INSTRUCTIONS TO CANDIDATES

General Instructions

1. This examination has a total of 100 points. It consists of a morning session (worth 60 points) and an afternoon session (worth 40 points).
   a) The morning session consists of 10 questions numbered 1 through 10.
   b) The afternoon session consists of 6 questions numbered 11 through 16.

The points for each question are indicated at the beginning of the question.

2. Failure to stop writing after time is called will result in the disqualification of your answers or further disciplinary action.

3. While every attempt is made to avoid defective questions, sometimes they do occur. If you believe a question is defective, the supervisor or proctor cannot give you any guidance beyond the instructions on the exam booklet.

Written-Answer Instructions

1. Write your candidate number at the top of each sheet. Your name must not appear.

2. Write on only one side of a sheet. Start each question on a fresh sheet. On each sheet, write the number of the question that you are answering. Do not answer more than one question on a single sheet.

3. The answer should be confined to the question as set.

4. When you are asked to calculate, show all your work including any applicable formulas. When you are asked to recommend, provide proper justification supporting your recommendation.

5. When you finish, insert all your written-answer sheets into the Essay Answer Envelope. Be sure to hand in all your answer sheets because they cannot be accepted later. Seal the envelope and write your candidate number in the space provided on the outside of the envelope. Check the appropriate box to indicate morning or afternoon session for Exam QFIQF.

6. Be sure your written-answer envelope is signed because if it is not, your examination will not be graded.

Recognized by the Canadian Institute of Actuaries.

Tournez le cahier d’examen pour la version française.
1. (5 points) Let \( \{I_t\} \) be a family of information sets over a period \([0, T]\), where \( T \) is finite. Let \( S_t \) be a stochastic process adapted to \( \{I_t\} \) and satisfying the following SDE

\[
dS_t = \sigma_t S_t dW_t
\]

where \( \sigma_t \) is a deterministic positive function and \( W_t \) is a standard Wiener process with respect to the probability measure \( \mathbb{P} \). The initial value of the process is \( S_0 = 1 \).

(a) (1 point) Verify that

\[
S_t = \exp\left(-\frac{1}{2} \int_0^t \sigma_u^2 du + \int_0^t \sigma_u dW_u\right)
\]

(b) (1 point) Identify a probability measure \( \mathbb{Q} \) that is equivalent to \( \mathbb{P} \) such that the following process

\[
W_t^Q = \int_0^t \sigma_u dW_u - W_t
\]

is a standard Wiener process with respect to \( \mathbb{Q} \).

Let \( X_t = S_t^{-1} \) for parts (c) and (d) below.

(c) (2 points) Show that \( X_t \) is a martingale under \( \mathbb{Q} \).

(d) (1 point) Show that for any constant \( K > 0 \)

\[
\mathbb{E}^\mathbb{P}[\max(0, S_T - K)] = K \mathbb{E}^\mathbb{Q}[\max(0, K^{-1} - X_T)]
\]
2. (5 points) Let $W_t$ be a standard Wiener process in a probability space $(\Omega, F, \mathbb{P})$. Consider a non-dividend paying equity $S_t$ and a risk-free savings account $B_t$, which evolve at time $t$ according to the following processes:

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

$$dB_t = rB_t dt$$

where $\mu, \sigma$, and $r$ are all constants.

(a) (0.5 points) State the three conditions for a stochastic process to be a martingale.

(b) (1 point) Determine the necessary condition on $\mu, \sigma$, and $r$ in order for the stochastic process $Y_t = \frac{B_t}{S_t}$ to be a martingale under $\mathbb{P}$.

Now assume that the condition in part (b) is not satisfied.

(c) (1.5 points) Find, using the Girsanov theorem, a probability measure $\widetilde{\mathbb{P}}$ and a standard Wiener process $\widetilde{W}_t$ under $\widetilde{\mathbb{P}}$ such that $Y_t$ is a martingale under $\widetilde{\mathbb{P}}$.

Let $\mathbb{Q}$ be the risk-neutral measure and $C_t$ be the price at time $t$ of an Arithmetic Asian call option which can be exercised at time $T \geq t$ with payout equal to

$$\max \left( \frac{1}{T} \int_0^T S_u du - K, 0 \right)$$

where $K$ is the strike price.

(d) (2 points) Prove that for $t \leq T$ the process $e^{r(T-t)}C_t$ is a martingale under $\mathbb{Q}$ by verifying that it satisfies the three conditions in part (a).
3. (7 points) Let \( r_s \) be the stochastic short rate and \( \mathbb{Q} \) the classic risk-neutral measure with the bank account numeraire \( B_t = e^{\int_0^t r_s ds} \).

For any given \( T > 0 \) denote by \( P(t,T) \) the price at time \( t \leq T \) of a default-free pure discount bond with maturity \( T \) in a no-arbitrage setting. That is, \( P(t,T) = \mathbb{E}^\mathbb{Q}_t \left[ e^{-\int_t^T r_s ds} \right] \)

Using the Girsanov theorem we have a forward equivalent martingale measure \( \mathbb{Q}^T \) with corresponding numeraire \( P(t,T) \) such that for any payoff \( V_T \),

\[
\mathbb{E}^\mathbb{Q}_t \left[ D(t,T) V_T \right] = P(t,T) \mathbb{E}^{\mathbb{Q}^T}_t \left[ V_T \right]
\]

where and henceforth \( D(t,T) = e^{-\int_t^T r_s ds} \) denotes the discount factor.

For any pair \( S, T \) with \( S < T \) denote by \( F(t, S, T) \) the simply-compounded forward rate prevailing at time \( t \leq S \) over the time period \([S, T]\).

(a) (0.5 points) Explain why \( F(t, S, T) \) is a \( \mathbb{Q}^T \)-martingale for any pair \( S, T \) with \( S < T \).

Consider a payer forward start swap where the payments begin at time \( T_0 \) and end at time \( T_N = T_0 + N \delta \), receiving the prevailing LIBOR spot rate \( L(T_i, T_i) = F(T_i, T_i, T_{i+1}) \) and paying a fixed rate \( K > 0 \) at times \( T_{i+1} = T_0 + (i+1) \delta \) for \( i = 0, 1, \ldots, N-1 \).

(b) (2 points) Prove that at valuation time \( t \) the price \( SW_t \) of the swap is given by

\[
SW_t = \mathbb{E}^\mathbb{Q}_t \left[ \sum_{i=0}^{N-1} L(T_i, T_{i+1}) D(t, T_{i+1}) \right] - K \sum_{i=0}^{N-1} P(t, T_{i+1})
\]

For simplicity assume that \( \delta = 1 \) from now on.

(c) (2.5 points) Derive the price \( SW_t \) in terms of \( K \) and \( P(t, T_0), \ldots, P(t, T_N) \) by evaluating the expectation in part (b).

Hint: Apply the aforementioned Equality (\( * \)) (with suitable choices for \( V_T \)) and use the result of part (a).
Denote by $K(t)$ the forward swap rate at valuation time $t \leq T_0$, that is, the value of $K$ such that $SW_t = 0$.

(d) **(0.5 points)** Prove that

$$K(t) = \frac{P(t, T_0) - P(t, T_N)}{A(t)}$$

where and henceforth $A(t) = \sum_{i=0}^{N-1} P(t, T_{i+1})$

The holder of a $T_0 \times T_N$ payer swaption with strike $K$ has the option, but not the obligation, to enter at time $T_0$ a swap that receives the LIBOR spot rate and pays a fixed rate $K$ at times $T_{i+1} = T_0 + (i+1)$ for $i = 0, 1, ..., N-1$.

Denote by $[x]^+ = \max(0, x)$. Then the payoff at time $T_0$ of a $T_0 \times T_N$ payer swaption can be written as follows:

$$\text{Payoff} = \left[ 1 - P(T_0, T_N) - K \sum_{i=0}^{N-1} P(T_0, T_{i+1}) \right]^+$$

Denote by $\mathbb{Q}^t$ an equivalent martingale measure with corresponding numeraire $A(t)$. Similar to the aforementioned Equality $(\ast)$ we have for any payoff $V_T$,

$$\mathbb{E}^Q_t \left[ D(t, T) V_T \right] = A(t) \mathbb{E}^Q_t \left[ \frac{V_T}{A(T)} \right] \quad (***)$$

(e) **(1.5 points)** Prove that the price $PS_t$ of the payer swaption at time $t \leq T_0$ is

$$PS_t = A(t) \mathbb{E}^Q_t \left[ \left( K(T_0) - K \right)^+ \right]$$
4. (5 points) Assume that the price $S_t$ at time $t \geq 0$ of a non-dividend paying stock follows the stochastic differential equation (SDE)

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

where $\mu$ and $\sigma$ are positive constants, and $W_t$ is a standard Wiener process.

Consider a derivative security on the stock. At every time $t$, the derivative security has a value of

$$V_t = e^{ct} (S_t)^k$$

where $c$ and $k$ are constants.

Assume that the current time is 0.

(a) (1.5 points) Show that for $t > 0$

$$\mathbb{E}[V_t] = (S_0)^k e^{\left[k\mu + \frac{1}{2}k(k-1)\sigma^2 + c\right]}$$

(b) (2 points) Show, using Ito’s lemma, that $V_t$ follows the process:

$$\frac{dV_t}{V_t} = \left[k\mu + \frac{1}{2}k(k-1)\sigma^2 + c\right] dt + k\sigma dW_t$$

Assume that there is no arbitrage and the risk-free interest rate is a constant $r$.

(c) (1.5 points) Show that

$$c = (1-k)\left(r + \frac{\sigma^2}{2}k\right)$$
5. (6 points) Denote by $Z(0)$ the vector of discount factors for various maturities $T_i$ (in years) at time 0

$$Z(0) = \begin{pmatrix} Z(0, T_1) \\ Z(0, T_2) \\ \vdots \\ Z(0, T_n) \end{pmatrix}$$

Denote by $P(0)$ the column vector of bond prices available for various maturities $T_i$ at time 0

$$P(0) = \begin{pmatrix} P_c(0, T_1) \\ P_c(0, T_2) \\ \vdots \\ P_c(0, T_n) \end{pmatrix}$$

Denote by $C$ the cash flow matrix:

$$C = \begin{pmatrix} c^1(T_1) & c^1(T_2) & \cdots & c^1(T_n) \\ c^2(T_1) & c^2(T_2) & \cdots & c^2(T_n) \\ \vdots & \ddots & \vdots \\ c^n(T_1) & c^n(T_2) & \cdots & c^n(T_n) \end{pmatrix}$$

where each row $i$ of $C$ corresponds to the cash flows of bond $i$ for all the $n$ different maturities $T_1, T_2, \ldots, T_n$ ($T_1 < T_2 < \cdots < T_n$) and each column $j$ describes all the cash flows that occur on that particular maturity $T_j$ from the $n$ bonds.

(a) (0.5 points) Express the relation among $Z(0)$, $C$ and $P(0)$ in matrix format.

Given the following U.S. Treasury bonds (with face value at $100) traded in the market:

<table>
<thead>
<tr>
<th>Treasury Bond</th>
<th>Maturity (year)</th>
<th>Coupon</th>
<th>Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bond 1</td>
<td>0.5</td>
<td>0%</td>
<td>$98.50</td>
</tr>
<tr>
<td>Bond 2</td>
<td>1</td>
<td>4%</td>
<td>$97.25</td>
</tr>
<tr>
<td>Bond 3</td>
<td>1.5</td>
<td>6%</td>
<td>$96.00</td>
</tr>
</tbody>
</table>

(b) (1 point) Identify all elements of $C$ and $P(0)$.
5. Continued

(c) \( (0.5 \text{ points}) \) Express \( Z(0) \) in terms of \( C \) and \( P(0) \) from parts (a) and (b).

Suppose that, in addition to the three Treasury bonds listed above, two more Treasury bonds with similar maturities but different coupon rates are actively traded at different prices, as shown below:

<table>
<thead>
<tr>
<th>Treasury Bond</th>
<th>Maturity (year)</th>
<th>Coupon</th>
<th>Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bond 4</td>
<td>1</td>
<td>5%</td>
<td>$98.75</td>
</tr>
<tr>
<td>Bond 5</td>
<td>1.5</td>
<td>7%</td>
<td>$97.50</td>
</tr>
</tbody>
</table>

(d) \( (2 \text{ points}) \)

(i) Describe a regression-based method to extract the discount factors from all these Treasury bonds available in the market.

(ii) Express \( Z(0) \) in matrix format using the augmented \( C \) and \( P(0) \).

(e) \( (2 \text{ points}) \)

(i) Describe the Nelson Siegel curve fitting method to extract discount factors from coupon bonds.

(ii) Describe one advantage and one disadvantage of the Nelson Siegel method relative to the bootstrapping based methodology used in part (c).
6.  (6 points) The following chart shows the Euro Absolute/Normal volatilities (Left-hand side, LHS) and Black volatilities (Right-hand side, RHS) reflected in swaptions for different strikes:

(a)  (1 point) Compare the stochastic differential equation (SDE) for the forward swap rates underlying the absolute/normal swaption formula with the SDE for the forward swap rates underlying the Black swaption formula.

(b)  (1 point)

(i)  Compare behaviors of absolute/normal volatility and Black volatility exhibited at different strikes.

(ii)  Explain the reason for the differences.

(c)  (1 point) Describe the adjustment in the SDE for the Displaced Lognormal swaption formula.

(d)  (1.5 points) Compare and contrast swaption implied volatility modeling by giving:

(i)  One advantage of the Displaced Lognormal formula over the Black formula

(ii)  One advantage of the Black formula over the Displaced Lognormal formula
6. Continued

(e) (1.5 points) Assess the use of absolute/normal volatilities and the Black volatilities from the following perspectives:

(i) Correlation with yield curve

(ii) Evaluating and validating quality of fit to market prices
7. (5 points) Your firm sold a European call option on a non-dividend paying stock and immediately started delta-hedging the option until its maturity.

Let

- \( t \) = time variable (\( t = 0 \) indicates inception of the hedge program)
- \( r \) = continuously compounded risk-free interest rate
- \( \sigma_R \) and \( \sigma_I \) be the underlying stock’s realized and implied volatility, respectively
- \( V_{R,t} \) and \( V_{I,t} \) be the value of the option at time \( t \) based on the Black-Scholes-Merton model using \( \sigma_R \) and \( \sigma_I \), respectively
- \( PV_t \) = present value of your firm’s hedging gain/loss from the inception to some future time \( t \) (up to the option’s maturity time \( T \))

Assume that

- Your firm knows the value of \( \sigma_R \)
- \( \sigma_R > \sigma_I \)
- Your firm uses \( \sigma_R \) to rebalance the stock position in its delta-hedging program

Based on your firm’s hedging strategy, it is shown that for \( t \leq T \)

\[
PV_t = V_{R,0} - V_{I,0} - e^{-rt} (V_{R,t} - V_{I,t})
\]

(a) (1 point) Determine the least upper bound \( \text{SUP}_t \) of \( PV_t \) for a given time \( t \leq T \). Justify your answer.

Let \( K \) be the option’s strike price and \( N(x) \) be the standard normal distribution function.

(b) (4 points) Prove that the largest lower bound \( \text{INF}_t \) of \( PV_t \) for a given time \( t \leq T \) is

\[
\text{INF}_t = V_{R,0} - V_{I,0} + Ke^{-r(T-t)} \left[ 1 - 2N \left( \frac{\sigma_R - \sigma_I}{\sqrt{T-t}} \right) \right]
\]

Hint: Use the first and second derivatives for extrema.
8. \((6 \text{ points})\) You have three call options on the same stock with different time-to-
maturities: 1 year, 1 month, and 1 week. Figure 1 through 4 below illustrate how your
options’ Greeks vary with the underlying stock price and time-to-maturity based on
Black-Scholes-Merton model. In each figure, the three lines (dotted, solid, and dashed)
represent the same Greek (either Delta, Gamma, Vega, or Theta). All three lines in each
figure are shown with the same strike price, interest rate, dividend yield, and implied
volatility.

![Figure 1: Stock option Greek (option strike price = $100)](chart1)

![Figure 2: Stock option Greek (option strike price = $100)](chart2)
8. Continued

(a) (1.5 points) Determine which figure corresponds to Delta, Gamma, Vega, and Theta, respectively. Justify your answers.

(b) (1 point) Determine which line (dotted, solid or dashed) corresponds to time-to-maturity of 1 year, 1 month, and 1 week, respectively, by analyzing any one of the four figures. Briefly justify your answers based on each Greek’s characteristics.

You are given the following for a European call option on a non-dividend-paying stock:

- The implied volatility (standard deviation) = 20%
- The time-to-maturity = 6 months
- The stock price = $90
- Vega = 0.207, expressed as the sensitivity of option price change with respect to 1 percentage point change in the level of the volatility (from 20% to 21%)

(c) (1 point) Calculate the option’s Gamma.

Question 8 continued on the next page.
8. Continued

You are given the following information on a put option on a non-dividend paying stock.

<table>
<thead>
<tr>
<th>End of Week</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stock Price</td>
<td>90</td>
<td>92</td>
<td>90</td>
</tr>
<tr>
<td>Option Price</td>
<td>2.17</td>
<td>2.03</td>
<td></td>
</tr>
<tr>
<td>Delta</td>
<td>-0.26</td>
<td>-0.26</td>
<td></td>
</tr>
<tr>
<td>Gamma</td>
<td>0.0255</td>
<td>0.0264</td>
<td></td>
</tr>
</tbody>
</table>

Assume that

- At the end of week 0, you sold 100 of these options and immediately adopted delta-hedging strategy.
- At the end of week 2, you liquidated all of your options and shares of the stock.
- There were no changes in interest rate and the implied stock price volatility during the two-week period.

(d) (2 points) Estimate the profit or loss from delta-hedging strategy at the end of two weeks if

(i) You rebalanced your hedge only at the end of each week

(ii) You never rebalanced your hedge

(e) (0.5 points) Explain why the option price changed even though the stock price was unchanged from the end of week 0 to the end of week 2.
9. (7 points) You have been asked to compare the prices of zero-coupon bonds to the Treasury STRIPS data for various maturities at time 0 under the single factor Vasicek model. You are given:

- The SDE of the short rate for the single factor Vasicek model is given by 
  \[ dr_t = \gamma^* (\bar{r} - r_t) dt + \sigma dX_t \] 
  where \( X_t \) is a standard Wiener process.
- Each STRIPS has face value of 100.
- \( Z_{Vasicek} (r_0, 0; T_i) \) is the Vasicek model price at time 0 of STRIPS with maturity \( T_i, i = 1, 2, \ldots \), given the initial rate \( r_0 \).

(a) (1.5 points) Describe a method to estimate parameters \( \gamma^*, \bar{r}, \) and \( \sigma \).

(b) (1.5 points) Show that 
  \[ r_t = \bar{r} + (r_0 - \bar{r}) e^{-\gamma^* t} + \sigma e^{-\gamma^* t} \int_0^t e^{\gamma^* s} dX_s. \]

You obtain estimates \( \gamma^* = 0.029, \bar{r} = 2.09\% \), and \( \sigma = 1.78\% \).

In addition, you are given \( r_0 = 1.8\% \) along with the following information:

<table>
<thead>
<tr>
<th>Time to maturity (( \tau ))</th>
<th>( A(0, \tau) )</th>
<th>( B(0, \tau) )</th>
<th>( Z_{Vasicek} (r_0, 0; \tau) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.139</td>
<td>-0.00031</td>
<td>1.12039</td>
<td>94.6128</td>
</tr>
<tr>
<td>3.139</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(c) (1.5 points)

(i) Calculate \( Z_{Vasicek} (r_0, 0; 1.139) \).

(ii) Calculate \( A(0, 3.139) \) and \( B(0, 3.139) \) based on the estimates given above.

Consider a portfolio with two zero-coupon bonds, long one unit of a bond with maturity date \( T_1 \) and short \( \Delta \) units of another bond with maturity date \( T_2 > T_1 \).

Let \( Z_{Vasicek} (r_t, t; T_1) \) and \( Z_{Vasicek} (r_t, t; T_2) \) be the two prices with the Vasicek model for the \( T_1 \) and \( T_2 \) maturity dates, respectively, at time \( t \) when the current interest rate is \( r_t \).
9. Continued

Denote the value of this portfolio at time $t$ by

$$\Pi (r_i, t) = Z^{\text{Vasicek}} (r_i, t; T_1) - \Delta Z^{\text{Vasicek}} (r_i, t; T_2)$$

(d) (1.5 points) Prove that if $\Delta$ makes the portfolio insensitive to $r_i$ at time $t$, then

$$\Delta = \frac{B(t, T_1) Z^{\text{Vasicek}} (r_i, t; T_1)}{B(t, T_2) Z^{\text{Vasicek}} (r_i, t; T_2)}$$

where $B(t, T) = \frac{1 - e^{-\gamma^*(T-t)}}{\gamma^*}$.

(e) (1 point) Calculate the $\Delta$ in part (d) at time 0 when $T_1 = 1.139$ and $T_2 = 3.139$, using the data given and results derived earlier.
10. (8 points) You are using Ho-Lee model for short rate \( r_t \):

\[
dr_t = \theta'_t \, dt + \sigma \, dX_t
\]

where \( \theta'_t \) is a deterministic function of \( t \) and \( \sigma \) is a positive constant, and \( X_t \) is a standard Wiener process.

(a) (2.5 points) Calculate the mean and the variance of \( M_t = \int_0^t X_s \, ds \).

Let \( Z(r_t, t; T) \) be the price of zero-coupon bond under the Ho-Lee Model at time \( t \) with maturity \( T \), satisfying the following partial differential equation with the terminal condition \( Z(r_T, T; T) = 1 \):

\[
rZ = \frac{\partial Z}{\partial t} + \frac{\partial Z}{\partial r} \theta'_t + \frac{1}{2} \frac{\partial^2 Z}{\partial r^2} \sigma^2
\]

(b) (2.5 points) Show that \( Z(r_t, t; T) = e^{-\frac{\sigma^2}{6} \left( \int_t^T \theta'_s \, ds + \sigma^2 (T-t)^3 \right)} \) by using the Feynman-Kac Theorem.

(c) (1 point)

(i) Identify the distribution of \( Z(r_t, t; T) \) and \( \frac{\ln Z(r_t, t; T)}{T-t} \).

(ii) Prove that the volatility of the changes \( dr_t \) in long-term bond yields equals the volatility of the changes \( dr_t \) in short-term interest rates.
10. Continued

You are given the following for a call option written on a coupon bond.

- Coupon rate is \( c \)
- Today is an ex-coupon date
- Maturity date of the bond is \( T_B \)
- Maturity date of the option is \( T_O \)
- Strike price is \( K \)
- Assume \( \theta^*_t = \theta \) for all \( t \)

(d) (2 points) Outline the steps of pricing the above call option, assuming the pricing takes place on an ex-coupon date.

**END OF EXAMINATION**
Morning Session
USE THIS PAGE FOR YOUR SCRATCH WORK