Exam QFIQF

AFTERNOON SESSION

Date:  Wednesday, October 30, 2019
Time:  1:30 p.m. – 3:45 p.m.

INSTRUCTIONS TO CANDIDATES

General Instructions

1. This afternoon session consists of 6 questions numbered 11 through 16 for a total of 40 points. The points for each question are indicated at the beginning of the question.

2. Failure to stop writing after time is called will result in the disqualification of your answers or further disciplinary action.

3. While every attempt is made to avoid defective questions, sometimes they do occur. If you believe a question is defective, the supervisor or proctor cannot give you any guidance beyond the instructions on the exam booklet.

Written-Answer Instructions

1. Write your candidate number at the top of each sheet. Your name must not appear.

2. Write on only one side of a sheet. Start each question on a fresh sheet. On each sheet, write the number of the question that you are answering. Do not answer more than one question on a single sheet.

3. The answer should be confined to the question as set.

4. When you are asked to calculate, show all your work including any applicable formulas. When you are asked to recommend, provide proper justification supporting your recommendation.

5. When you finish, insert all your written-answer sheets into the Essay Answer Envelope. Be sure to hand in all your answer sheets because they cannot be accepted later. Seal the envelope and write your candidate number in the space provided on the outside of the envelope. Check the appropriate box to indicate morning or afternoon session for Exam QFIQF.

6. Be sure your written-answer envelope is signed because if it is not, your examination will not be graded.

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Tournez le cahier d’examen pour la version française.
11.  

(5 points) Straddles and Strangles are common ways to hedge Vega exposure which shows strong sensitivity to the absolute level of volatility. However such hedges will not mitigate the risk of changes in the steepness of volatility skew and of its convexity. Thus to better reduce the Vega exposure of a position, higher-order derivatives such as \( dVega/dSpot \) and \( dVega/dVol \) need to be considered.

(a)  

(1 point) Describe the position of a long Straddle and a long Strangle and their payoffs with respect to underlying spot price.

(b)  

(1 point) Plot the graph of Delta of a long Strangle and justify your answers.

(c)  

(1 point) Describe the dynamics of Vega of a long Straddle and show why it can be used to hedge volatility risks.

Since your company has liabilities showing very high sensitivity of Vega with respect to spot, one actuarial student suggests that risk reversals can be used to hedge Vega with small changes of underlying spot.

(d)  

(1 point) Describe the Delta of risk reversals.

Assuming the student constructs risk reversals with a 110 call and a 90 put with same maturity.

(e)  

(1 point) Plot the dynamics of Vega with respect to underlying spot and assess if the actuarial student’s suggestion is appropriate for your company.
12. \(9 \text{ points}\) For any given \(T > 0\), denote by \(Z(t,T)\) the price at time \(t \leq T\) of the risk-free zero-coupon bond maturing at \(T\).

The LIBOR market model assumes that the forward rate \(f_n(t,\tau,T)\) follows a log-normal diffusion process under the \(T\)-forward risk neutral dynamics:

\[
\frac{df_n(t,\tau,T)}{f_n(t,\tau,T)} = \sigma_f(t) dW_t
\]

where \(\sigma_f(t)\) is deterministic. Thus the \(n\)-times compounded LIBOR \(r_n(\tau,T)\) at \(\tau\) with maturity \(T\) has a log-normal distribution:

\[
\ln(r_n(\tau,T)) \sim \text{Normal}\left(f_n(0,\tau,T), \int_0^\tau \left[\sigma_f(t)\right]^2 dt\right)
\]

A power call option pays at maturity time 1 the excess of the square of the 3-month LIBOR over strike:

\[
g = N \max\left(r_4(0.75,1)^2 - K, 0\right)
\]

where

- \(r_4(0.75,1)\) is the quarterly compounded LIBOR at time 0.75 with maturity at time 1.
- The strike price \(K = (3.12\%)^2\).
- The discount factor \(Z(0,1) = 0.9530\).
- Nominal value \(N\) is $1 million.
- The current quarterly compounded forward rate \(f_4(0,0.75,1) = 3.42\%\).
- The current forward caplet volatility \(\sigma_f^{\text{fwd}}(1) = 35\%\).
12. Continued

(a) \(3\) points Calculate the price of the power call option using the LIBOR market model.

Hint: If a variable \(x\) has a lognormal distribution with mean \(\bar{x} = \mathbb{E}[x]\) and variance \(\text{var}(\ln(x)) = \sigma_x^2\), then for any constant \(\alpha, x^\alpha\) also has a lognormal distribution with mean \(\mathbb{E}[x^\alpha] = \bar{x}^\alpha e^{\frac{(\alpha-1)\sigma_x^2}{2}}\) and variance \(\text{var}(\ln(x^\alpha)) = \alpha^2 \sigma_x^2\).

You just applied the forward risk neutral pricing methodology in part (a). To better understand its mechanics, you consider a one-factor HJM model with risk-neutral dynamics of the instantaneous forward rates \(f(t,T)\) as follows:

\[ df(t,T) = m(t,T)dt + \sigma e^{-\alpha(T-t)}dW_t, \]

where \(\sigma\) and \(\alpha\) are positive constants.

(b) \(1\) point Derive an expression for the drift \(m(t,T)\).

(c) \(1.5\) points Prove that the short rate \(r_t\) is given by

\[ r_t = f(0,t) + \frac{\sigma^2}{2\alpha^2}(1-e^{-\alpha t})^2 + \sigma \int_0^t e^{-\alpha(t-t')}dW_t. \]

(d) \(1\) point Derive the risk neutral dynamics \(\frac{dZ(t,T)}{Z(t,T)}\) in terms of \(r_t\) and the parameters \(\sigma\) and \(\alpha\).

Denote by \(P(t,T;S) = \frac{Z(t,T)}{Z(t,S)}\) the forward price of the bond with delivery at \(S\).

(e) \(2.5\) points Determine the \(\mathbb{Q}^S\) dynamics of \(P(t,T;S)\) where \(\mathbb{Q}^S\) is the equivalent measure associated with \(Z(t,S)\) as the numeraire.
13. (6 points) You are analyzing country ABC’s rate market, and are considering modeling using the following well-known models:

(i) Vasicek
(ii) Ho-Lee
(iii) One-factor Hull-White
(iv) Black-Derman-Toy

You first look at the term structure of rates implied by government zero-coupon bonds.

(a) (0.5 points) Identify which models give positive probability to negative rates.

(b) (1 point) Identify which models could not be fitted to the term structure of interest rates exactly and explain why.

Next consider the following one-factor Hull-White model:

\[ dr_t = \left( \theta_t - \gamma \tau r_t \right) dt + \sigma dX_t. \]

(c) (0.5 points) Write down an expression for the annualized volatility \( \sigma(\tau) \) of the changes \( dr_t(\tau) \) in long-term bond yield \( r_t(\tau) \) for the one-factor Hull-White model.

(d) (1.5 points) Show that \( \sigma(\tau) \) in part (c) increases as \( \tau \) increases when \( \gamma < 0 \) and it decreases as \( \tau \) increases when \( \gamma > 0 \).
13. Continued

You plot the average standard deviations of changes in zero-coupon bond yields for maturities between one month and five years using historical monthly data. Your plot is given below:

On the same chart you want to plot the expected fitting results obtained using:

- The Ho-Lee model.
- The one-factor Hull-White model.

(e) (2.5 points)

(i) Plot the volatility term structure for the above two models on a reproduction of the above graph.

(ii) Explain the relative positions of the three curves.

(iii) Describe the expected behavior of long-term bond yield under the one-factor Hull-White model.
14. (7 points) Company LMN sells a Guaranteed Minimum Accumulation Benefit product (GMAB):

- The policyholder can invest in a non-dividend paying index fund in a separate account;
- The balance of premium goes into a general account which will earn a fixed interest rate each year;
- All fees will be deducted from the general account, and the money invested in the separate account is guaranteed not to go below the initial money invested at the end of 5 years.

A consultant was hired by LMN to price the product. He assumed:

- A policy with 100,000 invested in the separate account
- The general account balance is sufficient to pay fees for the first 5 years
- Currently the index fund stands at 2,500
- 5-year interest rate is 3% per annum (continuous compounding)
- 5-year volatility (standard deviation) of the index fund return is 15% per annum

(a) (1.5 points) Calculate the installment of flat fees the company should charge the policyholder at the beginning of each year. Describe the assumptions you used for the calculation.

Let \( P \) be the price of a European put option on a non-dividend-paying stock with strike price \( K \) and expiry at \( T \). Denote by

\[
S = \text{the current price of the stock} \\
r = \text{the continuously compounded risk-free interest rate} \\
\sigma = \text{the volatility of the stock’s continuously compounded returns}
\]

\[
d_i = \frac{\ln \frac{S}{K} + \left( r + \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}}
\]

(b) (2 points) Show that

\[
\frac{\partial P}{\partial S} = -N(-d_1)
\]

where \( N(\cdot) \) is the cumulative normal distribution.
14. Continued

After one year

- LMN has been delta hedging the product on daily basis using the futures of the underlying index
- The interest rate and volatility have remained the same throughout the year.

The consultant summarizes the hedge performance for the year by attributing the profit & loss (P&L) after hedge to five Greeks (Delta, Gamma, Rho, Vega, and Theta).

(c) (2 points) Describe the five Greeks given above and their contributions to the P&L.

The two largest contributions in magnitude are 574 and -180.

(d) (0.5 points) Identify the two Greeks which contributed the above amounts to the P&L.

(e) (1 point) Identify whether each of the following statements is true or false. Briefly justify your answer.

(i) The Delta of a European put option that is out-of-the-money will converge to 0 as the expiration date approaches.

(ii) For an out-of-the-money option with the underlying having low volatility, if the volatility increases, the Delta of the option increases.
15. **(6 points)** You are assigned to develop the following applications of an Economic Scenario Generator (ESG) framework:

**Application 1** - Determine the fair value of the investment guarantees embedded in the company’s variable annuity products.

**Application 2** - Measure the effectiveness of the hedging strategies for the economic capital calculation purposes (where nested simulation is needed).

**Application 3** - Price a traded put option on a major equity index with early exercise, where closed-form formula for its price is not available.

(a) **(1.5 points)** Determine the type(s) of ESG simulation needed for each of the three applications above, and justify your choices.

Using a set of parameters, you have simulated the risk-free interest rates and equity returns for 5 years. The discount factors from the end of each year to time zero are as below:

<table>
<thead>
<tr>
<th>Scenario (i)</th>
<th>Year 1</th>
<th>Year 2</th>
<th>Year 3</th>
<th>Year 4</th>
<th>Year 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>93.70%</td>
<td>93.60%</td>
<td>90.40%</td>
<td>85.70%</td>
<td>82.70%</td>
</tr>
<tr>
<td>2</td>
<td>95.90%</td>
<td>92.10%</td>
<td>86.50%</td>
<td>82.30%</td>
<td>78.70%</td>
</tr>
<tr>
<td>3</td>
<td>97.50%</td>
<td>94.40%</td>
<td>88.30%</td>
<td>85.60%</td>
<td>84.40%</td>
</tr>
<tr>
<td>4</td>
<td>96.80%</td>
<td>90.70%</td>
<td>85.60%</td>
<td>84.50%</td>
<td>83.00%</td>
</tr>
<tr>
<td>5</td>
<td>97.10%</td>
<td>95.60%</td>
<td>91.90%</td>
<td>91.90%</td>
<td>86.07%</td>
</tr>
</tbody>
</table>

The equity accumulation factors $g_i(T)$ from time zero to the end of each projection year are as below, where $T$ is the projection year (accumulation factor is the future value of $1$ invested at time zero):

<table>
<thead>
<tr>
<th>Scenario (i)</th>
<th>Year 1</th>
<th>Year 2</th>
<th>Year 3</th>
<th>Year 4</th>
<th>Year 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>95.20%</td>
<td>106.00%</td>
<td>125.00%</td>
<td>141.90%</td>
<td>116.40%</td>
</tr>
<tr>
<td>2</td>
<td>100.80%</td>
<td>91.70%</td>
<td>111.60%</td>
<td>100.70%</td>
<td>125.40%</td>
</tr>
<tr>
<td>3</td>
<td>119.50%</td>
<td>103.10%</td>
<td>117.20%</td>
<td>99.80%</td>
<td>98.20%</td>
</tr>
<tr>
<td>4</td>
<td>75.50%</td>
<td>57.00%</td>
<td>61.20%</td>
<td>75.50%</td>
<td>90.20%</td>
</tr>
<tr>
<td>5</td>
<td>90.20%</td>
<td>70.50%</td>
<td>88.00%</td>
<td>105.70%</td>
<td>93.00%</td>
</tr>
</tbody>
</table>
The average present values of the equity accumulation factors are as below:

<table>
<thead>
<tr>
<th>Year 1</th>
<th>Year 2</th>
<th>Year 3</th>
<th>Year 4</th>
<th>Year 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k$</td>
<td>80.02%</td>
<td>89.26%</td>
<td>90.17%</td>
<td>86.55%</td>
</tr>
</tbody>
</table>

(b) \((0.5 \ \text{points})\) Calculate $k$ in the table above.

Assume that the time horizon is sufficiently long and the size of this scenario set is sufficiently large to perform the martingale test.

(c) \((1 \ \text{point})\) Assess whether this scenario set passes the martingale test.

A new set of equity returns is simulated and the following average present values of the equity accumulation factors are calculated (using the same discount factors provided above):

<table>
<thead>
<tr>
<th>Year 1</th>
<th>Year 2</th>
<th>Year 3</th>
<th>Year 4</th>
<th>Year 5</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>99.2%</td>
<td>103.3%</td>
<td>100.2%</td>
<td>100%</td>
</tr>
</tbody>
</table>

The accumulation factors $\tilde{g}_i(T)$ of the newly simulated equity returns can be obtained by applying an adjustment $s(T)$ to the old accumulation factors $g_i(T)$ (for scenarios $i = 1, 2, 3, 4, 5$)

$$\tilde{g}_i(T) = e^{s_i(T)} g_i(T)$$

(d) \((1 \ \text{point})\) Determine $s(5)$ and explain its purpose.

An exotic 5-year European Asian-style call equity option with fixed strike of $K$ has the following payoff at the end of the 5-year term (where $X_4$ and $X_5$ are the prices of underlying at the end of year 4 and 5 respectively):

$$\text{Payoff} = \max(\text{average}(X_4, X_5) - K, 0)$$

(e) \((2 \ \text{points})\) Determine the time zero price of the option with $K = 1$ using the appropriate scenario data above.
16. (7 points) You have been asked to apply principal component analysis (PCA) on a yield curve factor model. You are given the eigenvectors of the correlation matrix of interest rates of various terms. Rates have been normalized before the analysis.

<table>
<thead>
<tr>
<th>Term</th>
<th>Eigenvector $v_1$</th>
<th>Eigenvector $v_2$</th>
<th>Principal Components $P_i$</th>
<th>Eigenvalues $\lambda_i$</th>
<th>The cumulative portion of variation</th>
</tr>
</thead>
<tbody>
<tr>
<td>6-month</td>
<td>0.3294</td>
<td>-0.4278</td>
<td>1</td>
<td>6.7432</td>
<td></td>
</tr>
<tr>
<td>1-year</td>
<td>0.3459</td>
<td>-0.3992</td>
<td>2</td>
<td>0.9823</td>
<td></td>
</tr>
<tr>
<td>2-year</td>
<td>0.3506</td>
<td>-0.3514</td>
<td>3</td>
<td>0.1395</td>
<td></td>
</tr>
<tr>
<td>5-year</td>
<td>0.3634</td>
<td>-0.2319</td>
<td>4</td>
<td>0.9924</td>
<td></td>
</tr>
<tr>
<td>10-year</td>
<td>0.3702</td>
<td>0.2209</td>
<td>5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>15-year</td>
<td>0.3573</td>
<td>0.3730</td>
<td>6</td>
<td>0.0221</td>
<td></td>
</tr>
<tr>
<td>20-year</td>
<td>0.3553</td>
<td>0.3831</td>
<td>7</td>
<td>0.0027</td>
<td></td>
</tr>
<tr>
<td>30-year</td>
<td>0.3548</td>
<td>0.3816</td>
<td>8</td>
<td>0.0003</td>
<td></td>
</tr>
</tbody>
</table>

$\lambda_i$ and $v_i$ are eigenvalues and eigenvectors respectively of the principal components $P_i$ using correlation matrix, where $i=1, 2, ..., 7, 8$

$\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4 > \lambda_5 > \lambda_6 > \lambda_7 > \lambda_8 > 0$

Objective: The cumulative portion of variation explained by the principal components is greater than 96%.

(a) (0.5 points) Identify two benefits of PCA.

(b) (1 point) Calculate $\lambda_4$ and $\lambda_5$.

(c) (1 point) Determine the minimum number of principal components for meeting the given objective.

(d) (1 point) Interpret the meanings of the first and second principal components of your yield curve factor model.
16. Continued

Now you are asked to apply PCA analysis to model volatilities of equity index options at various strike prices. Implied volatilities for options at strike prices of 90%, 100%, 110%, and 120% of the prevailing at-the-money price are available for the past 5 years. Suppose that another team gives you the following linear model:

\[
\Delta (\sigma_K - \sigma_{\text{ATM}}) = c + \beta_1 X_{0.9} + \beta_2 X_{1.1} + \beta_3 X_{1.2}
\]

where \( X_i \) are non-standardized explanatory variables of the linear model for \( i = 0.9, 1.1, 1.2 \) (strike prices).

You found that the explanatory variables are highly collinear and decided to use PCA analysis to deal with multicollinearity. Thus you normalized the explanatory variables and obtained \( X^*_i \) that have mean 0 and variance 1. You then passed the variables \( X^*_i \) through PCA and obtained \( X^*_i = w_{i,1}P_1 + w_{i,2}P_2 + w_{i,3}P_3 \) (where \( i = 0.9, 1.1, 1.2 \)).

(e) (1 point) Describe the next steps of the orthogonalization of variables using PCA to deal with multicollinearity issue.

Furthermore, for options with maturity of \( \tau \) years an ordinary least squares regression gave rise to the following estimated model:

\[
\Delta (\sigma_K - \sigma_{\text{ATM}}) = P_1 + \left( \frac{1 - e^{-\gamma \tau}}{\gamma \tau} - \gamma \right) P_2 + \left[ 0.4 \left( \frac{1 - e^{-\gamma \tau}}{\gamma \tau} - \gamma \right) - e^{-\gamma \tau} \right] P_3
\]

You are given that for \( \tau = 2 \)

- \( X^*_{0.9} = 0.02P_1 + 0.064P_2 - 0.16P_3 \)
- \( \beta_1 = 1.2 \)
- Standard deviation of \( X_{0.9} \) is 10%

(f) (2 points) Determine \( \gamma \).

Your supervisor suggests that it is also possible to study volatility skew using daily changes in implied volatilities by strike or moneyness as inputs.

(g) (0.5 points) Critique your supervisor’s suggestion and explain why your model should be used.

**END OF EXAMINATION**

Afternoon Session