# Using Life Table Techniques to Model Mortality Rates for Small Populations

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Presented at the Living to 100 Symposium

Orlando, Fla.

January 4–6, 2017

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## Abstract

The study of human longevity has been a popular research topic due to the prolonging of life. However, the limited availability and poor quality of elderly data increase the difficulty of mortality modeling. It is particularly challenging if the size of the target population is small, and the parameter estimation of stochastic mortality models can be distorted. For example, the famous Lee-Carter model (Lee and Carter 1992) would have biased estimates for age-related parameters $\alpha_x$ and $\beta_x$ in the case of small populations. In this study, we aim to provide a possible solution to deal with the parameter estimation of mortality models when the population size is small.

We propose graduation methods to modify the parameters' estimates of mortality models, similar to the process of constructing life tables where mortality rates are smoothed to remove the irregularity of some observed values. The graduation methods, including Whittaker graduation and partial standard mortality ratio (SMR), will be applied to the Lee-Carter model to smooth the parameters' estimates and compared to the coherent Lee-Carter model (Li and Lee 2005). We use computer simulation to evaluate the proposed approach, and we find that it does have smaller fitting errors when the population size is small.

Keywords: small area estimation, standard mortality ratio, graduation methods, Lee-Carter model, longevity risk

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## 1. Introduction

Since people are living longer, life planning for the elderly has become a popular issue around the world. Among all topics, the study of the elderly's mortality rates and health receives a lot of attention. However, human life expectancy has been increasing rapidly, and in many countries (especially those with small populations and a rapid increase in longevity), data about the elderly have been limited in quantity and available period, which makes modeling mortality rates among the elderly difficult. For example, the famous Lee-Carter model (Lee and Carter 1992) would have biased estimates for age-related parameters $\alpha_x$ and $\beta_x$ in the case of small populations. According to our simulation, the bias is especially noticeable when the population size is 200,000 or less.

The following example demonstrates the influence of small populations. We first use Taiwan's mortality to derive the parameters of the Lee-Carter model. Suppose that the mortality rates follow the Lee-Carter model and the population structure is the same as that in Taiwan. We consider different population sizes, ranging from 10,000 to 5 million, and then simulate the random numbers of deaths, and then we apply them to the Lee-Carter model. To emphasize the influence of small populations, we show the estimation results only of population sizes not more than 200,000. Figure 1 shows the average biases of estimates of parameters $\alpha_x$ and $\beta_x$ via singular value decomposition. The biases of $\alpha_x$ estimates are especially noticeable and always larger than 0. In contrast, the biases of $\beta_x$ estimates can be positive or negative and seem to be around 0 on average. Note that the average biases are calculated based on 1,000 replications; the data period is 1991–2010, and the age range is 0–84 in the format of five-year age groups (17 groups).

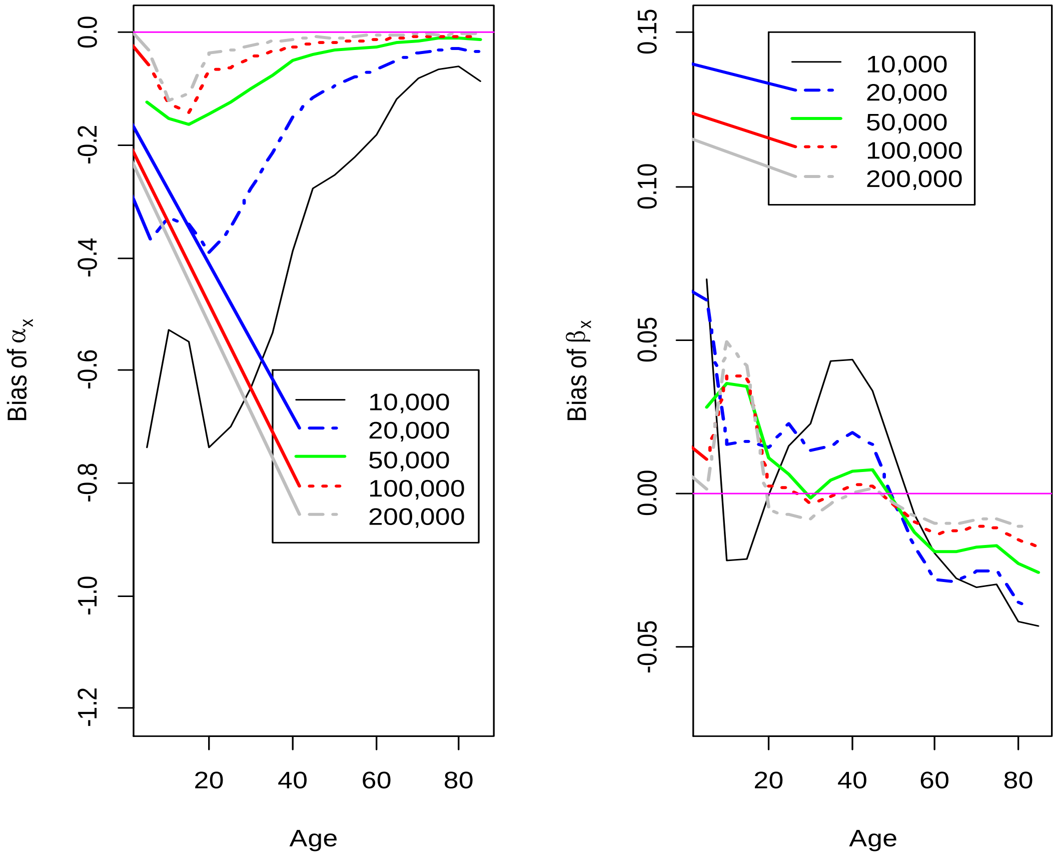


**Fig. 1. Bias of Parameters' Estimates of the Lee-Carter Model**

The biased estimates in the case of small populations probably are the main reason why many recent studies focus on modifying mortality models for small populations. Intuitively, increasing the sample size is the most efficient way to stabilize the parameter estimation of mortality models, and including the mortality data from neighboring areas (or areas with similar mortality profiles) is a natural choice. For example, the coherent Lee-Carter model by Li and Lee (2005) can reduce the estimation errors by referencing the mortality data from populations with similar mortality improvements. Of course, the Bayesian approach is another possibility for increasing the sample size, such as the Bayesian modification of Lee-Carter model by Wiśniowski et al. (2015). In a sense, the coherent Lee-Carter model and Bayesian methods increase the sample size of small populations. However, it is difficult to judge which populations have mortality profiles similar to that of the target population.

Dealing with estimating mortality rates of small populations is not new in the insurance industry, and actuaries often apply smoothing methods to reduce the fluctuations of age-specific mortality rates in constructing life tables. In fact, the graduation methods originally are designed to handle the problem of insufficient data, particularly for the elderly. Many traditional graduation methods enlarge the sample size by including data from adjacent ages, similar to those in the previous paragraph; two examples are moving weighted averages and the Whittaker method. We think it is possible to adapt the idea of graduation methods and use it to stabilize the parameter estimation of mortality models in the case of small populations.

In this study, we explore the possibility of combining graduation methods and the Lee-Carter model, adjusting the differences between the small and the reference populations. The reference population is used to provide smoothed mortality rates of the small population. Wang et al. (2012) showed that the Whittaker and partial standard mortality ratio (SMR) methods are effective in reducing bias and variation of mortality estimates if a proper reference population is chosen. We shall further evaluate if the idea of Wang et al. can be used to stabilize the parameter estimation of mortality models. Specifically, we assume that the mortality rates of small and reference populations satisfy the Lee-Carter model but their model parameters (or mortality improvements) are not the same.

Similar to the setting in Wang et al., under certain mortality scenarios, we use a computer simulation in Section 4 to evaluate whether the modification via graduation methods is valid with respect to the mean absolute percentage error (MAPE), compared with those from the Lee-Carter model and coherent Lee-Carter model. The results show that the Whittaker and partial SMR methods can improve the parameter estimation of the Lee-Carter model in the case of small populations.

## 2. Methodology

The idea behind the proposed approach is similar to that of using graduation methods to adjust irregular fluctuations in observed mortality rates. However, unlike the usual graduation methods, such as the moving average, the proposed adjustment of mortality rates is based on a reference population, similar to Bayesian graduation. Basically, we propose two graduation methods: partial standard mortality ratio (SMR) and the Whittaker method. We will introduce the proposed approach in this section and evaluate its performance in the next section.

The partial SMR (Lee 2003) is a modification of SMR, which is used to smooth mortality rates of small populations via the information from a large population, referencing the value of the SMR. The SMR, which is often used in epidemiology, is defined as follows:

$$\text{SMR} = \frac{\sum_x d_x}{\sum_x e_x} = \frac{\sum_x d_x}{\sum_x P_x \times m_x^R}, \tag{2.1}$$

where $d_x$ and $e_x$ are the observed and expected numbers of deaths at age $x$ for the small population, $P_x$ is the population size of age $x$ for the small population, and $m_x^R$ is the central death (or mortality) rate of age $x$ from the reference population. The SMR can be treated as a mortality index. If the SMR is larger (or smaller) than 1, then it usually indicates that the small population has a higher (or lower) overall mortality rate than the reference population.

The numbers of age-specific deaths in the small population often are not many, and the observed mortality rates fluctuate a lot and sometimes are even 0. The SMR can provide a possible guideline to fine-tune these mortality rates. For the partial SMR, the graduated mortality rates satisfy

$$v_x = u_x^* \times \exp\left( \frac{d_x \times \hat{h}^2 \times \log(d_x / e_x) + (1 - d_x / \sum d_x) \times \log(\text{SMR})}{d_x \times \hat{h}^2 + (1 - d_x / \sum d_x)} \right), \tag{2.2}$$

or the weighted average between raw mortality rates and SMR, where $\hat{h}^2$ is the estimate of parameter $h^2$ for measuring the heterogeneity (in mortality rates) between the small and reference populations, and $u_x^*$ is the mortality rate for age $x$ in the reference population.

The idea behind the partial SMR is similar to a credibility-weighted estimate for calculating the future premium (Klugman et al. 2012), where the estimate is a linear combination of recent observed loss and related reference information. The Bayesian graduation methods (e.g., Kimeldorf and Jones 1967) function in a similar format, and the updated (or posterior) estimates are also a linear combination of new observations and past experience (London 1985). The key is to choose appropriate weights and the proper reference population. Of course, the reference population should have larger population size in order to have smooth values of $u_x^*$.

To achieve satisfactory results, Lee (2003) suggests the weight of partial SMR:

$$\hat{h}^2 = \max\left( \frac{\sum\left((d_x - e_x \times SMR)^2 - \sum d_x\right)}{SMR^2 \times \sum e_x^2}\ ,\ 0 \right) \tag{2.3}$$

The larger $\hat{h}^2$ is, the larger the difference in age-specific mortality rates (i.e., mortality heterogeneity, or larger dissimilarity in shape between the age-specific mortality curve of the small population and that of the larger population). When the number of deaths is smaller, there will be greater weight from the large population, and the graduated mortality value equals $\mathrm{SMR} \times u_x^*$ when the number of deaths is 0. Lee mentioned that using the weight function $\hat{h}^2$ in Equation (2.3) usually has smaller mean square error (MSE) in mortality estimation. However, the derivation of $\hat{h}^2$ is through some sort of approximations, and it cannot guarantee to have the smallest MSE.

Alternatively, we can also use the Whittaker graduation method to stabilize the mortality rates of a small population, with a modification similar to the partial SMR. First, we calculate the age-specific ratio of mortality rates from the small population to those from the reference population, or define $s_x = u_x / u_x^*$, where $u_x$ is the observed mortality rate of age $x$ for the small population. Next, we apply the Whittaker graduation to the mortality ratio $s_x$ via minimizing the following objective function:

$$M = \sum_x w_x (r_x - r_x^*)^2 + h \sum_x (\Delta^z r_x^*)^2 , \tag{2.4}$$

where $r_x$ is the graduated mortality ratio, $w_x$ is the weight (or exposure) of age $x$, $h$ is a smoothing parameter, and $\Delta$ is the difference operator, or $\Delta f(x) = f(x + 1) - f(x)$. Finally, the graduated mortality rates of small population are $s_x \times u_x^*$. The choice of parameter $h$ is the key, as well as the choice of reference population, in applying the Whittaker ratio (namely) graduation.

Selecting the reference population is critical in applying the proposed graduation methods. This is also the case for applying the coherent Lee-Carter model, and choosing the appropriate group of coherent populations is important. In practice, selecting the populations with similar mortality profiles is not easy, and a natural choice is the whole nation (or nearby areas) if the small population is a subset of the nation. But the mortality differences within a country can be huge, even for neighboring cities. For example, in Taiwan, the largest difference in life expectancy between counties is more than 10 years (the city of Taipei versus Tai-tung County in the 2014 Taiwan Abridged Life Tables). It would be questionable to use the population of Taipei as the reference group for Tai-tung County. In the next section, we will use computer simulation to evaluate the proposed approach, with emphasis on the similarity between the small and reference populations.

## 3. Graduating Mortality Rates via the Reference Population

As mentioned previously, choosing the appropriate reference population is important. However, instead of searching for the perfect reference population, we want to use the similarity level between the small and reference populations to judge whether we should adjust the mortality rates of the small population via the reference population. In this section, we first evaluate the performance of graduation methods using varies similarity levels. In the next section, we will use the idea of graduation to modify the parameter estimation of the Lee-Carter model.

Suppose that there are seven scenarios for the mortality ratio $s_x$ between the small and reference populations, as shown in Figure 2. Various scenarios are designed to evaluate the effect of different graduation methods. The three scenarios in the left panel indicate that the mortality rates of the small and reference populations are similar, and we expect that the partial SMR would be a good choice for graduation. In contrast, the other four scenarios in the right panel assume that the mortality rates of the small and reference populations are different. For these four cases, the partial SMR might not be a good choice.

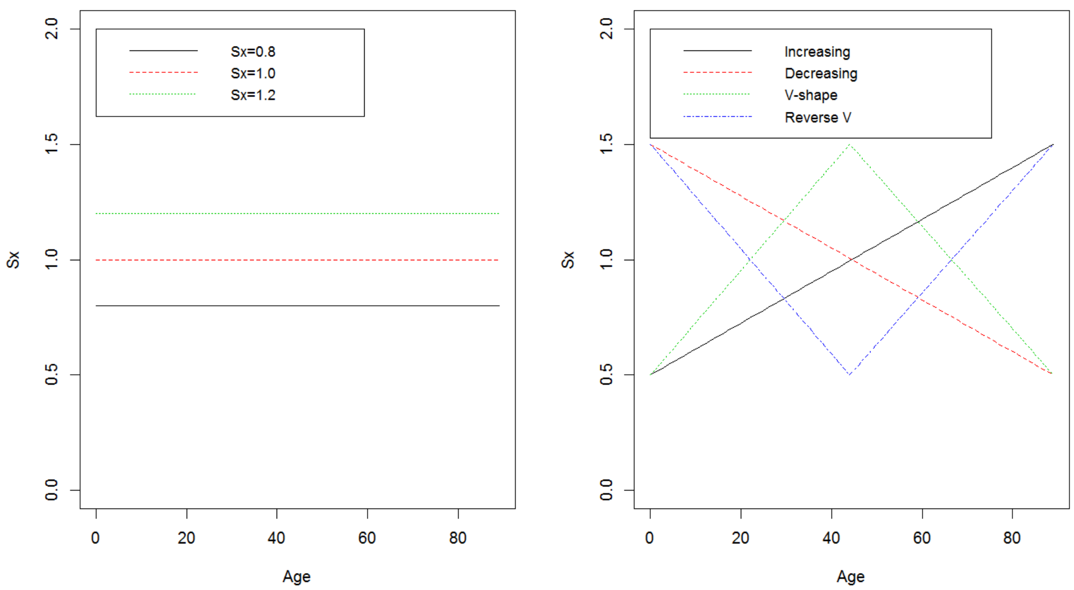


**Fig. 2. Seven Mortality Ratio Scenarios**

We use a computer simulation to evaluate the performance of partial SMR and Whittaker graduation. Specifically, we use Taiwan's female mortality data to fit the Lee-Carter model and treat the estimated parameters as the true values. The data period is 1991–2010, and the age range is 0–89, divided into five-year age groups of 0–4, 5–9, 10–14, … , 85–89, for a total of 19 groups. The size of the small population is set to be either 100,000 or 200,000, and the size of reference population is set to be either 2 million or 4 million. Also, the comparison criterion is based on the mean absolute percentage error (MAPE):

$$MAPE = \frac{1}{n}\sum_{i=1}^{n}\frac{\left|Y_i - \hat{Y}_i\right|}{Y_i}\times 100\%, \tag{3.1}$$

where $Y_i$ and $\hat{Y}_i$ are the observed and predicted values for observation $i$, $i$ = 1, 2, ..., $n$. According to Lewis (1982), a prediction with MAPE less than 10 percent is treated as highly accurate, and a MAPE greater than 50 percent is considered inaccurate.

Since the simulation results are similar for the cases where the reference population is larger than 2 million, we will only show the cases of 2 million. Tables 1 and 2 are the simulation results of cases where the small population is 100,000 and 200,000, for 1,000 simulation replications. Other than raw data and two proposed graduation methods, we also consider the case of Whittaker graduation to the observed mortality rates as a control group. For the Whittaker ratio and Whittaker graduation methods, the parameter $w_x$ is the exposure of age $x$, and the parameter $h$ is average exposure of all ages.

**Table 1. Mean Absolute Percentage Error (MAPE) of Graduation Methods: 100,000 vs. 2 Million**

| | $s_x$ = 0.8 | $s_x$ = 1 | $s_x$ = 1.2 | Increase | Decrease | V | Rev-V |
|---|---|---|---|---|---|---|---|
| Raw | 42.87% | 38.99% | 36.05% | 36.38% | 43.80% | 39.81% | 41.26% |
| Whittaker | 32.27% | 30.13% | 28.24% | 28.52% | 33.52% | 32.12% | 29.90% |
| Whittaker ratio | 25.18% | 21.65% | 19.70% | 19.88% | 27.32% | 21.54% | 25.18% |
| Partial SMR | 9.45% | 9.15% | 9.02% | 27.42% | 49.04% | 30.85% | 20.70% |

***Note:*** The cells with gray background are those with the smallest MAPE.

**Table 2. Mean Absolute Percentage Error (MAPE) of Graduation Methods: 200,000 vs. 2 Million**

| | $s_x$ = 0.8 | $s_x$ = 1 | $s_x$ = 1.2 | Increase | Decrease | V | Rev-V |
|---|---|---|---|---|---|---|---|
| Raw | 31.74% | 29.15% | 27.40% | 27.71% | 32.50% | 30.92% | 29.28% |
| Whittaker | 25.42% | 23.73% | 22.76% | 23.33% | 26.30% | 26.52% | 22.74% |
| Whittaker ratio | 20.22% | 17.77% | 15.92% | 15.88% | 21.31% | 17.57% | 19.91% |
| Partial SMR | 8.73% | 8.55% | 8.45% | 24.59% | 47.24% | 26.25% | 17.43% |

***Note:*** The cells with gray background are those with the smallest MAPE.

As expected, all the graduation methods have smaller MAPEs than those without graduation (except for the case of decreasing scenario). For the first three scenarios, in which the mortality rates of small and reference populations have the same proportion for all ages, the SMR can provide a very good approximate estimate of this proportion. Thus, the MAPEs of the partial SMR are much smaller than other methods. Heuristically speaking, taking the results in Table 1 as a demonstration, it is like treating the reference population as the small population when we apply the partial SMR, so the MAPEs of the raw data are about $\sqrt{20}$ times of those of partial SMR, i.e., 20 = 2,000,000/100,000.

For the other four mortality scenarios, where the mortality rates of small and reference populations are not very similar, the MAPEs of the Whittaker ratio generally are the smallest (except for the scenario of the reverse V shape). It seems that the Whittaker ratio is more robust than the partial SMR and the graduation results are not influenced much by different mortality scenarios. This probably can explain why the Whittaker method is still a popular choice of graduation methods.

Of course, we can conduct exploratory data analysis (EDA) to evaluate if the mortality rates of small and reference populations are similar. For example, the age-specific mortality ratios in Figure 2 are one of the EDA tools we can use. We suggest using the partial SMR if they look like the first three scenarios, but we are skeptical of using the partial SMR for the last four scenarios. In fact, we experimented with using different values of mortality ratios for the last four scenarios, such as changing the ratios of the increasing scenario from 0.5–1.5 to $(1-a)$–$(1+a)$ for $0 < a < 1$. We found that the MAPEs of the partial SMR are smaller than those of Whittaker ratio if $a$ < 0.3. In other words, if the small and reference populations are not very different, then the partial SMR is preferred. We should continue exploring whether we can modify the stochastic mortality models via graduation methods in the next section.

## 4. Modification of the Lee-Carter Model

In this section, we continue the discussion of applying the graduation methods to modify the Lee-Carter model. We first use the proposed approach to revise the mortality rates and then apply the graduated mortality rates to the Lee-Carter model. Again, we assume that the age-specific mortality rates satisfy the Lee-Carter model and consider two different settings: whether the small and reference populations have different $\alpha_x$ and the same $\beta_x$ or have the same $\alpha_x$ and different $\beta_x$. The roles of parameters $\alpha_x$ and $\beta_x$ are somewhat close to the intercept and slope, if we treat the Lee-Carter model as the regression equation. The influence of different $\beta_x$ is expected to be larger than that of different $\alpha_x$.

Also, two types of modifications are considered: graduate raw mortality rates first and then apply the Lee-Carter model, or apply the Lee-Carter model first and then graduate model-fitted mortality rates. To demonstrate the proposed approach, we should use the case where the small population is 100,000 and the reference population is 2 million. The results for the other combinations of small and reference populations are similar. Taiwan's female mortality data, similar to those in the last section, are used to fit the Lee-Carter model, and the estimated parameters are treated as the true values. The model comparison is based on the MAPE as well, based on 1,000 simulation runs.

We have two choices of graduation methods, the partial SMR and Whittaker ratio, and also two choices of mortality models, the Lee-Carter model or the coherent Lee-Carter model (namely, the Li-Lee model), as well as the order of mortality graduation and mortality model. Because there are quite a few choices of the proposed methods (order of mortality graduation, graduation methods, and mortality models), we only show those with smaller MAPEs. As for the reference group for evaluating the proposed approach, we choose the Lee-Carter and Li-Lee models, in addition to the raw observations.

The setting of different $\alpha_x$ and the same $\beta_x$ in the Lee-Carter model is similar to those in Figure 2, and again we adapt the same seven mortality scenarios. The MAPEs of the proposed methods and the reference group are shown in Table 3. Unsurprisingly, the MAPEs of Lee-Carter and Li-Lee models are obviously smaller than those of raw observations, since the mortality rates satisfy the Lee-Carter model. In addition, the MAPE's of Li-Lee model are always smaller than those of Lee-Carter model. It seems that the Li-Lee model is a fine modification to the Lee-Carter model when the small and reference populations share same slope but different intercept.

**Table 3. Mean Absolute Percentage Error (MAPE) of Lee-Carter Model Graduation: Different $\alpha_x$**

| | $s_x$ = 0.8 | $s_x$ = 1 | $s_x$ = 1.2 | Incr. | Decr. | V | Rev-V |
|---|---|---|---|---|---|---|---|
| Raw | 34.11% | 30.27% | 27.63% | 28.50% | 34.98% | 30.36% | 32.57% |
| Lee-Carter | 18.25% | 16.54% | 15.09% | 14.94% | 20.14% | 16.17% | 18.66% |
| Li-Lee | 12.44% | 11.76% | 11.45% | 12.14% | 12.55% | 12.89% | 11.33% |
| Partial SMR + Lee-Carter | 5.00% | 4.70% | 4.44% | 23.71% | 43.97% | 25.04% | 17.25% |
| Li-Lee + Whittaker | 10.87% | 10.69% | 10.66% | 11.46% | 10.49% | 12.53% | 9.67% |
| Lee-Carter + Whittaker | 16.34% | 15.04% | 14.05% | 14.06% | 17.41% | 15.39% | 16.32% |

***Note:*** The cells with gray background are those with the smallest MAPE.

The MAPEs of graduation methods vary quite a lot. For the first three mortality scenarios, using the partial SMR to graduate first and then apply the Lee-Carter model has the smallest MAPEs, but the MAPEs for the last four scenarios are almost the largest. In contrast, applying the mortality model (Lee-Carter or Li-Lee model) first and using Whittaker ratio graduation outperforms the Lee-Carter model. The treatment combination Li-Lee + Whittaker has the smallest MAPEs for the last four mortality scenarios, with noticeable improvements over the Lee-Carter and Li-Lee models in all mortality scenarios. It seems that the graduation methods can improve the mortality estimates in the case of a small population.

We continue the discussion for the case of same $\alpha_x$ and different $\beta_x$, and use a setting similar to the seven mortality scenarios in Figure 2. Following the same concept, we also set up seven scenarios for the parameter $\beta_x$ to describe the relationship between the small and reference populations. Let $C_x = \beta_x^s / \beta_x^r$, where $\beta_x^s$ and $\beta_x^r$ are the age-related slope parameters in the Lee-Carter model for the small and reference populations. Seven $\beta_x$ scenarios are as follows: $C_x = 0.8$, $C_x = 1.0$, $C_x = 1.2$, and $C_x$ is increasing, decreasing, V-shape, or reverse V-shape.

Table 4 shows the MAPEs of the reference groups and the proposed graduation methods for the case of same $\alpha_x$ and different $\beta_x$. Similar to the case of different $\alpha_x$ in Table 3, the Li-Lee model always has smaller MAPEs than the Lee-Carter model in all seven $\beta_x$ scenarios. It seems that even if the small and reference populations have quite different $\beta_x$, using the idea of coherent group to increase the population size still can reduce the estimation error of mortality rates for the small populations. This is not what we expected, but it indicates that increasing the population size is a feasible approach, even though the populations included do not have a mortality profile identical to that of the small population.

**Table 4. Mean Absolute Percentage Error (MAPE) of Lee-Carter Model Graduation: Different $\beta_x$**

| | $C_x$ = 0.8 | $C_x$ = 1.0 | $C_x$ = 1.2 | Increase | Decrease | V | Rev-V |
|---|---|---|---|---|---|---|---|
| Raw | 30.15% | 30.27% | 30.41% | 30.48% | 30.10% | 30.33% | 30.15% |
| Lee-Carter | 16.31% | 16.54% | 15.73% | 15.47% | 17.14% | 16.01% | 16.23% |
| Li-Lee | 12.28% | 11.76% | 12.22% | 12.46% | 13.05% | 12.49% | 12.67% |
| Partial SMR + Lee-Carter | 4.68% | 4.70% | 4.74% | 6.91% | 7.21% | 5.96% | 6.03% |
| Lee-Carter + Whittaker | 13.98% | 14.09% | 13.99% | 13.55% | 14.90% | 14.03% | 14.10% |

***Note:*** The cells with gray background are those with the smallest MAPE.

Unlike the case of different $\alpha_x$, using the partial SMR to graduate mortality rates and then apply the Lee-Carter model has the smallest MAPEs. It outperforms the Lee-Carter and Li-Lee models in all scenarios, with significant reductions in the estimation errors. However, the improvements of the treatment combination of applying the Lee-Carter model first and using the graduation methods are not as significant. The best treatment combination is to apply the Lee-Carter model first and use the Whittaker ratio to graduate. Although this combination's MAPEs are smaller than those of Lee-Carter model, they are always larger than those of Li-Lee model.

Based on the computer simulation, we found that the mortality graduation can improve the mortality estimation of the Lee-Carter model, if proper graduation methods are selected. But the selection depends on the characteristics of mortality rates. We suggest conducting exploratory data analysis for the mortality rates, and the information, such as mortality ratios, can provide a useful guideline to choose the appropriate graduation methods.

## 5. Conclusion and Discussion

Living longer is a common phenomenon of human beings in the 21st century, and the study of mortality rates is a popular research topic in many fields, such as demography and actuarial science. The mortality models are a common tool for modeling the mortality rates, but the model estimation tends to be distorted by small sample size. In addition to larger variance, parameters' estimates for the small populations often are biased. Quite a lot of modifications have been proposed to deal with the case of a small population. Three examples are the coherent Lee-Carter model by Li and Lee (2005), the Bayesian approach by Cairns et al. (2011) and the SAINT model by Jarner and Kryger (2011). Most of these modifications use mortality information from another population(s) as a reference to improve the model fitting.

Including another population as a reference is like increasing the sample size, and this probably is the most intuitive and effective way to deal with the model estimation for small populations. The idea of increasing sample size has been used by actuaries to construct life tables as well, and many graduation methods can be treated as increasing sample size from those with a similar mortality profile. In this study, we adapt the idea of graduation and propose a modification of the Lee-Carter model, also with information from a reference population. Two types of graduation methods are used in this study: the partial SMR (Lee 2003) and Whittaker ratio.

We consider two parameter settings for the Lee-Carter model: same $\alpha_x$ with different $\beta_x$ and different $\alpha_x$ with same $\beta_x$, and use computer simulation to evaluate the proposed approach. In general, the partial SMR modification has smaller estimation errors (with respect to MAPE) than the Lee-Carter and Li-Lee models, if the small and reference populations have similar mortality profiles. When the mortality rates of small and reference populations are not similar, the Whittaker ratio is a possible alternative choice of graduation methods. We think the graduation methods are a feasible approach for dealing with small populations and can effectively reduce the estimation errors of the Lee-Carter model.

We should continue exploring the graduation methods and use them to modify the mortality models. However, we only consider various settings of age-related parameters $\alpha_x$ and $\beta_x$ for the Lee-Carter model and do not consider the time-related parameter $\kappa_i$. Different functional forms of parameter $\kappa_i$ can create problems when the reference populations are included (e.g., quadratic for the small population and linear for the reference population). Of course, the interactive effects might also exist if two or three parameters ($\alpha_x$, $\beta_x$ and $\kappa_i$) are different, and this can distort or even ruin the effect of graduation.

Also, there can be more than one reference population, and of course, it is impossible that these populations are perfectly homogeneous in terms of mortality rates. It is more realistic to expect that some populations and the small population have similar mortality rates at younger ages, while other populations and the small population are similar at older ages. Then the concept of variable selection can be applied. We may develop similarity measures and use them to judge whether a reference population should be included. Further, it would be even better (but more difficult) if the selection of appropriate reference populations were age dependent.

Modifying the graduation methods for mortality models (other than the Lee-Carter mode) is also a possible direction for future study. If the parameters of mortality models are additive, such as the age-period-cohort model, we can use the graduation methods to adjust the parameter estimation one parameter at a time. However, if the parameters are not additive, the situation is expected to be more complicated. For example, the cohort modification to the Lee-Carter model by Renshaw and Haberman (2006) contains one component of age with time and one component of age with cohort. These two components are not linearly dependent and can cause problems of adjusting the age parameters associated with time and cohort.

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