Large models (many marginals)
  – Dimensionality reduction for correlation
• Correlation term structure
  – Relate term to incremental correlation

\[ g(s, t) = \frac{f(t) - \gamma^st_1 \gamma^st_2 f(s)}{\gamma^st_1 \gamma^st_2} \]

\[ \gamma^st_i + \gamma^st_i = 1 \]
Copula hybrids summary

• Model the forward, not spot, solves

• Solve the remaining measure issue by imposing Numeraire Corrections directly

• Goals checkup:
Contract Representation
Flows

Flow: obligation to deliver cash or physical asset

Leg: multiple flows

Swap: offsetting legs

Choice: Right to receive flows
Or further rights

Condition: Delegate choice to rule based on observables

Convenience: Null, Remainder, Reweight...

http://www.fincad.com/derivatives-resources/white-papers/optimal-architecture.aspx
Observables

Flow anatomy:

\[ \pm N \cdot \alpha \cdot E^M \left[ \frac{M(0)}{M(t)} X(s) \right] \]

Core language elements for \( X(s) \)

- \(+ / - \)
- \(\max, \min\)
- \(\text{if}(C, A, B)\) for condition \(C\) and results \(A\) and \(B\)
- \(\text{and, or, } <, <=, >, >=\)
- bind to observation date (time)
Example observables

Account value: $A(t)$

Ratchet: $\max( A_i )_{\{t_i\}}$

Rollup: $(1 + r)^i A_i$

Participation rate $r$

Return $y_i = \frac{A_i}{A_{i-1}} - 1$

Ratchet: $G(t_i) = G(t_{i-1})(1 + \max(ry_i, 0))$

Accumulation benefit: $\max(A_i, G_i)$
Example contracts

- **GMAB**: \( \text{flow}(\text{USD}, t, \max(A(t), G(t)) \mathbb{I}_{\tau > t}) \)
- **GMIB\( (t) \)**: \( \text{leg}(t, \max(A(t), G(t)) \max(R(t_i), k) \mathbb{I}_{\tau > t_i}) \)
- **GMIB with rollover option**:
  - \( \text{choice}(\text{GMIB}(t_1), \text{choice}(\text{GMIB}(t_2), \ldots)) \)
- **GMDB**: \( \text{leg}(s, \max(A(t_i), G(t_i)) \mathbb{I}_{t_{i-1} < \tau \leq t_i}) \)
- **GMWB**:  
  - Rational policyholder  
    - \( C_n = \text{choice}(\text{flow}(t_n, wG(t_n)) + \text{scale}(\text{GMIB}(t_n), 1 - w), \text{GMIB}(t_n)) \)
    - \( C_{n-1} = \text{choice}(\text{flow}(t_{n-1}, wG(t_{n-1})) + \text{scale}(C_n, 1 - w), C_n) \)
  - Policyholder behavior model  
    - No choices, model the fraction of fund units remaining
Analytic Risk
Fast, exact first-order exposures

- Portfolio of 250 trades
- 80% 10y Libor swaps
- Remainder:
  - CDS
  - USD-EUR xccy swaps
  - Swaptions, options,
- Desktop PC, core i7 CPU
- Sensitive to >400 quotes:
  - OIS, Libor, basis and xccy swaps, USD and EUR vol cube, equity vols, survival, FX spot ...
- 600x speed-up
- Typically 10-10,000x

Valuation Compiler
Compilation process

Source code \rightarrow Parser \rightarrow AST \rightarrow Semantic Analysis \rightarrow IR \rightarrow Code generation \rightarrow Program \rightarrow Execution

```
int f(int x) {
    int result = (x / 42);
    return result;
}
```

```
clang -Weverything -pthread -fPIC code.cpp
```
Compilation process

1. **Source code**
2. **Parser**
3. **AST**
4. **Semantic Analysis**
5. **IR**
6. **Code generation**
7. **Program**
8. **Execution**

- **Rules**

---

```
llvm ir output:
; ModuleID = 'test.c'
target triple = "x86_64-unknown-freebsd10.0"
; Function Attrs: nounwind
define i32 @f(i32 %x) #0 {
  %1 = alloca i32, align 4
  %result = alloca i32, align 4
  store i32 %x, i32* %1, align 4
  %2 = load i32* %1, align 4
  %3 = sdiv i32 %2, 42
  store i32 %3, i32* %result, align 4
  %4 = load i32* %result, align 4
  ret i32 %4
}
```

```
.text
.globl _Z1fi
.align 16, 0x90
.type _Z1fi, @function
_Z1fi:
 .cfi_startproc
 .Ltmp2:
  pushq %rbp
  .Ltmp3:
  .cfi_def_cfa_offset 16
  .Ltmp4:
  .cfi_offset %rbp, -16
  movq %rsp, %rbp
  movl $42, %eax
  movl %edi, -4(%rbp)
  movl -4(%rbp), %edi
  cltd
  movl -12(%rbp), %edi
  idivl %edi
  movl %eax, -8(%rbp)
```
flow( 1y3m, EUR, min( C, S(1y)*Libor3m ) )

<table>
<thead>
<tr>
<th>Observable</th>
<th>Process</th>
</tr>
</thead>
<tbody>
<tr>
<td>S</td>
<td>Black</td>
</tr>
<tr>
<td>Libor3m</td>
<td>Hull White 1f</td>
</tr>
</tbody>
</table>

\(<S, \text{Libor3m}> = -0.2\)
Valuation compiler

- **Source code**
  - Rules
  - Flow
    - 1y3m
    - EUR
  - Min
    - C
    - Multiply
    - Min
    - Multiply
    - Observe
    - S
    - 1y
    - Libor3m
  - Multiply
  - Simulate
    - S
    - 1.0
    - Libor3m
    - 1.0
    - Numeraire

Payoff operations
- Numeraire correction
- Lognormal
- HW 1f
- Constants
- Copula
- U(0,1)
Valuation compiler

<table>
<thead>
<tr>
<th>Observable</th>
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<tbody>
<tr>
<td>EUR Stock 1</td>
<td>Stoch Local Vol</td>
</tr>
<tr>
<td>EUR Stock N</td>
<td>CGMYe</td>
</tr>
<tr>
<td>USD Stock 1</td>
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<td>Quad. Gaussian 3f</td>
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<td>LMM</td>
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<tr>
<td>Counterparty survival</td>
<td>Hull White 1f</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
<\text{EUR } S_1, \text{ EUR } S_N> & = -0.5 \\
<\text{EUR } S_N, \text{ USD } S_M> & = 0.75 \\
<\text{EUR } S_1, \text{ Libor3m}> & = -0.2 \\
<\text{USD Libor, Euribor}> & = 0.8
\end{align*}
\]
Valuation compiler

1. Contract details
2. Model
   • Marginals
   • Copula
3. Valuation methodology
   • Eg, Simulation
   • Number of paths
4. Desired outputs
   • Value, Risk
   • Histogram, cash flows

1. Parallel computation
   • Multi-threading
   • Multi-process
2. Numeraire Corrections
3. Analytic Risk
4. Cross-platform support
Examples
Example – GMAB paths

- For a given contract, compare models path by path
- Ratchet guarantee with participation rate
- Four one-factor models, same Brownian sample for each
  - Lognormal as baseline
  - Single parameter local vol
  - Two jump-diffusions
- No rates exposure
Example – GMAB histogram

Valuation histogram for a GMAB with 80% participation rate and ratchet guarantee

- Lognormal
- Merton Jump Diffusion
- Extended CGMY
- Dupire Local Volatility
Example – rollover options

- Ratchet guarantee with 80% participation rate as before
- Notional is guarantee level at 10y, coupon is greater of 5y swap rate at 10y and a fixed strike
- Mortality assumption: Γ-Gompertz curve*
- 10 year accumulation phase with death guarantee
- Subsequently, 30 year annuity stream
- Option to start annuity each year from year 10 to 20

* http://paa2012.princeton.edu/papers/121013
Correlation

Effect of correlation on ratchet GMB with death benefit

- Relative value vs. Fund value - 5y swap rate correlation
- BS - HW1f
- BS - HW2f
- Merton JD - HW1f
- Merton JD - HW2f
- CGMYe - HW1f
- CGMYe - HW2f
- Dupire - HW1f
- Dupire - HW2f
Convergence

Valuation convergence for a ratchet GMIB with death benefit

Relative value

Number of paths

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Conclusion

Hybrid modeling is hard

Numeraire corrections \rightarrow Copula hybrid modeling

Copula hybrids + Flows and observables + Compiler ideas = Flexibility + Model-contract combinatorics
State-of-the-art Hybrid Modeling for Fixed Indexed Annuities and Variable Annuities

Russell Goyder
Equity-Based Insurance Guarantees Conference
Chicago, November 17th 2014
References


(BK, forshadowing result that insurance industry was mispricing complex options) (stock disappears somewhere?)


References


