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Abstract

We consider variable payout annuities (VPAs) as a special case of a group self-annuitization scheme. The VPAs are adjusted each year to reflect the investment and mortality experience of the group. We first develop the adjustment factor formula. We then consider the value of the VPA to a retiree with constant relative risk aversion, who may invest her retirement wealth in any combination of the VPA, a fixed annuity, stocks and risk free bonds. We find that using CRRA utility the VPAs represent a major part of the retiree's 'optimal' portfolio. However, when we look at the distribution of income paths under the optimal strategy, we find that it is inconsistent with the reasonable risk preferences of retirees. We adjust the utility function to allow for a fixed floor to the income stream, and find that the role of the VPA in this case is reduced, though still significant. We also consider the case where the retiree wishes to avoid the risk of substantive annual decline in income, and again find a more restricted role for the VPA. Finally, we discuss the results, and the appropriateness of the utility maximization approach, in the light of the qualitative information on risk attitudes from a recent survey of US retirees.

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1 Group Self Annuitization Schemes

Group self-annuitization (GSA) schemes allow individuals to pool some or all of their retirement fund assets with other individuals, with a view to providing income in retirement through a risk sharing arrangement. Each year the income of the surviving members is adjusted to reflect the investment experience of the pooled fund, or the mortality experience of the group, or both.

For individual retirees the GSA offers some of the benefits of an annuity at (potentially) less cost than through a fixed annuity purchased from an insurance company. Furthermore, if investments perform above expectations, and longevity is adequately anticipated, then the extra return in a GSA scheme is returned to the participants, whereas for a fixed annuity, any excess investment income would not increase benefits. This upside opportunity may be an attraction for participants, and it has been suggested (for example, by Maurer *et al.* (2013)) that GSAs could increase annuitization of retirement benefits. However, there is also a downside risk; adverse investment or mortality experience could result in volatile or decreasing annuity payments over time.

In this paper, we assess the value of a GSA-type annuity within a retiree's portfolio. We note that variants of these schemes are available within some employer sponsored DC pension plans. For example, a GSA features in the University of British Columbia (UBC) pension plan¹. Under the UBC version, the yearly amount of the annuity is computed based on an assumed mortality table and an assumed interest rate, which can be selected by the participant to be 4% or 7% per year. The group of retirees share the investment risk and the mortality experience. Every year the annuity payments are recomputed on the same valuation basis (4% or 7%) given the funds available, which depend on the investment return on the fund, the mortality experience of the group, and the cash paid out as annuity payments during the year. We use the term *variable payout annuity* (VPA) for this type of GSA annuity.

Intuitively, this arrangement seems somewhat risky for the retiree, unless she has significant other stable income. The UBC plan results available for the period 1996 to 2013 show that the retirees selecting the GSA option have had a volatile ride. For example, in 2009 the payments in the UBC plan were reduced by 17.4% for the 4% option, and by 19.8% for the 7% option.

The most common approach to assessing the value of different annuitization options in the academic literature is to maximize the expected discounted utility of the retiree's consumption. The seminal paper of Yaari (1965) demonstrated that under certain fairly restrictive assumptions, a retiree should annuitize all their liquid wealth at retirement.

¹See UBC Faculty Pension Plan (2013).

Key assumptions required for this result include (i) no bequest motive; (ii) no loading in the annuity price, (iii) a single time point for the purchase (or not) of annuities, and (iv) a constant relative risk aversion (CRRA) utility function satisfying time-separability². Subsequently, researchers have relaxed some of these assumptions. A bequest motive may be introduced to the utility calculation, resulting in partial annuitization (generally, full annuitization of all funds less the bequest amount). The possibility of delaying the annuitization decision, or gradually annuitizing, has been explored by, for example, Milevsky and Salisbury (2006) and Kingston and Thorp (2005). However, the broad approach of these papers is the same. An annuitization strategy is deemed optimal if it maximizes the expected discounted CRRA utility of the consumption stream. It is assumed that each year the consumption is fully controllable by the retiree.

Different types of GSAs have been studied previously. Hanewald *et al.* (2013) use Monte Carlo simulation to analyse different portfolios that include immediate and deferred annuities, fixed and inflation-indexed annuities, group self-annuitization and individual self-annuitization; they do not formally optimize over all possible portfolio combinations, but instead consider a fixed set of investment strategies and find the best performing in terms of the expected discounted utility. Their GSA shares mortality risk, but not investment risk. The GSA in Horneff *et al.* (2010a) is similar to the one we study. One major difference between our work and theirs is that they assume the retiree's investment options comprise stocks, bonds and VPAs, whereas we consider a retiree choosing between a VPA and a fixed annuity, as well as maintaining the option to invest in stocks and bonds. In other words, unlike Horneff *et al.* (2010a), but similarly to Hanewald *et al.* (2013), we are interested in the relative attractions of fixed and variable payout annuities, but we differ from Hanewald *et al.* (2013) by considering VPAs which incorporate shared investment and mortality risk.

In this paper we show some results of our analysis of a VPA scheme using the standard CRRA utility maximization approach. The results of the dynamic optimization give an optimal investment and consumption strategy for a retiree who has access to both a GSA scheme, offering a VPA with pooled investment and mortality risk, and a fixed whole-life annuity offered by an insurer, who charges a loading for risk and profit. We assume that the annuitization decision must be made at retirement. We note that this single decision point is realistic for a GSA offered by a pension plan sponsor, but is not realistic for the fixed annuity, which could be purchased at any date. Despite this constraint, the results do give an indication of the relative attractions of the two annuity types under the CRRA

²Time-separability means that past consumption does not impact the utility of current and future consumption – the utility of consuming, say, C at t is the same whether all the past consumption has been at a rate of $10C$, or at a rate of $0.10C$. An alternative hypothesis involves *habit formation*, which allows for the possibility that people prefer not to see their income decline.

utility measure.

However, when we examine the ‘optimal strategy’ in more detail, we find that the results of the optimization do not appear to provide reasonable guidelines for retirees in practice. Our results show that there are substantial and unnecessary risks for a retiree who follows the ‘optimal strategy’.

2 Variable Payout Annuities

In this section, we introduce the variable payout annuity product in more detail. A VPA is a life annuity with payments that vary depending on the performance of the fund, relative to pre-specified interest and mortality rates. In this work, we assume that it is a type of annuity offered to members of a DC pension plan. The evolution of the annuity payments depends on the performance of the assets allocated to the VPA and on the mortality experience of the participants.

2.1 Example

We first work through a numerical example to show how benefits are determined.

Suppose we have 1000 new retirees, each age 65. Each deposits 200,000 into a VPA fund. The administrator calculates the annuity factor using an effective rate of interest of 7% per year, and using CPM2014 (females) mortality without generational adjustment. We assume, for simplicity, that payments to retirees are made annually, at the start of the year.

Let N_t denote the number of survivors at t , so that $N_0 = 1000$. Let F_t denote the aggregate fund at t before the annuity payments, and let F_{t+} denote the fund after the annuity payments.

The benefit per person at t (which, in this example, is the same for all surviving participants) is calculated as

$$B(t) = \frac{F_t}{N_t} \frac{1}{\ddot{a}_{65+t}}$$

and then $F_{t+} = F_t - N_t B(t)$.

So, working in \$000s, we have an initial fund $F_0 = 200\,000$, and the annuity factor applied to determine the benefit is $\ddot{a}_{65} = 11.6431$. This gives a benefit of 17.178 for each of the 1000 participants at $t = 0$. The fund immediately after the benefit payments is now

$$F_{0+} = F_0 - N_0 B(0) = 182,822.$$

The fund is invested for a year, and earns a rate of interest that is uncertain. Suppose in this case the fund earns, say, $R_1 = 3.5\%$ in the first year. Then $F_1 = F_{0+}(1.035) = 189,221$. Suppose also that 6 participants die during the first year. Then $N_1 = 994$, and since $\ddot{a}_{66(7\%)} = 11.4525$, the benefit to each surviving participant is

$$B(1) = \frac{189,221}{994(11.4525)} = 16.622$$

The adjustment factor, j_t , is the proportionate increase in benefit at t , reflecting the investment and mortality experience of the year from $t - 1$ to t . That is,

$$1 + j_t = \frac{B(t)}{B(t-1)}$$

so in this case, $j_1 = 16.622/17.178 - 1 = -0.032$.

In the second year, assume the fund earns 8% , and 2 participants die. We also have $\ddot{a}_{67(7\%)} = 11.2536$ so that

$$F_{1+} = F_1 - N_1 B(1) = 172,699 \quad F_2 = F_{1+}(1.08) = 186,515$$

$$N_2 = 992 \quad B(2) = \frac{186,515}{992(11.2536)} = 16.707$$

and

$$j_2 = \frac{16.707}{16.622} - 1 = 0.005.$$

Note that, if the mortality and interest experience exactly match the annuitization function parameters, that is, the return on funds is $i = 7\%$, and the number of deaths exactly follows the CPM females mortality table, then we would have $j_t = 0$ for all t , and the benefit would be level.

We now consider an example where the participants have different initial ages and different benefits.

Suppose now that at the inception of the fund we have 700 participants age 65 at entry, each with an investment of 200,000, and 300 participants age 66 at entry, each with an investment of 400,000. All other assumptions remain the same.

Let $x_k + t$ denote the age of the k th life at t , so that $x_k + t = 65 + t$ for $k = 1, 2, \dots, 700$

and $x_k + t = 66 + t$ for $k = 701, 702, \dots, 1000$. Let $B_{65}(t)$ denote the annuity payout at t a survivor who entered at age 65, and let $B_{66}(t)$ denote the payout to a survivor from the age 66 entry group. Then, in \$000's,

$$B_{65}(0) = 200/\ddot{a}_{65} = 17.178 \quad \text{and} \quad B_{66}(0) = 400/\ddot{a}_{66} = 34.927.$$

$$F_0 = 700 \times 200 + 300 \times 400 = 260\,000$$

$$\text{and also note that } F_0 = 700B_{65}(0)\ddot{a}_{65} + 300B_{66}(0)\ddot{a}_{66} = 260,000$$

The last two lines demonstrate that the fund can be calculated retrospectively, by accumulating the assets invested, after deducting annuities paid out, or prospectively, by valuing the future annuity payments using the annuitization assumptions. We use these two equations for the fund to determine the adjustment factor at each year end. Note that we assume that the adjustment factor is the same for all surviving participants. It does not vary by age or amount of annuity.

So, at $t = 1$ assume, as before, that $R_1 = 3.5\%$, and assume also that 4 lives died from the age 65 entry group, and 2 lives from the age 66 entry group.

Then the two equations for the fund F_1 are derived as:

$$\begin{aligned} F_{0+} &= F_0 - (700B_{65}(0) + 300B_{66}(0)) \\ &= 237,498 \end{aligned}$$

$$F_1 = F_{0+}(1.035) = 245,810$$

$$\begin{aligned} \text{and } F_1 &= 696B_{65}(1)\ddot{a}_{66} + 298B_{66}(1)\ddot{a}_{67} \\ &= 696B_{65}(0)(1 + j_1)\ddot{a}_{66} + 298B_{66}(0)(1 + j_1)\ddot{a}_{67} \\ &= (1 + j_1)254\,051 \end{aligned}$$

$$\Rightarrow j_1 = -0.032$$

and we carry on equating the retrospective and prospective fund values to determine the adjustment factors, and hence the adjusted benefits.

If the two groups experience mortality exactly following the annuity table rates, and the fund earns exactly the 7% assumed in the annuity factors, then the benefits will stay level, as before.

2.2 The Adjustment Factor Formula

We now generalize the result in the example to derive the adjustment factor formula.

Let $B_k(t)$ denote the benefit paid at t to the k th life, assuming survival to t .

Let $x_k + t$ denote the age of the k th life at t .

Let \mathcal{A}_t denote the survival set at t , that is, $k \in \mathcal{A}_t$ if and only if the k th life is alive at t .

Then for $t = 1, 2, \dots$, retrospectively,

$$\begin{aligned}
 F_{t-1} &= \sum_{k \in \mathcal{A}_{t-1}} B_k(t-1) \ddot{a}_{x_k+t-1} \\
 F_{t-1+} &= F_{t-1} - \sum_{k \in \mathcal{A}_{t-1}} B_k(t-1) \\
 F_t &= F_{t-1+} (1 + R_t) \\
 \Rightarrow F_t &= \left(\sum_{k \in \mathcal{A}_{t-1}} B_k(t-1) a_{x_k+t-1} \right) (1 + R_t).
 \end{aligned}$$

Also, prospectively

$$\begin{aligned}
 F_t &= \sum_{k \in \mathcal{A}_t} B_k(t) \ddot{a}_{x_k+t} \\
 &= (1 + j_t) \left(\sum_{k \in \mathcal{A}_t} B_k(t-1) \ddot{a}_{x_k+t} \right)
 \end{aligned}$$

Equating the retrospective and prospective values for F_t we have

$$1 + j_t = \frac{\left(\sum_{k \in \mathcal{A}_{t-1}} B_k(t-1) a_{x_k+t-1} \right) (1 + R_t)}{\left(\sum_{k \in \mathcal{A}_t} B_k(t-1) \ddot{a}_{x_k+t} \right)}$$

Recall (see, eg, Dickson *et al.* (2013), equation (5.11))

$$a_{x+t-1} = \ddot{a}_{x+t-1} - 1 = \frac{p_{x+t-1} \ddot{a}_{x+t}}{1 + i}$$

where i is the interest rate assumption used in the annuity factors \ddot{a}_{x+t} . This gives

$$1 + j_t = \left(\frac{\sum_{k \in \mathcal{A}_{t-1}} B_k(t-1) p_{x_k+t-1} \ddot{a}_{x_k+t}}{\sum_{k \in \mathcal{A}_t} B_k(t-1) \ddot{a}_{x_k+t}} \right) \left(\frac{1 + R_t}{1 + i} \right) \quad (1)$$

This form shows the two components of the adjustment factor. The first is a weighted mortality ratio. The numerator shows the expected fund at t , given the survivor group at $t-1$, assuming the benefit is unchanged from $t-1$. That is, let E_t denote the expectation given the information (i.e. survivor group) at t . Then

$$\sum_{k \in \mathcal{A}_{t-1}} B_k(t-1) p_{x_k+t-1} \ddot{a}_{x_k+t} = E_{t-1} \left[\sum_{k \in \mathcal{A}_t} B_k(t-1) \ddot{a}_{x_k+t} \right]$$

The denominator is the actual cost of the annuity payments, given the survival group at t , and assuming the benefit is unchanged from $t-1$. The ratio is an expected/actual survival ratio, weighted by the individual annuity values. This term is greater than 1 if there are more deaths than expected, or if deaths are concentrated in the higher annuity groups, so that the sum of the annuity values in the survivor set \mathcal{A}_t (the denominator) is less than its expected value at $t-1$ (the numerator).

The second term in equation (1) is an adjustment for the investment experience. It is greater than 1 if the actual return, R_t is greater than the annuitization interest rate, i .

Notice that if we assume that everybody retires at the same age, x , say, then equation (1) simplifies to

$$1 + j_t = \frac{\left(\sum_{k \in \mathcal{A}_{t-1}} B_k(t-1) \ddot{a}_{x+t} \right)}{\left(\sum_{k \in \mathcal{A}_t} B_k(t-1) \ddot{a}_{x+t} \right)} \frac{1 + R_t}{1 + i} \quad (2)$$

$$= \frac{p_{x+t-1}}{p_{x+t-1}^*} \frac{1 + R_t}{1 + i} \quad (3)$$

where we define the weighted survival rate in the t th year for the group as

$$p_{x+t-1}^* = \frac{\sum_{k \in \mathcal{A}_t} B_k(t-1)}{\sum_{k \in \mathcal{A}_{t-1}} B_k(t-1)}$$

which is the survival rate weighted by the annuity payment. Our equation (3) is identical to equation (4) from Piggott *et al.* (2005).

Equation (1) can easily be adapted to an open fund, rather than the closed group assumed above. Assuming new entrants are permitted at each year end, we would adjust \mathcal{A}_t , after the adjustment factor j_t has been calculated, so that \mathcal{A}_{t+} is the group of all participants at t , including new entrants. Then

$$1 + j_t = \frac{\sum_{k \in \mathcal{A}_{(t-1)+}} B_k(t-1) p_{x_k+t-1} \ddot{a}_{x_k+t}}{\sum_{k \in \mathcal{A}_t} B_k(t-1) \ddot{a}_{x_k+t}} \frac{1 + R_t}{1 + i} \quad (4)$$

and similarly

$$1 + j_{t+1} = \frac{\sum_{k \in \mathcal{A}_{t+}} B_k(t) p_{x_k+t} \ddot{a}_{x_k+t+1}}{\sum_{k \in \mathcal{A}_{t+1}} B_k(t) \ddot{a}_{x_k+t+1}} \frac{1 + R_{t+1}}{1 + i} \quad (5)$$

Observe that the adjustment factor equation (5) does not take the retirees entering the group at $t + 1$ into account since they enter at the end of the period.

2.3 Adjustment factor with systematic mortality improvements

In this section, we derive the adjustment factor when the group is open to new entrants and when the adjustment factor takes systematic mortality improvements into account by changing the mortality rates used in the annuity factor. To derive the mortality adjustment factor incorporating systematic mortality improvements, we first introduce some notation and further assumptions.

We denote by ${}_s p_{x,t}$ the probability, measured at time t , that a life aged x at time t survives s more years. When $s = 1$, we omit the subscript s so that $p_{x,t}$ denotes the probability measured at time t that a life aged x at t survives at least one year. We denote by $\ddot{a}_{x,t}$ the annuity factor for a life aged x measured at time t :

$$\ddot{a}_{x,t} = \sum_{u=0}^{\infty} v^u {}_u p_{x,t}.$$

Then

$$1 + j_t = \left(\frac{\sum_{k \in \mathcal{A}_{t-1+}} p_{x_k+t-1,t-1} B_k(t-1) \ddot{a}_{x_k+t,t-1}}{\sum_{k \in \mathcal{A}_t} B_k(t-1) \ddot{a}_{x_k+t,t}} \right) \left(\frac{1 + R_t}{1 + i} \right) \quad (6)$$

This shows that the longevity adjustment factor is now the sum of annuity factors under the old and the new mortality assumptions, weighted by the number of retirees at each age and the amount of their annuity payment.

3 The annuity decision

The main question we address in this paper is whether retirees should participate in the GSA. We consider a retiree who has a pool of liquid assets at retirement, and who can allocate her funds at that time between the following investments.

- The VPA described in Section 2, but with annuity interest rate $i = 0.03$. The VPA fund is invested in a mix of risk-free and risky assets. The proportion of risky assets in the fund is assumed to be $\alpha_V = 40\%$ (we also consider 25% and 60% as alternatives). The risky asset prices are lognormally distributed, with mean annual log-return $\mu = 4.078\%$ and volatility $\sigma = 18.703\%$. The expected annual return is 6%.

We do not model the effects of idiosyncratic mortality experience. That is, we assume the VPA group is sufficiently large that idiosyncratic risk is fully diversified, and therefore the mortality of the group, follows the assumed rates. We assume the retiree also experiences the same mortality, and that her subjective mortality probabilities are the same as the group rates.

We do allow for longevity risk, which changes the annuity factors used in the adjustment factor calculations, and in the subsequent mortality experience. We allow the mortality rates for the group to vary stochastically, following the two-factor Cairns, Blake and Dowd (CBD) model, introduced in Cairns *et al.* (2006). Within each year, we assume mortality experience exactly matches the rates generated by the CBD model (thus ignoring idiosyncratic, or diversifiable risk). Details of the mortality model and parameters are given in Appendix A.

- A fixed annuity purchased from an insurer. We assume the annuity is priced using the same interest and mortality assumption as the GSA, but that the insurer applies a loading for profit and contingencies. The loading factor is denoted λ , and we explore a range of values, from 0% to 10%.
- Risk free bonds, assumed to generate returns of 2% per year, effective, and
- An equity fund generating the same returns as the risky assets in the VPA fund.

We assume that the annuitization decision is made only once, at retirement, which is assumed to occur at age 65. Subsequently, the retiree can rebalance her liquid assets (those invested in the money market and the mutual fund, including any excess annuity income not consumed).

3.1 Evolution of the retiree's wealth

A new retiree has wealth A_0 to divide between the four assets.

The proportions of initial wealth invested in balanced fund, the fixed annuity and the variable annuity are denoted by ω_B , ω_F and ω_V , respectively. The remaining wealth is invested in risk free bonds.

Denote the liquid wealth of the retiree at time t by W_t .

We define B^F to be the annual income from the fixed annuity, and B_t^V to be the retiree's income at time t from the VPA. The total annuity income at t is

$$B_t = B_t^V + B^F.$$

At $t = 0$, we obtain B^F and B_0^V by dividing the amount invested in each annuity by \ddot{a}_{65}^F and \ddot{a}_{65}^V , respectively. The difference between the annuity factors arises from the insurer's loading λ , such that

$$\ddot{a}_{65}^F = (1 + \lambda)\ddot{a}_{65}^V.$$

Thus, starting with an accumulated amount at retirement A_0 , the annuity payments and liquid wealth at time 0, after investment decisions are made, are given by

$$B^F = \frac{\omega_F A_0}{\ddot{a}_{65}^F}$$

$$B_0^V = \frac{\omega_V A_0}{\ddot{a}_{65}^V}$$

$$B_0 = B^F + B_0^V$$

$$W_0 = A_0(1 - \omega_F - \omega_V) + B_0,$$

At times $t = 1, 2, \dots, 54$, the only investment decision that the retiree must make is how to divide her non-annuitized wealth, W_t , between the risk free asset and the equity fund. We denote by ω_t the proportion of the wealth invested in the equity fund at t . Let ω denote the set of portfolio control variables, $\{\omega_B, \omega_V, \omega_F, \omega_1, \omega_2, \dots, \omega_T\}$.

Investments in the risk free asset are assumed to earn $r = 0.02$ per year. The return on the equity fund in $(t, t + 1)$ is denoted R_t^{Eq} , where $1 + R_t^{Eq} \sim \text{logN}(\mu = 0.04078, \sigma = 0.18703)$. Hence, the return on the non-annuitized wealth during the year starting at time t , denoted R_t^w , is given by

$$R_t^w = r + \omega_t(R_t^{Eq} - r).$$

To determine the adjustment factors we assume that a proportion α_V of the VPA fund is invested in the risky asset, earning R_t^{Eq} in the t th year while the rest is in the risk-free asset.

After one period, the total liquid wealth, W_1 , and the annuity income, B_1 , are given by

$$B_1^V = B_0^V(1 + j_1)$$

$$B_1 = B_1^V + B^F$$

$$W_1 = (W_0 - C_0)(1 + R_1^w) + B_1,$$

where C_0 is the amount consumed at time 0 and $(1 + j_1)$ is the first year adjustment factor for the VPA.

For $t = 1, 2, \dots, T$, the total wealth W_t and the annuity income B_t evolve according to the following equations.

$$B_t^V = B_{t-1}^V(1 + j_t),$$

$$B_t = B_t^V + B^F,$$

$$W_t = (W_{t-1} - C_{t-1})(1 + R_t^w) + B_t,$$

where j_t is a function of the return on the VPA fund from time $t - 1$ to t , and of the mortality experience of the VPA fund members, and C_t is the consumption selected by the retiree at t .

3.2 Example of retiree wealth process

We illustrate this process with a numerical example.

We consider a retiree who joins the VPA at age 65 with a fund of $A_0 = 1,000,000$. We will review her income under five different strategies for investment. The income is conditional on the retiree's survival to the start of each year, for a maximum of 30 years.

Regardless of her investment strategy, her consumption at the start of each year will be the lesser of the target consumption, given in Table 1, and the liquid assets available (she is not permitted to borrow). Her target consumption is assumed to be \$55,000 at retirement, increasing by 2% each year to allow for inflation. The target consumption values are arbitrary, chosen to illustrate the process.

The asset returns, adjustment factors and target consumption for each year are given in Table 1. The equity returns are random draws from the lognormal risky asset distribution, and the adjustment factors are simulated assuming the group mortality exactly follows the Cairns, Blake and Dowd (2006) longevity model, described in Appendix A. Risk free returns are assumed to be 2% throughout.

The annuity factor for the initial payment from the VPA is $\ddot{a}_{65} = 14.3896$, and the annuity factor for the fixed annuity payment is $\ddot{a}_{65}^F = (1.1)14.3896 = 15.8286$.

We consider five different investment strategies for the retiree:

Strategy A Invest all starting assets in equities. Any excess income after meeting target consumption remains invested in equities.

Strategy B Invest all starting assets in the VPA. Any excess income after meeting target consumption is invested at the risk free rate.

Strategy C Invest all starting assets in the fixed annuity. Any excess income after meeting target consumption is invested at the risk free rate.

Strategy D Invest 60% of starting assets in the VPA, 20% in the fixed annuity, 15% in equities and 5% in the money market. Any excess income after meeting target consumption is invested 75% in equities and 25% at the risk free rate.

Strategy E Invest 80% of starting assets in the VPA, 20% in the fixed annuity, none in equities or the money market. Any excess income after meeting target consumption is invested at the risk free rate.

The resulting consumption patterns are given in Table 2. We also give the amount available for bequest for a retiree dying in each year up to age 95 in Table 3. This is the balance of liquid wealth available at the time of death.

In the table headers FA denotes the fixed annuity, RF denotes the risk free asset, and Eq denotes equities.

We summarize the advantages and disadvantages of each strategy, based on this single economic scenario.

Time t	Return on Equities	Return on VPA Fund	Adjustment factor, j_t	Target Consumption
1	-0.1331	-0.0412	-0.0562	55,000
2	0.1197	0.0599	0.0436	56,100
3	0.1398	0.0679	0.0416	57,222
4	-0.2392	-0.0837	-0.1556	58,366
5	0.1045	0.0538	0.0137	59,534
6	0.0431	0.0292	0.0064	60,724
7	-0.1268	-0.0387	-0.0793	61,939
8	0.2981	0.1312	0.1455	63,178
9	0.0260	0.0224	-0.0016	64,441
10	-0.0276	0.0010	-0.0178	65,730
11	0.0353	0.0261	-0.0009	67,045
12	0.1985	0.0914	0.0734	68,386
13	-0.1662	-0.0545	-0.0811	69,753
14	0.1482	0.0713	0.0536	71,148
15	0.3215	0.1406	0.1426	72,571
16	-0.0477	-0.0071	-0.0235	74,023
17	0.0407	0.0283	-0.0305	75,503
18	-0.2700	-0.0960	-0.1325	77,013
19	-0.0834	-0.0214	-0.0399	78,554
20	-0.1863	-0.0625	-0.0927	80,125
21	0.0142	0.0177	-0.0350	81,727
22	-0.1808	-0.0603	-0.0781	83,362
23	0.1509	0.0724	0.0468	85,029
24	0.1061	0.0544	-0.0021	86,729
25	0.0372	0.0269	-0.0126	88,464
26	0.0076	0.0150	-0.0413	90,233
27	0.6066	0.2546	0.1809	92,038
28	0.0508	0.0323	-0.0155	93,879
29	-0.0138	0.0065	-0.0611	95,756
30	0.2301	0.1040	0.0951	97,671

Table 1: Scenario information for wealth and consumption process example.

Time t	Strategy A	Strategy B	Strategy C	Strategy D	Strategy E
	100% Eq	100% VPA	100% FA	60%VPA 20%FA 15% Eq, 5% RF	80% VPA 20% FA
1	55,000	55,000	55,000	55,000	55,000
2	56,100	56,100	56,100	56,100	56,100
3	57,222	57,222	57,222	57,222	57,222
4	58,366	58,366	58,366	58,366	58,366
5	59,534	59,534	59,534	59,534	59,534
6	60,724	60,724	60,724	60,724	60,724
7	61,939	61,939	61,939	61,939	61,939
8	63,178	63,178	63,178	63,178	63,178
9	64,441	64,441	64,441	64,441	64,441
10	65,730	65,730	65,730	65,730	65,730
11	67,045	67,045	67,045	67,045	67,045
12	68,386	68,386	68,386	68,386	68,386
13	69,753	69,753	69,753	69,753	69,753
14	71,148	71,148	71,148	71,148	57,222
15	72,571	72,571	72,571	72,571	57,629
16	74,023	74,023	67,236	74,023	62,123
17	70,678	75,503	63,177	75,503	59,970
18	-	77,013	63,177	77,013	59,536
19	-	78,554	63,177	55,058	53,447
20	-	66,621	63,177	48,287	51,106
21	-	53,911	63,177	44,982	47,362
22	-	52,024	63,177	43,850	46,686
23	-	47,961	63,177	41,412	43,445
24	-	50,206	63,177	42,759	44,481
25	-	50,100	63,177	42,696	44,997
26	-	49,469	63,177	42,317	44,657
27	-	47,426	63,177	41,091	43,951
28	-	56,005	63,177	46,239	50,546
29	-	55,137	63,177	45,718	50,347
30	-	51,768	63,177	43,697	49,203

Table 2: Consumption for the wealth and consumption process example

Year <i>t</i>	Strategy A	Strategy B	Strategy C	Strategy D		Strategy E
	100% Eq	100% VPA	100% FA	60%VPA 15% Eq, 5% RF	20%FA	80% VPA 20% FA
1	945,000	14,495	8,177	199,332		13,231
2	763,121	24,274	15,418	176,320		21,365
3	797,244	35,986	21,681	189,513		29,644
4	850,332	49,636	26,926	207,378		38,480
5	587,399	51,298	31,108	160,435		39,965
6	588,058	52,627	34,183	162,338		41,031
7	551,464	53,159	36,104	155,945		40,506
8	418,361	47,592	36,826	125,281		35,173
9	478,633	48,877	36,298	140,976		32,500
10	425,347	48,797	34,471	130,138		27,764
11	346,563	46,248	31,293	111,798		19,916
12	290,411	42,251	26,711	97,645		10,052
13	278,304	41,464	20,669	96,425		993
14	160,902	33,742	13,111	63,933		0
15	112,176	27,798	3,979	50,994		0
16	74,218	29,688	0	47,372		0
17	0	28,364	0	27,197		0
18	0	23,260	0	6,591		0
19	0	7,060	0	0		0
20	0	0	0	0		0
21	0	0	0	0		0
22	0	0	0	0		0
23	0	0	0	0		0
24	0	0	0	0		0
25	0	0	0	0		0
26	0	0	0	0		0
27	0	0	0	0		0
28	0	0	0	0		0
29	0	0	0	0		0
30	0	0	0	0		0

Table 3: Bequest for the wealth and consumption process example

- Strategy A, with 100% of the fund invested in equities, generates consumption at target levels for 16 years, but there are no funds left, and therefore no income, for retirees who survive more than 17 years, which appears to be an unacceptable risk. For retirees who die in the first 15 years there is a substantial bequest available.
- Strategy B, with 100% of the fund invested in the VPA will not run out. The target consumption is met for the first 19 years, but not at all thereafter. The income is quite variable – there is a 39% difference between the income in the 23rd year compared with the 19th year. There is a modest bequest during the first 18 years.
- Strategy C, with 100% of the fund invested in the fixed annuity, meets consumption targets for the first 15 years, because there is excess income in the first 7 years that can supplement the fixed annuity payment for the following 8 years. Subsequently, the consumption falls back to the level income generated by the annuity. The bequest potential under this strategy is small.
- Strategy D, with a mix of the VPA, fixed annuity, equities and bonds, meets the target consumption for the first 18 years, but subsequently generates quite volatile income. The income in the 27th year (assuming the retiree survives) is only 75% of the starting income level. There is a significant bequest available on early death.
- Strategy E, with a mix of VPA and fixed annuity performs similarly to strategy B (unsurprisingly), but the target consumption is only met for the first 13 years; subsequent consumption levels are smoothed compared with strategy B, and the bequests are smaller.

The classical approach to deciding the best investment/annuitization strategy does not involve target consumption; instead, the consumption levels are treated as fully controllable by the retiree. In the next section, we describe the classical approach in more detail.

3.3 Modelling retiree utility

The classical approach to the annuitization decision involves assigning a utility function to the retiree, and optimizing the expected utility of all future consumption, discounted at a ‘subjective’ discount rate. The discount rate reflects the retiree’s own time preference. The assumed objective is to select both an investment/annuitization strategy and, simultaneously, a path of consumption levels, that maximizes the expected value of the utility of the retiree’s subjectively discounted consumption. The expectation is taken with respect to the randomness of the future lifetime and of the future income.

Under this standard approach, we assume that the consumption of the retiree is entirely flexible and is one of the controls under a dynamic optimization³. The other control variables are the proportions of wealth invested in the different asset types. This is quite different to the exercise in the previous example, where we determined a target consumption level, and explored whether any of the strategies could meet the target levels. We assume that the retiree has no bequest motive.

Consistent with most of the annuitization literature, we assume that the retiree's risk preferences are represented by a constant relative risk aversion (CRRA) utility function, with a parameter of relative risk aversion $\gamma > 0$. CRRA utility is represented by a power function for $\gamma \neq 1$, or a log function for $\gamma = 1$. For arbitrary constants $a, b > 0$, we have

$$U(c) = \begin{cases} a + \frac{b}{1-\gamma}c^{1-\gamma} & \text{for } \gamma \neq 1 \\ a + b \log(c) & \text{for } \gamma = 1. \end{cases} \quad (7)$$

CRRA utility is chosen largely for its tractability. However, it may not be the best choice for the annuitization problem. For example, CRRA implies that utility depends on proportional changes in wealth, not on absolute values. If we assume all individuals in the group have the same risk aversion parameter γ , then we are assuming that an individual with a pension of \$20,000 has the same aversion to a 10% drop in income as an individual with a pension of \$200,000.

We use a subjective time preference discount factor of 96%. This means that at the optimization date, a payment projected t years ahead would be multiplied by $(0.96)^t$ in the expected utility calculation.

We assume a risk aversion parameter $\gamma = 2$. This is similar to other researchers, but differs quite substantially from Maurer *et al.* (2013), who use $\gamma = 5$. This is a significant difference. To illustrate, consider an individual who risks losing 80% of their wealth with a probability of 1%. The individual would pay a premium of 40% of their wealth for full insurance with $\gamma = 5$, but only 4% of their wealth with $\gamma = 2$. The decision not to use the Maurer *et al.* (2013) assumption was based on empirical research, for example in Maier and R uger (2010), indicating that $\gamma = 2$ is a more realistic assumption.

To illustrate the utility calculation, we use the results from Table 2. It makes no difference to the relative utility results if we scale the consumption c or add a constant, so, for presentation purposes, we use the following CRRA utility for consumption in year t ,

³A dynamic optimization, in this context, means an optimization involving decisions at different time points which depend on the evolving underlying stochastic processes.

conditional on survival,

$$\begin{aligned}
 U_t(c_t) &= 2 + \frac{(c_t \times 10^{-5})^{1-\gamma}}{1-\gamma} \\
 &= 2 - \frac{10^5}{c_t} \text{ for } \gamma = 2
 \end{aligned}$$

where c_t is the consumption at t .

For an individual scenario, such as that used in the example in Section 3.2, the total discounted lifetime utility, allowing for survival, is

$$U(\mathbf{c}) = \sum_{t=0}^{45} U_t(c_t) \beta^t {}_t p_{65}$$

To find the expected discounted lifetime utility, we could simulate over a large number of scenarios for the equity returns, and take the mean of the resulting discounted future utility values, or, we may proceed analytically if the problem is sufficiently tractable.

In Table 4 we show utility of consumption in each year ($U_t(c_t)$) for the example in Section 3.2, together with the total discounted lifetime utility, allowing for survival, for each of the five investment strategies.

We note that CRRA utility is not defined (equal to $-\infty$) when the consumption falls to zero, so the all-equity Strategy A is dominated by all the other strategies when the equity fund runs out, as it does in this scenario. Considering the other four strategies, given the target consumption in the example, the utility is maximized under strategy C (the all-fixed-annuity option).

The preferences are quite sensitive to the parameters. In Table 5 we show the discounted utility for the example from Section 3.2, for different values of the risk aversion parameter γ , and the subjective discount factor, β . We omit strategy A as the utility is undefined for all values of γ and β . The effect of decreasing β is to reduce the impact of the older age consumption levels. For $\gamma = 2$, comparing the case where $\beta = 0.90$ to the base case, $\beta = 0.96$, we see that the preference has changed to Strategy B (the 100% VPA strategy) as the impact of lower consumption in later life under that strategy is more heavily discounted. When $\beta = 0.99$, the preference swings more decisively to Strategy C, the fixed annuity strategy.

The effect of increasing γ is to increase the risk aversion, so that for all three values of β , the steady income from the fixed annuity strategy is preferred to all other strategies. However, this is only one scenario. The picture could be different when considering the

Year t	Strategy A	Strategy B	Strategy C	Strategy D		Strategy E
	100% Eq	100% VPA	100% FA	60%VPA 20%FA 15% Eq, 5% RF	82% VPA 18% FA	
1	0.1818	0.1818	0.1818	0.1818	0.1818	
2	0.2175	0.2175	0.2175	0.2175	0.2175	
3	0.2524	0.2524	0.2524	0.2524	0.2524	
4	0.2867	0.2867	0.2867	0.2867	0.2867	
5	0.3203	0.3203	0.3203	0.3203	0.3203	
6	0.3532	0.3532	0.3532	0.3532	0.3532	
7	0.3855	0.3855	0.3855	0.3855	0.3855	
8	0.4172	0.4172	0.4172	0.4172	0.4172	
9	0.4482	0.4482	0.4482	0.4482	0.4482	
10	0.4786	0.4786	0.4786	0.4786	0.4786	
11	0.5085	0.5085	0.5085	0.5085	0.5085	
12	0.5377	0.5377	0.5377	0.5377	0.5377	
13	0.5664	0.5664	0.5664	0.5664	0.5664	
14	0.5945	0.5945	0.5945	0.5945	0.2524	
15	0.6220	0.6220	0.6220	0.6220	0.2648	
16	0.6491	0.6491	0.5127	0.6491	0.3903	
17	0.5851	0.6756	0.4171	0.6756	0.3325	
18	$-\infty$	0.7015	0.4171	0.7015	0.3203	
19	$-\infty$	0.7270	0.4171	0.1837	0.1290	
20	$-\infty$	0.4990	0.4171	-0.0709	0.0433	
21	$-\infty$	0.1451	0.4171	-0.2231	-0.1114	
22	$-\infty$	0.0778	0.4171	-0.2805	-0.1420	
23	$-\infty$	-0.0850	0.4171	-0.4147	-0.3018	
24	$-\infty$	0.0082	0.4171	-0.3387	-0.2482	
25	$-\infty$	0.0040	0.4171	-0.3422	-0.2224	
26	$-\infty$	-0.0215	0.4171	-0.3631	-0.2393	
27	$-\infty$	-0.1085	0.4171	-0.4336	-0.2752	
28	$-\infty$	0.2145	0.4171	-0.1627	0.0216	
29	$-\infty$	0.1863	0.4171	-0.1873	0.0138	
30	$-\infty$	0.0683	0.4171	-0.2885	-0.0324	
$U(c)$	$-\infty$	4.76	4.86	4.12	3.68	

Table 4: Utility of future year consumption, conditional on survival, for the example in Section 3.2, with total discounted utility for one investment scenario.

	Strategy B	Strategy C	Strategy D	Strategy E
$\gamma = 2$				
$\beta = 0.90$	2.78	2.76	2.61	2.42
$\beta = 0.96$	4.76	4.86	4.12	3.68
$\beta = 0.99$	6.43	6.74	5.19	4.57
$\gamma = 5$				
$\beta = 0.90$	0.42	0.62	-0.81	-1.25
$\beta = 0.96$	1.65	3.04	-3.33	-3.74
$\beta = 0.99$	2.08	5.28	-7.77	-7.58

Table 5: Utility of consumption for Section 3.2 example, for different risk aversion and discount parameters.

expectation over all possible investment scenarios.

4 Solving the Optimization Problem

In this section we will give an outline of the numerical procedure for finding the optimal solution to the stylized annuitization problem, where the level of consumption, and the allocation to the different investment options are control variables that the retiree can set to maximize her expected discounted utility of consumption.

Let $H(t, W_t, B)$ denote the maximum expected future discounted utility of consumption at t , given wealth W_t at t , and annuity income B per year, for a retiree who is alive at t . Then

$$H(0, W_0, B_0) = \max_{\boldsymbol{\omega}, \mathbf{C}} E \left[\sum_{t=0}^K \beta^t U(C_t) \right] \quad (8)$$

where $\boldsymbol{\omega}$ is a vector of portfolio control variables, $\{\omega_B, \omega_V, \omega_F, \omega_1, \omega_2, \dots, \omega_T\}$, and $\mathbf{C} = (C_0, C_1, \dots, C_T)$ is the vector of the consumption control variables. K is the random curtate future lifetime of the retiree.

Optimizing over all these variables simultaneously is too complex for standard optimization methods, but the CRRA utility allows us to optimize iteratively, as it is time-separable. This allows us to build a grid of possible values for H at different times, starting from the last possible survival date, and moving back to the retirement date, for

a range of possible values of W_t and B_t , as follows.

Suppose \mathcal{K} is the maximum value for K . At time \mathcal{K} we know that $H(\mathcal{K} + 1, W_{\mathcal{K}+1}) = 0$ for any $W_{\mathcal{K}+1}$, as any life alive at \mathcal{K} dies before $\mathcal{K} + 1$ and there is no utility from bequests. Since this is the only known value for the derived utility function, we begin our optimization from the last possible annuity payment date, \mathcal{K} . At that time it is optimal for any surviving retiree to consume all her remaining wealth, giving

$$H(\mathcal{K}, W_{\mathcal{K}}, B_{\mathcal{K}}) = \max_{\omega_{\mathcal{K}}, C_{\mathcal{K}}} \{U(C_{\mathcal{K}})\} = U(W_{\mathcal{K}}) \quad (9)$$

Since we cannot know the value of the wealth process at time \mathcal{K} (it will depend on the optimal controls during earlier periods), we calculate and store the derived utility function $H(\mathcal{K}, W_{\mathcal{K}})$ for a range of different feasible values of $W_{\mathcal{K}}$. Then, we move back one period and, for a range of feasible values of $W_{\mathcal{K}-1}$ and $B_{\mathcal{K}}$, solve for the optimal controls $\omega_{\mathcal{K}-1}$ and $C_{\mathcal{K}-1}$ that will generate

$$\begin{aligned} H(\mathcal{K} - 1, W_{\mathcal{K}-1}) &= \max_{\omega_{\mathcal{K}-1}, C_{\mathcal{K}-1}} U(C_{\mathcal{K}-1}) + E_{\mathcal{K}-1} [\beta H(\mathcal{K}, W_{\mathcal{K}})] \\ &= \max_{\omega_{\mathcal{K}-1}, C_{\mathcal{K}-1}} U(C_{\mathcal{K}-1}) + \beta E_{\mathcal{K}-1} [U(W_{\mathcal{K}})] \end{aligned}$$

Note that the expectation E_t allows for mortality from t to $t + 1$, assuming survival at t , as well as allowing for random investment returns, and the random adjustment factor for the VPA.

The procedure is repeated until we find the controls at time 0. We describe the optimization procedure in more detail in Appendix B. More information about the general methodology is given, for example, in Pennacchi (2008).

We assume that improvements are observable in the group's mortality experience and that these improvements are reflected in the VPA payments. To model the mortality process, we use the two-factor Cairns, Blake Dowd (CBD) model, introduced by Cairns *et al.* (2006), and also used by Maurer *et al.* (2013). Details are given in Appendix A.

We assume a retirement age of 65 and let $t = 0$ denote the initial retirement date.

4.1 Mortality and annuity updating assumptions

There are a range of different possible assumptions for allowing for idiosyncratic and systematic mortality variation. Under the CBD mortality model, the mortality rates at any time t are a function of a bivariate random process, A_t . The randomness in A_t allows for systematic mortality risk; all lives at all ages experience mortality rates which are a

λ	0.00	0.05	0.075	0.100
ω_V	0.24	0.66	0.86	1.0
ω_F	0.76	0.34	0.14	0.0
B_0 (\$000s)	69.5	68.4	68.8	69.5
B_0^V (\$000s)	16.7	22.5	59.8	69.5
B_0^F (\$000s)	52.8	45.9	9.0	0.0

Table 6: Optimal allocation proportions, and associated annuity income per \$1 million of wealth at retirement, for different margins λ , and with 40% risky assets in the VPA fund.

deterministic function of A_t . Idiosyncratic mortality risk is the variation in experience given the mortality rates, arising from the binomial distribution of the annual number of deaths in each cohort.

In determining an optimal strategy for an individual investor, we may allow for idiosyncratic and systematic risk. In practice, we assume that the groups comprising the GSA members are very large, which means that the idiosyncratic risk will not be significant (it will be diversified away). We therefore assume in the following results sections that there is no idiosyncratic risk.

The CBD model allows for an evolving mortality model, with both deterministic trend and random innovation, impacting all ages (through a logit function). This is the systematic mortality/longevity risk model, and it will impact the adjustment factor. We use equation (6), which assumes that the GSA sponsor recalculates annuities each year as the systematic changes emerge. The adjustment factor will then reflect the difference each year between the new annuity rate and the rate calculated at the end of the previous year. In practice, annuities may be updated less frequently, which would increase the variability in the adjustment factor.

4.2 Numerical Results of Utility Maximization

In Table 6, we present a summary of the results of the utility optimization process. Recall that ω_V is the proportion invested in the VPA, and ω_F is the proportion in the fixed annuity. B_0 is the initial payment, per \$100 invested in the annuities. B_0^V and B_0^F give the initial payments from the VPA and the fixed annuity, respectively, for an initial investment of \$1,000,000.

We note some key features from the results summarized in Table 6.

- It is always optimal (using these models and assumptions) to invest all of the retire-

ment funds in a combination of the fixed and variable annuity; that is, $\omega_V + \omega_F = 1$.

- Under the annuity designs studied, it is always optimal to consume the full annuity payment each year – that is, $C_t = B_t$ for all t . This means that the utility maximization never indicates that the retiree should maintain a liquid reserve⁴.
- As the cost of the fixed annuity increases, the retiree optimizes the utility of her consumption by investing a greater part of her initial wealth in the VPA. For $\lambda \geq 0.1$, the retiree prefers to invest her entire wealth in the VPA.
- Even when the fee load is high, the initial payments are quite similar. The distribution of the payments throughout retirement is however quite different under the different values for ω_V , as we demonstrate in the next section.

In the base case we assume that 40% of the VPA sub-fund is invested in the risky asset. This corresponds to the average exposure obtained by Maurer *et al.* (2013). In Table 7 we show the optimal investment proportions for the fixed and variable annuity, for different loading factors, where the equity proportion in the VPA fund is 25% and 60% respectively. In the 25% case, the expected return on the VPA fund is equal to the interest rate assumed in the annuity factor, so that, on average, the contribution to the adjustment factor from investment returns is 1.0. The 60% case reflects a relatively standard pension asset mix.

In all cases, we still find that the optimal strategy is to annuitize all assets ($\omega_V + \omega_F = 1$), and to consume all income each year.

Maurer *et al.* (2013) assume that the equity proportion is at the discretion of the retirees. Suppose a retiree is given the option of three VPA funds, with 25%, 40% or 60% equity investment. For each of the values of λ considered, the retiree achieves a higher expected discounted utility in the case where 60% of the VPA fund is invested in equities. We note though, that in the case $\lambda = 0.1$, the extra risk from the equity exposure in the VPA fund is offset by the reduced proportion of wealth invested in the VPA.

A more aggressively invested VPA fund might be accompanied by a more aggressive VPA annuity assumption – recall the UBC annuity rates used 4% or 7% annually. Using our model assumptions and parameters, the expected return on the VPA fund with 60% equity investment is 4.4% per year. In Table 8 we show the optimal strategies for different values of λ where the annuity interest rate used for the fixed payout annuity remains at 3% p.y., but the annuity rate used for the VPA is increased to 6% per year. In this case, the

⁴Note that this outcome arises because the annuitization decision in our setting is made at retirement. Other researchers find that if gradual annuitization over the retirement period is permitted, it may be optimal to maintain some liquid wealth in the early retirement years. See, for example, Milevsky and Young (2007) and Horneff *et al.* (2010b)

λ	0.00	0.05	0.075	0.100
$\alpha_V = 25\%$				
ω_V	0.0	0.45	0.86	1.0
ω_F	1.0	0.55	0.14	0.0
$\alpha_v = 60\%$				
ω_V	0.40	0.61	0.71	0.80
ω_F	0.60	0.39	0.29	0.20

Table 7: Optimal allocation proportions, and associated annuity income per \$1 million of wealth at retirement, for different margins λ , and with 40% or 60% equities in the VPA fund.

λ	0.00	0.05	0.075	0.100
ω_V	0.36	0.51	0.57	0.63
ω_F	0.67	0.49	0.43	0.37
B_0 (\$000s)	76.7	78.1	78.9	79.9
B_0^V (\$000s)	32.3	45.7	51.1	56.5
B_0^F (\$000s)	44.4	32.4	21.8	23.4

Table 8: Optimal allocation proportions, and associated annuity income per \$1 million of wealth at retirement, with 60% risky assets in the VPA fund, 6% VPA annuity interest rate.

initial annuity payment under the VPA will be significantly increased, but this is offset by reduced adjustment factors. Given the choice between the 6% annuity interest rate, and the 3% annuity interest rate, the 3% case generates higher expected discounted utility, although this assumes the retiree experiences the group mortality rates. A retiree with lower future lifespan might benefit from the higher initial payouts. Similarly, a retiree with a lower subjective discount factor (that is, one who places a higher weight on the immediate future) could get higher utility from the 6% interest rate VPA.

We noted above that an appropriate value for the risk aversion parameter, γ is not widely agreed by researchers in the area. We have used $\gamma = 2.0$ above, consistently with the empirical study in Maier and Ruger (2010) for example. However, other authors, including the influential work of Maurer *et al.* (2013), assume a much stronger risk aversion parameter, of $\gamma = 5.0$. In Table 9, we show the optimal values for the VPA annuity proportions for $\gamma = 5.0$. We assume that the VPA and the fixed annuity use an annuitization interest rate of 3%. Similarly to the case where $\gamma = 2.0$, the optimal strategy is to invest all the initial wealth in a combination of the VPA and the fixed annuity, so we only show

λ	0.00	0.05	0.075	0.100
$\alpha_v = 0.25$	0.00	0.21	0.38	0.54
$\alpha_v = 0.40$	0.11	0.30	0.37	0.45
$\alpha_v = 0.60$	0.16	0.25	0.30	0.34

Table 9: Optimal proportion of initial wealth invested in the VPA (ω_V) for risk aversion coefficient $\gamma = 5.0$.

the proportion of initial wealth invested in the VPA in this table. As we would expect, increasing the risk aversion decreases the proportion of wealth invested in the VPA in all cases. It is worth noting however, that for each of these values of λ , for a retiree who may choose the equity proportion of the VPA fund, the highest expected utility arises from the 60% equity fund.

5 Projecting the income paths under the optimal strategy with CRRA utility

If the framework for the optimization is appropriate, then the results of following the optimal strategy should appear reasonable. To investigate this further, we use stochastic simulation to explore the possible income streams generated by the optimal strategies derived in the previous section, for an individual retiree. We use the same mortality and investment models here as in the optimization process.

Using Monte Carlo simulation, we project 10,000 income paths through retirement, conditional on the retiree being alive at each age, for a retiree with wealth of \$1,000,000 at retirement (age 65). We use a loading factor of $\lambda = 0.1$ for the fixed annuity, since this is a plausible margin, (and is the one assumed by Milevsky (2001)), and assume the VPA fund equity proportion is $\alpha_v = 0.4$. For this case the optimal strategy from Table 7 is to invest 100% of initial wealth in the VPA. The first year's income for the annuitant is \$69,495. Subsequently, the income from the VPA can be quite volatile. In Figure 1 we show the 5%, 50% and 95% quantiles from the Monte Carlo projection. We also plot 50 individual sample paths.

We see that the median annual payment is reasonably flat; however the 5th percentile falls to around 60% of the initial income for a retiree who survives. That is, a retiree who follows the optimal strategy faces a 5% probability that income will fall below 60% of the initial amount by age 95, if she survives that long. It is interesting to compare the outcomes in Figure 1 with a suboptimal choice. In Figure 2, we show the paths

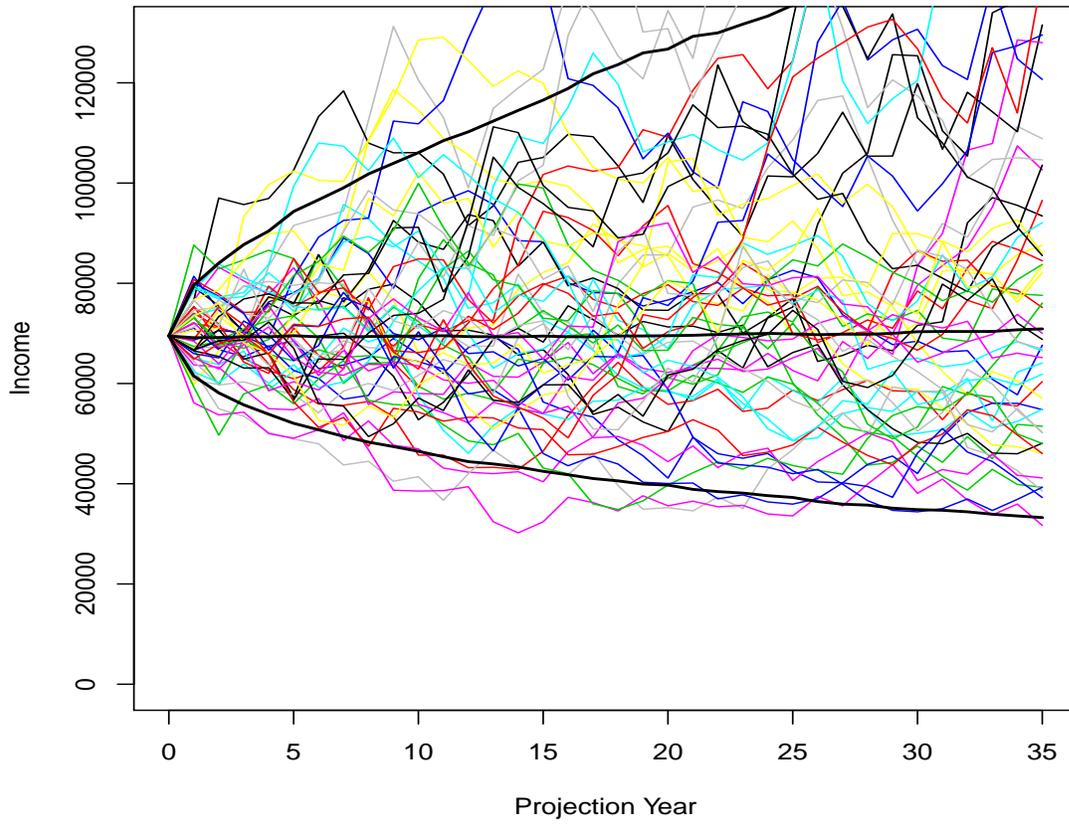


Figure 1: 5%, 50% and 95% quantiles of the annual payments during retirement, conditional on survival, with 50 individual paths; $\lambda = 0.1$, $\alpha_V = 0.25$, $\omega_V = 1.0$.

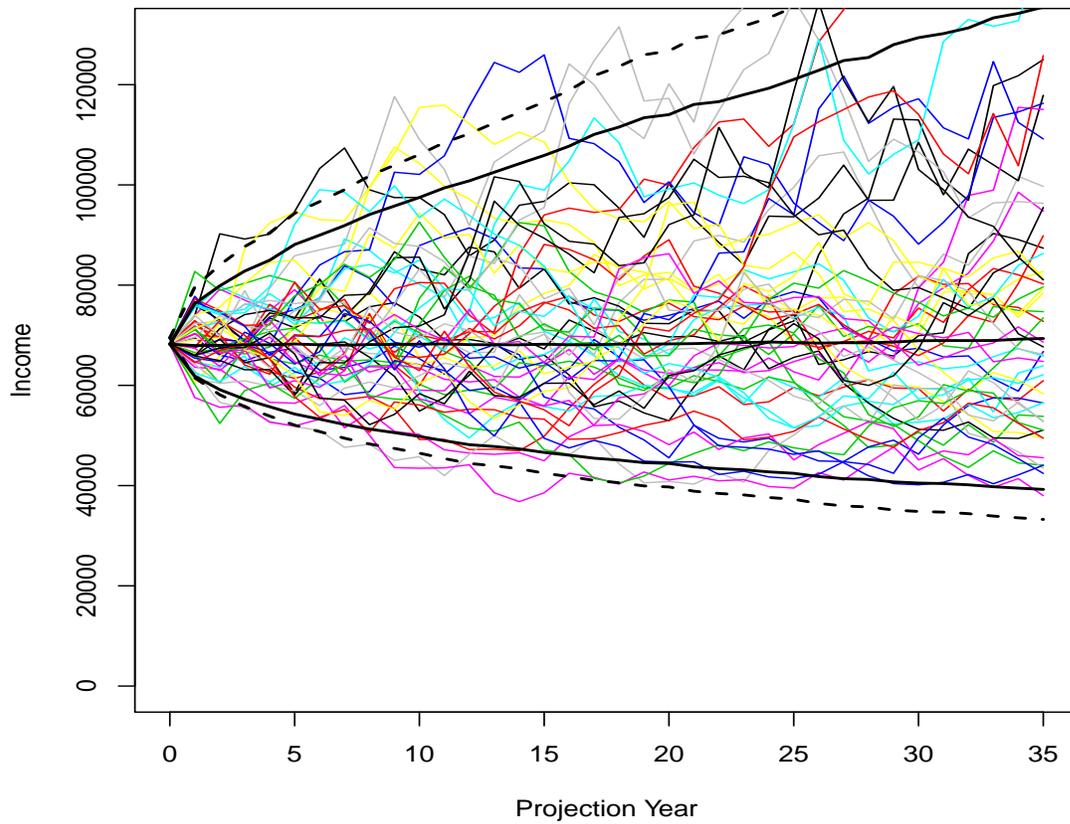


Figure 2: 5%, 50% and 95% quantiles of the annual payments during retirement , conditional on survival, with 50 individual paths; $\lambda = 0.1$, $\alpha_V = 0.25$, $\omega_V = 0.8$, with 5% and 95% quantiles from Figure 1

for the same investment/mortality scenarios, and for $\lambda = 0.10$ and $\alpha_V = 0.4$, as before, but we consider a strategy with 20% of wealth invested in the fixed annuity, and 80% invested in the VPA. The dashed lines are the 5% and 95% quantiles from Figure 1. The optimal strategy, in Figure 1, has more upside potential, but also a lower 5% quantile, compared with the sub-optimal strategy in Figure 2. To be more precise, at age 90 the 5% quantile of income under the optimal strategy is 37,600, which is a drop of 46% from the starting income of 69,495. Under the sub-optimal strategy, the starting income is 68,231, and the 5% quantile of income at age 90 is 42,700. The reason that the second strategy is dominated by the first is that the upside potential (represented by the higher 95% quantile) balances the downside risk in the utility calculation. Even though the utility function gives more weight to downside risk than upside potential, a strategy offering significant upside and downside variability may be preferred to a strategy which has better downside protection, if the upside potential in the first case is large enough. However, empirical studies (eg Greenwald and Associates (2013)) show that the strongest consideration for many retirees is fear of declining income proving inadequate to meet their needs. This consideration is not well accommodated in the CRRA utility approach. We can adapt the CRRA utility approach to require a minimum income level to be maintained, within the utility maximization framework. This is a form of *habit formation* utility; see MacDonald *et al.* (2013) and Pollak (1970).

6 Maximizing utility of excess consumption over a floor

In this section we use a version of a hyperbolic absolute risk aversion (HARA) utility function. This function is used by Kingston and Thorp (2005), as one of the simplest ways to introduce a form of habit formation. The version of HARA utility that we use measures the utility of *excess consumption* over a specified floor level F . Thus, the utility of a consumption level c is given by

$$U^{\text{HARA}}(c) = \frac{\max(0, (c - F))^{1-\delta}}{1 - \delta}, \quad \delta \neq 1. \quad (10)$$

In our case, the consumption floor F could represent necessary expenses that the retiree incurs every year (housing and medical care, for example). This utility function is similar to CRRA utility (with a shift of variable) and retains the tractability of the CRRA approach.

We consider, as before, a retiree with wealth at retirement of W_0 . We introduce a floor

consumption level, denoted F . Since the utility function would fall to $-\infty$ if the consumption falls below the floor, the retiree must invest enough of her wealth in the fixed annuity to secure the floor level income. This establishes a minimum value for ω_F , which varies for different values of λ . That is, given the annuity factor for the fixed annuity of $(1 + \lambda)\ddot{a}_{65}$, the minimum value for ω_F is given by

$$\begin{aligned} \frac{W_0 \omega_F}{(1 + \lambda)\ddot{a}_{65}} &\geq F \\ \Rightarrow \omega_F &\geq \frac{F(1 + \lambda)\ddot{a}_{65}}{W_0} \end{aligned}$$

So, for example, if we assume the annuity factor is calculated at 3%, which gives $\ddot{a}_{65} = 14.38955$, and also assume $W_0 = 1,000,000$, and $F = 35,000$, the minimum value for ω_F is $0.504(1 + \lambda)$.

We note that the δ in equation (10) is not quite the same as the γ in equation (7), though both measure risk aversion. To compare the results of our analysis in this section with the earlier section, we set δ to give, approximately, the same relative risk aversion using the HARA utility as we used for the CRRA utility. The relative risk aversion for utility $U(c)$ is defined as

$$R_R(c) = -c \frac{U''(c)}{U'(c)}$$

In the CRRA case (as the name implies), the relative risk aversion is constant for all c at $R_R(c) = \gamma$. For the HARA utility we have

$$R_R(c) = \frac{c\delta}{c - F}$$

Although this is not constant with respect to c , we know that the initial consumption is approximately

$$C^* \approx \frac{W_0}{\ddot{a}_{65}(1 + \lambda/2)}$$

which is the exact figure if the fund is evenly split between the fixed annuity and the VPA. To give an approximate match of relative risk aversion for the HARA as for the CRRA, we use

$$\frac{C^*\delta}{C^* - F} = \gamma$$

This needs further adjustment; because we want the floor to be a hard constraint, we need $\delta \geq 1$, so where the maintaining the CRRA parameter at issue equal to 2.0 would give $\delta < 1$, we set $\delta = 1$, and use log utility.

In Figures 3 and 4 we show the 5%, 50% and 95% quantiles for income paths using the optimal strategy for HARA utility, with parameters $\lambda = 0.1$, $W_0 = 1,000,000$, $F = 35,000$, $\gamma = 2.0$, which gives $C^* = 66,186$ and $\delta = 0.942$. Figure 3 is for a VPA fund with 40% in equities and Figure 4 assumes 60% of the VPA fund is invested in equities. In both cases, the minimum value for ω_F is 0.55, and this is also the optimal value. The dashed lines are the 5%, 50% and 95% quantiles from Figure 1. We see that both upside and downside variability are significantly constrained using HARA utility.

The initial income in both cases is 66,000, compared with 69,500 in the CRRA case. The 5% quantile at age 90 in the HARA case is 51,700 for $\alpha_V = 0.4$ and 49,000 for $\alpha_V = 0.6$, compared with 37,600 in the CRRA case.

The HARA approach will give more realistic income paths for retirees who require a minimum guaranteed income level to meet fixed expenses. We note that the proportion of assets invested in the VPA is reduced significantly when we use HARA utility instead of CRRA utility. Nevertheless, the ‘optimal’ strategy still includes a substantial proportion of the VPA.

7 Maximizing utility with constraint on declining income

In the previous two sections we found that, in all cases, the retiree’s expected discounted utility is maximized by annuitizing her entire wealth at retirement (in some combination of the VPA and the fixed annuity) and consuming all the income each year. The complex framework that allows for a myriad of choices of consumption and asset allocations, in each case reduces to the much simpler selection of the split of the initial wealth between the VPA and the fixed annuity. In this section, we take advantage of that simplification, to consider a more dynamic optimization objective. We now **assume** that the retiree will annuitize all her wealth (or, equivalently, that we are only concerned with the wealth that she chooses to annuitize), and also that she consumes all her income each year. The only control variable remaining is ω_V , the proportion of wealth invested in the VPA.

This simplified structure means that we can use monte carlo simulation to search for the optimal value of ω_V , without requiring the strict framework of time separable, static preferences.

We apply this method to consider a more dynamic form of habit formation. Suppose

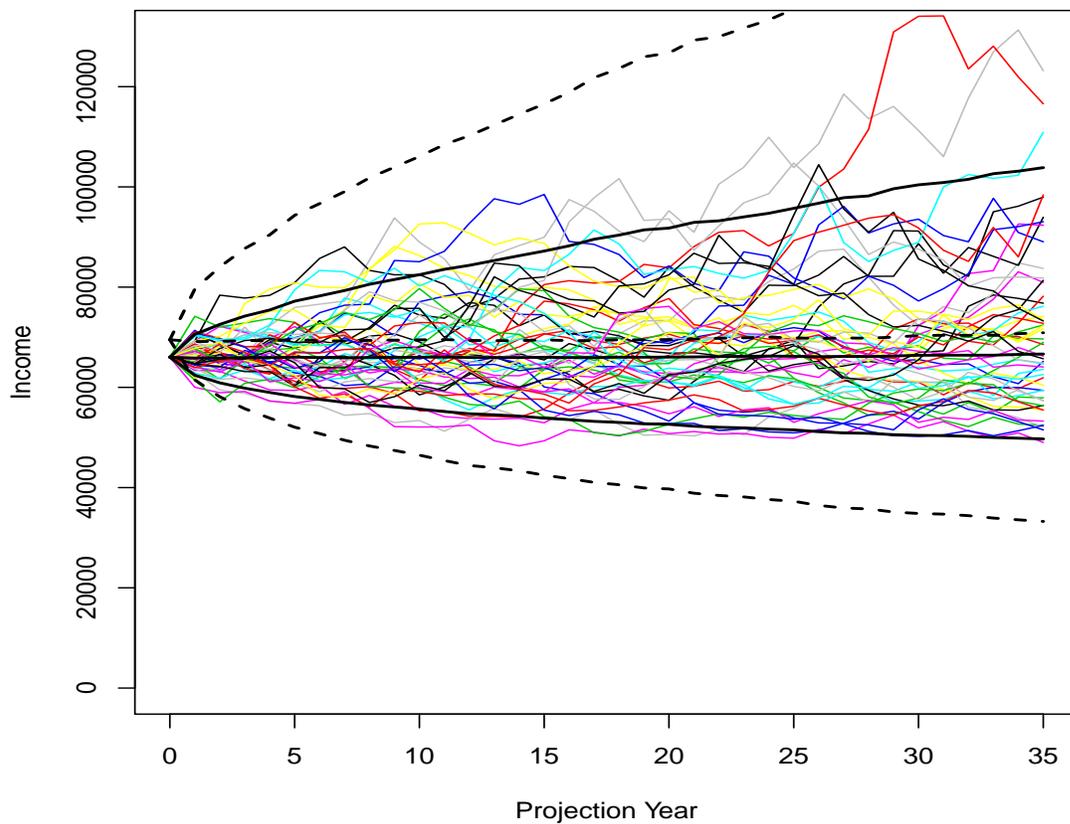


Figure 3: HARA utility; 5%, 50% and 95% quantiles of the annual payments during retirement, conditional on survival, with 50 individual paths; $\lambda = 0.1$, $\alpha_V = 0.4$, $\omega_V = 0.45$, with 5%, 50% and 95% quantiles from Figure 1

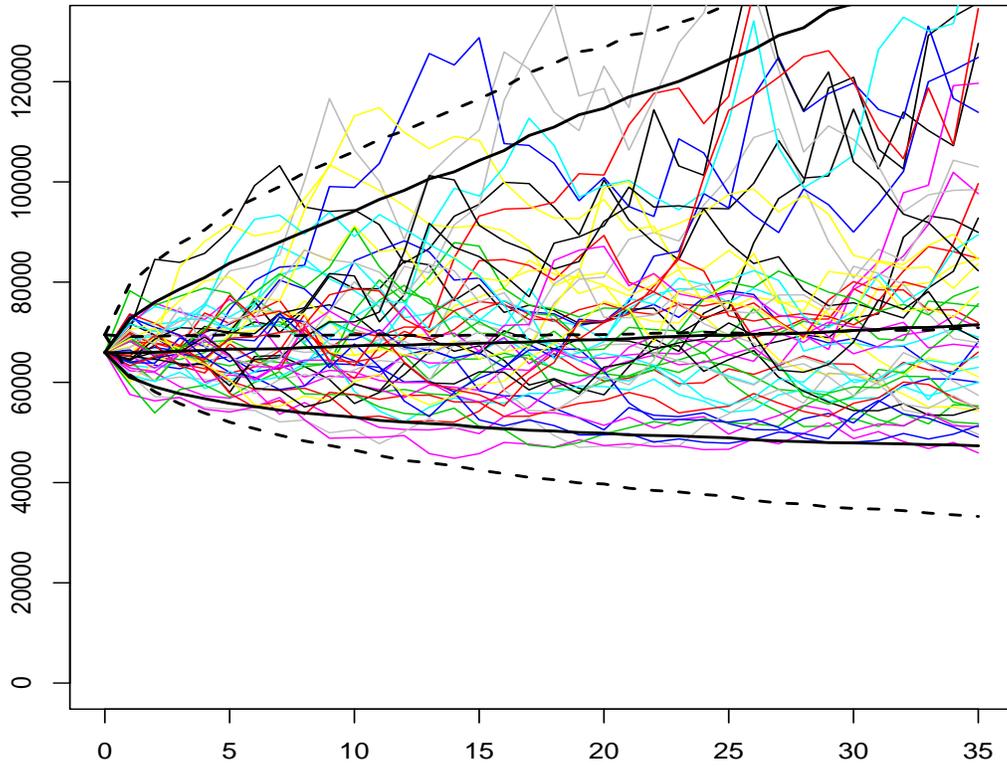


Figure 4: HARA utility; 5%, 50% and 95% quantiles of the annual payments during retirement, conditional on survival, with 50 individual paths; $\lambda = 0.1$, $\alpha_V = 0.6$, $\omega_V = 0.45$, with 5%, 50% and 95% quantiles from Figure 1

k	0.8	0.85	0.9
$\alpha_V = 0.4$	0.34	0.21	0.11
$\alpha_V = 0.6$	0.25	0.15	0.06

Table 10: Proportion of initial wealth invested in the VPA, ω_v , with maximum drop in consumption of $1 - k$; CRRA parameter $\gamma = 0.2$, fixed annuity loading $\lambda = 0.1$.

a retiree wishes to avoid a drop in income of more than $100k\%$ from the previous year. We can adapt the HARA utility above, but let the floor be set each year at k times the previous year's consumption.

The results are shown in Table 10. For these calculations we set the risk aversion coefficient at $\delta = 2.0$. We see that the VPA proportion ranges from 6%, for a retiree who wishes to avoid a drop of more than 10% in her income, and whose VPA fund is 60% invested in equities, to 34% for the retiree who wishes to avoid a drop of more than 20%, and whose VPA fund is 40% invested in equities.

8 Empirical evidence for retirees' risk preferences

Summarizing the utility-based results of the previous sections, we have:

- The classical CRRA approach to the annuitization decision, with a realistic loading factor of 0.1 for the price of a fixed annuity, indicates that a retiree with a CRRA coefficient of 2.0, should invest all their wealth in the VPA if the equity proportion in the VPA fund is 40%, or 80% of their wealth if the equity proportion in the VPA fund is 60% – and, given a choice between the 40% and 60% funds, the retiree should choose 60%. If the retiree's CRRA coefficient is 5.0, the proportion in the VPA would decrease to 45% for the 40% equity fund, or 34% for the 60% equity fund.
- Introducing a floor of \$35,000 to the utility calculation reduces the attractiveness of the VPA, and the utility maximization criterion indicates that the retiree should only invest 55% of their assets in the VPA, whether the VPA fund is 40% or 60% invested in equities.
- If we introduce a requirement that penalizes significant income decreases, the retiree with, for example, an aversion to a 15% drop in income should use this methodology, invest 21% of their initial wealth in the VPA, if the fund is 40% in equities, or 25% for the 60% equity fund.

So, we have a very wide range of possible values for the ‘optimal’ VPA investment. The key question is which, if any, of these utility functions and risk aversion parameters adequately describes a retiree’s risk preferences? In the CRRA case, we define the ‘optimal strategy’ in terms of the maximum discounted utility of total consumption, which implicitly assumes that retirees place significant value on the possibility (if slim) of windfall profits, even though this is only achievable by risking a substantial decline in income. This does not appear to us to be a realistic or appropriate objective function for most retirees, which leads us to question the value of an analysis based on CRRA utility maximization.

For some insight into retirees attitudes to risk, we consider the survey sponsored by the Society of Actuaries (SOA) (Greenwald and Associates (2013)). Retirees in the USA were asked about their attitude to financial security in retirement. The biggest concerns were as follows.

- The value of savings and investments might not keep up with inflation.
- There might not be enough money to pay for adequate health care.
- There might not be enough money to pay for a long stay in an nursing home.
- The retiree’s savings might be exhausted.
- The retiree might not be able to maintain a reasonable standard of living for the rest of their life.
- The retiree may become unable to manage their finances.

We note that the statements are all defensive. There is no evidence from this survey that retirees are interested in chasing high equity returns with their retirement funds, even though that is an optimal strategy under CRRA utility. Maintaining purchasing power is a much more modest growth objective. However, a fixed payout annuity fails to achieve inflation indexing, which could make the VPA more attractive⁵.

The fear of future health care costs is likely to be a major driver of liquidity preference, and this is not commonly reflected in the utility functions used in the mainstream annuitization literature⁶.

The fear of exhausting savings would be alleviated with annuitization. Similarly, since it is likely that in extreme old age there would be little incentive (and perhaps not much

⁵Inflation indexed annuities are still not widely used, perhaps because of conservative pricing assumptions relative to fixed payout annuities. See <http://www.investopedia.com/articles/05/inflationprotectannuity.asp>.

⁶An exception is Peijnenburg *et al.* (2013).

opportunity) to save money from income, the complexity of financial management under full annuitization is significantly simpler than the case where a retiree has an investment portfolio of stocks and bonds. It is interesting to note that the concern about managing money in old age is well-founded. Asp *et al.* (2012) demonstrate that the elderly are increasingly vulnerable to deception and fraud, due to a deterioration in the neurological mechanisms required in scepticism and disbelief. However, once again the reasonable concerns identified in the survey are not captured well by the utility functions used here or in the broader classical annuitization literature.

One more relevant piece of empirical evidence relates to individuals' subjective survival probability assessment. A meta-survey by O'Connell (2011) shows that there is systematic underestimation of future lifespan, with men typically underestimating their expected lifespan by around 4 years, and women by around 6 years. The difference between actual lifetime and the subjective estimation will, of course, be substantially greater for those who live well beyond the expected future lifetime. The annuitization and investment decisions made by retirees are likely to reflect this underestimation, which will make annuity prices seem very high. The market annuitization factor (\ddot{a}_{65}) for a 65-year-old female, with \$1 million to invest, is currently around 17.5 for a fixed, level annuity, without guarantee. So fully annuitizing the \$1 million fund would generate annual income of around 57,000 for life. A retiree who believes that they will live for exactly 20 years, and that they can achieve a 6% per year return on assets, would anticipate an income of over 80,000 per year. In fact, it is easy to find advice online that utilizes this type of calculation, to deliver the conclusion that life annuities are not a suitable investment for most retirees. For example, a well-known financial website⁷ declares that

“We don't recommend an allocation to annuities for any portion of your portfolio. We believe an age-appropriate allocation to bonds provides a similar boost to the likelihood you will have sufficient assets in retirement.”

Taking all of this evidence into consideration, it seems unlikely to us that a retiree would choose to invest all their retirement wealth in the VPA, despite the results from Table 6. The reasons may be rational, for example, ensuring adequate annual income into extreme old age, or irrational, as where the retiree significantly underestimates her future potential lifespan, or a mixture of the two. In any case, it does not appear that the CRRA does a good job of describing how retirees behave, nor does it succeed in describing how retirees *should* behave. We note further that there is little agreement in the annuitization literature about a suitable value for the CRRA parameter γ . Although the empirical evidence in Maier and R uger (2010) points to a CRRA parameter of around $\gamma = 2.0$, there

⁷Downloaded from <http://www.forbes.com/sites/davidmarotta/2012/08/27/the-false-promises-of-annuities-and-annuity-calculators/> on 9/10/2015.

is significant support in the literature for other values; Maurer *et al.* (2013), Donnelly *et al.* (2013) and Horneff *et al.* (2010b) use $\gamma = 5.0$; Mitchell and Moore (1998) suggest that values between 0 and 2 are appropriate; Milevsky and Young (2007) use values of 1, 2 and 5. We suggest that this lack of consensus may be a result of the fact that CRRA utility does not model risk aversion of retirees in a realistic, or reasonable manner. That is, the single value of γ that describes retirees' risk preferences does not exist, because risk preferences cannot realistically be captured with the CRRA utility. And yet, researchers considering the annuitization puzzle still extensively adopt the CRRA model, with little or no consideration of its appropriateness.

9 Conclusion

In this paper, we used dynamic programming to obtain the optimal investment and consumption strategy for a retiree whose choices at retirement include the VPA, a fixed annuity, and self-annuitization, and then used Monte Carlo simulation to test the optimal strategies to see what risks remained when expected utility is maximized. We considered three different utility functions: CRRA, which is the most popular amongst researchers in this area, a form of HARA which is effectively CRRA with a floor, and a form of HARA which limits the proportionate reduction in income in successive years.

We find that the VPA does improve expected utility of consumption in almost all cases. However, the optimal proportion of funds invested in the VPA differs widely for the different utility functions and risk aversion parameters. When we step back and consider the survey evidence of retirees' risk attitudes, we find that the CRRA results are not consistent with the survey evidence, but that using the VPA to achieve equity exposure, in conjunction with a fixed annuity, could be a reasonable decision, particularly if inflation indexed annuities are not available or are priced with very high loadings. However, under the constraints of our framework, once the VPA investment is made the funds cannot be withdrawn, which means that the equity portion of the retiree's portfolio will never decrease below that represented by the equities in the VPA fund.

The present analysis does not take idiosyncratic mortality risk into account. GSA schemes may be offered to smaller, open groups, whose mortality experience may depart from assumptions because of group size. GSA schemes could also be offered to closed groups that would shrink through time. These characteristics would lead to more volatile payments, and would thus increase the riskiness of the GSA scheme.

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A Mortality Model

Under the CBD model, The logit of the conditional mortality rate $q_{x,t} = 1 - p_{x,t}$ is

$$\text{logit } q_{x,t} = \log \frac{q_{x,t}}{1 - q_{x,t}} = A_{0,t} + A_{1,t}x,$$

and the two dimensional process $A_t = (A_{0,t}, A_{1,t})^T$ is given by

$$A_{t+1} = \tau + A_t + VZ_{t+1},$$

where $V^T V = \Sigma$ is the covariance matrix and Z_{t+1} is a standard normal random variable. We use the following parameters obtained Maurer *et al.* (2013):

$$A_0 = \begin{bmatrix} -10.1502416 \\ 0.0904819 \end{bmatrix}$$

$$\tau = \begin{bmatrix} -0.0337497 \\ 0.0003242 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} 0.0019766 & -0.0000291 \\ -0.0000291 & 0.0000006 \end{bmatrix}.$$

We also assume all lives expire by age 111, so we set $q_{110,t} = 1$ for all t .

B Solving the optimization problem using dynamic programming

B.1 The optimization process

In this section, we explain in greater detail how to solve the optimization problem through dynamic programming. Our optimization problem has three state variables: W_t , B_t^V and B_t^F . However, to illustrate the method, we assume only one state variable here, omitting the annuity payments B^V and B^F . The technique presented can easily be extended to higher dimensions.

We have $W_t = (W_{t-1} - C_{t-1})((1 + r) + \omega_{t-1}(R_{t-1}^B - r))$. Consider the objective function

$$H(0, W_0) = \max_{\omega_0, C_0} \{U(C_0) + E_0 [\beta H(1, W_1)]\}, \quad (11)$$

where ω and C are the controls we want to solve for. More generally, let

$$H(t, W_t) = \max_{\omega_t, C_t} \{U(C_t) + E_t [\beta H(t + 1, W_{t+1})]\}. \quad (12)$$

Notice that the function $H(t, W_t)$ is always the maximized future discounted expected utility. We assume that the utility function is time-separable. In other words, the optimal consumption at a given time is independent of past consumption except through the process W_t . This allows us to treat each period, recursively, from end to start. At a given time t , in the one-variable problem, the optimal controls are only dependent on W_t . In other words, we can construct a set of values for W_t , together with the optimal values for the control variables given W_t .

However, we cannot entirely solve the problem at each t since we do not know the value of the function $H(t + 1, W_{t+1})$. Generally, it is only possible to write this function in analytical form at the year end following the last possible curtate survival date, which we have denoted $\mathcal{K} + 1$, that is, $H(\mathcal{K} + 1, W_{\mathcal{K}+1}) = 0$ for any $W_{\mathcal{K}+1}$, as it is assumed all lives have died by $\mathcal{K} + 1$ (and there is no bequest motive). Since this is the only known value for the derived utility function, we begin our optimization from the second-to-last period \mathcal{K} . At that time, given our assumption that no lives survive to $\mathcal{K} + 1$, it is optimal to consume all remaining wealth, giving

$$H(\mathcal{K}, W_{\mathcal{K}}) = \max_{\omega_{\mathcal{K}}, C_{\mathcal{K}}} \{U(C_{\mathcal{K}}) + E_{\mathcal{K}}[H(\mathcal{K} + 1, W_{\mathcal{K}+1})]\} = U(W_{\mathcal{K}}) \quad (13)$$

Since we cannot know the value of the wealth process at time \mathcal{K} (it will depend on the optimal controls during periods 1 to \mathcal{K}), we calculate and store the derived utility function $H(\mathcal{K}, W_{\mathcal{K}})$ for a range of different values of $W_{\mathcal{K}}$. These are chosen to represent the range of feasible values for $W_{\mathcal{K}}$. Then, we move back one period and, again, for a range of values of $W_{\mathcal{K}-1}$, solve for the optimal controls $\omega_{\mathcal{K}-1}$ and $C_{\mathcal{K}-1}$ that will maximize

$$U(C_{\mathcal{K}-1}) + E_{\mathcal{K}-1} [\beta H(\mathcal{K}, W_{\mathcal{K}})] = U(C_{\mathcal{K}-1}) + \beta E[U(W_{\mathcal{K}})]$$

However, this time, we only know $H(\mathcal{K}, W_{\mathcal{K}})$ for the selected discrete values that we used for $W_{\mathcal{K}}$. Our candidate controls $\omega_{\mathcal{K}-1}$ and $C_{\mathcal{K}-1}$ will most likely not return one of the values $W_{\mathcal{K}}$ for which we have calculated $H(\mathcal{K}, W_{\mathcal{K}})$. Thus, we have to interpolate from the values we know to approximate the derived utility function for any value $W_{\mathcal{K}}$. This will allow us to obtain the optimal controls at time \mathcal{K} . The same procedure is repeated until we find the controls at time 0.

Here is the algorithm that is followed to obtain the optimal controls for a problem with $\mathcal{K} + 1$ periods, using n discrete values for each W_t .

1. Build a grid of values of W_t at which the derived utility function will be calculated. This grid will have n rows and $\mathcal{K} + 1$ columns. Each column represents a vector of possible wealths at a given time.
2. Build another grid of the same size to store the values of $H(t, W_t)$. Fill the last column with zeros, since we assume no bequest function.
3. Build two other grids of the same size to store the optimal values of ω and C_t at each time, for different wealths.
4. For each column $t = \mathcal{K}$ to 1, apply the following to each element $i = 1$ to n of the column:
 - (a) Given wealth W_t^i , find the optimal controls ω_t^i and C_t^i . Note that the function to optimize will use interpolation to calculate the value of the derived utility function one period later.
 - (b) Store the optimal controls and the derived utility in the corresponding grid.
5. Now the grids are filled out and the first period needs to be solved.
6. Given wealth W_0 , find the optimal controls ω_0 and C_0 . Again, the function to optimize will use interpolation to calculate the value of the derived utility one period later.

To apply this method to our optimization problem, we need to extend it to the case where there are three state variables. Hence, instead of having a vector of values W_t and its associated vector H_t at each time t , we have a four-dimensional array with values W_t , B_t^V and B_t^F at each time t (denote by n_W , n_{LV} and n_{LFA} the number of values of W_t , B_t^V and B_t^F that are considered, respectively). The interpolation that needs to be performed to solve the problem at each data point is thus 3-dimensional. This method extends quite easily to multiple dimensions. However, the number of data points at which the derived utility function must be calculated is multiplied ($n_W \times n_{LV} \times n_{LFA}$ instead of n), and the interpolation can become computationally burdensome.

B.2 Simplifying the optimization problem by normalizing

The normalization described in this section was inspired by Hubener *et al.* (2014).

The optimization results presented in Section 4.2 were obtained using the dynamic programming method described above. However, to increase the efficiency of the program, the number of dimensions was reduced from three to two by normalizing with respect

to B_t . That is, instead of working with the variables W_t , B_t^V , B^F , and C_t , we use the normalized variables

$$w_t = \frac{W_t}{B_t}, \quad \rho_t = \frac{B^F}{B_t}, \quad \text{and} \quad c_t = \frac{C_t}{B_t}.$$

This simplification is possible because once the initial investment choice is made, the optimization problem is homothetic in the total annuity payment. This effectively means that the absolute amount of the payment does not impact the utility maximizing strategy, so that working with B_t^F , B_t^V and c_t gives the same results as working with W_t , B_t^V , B^F and C_t for any B_t . This is very similar to the normalization by the labor income used by Maurer *et al.* (2013). We demonstrate this more formally here.

We need to show that

$$H(t, W_t, B_t^V, B_t^F, N_t) = B_t^{1-\gamma} h(t, B_t^V, B_t^F, N_t) \quad (14)$$

for some function $h(\cdot)$.

We show this by backwards induction. To make the proof easier to read, we will omit the arguments in H and h other than the time variable.

At $\mathcal{K} + 1$, $H(\mathcal{K} + 1) = 0$, so the result holds trivially. In the penultimate period, we have

$$\begin{aligned} H(\mathcal{K}) &= U(W_{\mathcal{K}}) \\ &= \frac{W_{\mathcal{K}}^{1-\gamma}}{1-\gamma} \\ &= B_{\mathcal{K}}^{1-\gamma} h(\mathcal{K}), \end{aligned}$$

where $h(t) = \frac{B_t^{1-\gamma}}{1-\gamma}$.

Now, assume that for some $t + 1$, $1 \leq t + 1 \leq \mathcal{K} + 1$ the result in equation (14) holds.

Then consider the function at t .

$$H(t) = \max_{\omega_t, C_t} \{U(C_t) + E_t [\beta H(t+1)]\}$$

$$\text{and } U(C_t) = \frac{C_t^{1-\gamma}}{1-\gamma} = \frac{(B_t c_t)^{1-\gamma}}{1-\gamma} = B_t^{1-\gamma} U(c_t)$$

$$\Rightarrow H(t) = \max_{\omega_t, c_t} \{B_t^{1-\gamma} U(c_t) + E_t [\beta B_{t+1}^{1-\gamma} h(t+1)]\} \quad (\text{using the inductive hypothesis}).$$

$$\text{Now } B_{t+1} = B_t(1 + \rho_t j_t),$$

since

$$\rho_t = \frac{B_t^V}{B_t}.$$

Then

$$H(t) = \max_{\omega_t, c_t} \{B_t^{1-\gamma} U(c_t) + E_t [\beta B_t^{1-\gamma} (1 + \rho_t j_t)^{1-\gamma} h(t+1)]\}$$

$$\Rightarrow H(t) = B_t^{1-\gamma} h(t) \quad \text{where}$$

$$h(t) = \max_{\omega_t, c_t} \{U(c_t) + E_t [\beta (1 + \rho_t j_t)^{1-\gamma} h(t+1)]\}$$

Note that, as required, $h(t)$ is a function of w_t and ρ_t , but not of W_t , B_t^V or B^F . Using this normalization we can perform the optimization problem using dynamic programming with two-dimensional grids.

To further accelerate the computation, we use the method described in Section 5.1 of Carroll (2011) to calculate the expectation of functions of lognormal random variables. In this method, the lognormal distribution is discretized and the integral is approximated by a sum, in which each term represents an interval of equal probability.

To obtain the results presented in Section 4.2, we discretized the space of state variables (w_t, ρ_t) over a grid of size 89×101 , where the distances between the gridlines in w_t increase as w_t increases. Larger grid sizes were explored, with similar results.