SOCIETY OF ACTUARIES
Quantitative Finance and Investment Core

Exam QFICORE

MORNING SESSION

Date: Wednesday, November 1, 2017
Time: 8:30 a.m. – 11:45 a.m.

INSTRUCTIONS TO CANDIDATES

General Instructions

1. This examination has a total of 100 points. It consists of a morning session (worth 60 points) and an afternoon session (worth 40 points).

   a) The morning session consists of 10 questions numbered 1 through 10.

   b) The afternoon session consists of 8 questions numbered 11 through 18.

   The points for each question are indicated at the beginning of the question.

2. Failure to stop writing after time is called will result in the disqualification of your answers or further disciplinary action.

3. While every attempt is made to avoid defective questions, sometimes they do occur. If you believe a question is defective, the supervisor or proctor cannot give you any guidance beyond the instructions on the exam booklet.

Written-Answer Instructions

1. Write your candidate number at the top of each sheet. Your name must not appear.

2. Write on only one side of a sheet. Start each question on a fresh sheet. On each sheet, write the number of the question that you are answering. Do not answer more than one question on a single sheet.

3. The answer should be confined to the question as set.

4. When you are asked to calculate, show all your work including any applicable formulas. When you are asked to recommend, provide proper justification supporting your recommendation.

5. When you finish, insert all your written-answer sheets into the Essay Answer Envelope. Be sure to hand in all your answer sheets because they cannot be accepted later. Seal the envelope and write your candidate number in the space provided on the outside of the envelope. Check the appropriate box to indicate morning or afternoon session for Exam QFICORE.

6. Be sure your written-answer envelope is signed because if it is not, your examination will not be graded.

Tournez le cahier d’examen pour la version française.

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****BEGINNING OF EXAMINATION****
Morning Session

1. (*4 points*)

Let $\{W_t : t \geq 0\}$ be a standard Wiener process under a measure with filtration $\{F_t\}$.

(a) (*0.5 points*) Find the correlation coefficient of W_{t_1} and W_{t_2} for $0 < t_1 \leq t_2$.

(b) (*2 points*) Find the variance of $\int_0^t W_s ds$.

(c) (*1.5 points*) Find the deterministic function m_t with $m_0 = 1$ such that $E\left[m_t e^{\lambda W_t} \middle| F_s \right] = m_s e^{\lambda W_s}$ for all $s \leq t$ and a given constant λ.

2. (*7 points*) Let S_t be the price of a stock at time *t*, which follows the process:

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

where μ and σ are positive constants, and W_t is a standard Wiener process.

Consider an Asian option with value V_t at time *t*. $V_t = V(t, S_t, A_t)$ is a function of the historical average stock price A_t, where $A_t = A(t, S_t) = \frac{1}{t}\int_0^t S_u du$.

(a) (*1 point*) Show that A_t follows the process for $t > 0$:

$$dA_t = \frac{1}{t}(S_t - A_t)dt$$

Assume a constant risk-free rate $r \geq 0$.

(b) (*3.5 points*) Show that *V* satisfies the following PDE:

$$\frac{\partial V}{\partial t} + \frac{S-A}{t}\frac{\partial V}{\partial A} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

Now assume a discrete time sequence $0 = t_0 < t_1 < ... < t_n$, $B_n = \frac{1}{n+1}\sum_{i=0}^{n} S_{t_i}$

Consider a European call option with fixed strike *K*, price *P*, and payoff $\max(0, B_n - K)$ at expiration date t_n.

(c) (*2.5 points*) Show that $P \leq \frac{1}{n+1}\sum_{i=0}^{n} C_i$ where C_i is the price of a European option with strike price *K* and maturity date t_i.

3. (*9 points*) A foreign currency denominated asset S_t and the foreign exchange rate f_t (price of one unit of foreign currency in domestic currency) follow the stochastic differential equations (SDE) below:

$$dS_t = \mu S_t dt + \sigma_s S_t dW_{1t}$$
$$df_t = \gamma f_t dt + \sigma_f f_t dW_{2t}$$

where W_{1t} and W_{2t} are correlated standard Wiener processes with $E\left(dW_{1t}dW_{2t}\right) = \rho dt$ under the real-world measure $\mathbb{P}$ and $\mu, \gamma, \rho, \sigma_s$, and σ_f are constants such that

$$-1 < \rho < 1,$$
$$\sigma_s > 0 \text{ and}$$
$$\sigma_f > 0.$$

Given continuous functions $g(u)$ and $h(u)$ such that

$$g^2(u) + h^2(u) + 2\rho g(u)h(u) > 0$$

Define

$$Z_t = \int_0^t \frac{g(u)}{\sqrt{g_{(u)}^2 + h_{(u)}^2 + 2\rho g(u)h(u)}} dW_{1u} + \int_0^t \frac{h(u)}{\sqrt{g_{(u)}^2 + h_{(u)}^2 + 2\rho g(u)h(u)}} dW_{2u}$$

(a) (*2.5 points*)

(i) State conditions for a process to be a standard Wiener process.

(ii) Show that Z_t is a standard Wiener process under $\mathbb{P}$ by verifying that Z_t satisfies each of the conditions in part (a)(i).

A risk-free bond is also traded in the foreign economy. Both the domestic and foreign risk-free rates r_D and r_F are constants.

Let $Y_t = S_t f_t$ be the value of S_t in the domestic currency.

(b) (*1 point*) Show that $dY_t = \left(\gamma + \mu + \rho\sigma_s\sigma_f\right)Y_t dt + \sigma_s Y_t dW_{1t} + \sigma_f Y_t dW_{2t}$.

3. Continued

Define $W_t = \frac{1}{\zeta}\left(\sigma_s W_{1t} + \sigma_f W_{2t}\right)$ where $\zeta = \sqrt{\sigma_s^2 + \sigma_f^2 + 2\rho\sigma_s\sigma_f}$.

Then from parts (a) and (b) $dY_t = \left(\gamma + \mu + \rho\sigma_s\sigma_f\right)Y_t dt + \zeta Y_t dW_t$ where W_t is a standard Wiener process under $\mathbb{P}$.

(c) (*1.5 points*)

(i) Show that $dY_t = r_D Y_t dt + \zeta Y_t d\tilde{W}_t$ where $\tilde{W}_t$ is a standard Wiener process under an equivalent martingale measure $\mathbb{Q}$ such that

$$d\tilde{W}_t = dW_t + \eta dt.$$

(ii) Solve for η in terms of the given constants.

Suppose that both markets are free of arbitrage.

(d) (*3 points*) Derive the SDEs followed by f_t and S_t under the measure $\mathbb{Q}$.

(e) (*1 point*) Calculate the price in domestic currency of a call on asset S_t with strike price $K = 100$ (domestic currency) and maturity $T = 1$ given the following:

$$\sigma_s = 0.1, \sigma_f = 0.15, \rho = 0.2, r_D = 0.06, S_0 = 100 \text{ and } f_0 = 1.05.$$

4. (*6 points*) Let B_t be a standard Brownian motion. Consider the stochastic differential equation (SDE):

$$dX_t = -g\left(X_t, t\right)dt + dB_t, \text{ where } g\left(X_t, t\right) = \frac{X_t}{1-t}$$

For $0 \le t < 1$ and $X_0 = 0$.

(a) (*2 points*) Verify, by using Ito's lemma, that the solution to the SDE can be expressed in the form

$$X_t = (1-t)\int_0^t \frac{dB_u}{1-u}$$

(b) (*2.5 points*) Show that for any $0 \le s \le t < 1$ the covariance between X_s and X_t is given by

$$Cov\left(X_s, X_t\right) = s - st$$

(c) (*1.5 points*) Show that the process X_t does not have independent increments.

5. (*7 points*) Let B_t be a standard Brownian motion. Consider the stochastic differential equation (SDE):

$$dX_t = (\alpha - \beta X_t)\,dt + \sigma dB_t,$$

where α, β and σ are constant parameters with $\beta > 0$ and $X_0 = x_0$. The solution is a mean-reverting process called the Ornstein-Uhlenbeck process and has been used to model the stochastic behavior of interest rates and currency exchange rates.

(a) (*2 points*) Verify that the solution to the SDE can be expressed in the form

$$X_t = e^{-\beta t}\left[x_0 + \frac{\alpha}{\beta}\left(e^{\beta t} - 1\right) + \sigma \int_0^t e^{\beta s}\,dB_s\right].$$

(b) (*3 points*) Derive a closed-form expression for $Cov(X_s, X_t)$ for $s \le t$.

(c) (*0.5 points)* Deduce from part (b) that

$$Var(X_t) = \frac{\sigma^2}{2\beta}\left(1 - e^{-2\beta t}\right).$$

(d) (*1 point*) Show that the limiting distribution of this process X_t as $t \to \infty$ is a normal distribution with mean $\frac{\alpha}{\beta}$ and variance $\frac{\sigma^2}{2\beta}$.

Suppose this process has been used to describe the stochastic behavior of interest rates.

(e) (*0.5 points*) Explain how the dynamics of the model are affected by the values of the parameters α and β.

6. (*4 points*) Consider a market of three freely traded assets $A, B,$ and C. At time 1, there are three states of the world, each with 1/3 probability of occurring, and the values of the three assets are as follow:

	ω_1	ω_2	ω_3
A_1	1	1	1
B_1	80	100	120
C_1	0	0	140

The time 0 asset values are given as follows:

A_0	0.8
B_0	70
C_0	28

(a) (*1 point*) Show that risk-neutral probabilities do not exist.

(b) (*1 point*) Construct an arbitrage portfolio which takes the position of -1 in asset *C*.

Now assume that the time 0 asset values observed are:

A_0	0.9
B_0	78
C_0	14

(c) (*1 point*) Calculate the risk-neutral probabilities of the three states of the world.

A fourth asset *D* with $D_0 = 15$ has the following values at time 1 in the three states of the world:

	ω_1	ω_2	ω_3
D_1	0	80	90

(d) (*1 point*) Construct an arbitrage portfolio using all 4 assets.

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7. (*7 points*) You are the risk manager of FJM Insurance and you are giving your new intern a training session on FJM's risk management strategy:

Insurance Contract:

- 10-year single-premium annuity contracts issued 5 years ago
- Total single-premium at issue = $100,000
- Guaranteed Minimum Accumulation Benefit (GMAB) rider with a guaranteed growth of 20% over the term of the contracts.
- Fees were charged at issue outside of the contract and no additional fees are charged on the contract.

Policyholder Account:

- Fund indexed to the equity index XYZ
- Current total fund value = $105,000.

Hedging Asset:

- Put option on the equity index XYZ to be held to its maturity in 2 years

Hedging strategy:

- Delta hedging using 1-month equity forwards; One forward contract is based on the value of the underlying index (i.e., multiplier =1)
- Rebalancing frequency: daily
- Trading limit: rebalance the position only if the mismatch is greater than 5% of the target in either direction

The rebalancing transaction aims to eliminate the mismatch. The only allowed residual mismatch would be due to the fact that number of contracts traded is rounded up to the nearest integer.

Market conditions on date 1:

- o Equity index XYZ value = 1,050
- o Partial treasury yield curve of zero coupon bonds, rates expressed with continuous compounding:

1 year	2 year	3 year	4 year	5 year
3.3%	3.5%	3.55%	3.6%	4%

- o Implied volatilities of traded equity options:

Term/Strike	1 year	2 year	3 year	4 year	5 year
Strike = 900	13%	22%	23%	24%	25%
Strike = 1050	15%	21%	22%	23%	24%
Strike = 1200	17%	23%	24%	25%	26%

7. Continued

You say to the intern: “At least three Black-Scholes assumptions will be violated if we are to use the Black-Scholes formula to implement our hedge strategy.”

(a) (*1.5 points*) Explain your statement for three assumptions that you have identified.

You are walking through a numerical example with your intern to illustrate how you project FJM’s current hedge strategy. The equity index value on date 2 is 900. A report of GMAB liability and asset deltas is presented below where GMAB liability at maturity is expressed as max(0, 1.2 * Initial Premium – Fund Value).

Delta	Rebalancing date 1	Rebalancing date 2
GMAB Liability	(i)	51.49
Equity Put Option (before rebalancing)	-19.05	-37.49
Equity Forward (after rebalancing)	(ii)	(iii)

(b) (*4 points*) Calculate the following using Black-Scholes:

(i) GMAB liability delta on date 1

(ii) Equity forward delta after rebalancing on date 1

(iii) Equity forward delta after rebalancing on date 2

Question 7 continued on next page.

7. **Continued**

You asked your intern to briefly describe how a Delta-Gamma neutral hedging strategy would work today. For simplicity, you assume no trading limit.

Here is the intern's answer:

> Step 1: Calculate Delta and Gamma of the liability as Delta(Liability) and Gamma(Liability), respectively;
> Step 2: Calculate Delta and Gamma of the equity put option as Delta(Put Option) and Gamma(Put Option);
> Step 3: Assume the Delta and Gamma of an available market-traded equity option is Delta(New Option) and Gamma(New Option), the number of new equity options needed is calculated as
>
> $$\frac{\text{Gamma(Liability)} - \text{Gamma(Put Option)}}{\text{Gamma(New Option)}}$$
>
> Step 4: The number of equity forwards needed to be traded is calculated as
>
> $$-\left(\text{Delta(Liability)}+\text{Delta(Put Option)}\right)$$
>
> We do not need to divide the above by Delta(Equity forward) because equity forward has delta = 1.

(c) (*1 point*) Assess if your intern made any mistakes in the strategy described above.

(d) (*0.5 points*) Describe the benefit a Delta-Gamma hedge strategy can bring to FJM.

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8. (*4 points*) You work for a hedge fund that employs a volatility arbitrage strategy which you are discussing with your colleague.

(a) (*0.5 points*) Recommend which volatility, actual volatility or implied volatility, should be used for a company that is required to report hedge positions on a mark-to-market basis.

Here are the prices of an 1-year European call option, calculated by the Black-Scholes formula and observed in the market.

Assume risk-free rate r = 3%, strike price K = 1000, and no dividend paid.

	Index value	**Market price**	**Black-Scholes price calculated given volatility σ**		
			20%	25%	30%
Day 1	1000	94.13	94.13	113.48	132.83
Day 2	1100	180.34	162.84	180.34	198.73
Day 3	900	97.49	54.36	75.75	97.49

Your colleague made the following statements:

- "The actual volatility is observed and calculated as the standard deviation of the equity index return during the last 10 years, because this is the exact representation of how volatile the equity index return is."
- "The implied volatility is derived from the posted prices of market-traded equity options."
- "If the actual volatility were always 20%, there would be arbitrage opportunities on each of the three days."

(b) (*1 point*) Assess the statements by identifying any mistakes your colleague made and explain why.

Your colleague made the following statement on the arbitrage strategy on Day 2:

- "I hedge using the implied volatility so that I can earn guaranteed profit over the term of the option if my belief becomes reality. The guaranteed amount can be calculated at inception of the position."

(c) (*0.5 points*) Identify and explain any mistakes your colleague made.

8. Continued

(d) (*2 points*) Describe how you would execute the arbitrage strategy by answering the following three parts (i)-(iii) if you hedged using actual volatility and you believed the actual volatility would be 23% on Day 2.

(i) Recommend whether to buy or sell the option.

(ii) Calculate the number of shares to trade.

(iii) Determine the guaranteed profit if your belief is proven to be correct.

9. (*5 points*) Under the Heath–Jarrow–Morton (HJM) approach, the dynamics of the instantaneous forward rate $F(t,T)$ is given by

$$dF(t,T) = \sigma(t,T)\left(\int_t^T \sigma(s,T)\,ds\right)dt + \sigma(t,T)\,dW_t$$

where $\sigma(t,T)$ is the forward rate volatility and W_t is a standard Wiener process under the risk-neutral measure.

(a) (*2 points*) Derive the SDE for the short rate r_t given that the forward rate volatility $\sigma(t,T) = \sigma$ is a positive constant.

(b) (*1.5 points*) Compare the Black-Scholes model with respect to arbitrage conditions to the following:

(i) The HJM

(ii) The classical approach (for interest rate modeling)

Consider the Vasicek model

$$dr_t = (\eta - \gamma r_t)\,dt + \beta^{\frac{1}{2}}\,dX_t$$

where X_t is a standard Wiener process, and $\eta, \gamma,$ and β are constants with $\gamma \neq 0$ and $\beta > 0$.

(c) (*1.5 points*) Calculate the price of 1-year bond with a principal of 100 and 4% semiannual coupon, given that $\eta = 0.01, \gamma = 0.1, \ \beta = 0.0004,$ and initial short rate $r_0 = 2\%$.

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10. (*7 points*) ABC, a telecommunication company, sponsors the ABC employees defined benefit pension plans covering all of its employees. You are a pension consultant with DEF investment consulting firm. The following information about the pension plan was provided by ABC:

- Pension assets of $150 million are invested in a bond portfolio matching the duration of the pension liabilities and earning on average 4% per annum.
- Employees over 50 are allowed to retire early. Most companies in the telecommunication industry do not allow early retirement.
- 35% of the participants are retirees. Industry average is about 25%.
- Pension Plan liabilities have a duration of 15.
- There is a pension fund shortfall of $15 million.
- Debt-to-Asset ratio for ABC is 0.6 vs. the industry average of 0.4.

(a) (*1.5 points*) Determine for each of the following three attributes whether ABC's ability to take risk is above or below the telecommunication industry average:

(i) sponsor financial condition

(ii) plan provisions

(iii) participant characteristics

X, Y, and Z make the following statements at the Pension Committee meeting:

X: "To avoid risk of market losses over the next year, we should limit ABC's investments to short term risk-free securities."

Y: "We can enhance the yield by leveraging the portfolio: borrowing funds with duration of 1 at 3% and invest the proceeds in the bonds identical to those held in the portfolio. "

Z: "ABC can use repos or options to leverage the portfolio."

(b) (*1 point*) Critique X's statement in light of the pension plan liabilities.

10. Continued

Assume that the strategy suggested by Y is implemented and the duration of the borrowed funds is 1.

(c) (*2 points*)

(i) Determine the amount ABC needs to borrow to increase the return from 4% to 5% assuming all invested funds earn 4% per annum.

(ii) Calculate the duration of the portfolio after borrowing.

(d) (*2 points*) Describe repurchase agreements and the six factors that will affect repo rates.

You are given the following information for the proposed strategy based on Z's comments.

- The underlying bond has a duration of 17 and a price of $1 million.
- ABC can get a 90-day call option on the bond with a price of $30,000.

(e) (*0.5 points*) Calculate the duration of the call option.

****END OF EXAMINATION****
Morning Session

USE THIS PAGE FOR YOUR SCRATCH WORK