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SOCIETY OF ACTUARIES®

Session 22 TS, Annuity Product Innovation–Pooled and Variable Annuities

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2018 SOA Life & Annuity Symposium

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Annuity Product Innovation: Pooled Annuities and Variable Annuities

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PART I: Pooled Annuities

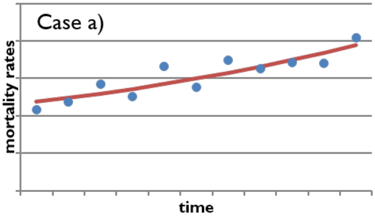
Agenda

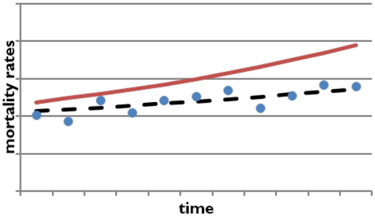
- Longevity risk
- Annuity Puzzle
- Longevity pooling product
- Conclusion

- Trends in underlying mortality rates are uncertain
- Systematic underestimation of how long people are going to live
  - by experts
  - by individuals
- Dangers
  - Individuals outlive their saving
  - Defined benefit pension plans guarantee retirement income for however long people live
  - Annuity providers have inadequate reserves

a) One individual may live longer or less than the average lifetime expected in the population
   - Mortality rates sometimes higher, sometimes lower than expected

b) The average lifetime of a population may be different from what is expected
   - Mortality rates are systematically above or below what is expected





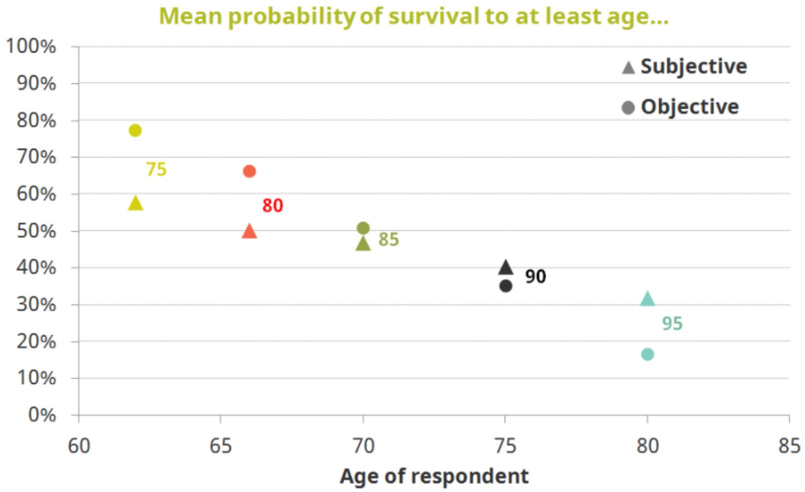
- Case a): Deviations around expected mortality rates
  - Random fluctuations, process risk, insurance risk, **idiosyncratic risk**
  - Individual mortality is involved (Usual pooling arguments)
- Case b): Deviations from expected mortality rates
  - **Systematic risk**
  - Aggregate mortality is involved (pooling arguments do not apply)
  - Mis-specification of the mortality rates: model risk
  - Biased assessment of the parameters: parameter risk

Longevity risk – Individuals

- Risk of not having sufficient financial resources at old ages
  - Reduce living standard
- People tend to underestimate their life expectancy
- People don't understand the variability in life expectancy

Longevity risk – Individuals

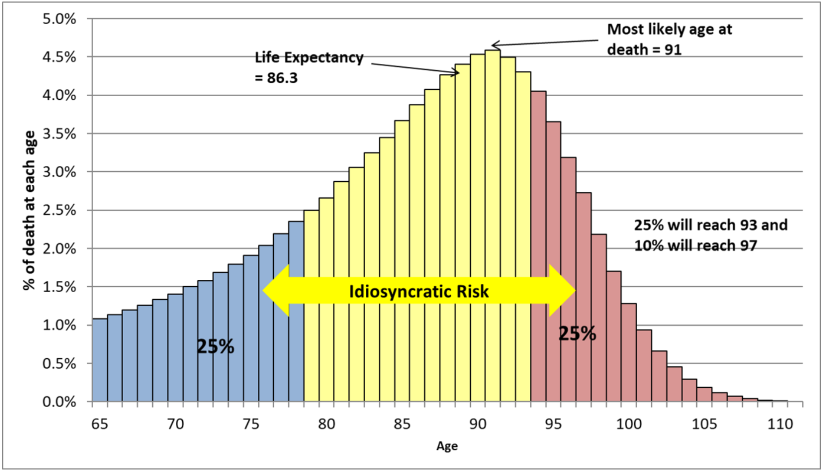
Survival probabilities perceptions by age: England and Wales men born 1930-39



Source: O'Dea and Sturrock (2018)

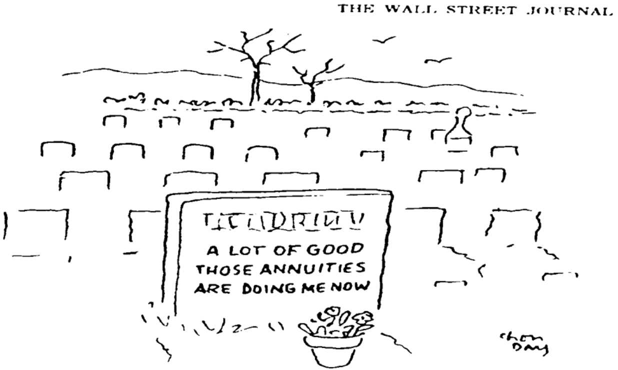
Longevity risk – Individuals

Expected distribution of deaths: USA Male Retiree age 65 in 2015



Source: RP-2014 mortality table with Lee-Carter mortality improvement based on USA population

- For individuals annuities are the only alternative for obtaining full coverage against longevity risk
- Yaari (1965) shows that they are optimal for a risk-averse utility-maximizing individual with no bequest



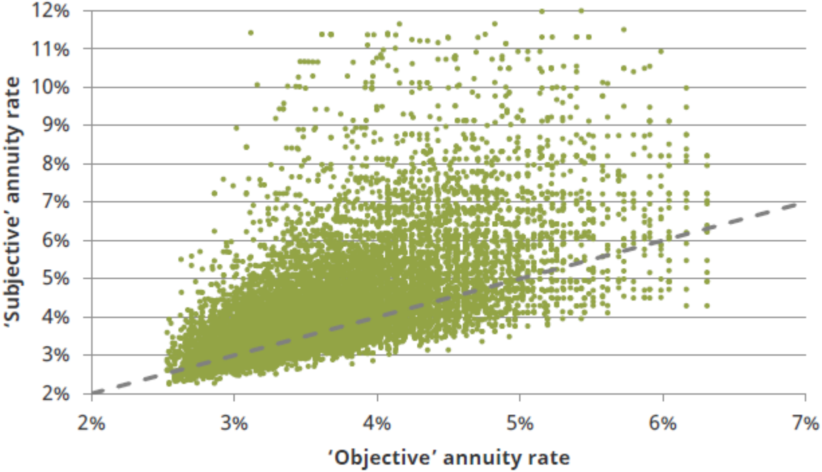
Annuity puzzle: Lack of demand of annuities

MacDonald et al. (2013) identify three groups of reasons for the absence of voluntary annuitization:

- Personal preferences and circumstances:
  - Loss in liquidity, loss in bequest
  - Benefit of delay
  - Low risk aversion, high personal discount rate
  - Short life expectancy
  - Other sources of guaranteed income
- Environmental limitations
  - Expensive pricing and poor financial market environment
  - Incomplete annuity market
  - Tax treatment
- Behavioral biases
  - Decision framing
  - Longevity gambling

Annuity puzzle: Price perception

Figure 4.1. Comparison of annuity rates based on 'subjective' and 'objective' survival curves, assuming 0% real interest rate



Source: O'Dea and Sturrock (2018)

Annuity product innovation

- Variable annuities
- Enhanced and impaired annuities
- Life care annuities
- **Pooled annuities**
- **Longevity linked annuities**

A good overview is given by:
Pitacco, Ermanno. 2017. "Life Annuities: Products, Guarantees, Basic Actuarial Models."
`http://www.cepar.edu.au/publications/working-papers/life-annuities-products-guarantees-basic-actuarial-models`.

Pooled annuity landscape

Product	Financial Risk	Longevity Risk	
		Idiosyncratic	Systematic
Life annuity	Provider	Provider	Provider
Systematic Withdrawal	Individual	Individual	Individual
Income Tontine	Provider	Pool	Pool
Group self-annuitization	Pool	Pool	Pool
Mortality-linked fund	Individual	Provider	Provider
Longevity-linked Annuity	Provider	Provider	Individual

Traditional Life Annuity

- An individual age x pays to an annuity provider and amount S to receive a life annuity consisting of annual benefits b at the end of every year as long as she/he is alive.
- The actuarial value of a whole life annuity is given by

$$a_x = \sum_{t=1}^{\omega - x} {}_h p_x (1+r)^{-t}$$

and

$$b = \frac{S}{a_x}$$

- The insurer takes **financial risk, systematic longevity risk, and idyosincratic longevity risk**
- The individual benefits from mutuality

Traditional Life Annuity: Mathematical reserve F_t

- Assume:
  - l_x individuals that purchase the annuity at time 0
  - l_{x+t} estimate (at time 0) of number of annuitants alive at time t
  - F_t reserve (fund) for a generic annuitant alive at time t
- The total reserve of the company must satisfy

$$l_{x+t+1}F_{t+1} = l_{x+t}F_t(1+r) - l_{x+t}b \qquad \text{with } F_0 = l_x S$$

- For an alive annuitant we have then

$$F_{t+1} = \quad F_t(1+r) \quad + F_t(1+r)\frac{l_{x+t} - l_{x+t+1}}{l_{x+t+1}} - b$$

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$$F_{t+1} = \underbrace{F_t(1+r)}_{\text{Financial credit}} + F_t(1+r)\frac{l_{x+t} - l_{x+t+1}}{l_{x+t+1}} - b$$

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Traditional life annuity: Mortality drag

We can re-write

$$F_{t+1} = F_t(1+r) + F_t(1+r)\frac{l_{x+t} - l_{x+t+1}}{l_{x+t+1}} - b$$

as

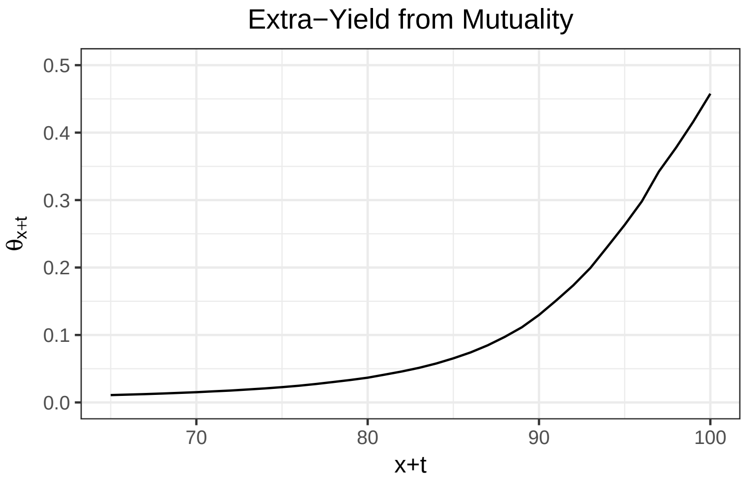
$$F_{t+1} = F_t(1+r)(1+\theta_{x+t}) - b$$

where

$$\theta_{x+t} = \frac{l_{x+t} - l_{x+t+1}}{l_{x+t+1}} = \frac{1}{p_{x+t}} - 1$$

is the **mortality drag** or **extra-yield from mutuality**.

Traditional life annuity: Mortality drag



Note: Based on RP-2014 mortality table with Lee-Carter improvement for a male age 65 in 2015

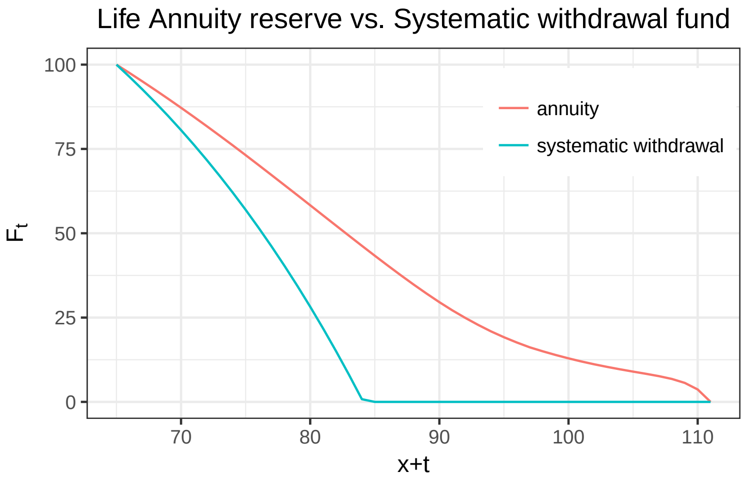
Income drawdown (Systematic Withdrawal)

- At the other end we have systematic withdrawal where individuals self-annuitize
  - **Individual takes financial risk and longevity risk**
  - Retain flexibility, liquidity and bequest
- If the yearly investment return in year t is given by R_t, then

$$F_{t+1} = F_t(1 + R_t) - b_{t+1}$$

- Individual has freedom to choose the annual benefit
  - Fixed amount: e.g. $b_t = b = \frac{S}{a_x}$
  - Percentage rule: e.g $b_t = 4\% F_t$
  - Variable according to mortality expectations: e.g $b_t = \frac{F_t}{a_{x+t}}$

Traditional Life annuity vs. Systematic Withdrawal



Note: Mortality: RP-2014 + LC, $x = 65$ in 2015; Interest rate $r = 4\%$

Income Tontine

- Assume that l_x retirees band together and each contributes an amount S.
- With the income $l_x S$ they buy from a financial institution a perpetuity (priced with interest rate r) that pays a constant amount B at the end of each year $t = 1, 2, \ldots$.

$$B = l_x S r$$

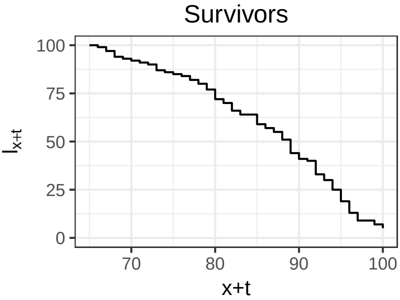
- Each year the amount B is divided among the survivors so that each individual alive at time t receives a benefit b_t:

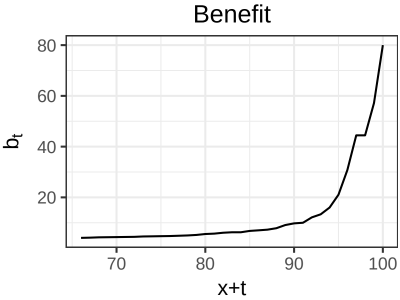
$$b_t = \frac{B}{l^*_{x+t}},$$

where l^*_{x+t} is the actual number of retirees alive at time t.

Income Tontine: Example

100 retirees aged 65 and each invests $S = \$100$ to buy a $r = 4\%$ perpetuity

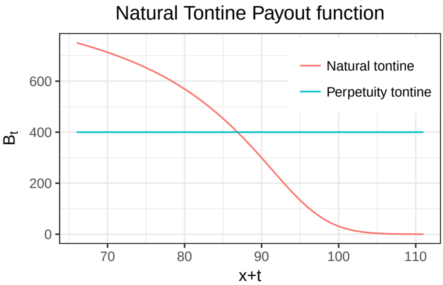




t	x	B	lx	b
0.00	65	400.00	100	4.00
10.00	75	400.00	85	4.71
20.00	85	400.00	59	6.78
35.00	100	400.00	5	80.00

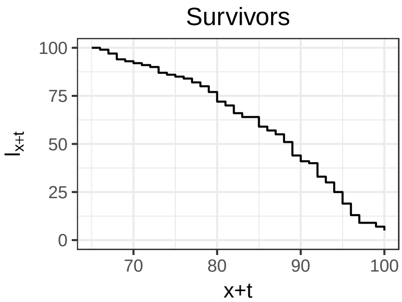
Natural Income Tontine

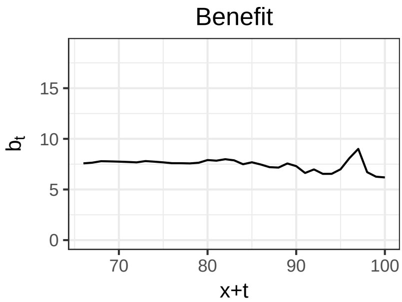
- Milevsky and Salisbury (2015) propose a natural income tontine where the total pool payments at time t, B_t, are proportional to the expected number of survivors at time t, l_{x+t}, so that, $B_t = S\frac{l_{x+t}}{a_x}$



- Milevsky and Salisbury (2015) show that such tontine is nearly optimal for a utility-maximizing individual with now bequest

Natural Income Tontine: Example with 100 members





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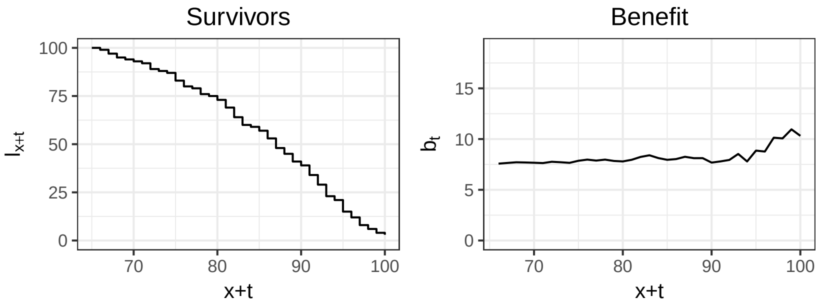

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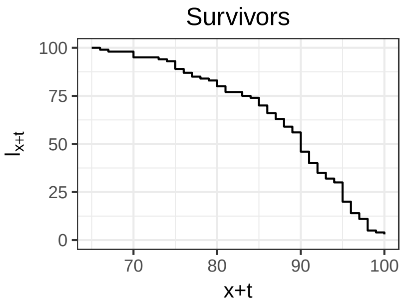

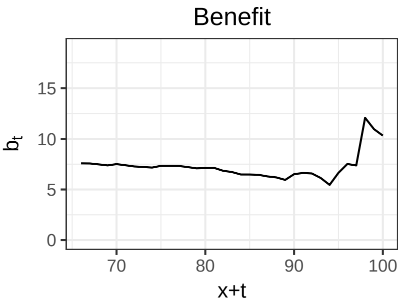

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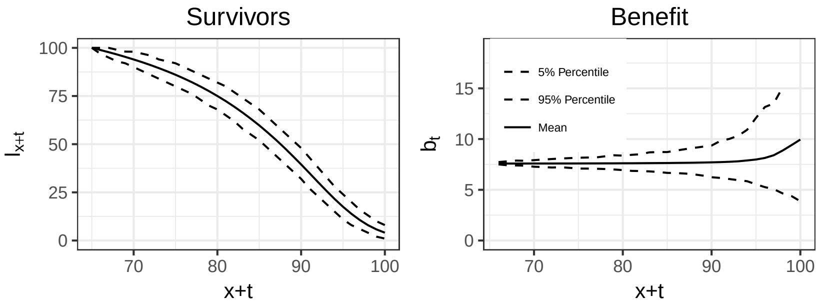
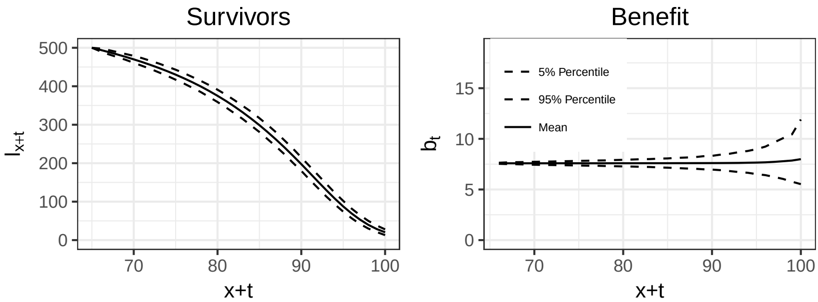
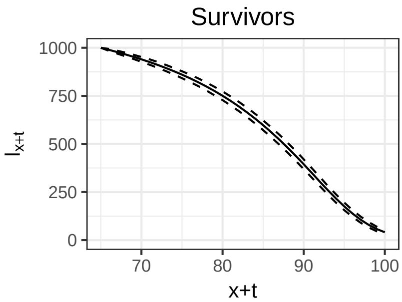
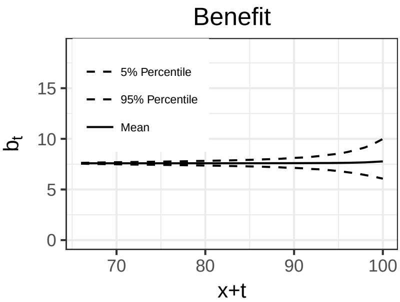


Natural Income Tontine: Example with 500 members

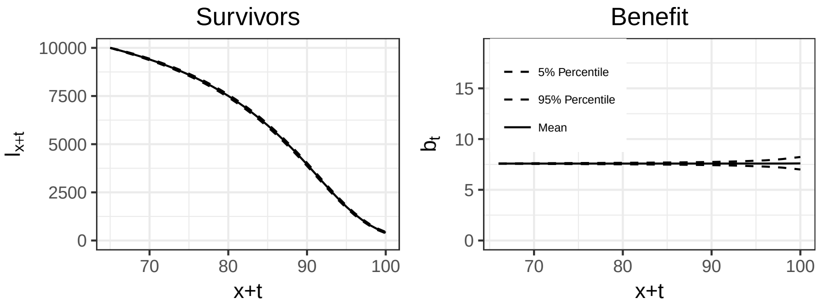


Natural Income Tontine: Example with 1000 members



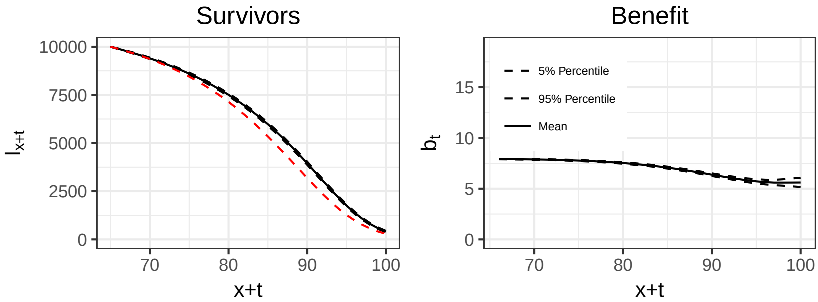


Natural Income Tontine: Example with 10000 members



Natural Income Tontine: Example with 10000 members

Impact of systematic longevity risk (e.g. ignoring improvements)



Natural Income Tontine

- It can easily be shown that under a natural income tontine the “fund” pertaining to an alive member of the pool at time t is given by:

$$F_{t+1} = F_t(1+r)(1+\theta^*_{x+t}) - b_{t+1},$$

where

$$\theta^*_{x+t} = \frac{l^*_{x+t} - l^*_{x+t+1}}{l^*_{x+t+1}} = \frac{1}{p^*_{x+t}} - 1$$

  - The provider takes financial risk
  - **Systematic and idyosincratic longevity risks are shared by the pool**

- The benefits at time t are given by

$$b_0 = \frac{S}{a_x}, \quad b_t = b_0 \frac{l_{x+t}}{l^*_{x+t}}, \quad b_{t+1} = b_t \frac{p_{x+t}}{p^*_{x+t}}$$

Group self-annuitization

- Piggott, Valdez, and Detzel (2005) propose group self-annuitization (GSA) where the **pool bears systematic and idiosyncratic longevity risk along with financial risk**.

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- In a GSA with l_x initial members the fund of an alive member of the pool at time t is given by:

$$F_{t+1} = F_t(1 + R_t)(1 + \theta^*_{x+t}) - b_{t+1},$$

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- The initial benefit is given by $b_0 = \frac{S}{a_x}$ and the benefits at time t by:

$$b_{t+1} = b_t \times \frac{1 + R_t}{1 + r} \times \frac{p_{x+t}}{p^*_{x+t}}$$

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- The initial benefit is given by $b_0 = \frac{S}{a_x}$ and the benefits at time t by:

$$b_{t+1} = b_t \times \underbrace{\frac{1+R_t}{1+r}}_{\text{Interest rate adjustment}} \times \underbrace{\frac{p_{x+t}}{p^*_{x+t}}}_{\text{Mortality adjustment}} \times \underbrace{\frac{a_{x+t}}{a^*_{x+t}}}_{\text{Expectation adjustment}}$$

Mortality-linked fund

- Donnelly, Guillén, and Nielsen (2013) propose a fund where **individuals are exposed only to financial risk and transfer the longevity risk** to the provider of the fund for a cost.

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- Donnelly, Guillén, and Nielsen (2013) propose a fund where **individuals are exposed only to financial risk and transfer the longevity risk** to the provider of the fund for a cost.

- In such fund the fund of a member at time t is given by:

$$F_{t+1} = F_t(1 + R_t)(1 + \theta_{x+t}) - b_{t+1},$$

where

$$\theta_{x+t} = \frac{l_{x+t} - l_{x+t+1}}{l_{x+t+1}} = \frac{1}{p_{x+t}} - 1$$

- Similar to a traditional annuity, the mortality credit is deterministic and is determined by the expected survival probabilities p_{x+t}.
- The provider controls the cost of the guaranteed mortality credit by choosing conservative p_{x+t}

Longevity-linked annuities

- Denuit, Haberman, and Renshaw (2011) propose longevity-linked annuity as a way to share longevity risk between the provider and the annuitants.
  - The **provider retains financial risk and idiosyncratic longevity risk**
  - **Individual bears systematic longevity risk**

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  - **Individual bears systematic longevity risk**

- In this contract, while alive the annuitant receives at the end of each year an amount b scaled by a longevity index:

$$b_t = b \times i_{x+t} = b \times \frac{l_{x+t}}{l^{ref}_{x+t}}$$

  - l_{x+t} : is the forecast number of survivors
  - l^{ref}_{x+t} is the observed number of survivors from a reference population (e..g the national population)

Longevity-linked annuities

- In an longevity-indexed annuity it can be shown that the reserve (fund) at time t of an alive annuitant is given by:

$$F_{t+1} = F_t(1+r)(1+\theta^{ref}_{x+t}) - b_{t+1},$$

where

$$\theta^{ref}_{x+t} = \frac{l^{ref}_{x+t} - l^{ref}_{x+t+1}}{l^{ref}_{x+t+1}} = \frac{1}{p^{ref}_{x+t}} - 1$$

- The mortality credit is based on the realized mortality of the reference population

Summary of Product Comparison

Most pooling and longevity-linked products build on the basic structure of a life annuity but differ on how they treat financial and longevity risk:

- Financial risk: taken by the provider (r), taken by the annuitant (R_t)
- Mortality drag: fixed (θ_x); based on the pool mortality (θ_x^*); based on the mortality of a reference population (θ_x^{ref})

Product	Fund (F_{t+1})	Benefits
Life annuity	$F_t(1+r)(1+\theta_{x+t}) - b_{t+1}$	$b_t = \frac{S}{a_x}$
Systematic Withdrawal	$F_t(1+R_t) - b_{t+1}$	b_t
Income Tontine	$F_t(1+r)(1+\theta^*_{x+t}) - b_{t+1}$	$b_{t+1} = b_t \frac{l_{x+t}}{l^*_{x+t}}$
GSA	$F_t(1+R_t)(1+\theta^*_{x+t}) - b_{t+1}$	$b_{t+1} = b_t \frac{1+R_t}{1+r} \frac{l_{x+t}}{l^*_{x+t}}$
Mortality-linked fund	$F_t(1+R_t)(1+\theta_{x+t}) - b_{t+1}$	b_t
Longevity-linked annuity	$F_t(1+r)(1+\theta^{ref}_{x+t}) - b_{t+1}$	$b_{t+1} = b_t \frac{l_{x+t}}{l^{ref}_{x+t}}$

Issues around logevity risk pooling

- Flexibility and inclusion of other assets
  - Donnelly, Guillén, and Nielsen (2014), Donnelly and Young (2017)
- Mixing cohorts
  - Piggott, Valdez, and Detzel (2005), Qiao and Sherris (2013)
- Equity, Fairness and Solidarity
  - Donnelly (2015), Milevsky and Salisbury (2016)
- Heterogeneity
  - Work in progress as part of the SOA CAE grant
- Investment strategy
  - Work in progress as part of the SOA CAE grant

PART II: Variable Annuities

Variable Annuities

- A variable annuity is a contract between an insurance company and a policyholder.
- The insurance company agrees to make periodic payments to the policyholder in future (mainly post retirement).
- The policyholder purchases a variable annuity by paying either a single premium payment or a series of payments.
- Unlike traditional mutual funds and life insurance products, variable annuity contracts come with embedded guarantees which protect the policyholder's savings against unanticipated outcomes.
- Some of the advantages of variable annuities include
  - Tax-deferred earnings,
  - Tax-free transfers across a variety of investment options,
  - Death benefit protection options,
  - Living benefit protection options,
  - Lifetime income options.

Variable Annuities cont...

- Variable Annuities (VAs) were first introduced in the early 1950s and can be underwritten for the accumulation phase, annuity phase or untimely death of the policyholder.
- VAs can be categorised into two major groups (Ledlie et al. 2008):
  - Guaranteed Minimum Death Benefit introduced in 1980s.
  - Guaranteed Minimum Living Benefits introduced in late 1990s.
    - GMAB - minimum guarantee at maturity
    - GMIB - minimum guaranteed income periodically
    - GMWB - minimum withdrawal guarantee until the initial premium is recovered
    - GLWB - minimum lifetime withdrawal.

Variable Annuities cont...

- Premiums paid when purchasing variable annuities are usually invested in various subaccounts with different characteristics and investment strategies.
- Variable annuity subaccounts include actively managed portfolios, exchange-traded funds, index-linked portfolios alternative investments and other quantitative-driven strategies.
- VA industry is large and still expanding:
  - US$1.35 trillion in the U.S. as of 2008 (Condron 2008).
  - US$1.96 trillion in the U.S. as of third quarter of 2017 (IRI 2017).
  - On a year-over-year basis, assets were up 1.9%, from US$1.92 trillion at the end of the third quarter of 2016, as positive market performance outweighed the impact of lower sales and negative net flows (IRI 2017).
- The riders have varying popularity:
  - 59% elected GLWB, 26% GMIB, 3% GMAB and 2% GMWB as of 2011 (Fung et al. 2014).
  - Death benefits are usually given as an additional rider ‘for free’ (Moenig and Bauer 2017).

Variable Annuities Pricing

- The greater part of the literature has focused on the pricing of riders embedded in VAs, with a recent spike of interest in hedging.
- Pricing in the VA context: Find the **regular** fair fee, as a percentage of the underlying fund, that covers the guarantees.
- The fee is usually paid while the rider is active.
- Main areas of focus have been:
  - Underlying fund dynamics
  - Policyholder behavior
  - Computational aspects

Underlying Fund Dynamics

- Most seminal papers assume that the underlying follows a Geometric Brownian Motion (GBM) (Milevsky and Posner 2001; Bauer et al. 2008)
- As a step towards considering a more realistic framework, regime-switching (RS) models have been proposed (Hardy 2001)
- However, GBM and RS do not capture full empirical properties of asset return distributions such as heavy tails, skewness and kurtosis.
- Levy processes have been proposed to address the shortcomings of GBM and RS (Chen et al. 2008; Bacinello et al. 2011; Kélani and Quittard-Pinon 2015; Bacinello et al. 2014)
- Stochastic volatility, or stochastic interest rates have also been considered too (Peng et al. 2012; Bacinello et al. 2011; Kling et al. 2011; Kang and Ziveyi 2018).

Policyholder behavior and frictions

- Commonly, pricing frameworks assume two main policyholder behavior:
  - **Static**: this is where pre-specified contract characteristics are followed; $\rightarrow$ this has European option-like features
  - **Dynamic**: This is where a policyholder behaves in a way that maximizes the value of the contract (including surrender) $\rightarrow$ this has American option-like features
- In practice, pricing will be affected by taxes and management fees too (Moenig and Bauer 2016, 2017)

Policy features

To dis-centivize surrender or dynamic behavior, various features are added to the contracts (Moenig and Zhu 2016):

- **surrender schedule**: within a certain number of years, lapsing will incur a surrender fee
- **roll-up guarantee**: the guaranteed minimum amount increases by a fixed percentage each year
- **ratchet-type guarantee / automatic annual step-up**: the guarantee is equal to the maximum of the values of the VA account at previous anniversary dates
- **state-dependent fee**: the fee for the guarantee is only paid if the account value is close to being in the money
- **enhanced earnings**: an additional earnings feature which provides an additional payout

Typical Underlying Fund Dynamics - GBM

- The policyholder's premium is normally invested in a fund consisting of units of an underlying asset, $S = (S_t)_{0 \leq t \leq T}$, whose risk-neutral evolution is governed by the geometric Brownian motion process

$$dS_t = rS_t dt + \sigma S_t dW_t, \tag{1}$$

  where $r > 0$ and $\sigma > 0$ are the risk-free interest rate and the volatility of the underlying asset, respectively.

- The fund value at time t is denoted as

$$F_t = e^{-ct} S_t, \tag{2}$$

  where c denotes management fees, hence

$$dF_t = (r - c)F_t dt + \sigma F_t dW_t. \tag{3}$$

- In the event of the guarantee being terminated early, the resulting benefit fund value for that component is $(1 - \kappa_t)F_t$ where κ_t is a surrender charge.

Computational aspects

- Monte Carlo based methods are commonly used to approach the complex policy features of the contract.
- However, to get the desired accuracy, high number of scenarios are needed.
- Recently, there has been increasing focus on computationally efficient methods:
  - Fast-Fourier Transform (Kélani and Quittard-Pinon 2015; Bacinello et al. 2014)
  - Fourier Space Time-Stepping (Ignatieva et al. 2016)
  - Fourier-COS method (Alonso-García et al. 2017)
  - Grid based approaches such as Method of lines algorithm (Kang and Ziveyi 2018).

Functional forms of Variable Annuity Riders - GMMB/GMAB

- The payoff of a GMMB at maturity can be represented as

$$\vartheta(F_T) = \max(F_T, G_T), \tag{4}$$

where

$$G_T = \begin{cases} G & \text{if the guarantee is fixed} \\ Ge^{\delta T} & \text{if the guarantee is rolled up at a rate of } \delta \\ \left(\prod_{j=0}^{T} F_j\right)^{\frac{1}{T+1}} & \text{if it is a ratchet geometric average guarantee} \\ \frac{1}{T+1}\sum_{j=0}^{T} F_j & \text{if it is a ratchet arithmetic average guarantee,} \end{cases}$$

- Graphically, this can be represented as

- The value of a GMMB rider can be represented as

$$C_M(t,T,F_t) = e^{-r(T-t)}\mathbb{E}_t^Q\left[\vartheta(F_T)|\mathcal{F}_t\right] = e^{-r(T-t)}\int_{-\infty}^{\infty}\vartheta(e^x)f(x)dx \qquad (5)$$

where $f(x)$ is the transition density function of the underlying process.

- Noting that the density function is a Fourier transform of the characteristic function and letting $\vartheta(e^x) \equiv h(x)$ yields

$$C_M(t,T,F_t) = \frac{e^{-r(T-t)}}{2\pi}\int_{-\infty}^{\infty}\phi(t,T,z)\hat{h}(z)dz,$$

- For completeness, the Fourier transform of the density function can be represented as

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-ixz}\phi(t,T,z)dz.$$

GMMB Valuation - Case with Surrender Features

- We assume an exponentially decreasing surrender fee structure on the guarantee implying that the fund value of the guarantee component is $(1-\kappa_t)F_t = e^{-\kappa(T-t)}F_t$
- The variable annuity contract at anytime prior to maturity can be represented as an optimal stopping problem such that

$$C(t,F) = \operatorname*{ess\,sup}_{t\le\tau^*\le T} \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{\tau^*} r_s ds} g(\tau^*, F_{\tau^*})|\mathcal{F}_t\right], \qquad (6)$$

where

$$g(t,F_t) = \begin{cases} e^{-\kappa(T-t)}F_t, & t<T \\ \max(F_t, G), & t=T \end{cases}$$

and the supremum is taken over all stopping times, τ^*.

- Using similar arguments to those presented in Jacka (1991) and Peskir and Shiryaev (2006), the optimal stopping problem in equation (6) is equivalent to the free boundary problem

$$\frac{\partial C}{\partial t} + (r - c)F\frac{\partial C}{\partial F} + \frac{1}{2}\sigma^2 F^2 \frac{\partial^2 C}{\partial F^2} - rC = 0, \tag{7}$$

where $0 < F < b(t)$, with $b(t)$ being the optimal surrender boundary. The PDE (7) is solved subject to boundary and terminal conditions

$$C(T, F) = \max(F, G), \tag{8}$$
$$C(t, b(t)) = e^{-\kappa(T-t)} b(t), \tag{9}$$
$$\lim_{F \to b(t)} \frac{\partial C}{\partial F} = e^{-\kappa(T-t)}, \tag{10}$$
$$C(t, 0) = e^{-r(T-t)} G. \tag{11}$$

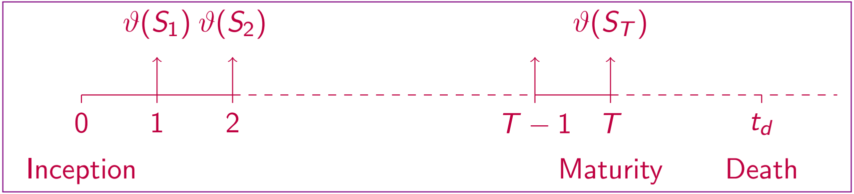
- The PDE (7) can be solving using a variety of techniques such as grid-based approaches the method of lines (Kang and Ziveyi 2018) or numerical integration (Shen et al. 2016).

Guaranteed Minimum Income Benefit - GMIB

- The policyholder is guaranteed a minimum level of income stream, G at periodic intervals as long as he or she stays alive, until maturity T.
- The value of the GIMB can be represented as

$$C_I(t, T, S_t) = \sum_{j=t+1}^{T} C_M(t, j, S_t)$$

- A stream of benefit payments $\vartheta(S_1), \vartheta(S_2), \ldots, \vartheta(S_T)$ until maturity or death can be expressed as

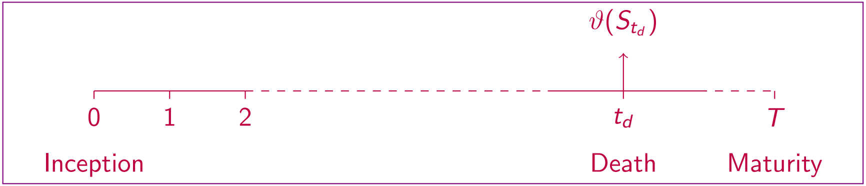


Guaranteed Minimum Death Benefit - GMDB

- The policyholder's beneficiaries are paid a guaranteed minimum level of benefit in the event of the policyholder's death before the maturity of the contract. Assuming that the benefit is paid immediately upon death, the value of a GMDB rider is given by

$$
\begin{aligned}
C_D(t,F) &= \mathbb{E}_t^{Q}\left[e^{-\int_t^{\tau_x} r ds}\vartheta(F_{\tau_x})\mathbb{1}_{\{t_d \le T-t\}}\middle|\mathcal{F}(t)\right] \\
&= \int_t^T \mathbb{E}_t^{\mathbb{Q}}\left[\mu(x+u)e^{-\int_t^u \mu(x+s)ds}\right] C_M(t,F_u)du,
\end{aligned}
$$

  with x being the age of the policyholder at inception of the contract.
- Graphically, this can be represented as



- A GMDB contract is a byproduct of a GMMB ⇒ usually given as a free benefit to holders of variable annuity contracts Moenig and Bauer (2017).

Guaranteed Minimum Withdrawal Benefit - GMWB

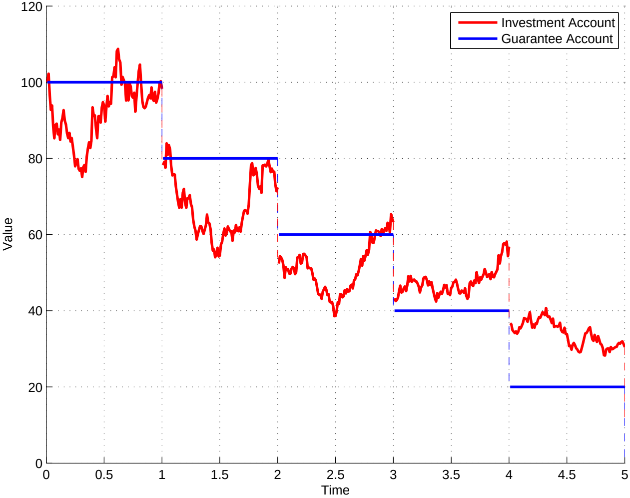
- At inception the policyholder pays a lump sum to the insurer, which becomes the initial balance of the two accounts forming a VA contract, namely, the investment account, $W(t)$ and the guarantee account, $A(t)$.
- Every time the policyholder withdraws a specified amount, denoted by γ_t, the two account values ($W(t)$ and $A(t)$) decrease by γ_t as well.
- γ_t can either be static or dynamic depending on contract specifications.
- The policyholder is able to make withdrawals as long as the guarantee account value is above zero, regardless of the performance of the $W(t)$.
- At maturity, the policyholder receives the larger of the investment account balance and the guarantee account balance, less any fees.
- At inception of the contract, the two account are equal, that is $W(0) = A(0)$

GMWB cont...

- The balance of the guarantee account at any given time can be represented as
$$A(t) = A(0) - \int_0^t \gamma(s)ds, \quad 0 \le \gamma(s) \le G,$$
with G being the contractually agreed withdrawal rate.
- Excess withdrawals above G attract a penalty fee, which we denote here as κ. The net amount received by the policyholder becomes
$$f(\gamma) = \begin{cases} \gamma_t, & 0 \le \gamma_t \le G \\ G + (1-\kappa)(\gamma_t - G), & \gamma_t > G \end{cases}$$
- The investment account evolves according to
$$dW(t) = (r-c)W(t)dt + \sigma W(t)dBt + dA(t), \quad W(t) > 0.$$
- The value of a VA contract embedded with a GMWB rider can then be represented as
$$V(t, W, A) = \sup_{\gamma} \mathbb{E}_t^Q \left[e^{-r(T-t)} \max(W(T), A(T)) + \int_t^T e^{-r(u-t)} f(\gamma_u) du \right].$$

GMWB cont...

- Example path of the investment and guarantee accounts for a five-year GMWB.



Efficient Valuation Techniques

- Numerical integration presented in Sherris, Shen and Ziveyi (2016) for valuing a GMMB rider with early surrender features.
- Comprehensive framework for valuing Guaranteed minimum benefits using the Fourier Space Time-Stepping (FST) approach presented in Ignatieva, Song and Ziveyi (2016)
- Method of line approach which is a mesh-based algorithm for solving free-boundary problems like equation (7) as presented in Kang and Ziveyi (2017)
- FST approach for valuing GMWB riders as presented in Ignatieva, Song and Ziveyi (2016)
- Fourier-Cosine approach is presented in Alonso-García, Wood and Ziveyi (2017) for valuing VA with a GMWB rider.
- What if the underlying fund consists of more than one underlying asset? Two asset case presented in Da Fonseca and Ziveyi (2015) who use the Fast Fourier transform (FFT) algorithm.

Hedging Initiatives

- In 2008, the total market capilisation of the top 10 insurers in the US decreased by 53% (McKinsey & Company, 2009) with VA losses amounting to $36 billion.
- Providers need to be well prepared for unexpected surrender/lapse of VA contracts.
- The frameworks developed in literature (eg. Alonso-García et al., 2017 and Kang and Ziveyi, 2017) all consider rational policyholder behaviour.
- Increasing literature on hedging the net liability as presented earlier.
- Need for incorporating realistic surrender behaviour and taxes in the valuation and hedging frameworks!
- Other issues to consider include:
  - Basis Risk arising from underlying fund and hedging instruments
  - Liquidity of the hedging instruments
  - Counterparty risk in cases of OTC contracts.

Questions and Comments?

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