1. **Learning Objective:**
4. The candidate will understand how to apply the fundamental techniques of reinsurance pricing.

**Learning Outcome:**
(4b) Calculate the price for a property per risk excess treaty.

**Source:**
Basics of Reinsurance Pricing, Clark

**Solution:**
(a) Explain why the exposure curve allows for exposure above 100% of the insured value.

There can be additional coverages that lead to total payments above the insured value. An example is business interruption coverage.

(b) Calculate the expected loss in the requested layer assuming an expected loss ratio of 50%.

For an insured value of 200K, the layer 100K to 400K is 50% to 200%. The exposure factors are 70% and 100%. The expected loss is 50% x 800K x (100% – 70%) = 120K.

For an insured value of 500K, the layer is 20% to 80%. The exposure factors are 49% and 85%. The expected loss is 50% x 1000K x (85% – 49%) = 180K.

For an insured value of 1000K, the layer is 10% to 40%. The exposure factors are 37% and 64%. The expected loss is 50% x 1200K x (64% – 37%) = 162K.

The total is 462K = 462,000.

(c) State the key assumption underlying the use of a single exposure curve to price this treaty.

The same exposure curve applies, regardless of the insured value.
1. Continued

(d) Calculate the revised expected loss in the layer.

Only the third case is affected. The insured value is now 500K and the subject premium is now 600K. The layer is 20% to 80% of the revised insured value. The exposure factors are 49% and 85%. The expected loss is $50\% \times 600K \times (85\% - 49\%) = 108K$. The revised total is 408K or 408,000.
2. Learning Objective:
5. The candidate will understand methodologies for determining an underwriting profit margin.

Learning Outcome:
(5b) Calculate an underwriting profit margin using the capital asset pricing model.

Source:
Ratemaking: A Financial Economics Approach, D’Arcy and Dyer

Solution:
(a) Calculate the funds generating coefficient estimate, $k$.

\[ k = 0.40(0.9) + 0.35(1.2) + 0.25(1.5) = 1.155. \]

(b) Calculate the underwriting beta.

\[ \beta = -1.155(-0.2) = 0.231. \]

(c) Calculate the underwriting profit margin.

\[ \text{UPM} = -1.155(0.02) + 0.231(0.10 - 0.02) = -0.00462. \]

(d) Indicate whether the underwriting profit margin would be higher, lower, or the same if taxes were not ignored.

It is not possible to know because the direction depends on the relationship between the tax rates on investment income and on underwriting income.

(e) Provide two criticisms of models that apply the Capital Asset Pricing Model to insurance.

- CAPM only covers risks that vary with market returns. It ignores unique insurance risks such as catastrophes.
- The insurance market cannot simply be appended to the stock market.
3. **Learning Objective:**
1. The candidate will understand how to use basic loss development models to estimate the standard deviation of an estimator of unpaid claims.

**Learning Outcomes:**
(1e) Apply a parametric model of loss development.
(1f) Estimate the standard deviation of a parametric estimator of unpaid claims.

**Source:**
LDF Curve Fitting and Stochastic Reserving: A Maximum Likelihood Approach, Clark

**Solution:**
(a) Write the loglikelihood function as a function of $\theta$ and $ELR$.

$$l = \sum c \ln(\mu) - \mu$$
$$= 3500 \ln[10,000ELR(1-e^{-6/\theta})] - 10,000ELR(1-e^{-6/\theta})$$
$$+ 5000 \ln[15,000ELR(1-e^{-6/\theta})] - 15,000ELR(1-e^{-6/\theta})$$
$$+ 2000 \ln[10,000ELR(e^{-6/\theta} - e^{-18/\theta})] - 10,000ELR(e^{-6/\theta} - e^{-18/\theta})$$

(b) Estimate the reserve for the two accident years combined.

Estimated ultimate losses are $(10,000 + 15,000)(0.58175) = 14,544$.
Estimated reserve is $14,544 - 5,500 - 5,000 = 4,044$.

(c) Estimate the scale factor, $\sigma^2$.

$G(6) = 0.58445$ and $G(18) = 0.92824$.

The three estimated increments are
$10,000(0.58175)(0.58445) = 3400$,
$10,000(0.58175)(0.92824 - 0.58445) = 2000$, and
$15,000(0.58175)(0.58445) = 5100$. The estimated scale factor is

$$\sigma^2 = \frac{1}{3-2} \left[ \frac{(3500 - 3400)^2}{3400} + \frac{(2000 - 2000)^2}{2000} + \frac{(5000 - 5100)^2}{5100} \right] = 4.902.$$

(d) Estimate the process standard deviation of the reserve in part (b).

The estimated process variance is $4.902(4,044) = 19,824$. The estimated process standard deviation is the square root, 141.
4. Learning Objective:
1. The candidate will understand how to use basic loss development models to estimate the standard deviation of an estimator of unpaid claims.

Learning Outcomes:
(1c) Identify alternative models that should be considered depending on the results of the tests.
(1d) Estimate the standard deviation of a chain ladder estimator of unpaid claims.

Sources:
Measuring the Variability of Chain Ladder Reserve Estimates, Mack
Testing the Assumptions of Age-to-Age Factors, Venter

Solution:
(a) Demonstrate that the proportionality constants $\alpha_1^2$, $\alpha_2^2$, and $\alpha_3^2$ are 937.5, 300, and 96, respectively.

$$\alpha_1^2 = \frac{1}{2} \left[ 8,000 \left( \frac{15,000}{8,000} - 1.5 \right)^2 + 10,000 \left( \frac{15,000}{10,000} - 1.5 \right)^2 + 12,000 \left( \frac{15,000}{12,000} - 1.5 \right)^2 \right]$$

$$= 937.5$$

$$\alpha_2^2 = \frac{1}{2} \left[ 15,000 \left( \frac{21,000}{15,000} - 1.3 \right)^2 + 15,000 \left( \frac{18,000}{15,000} - 1.3 \right)^2 \right] = 300$$

$$\alpha_3^2 = \frac{300^2}{937.5} = 96$$

(b) Demonstrate that the standard error of the reserve estimator for accident year 3 is 3,432.

$$15,000(1.3)(1.1) \left[ \frac{300}{1.3^2} \left( \frac{1}{15,000} + \frac{1}{15,000+15,000} \right) + \frac{96}{1.1^2} \left( \frac{1}{15,000(1.3)} + \frac{1}{21,000} \right) \right]^{0.5}$$

$$= 3,432$$

(c) Explain why this assumption might be appropriate.

Commentary on Question:
One of the following is sufficient.

- A change in the claims handling process; or
- An abrupt change in a cost index after the second calendar year.
4. Continued

(d) Calculate the values of \( f(1) \), \( f(2) \), \( f(3) \) and \( f(4) \) that minimize the sum of the squared residuals.

The incremental triangle is:

<table>
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<th>8,000</th>
<th>7,000</th>
<th>6,000</th>
<th>2,100</th>
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</thead>
<tbody>
<tr>
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<td>3,000</td>
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<tr>
<td>12,000</td>
<td>3,000</td>
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</tr>
<tr>
<td>14,000</td>
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</tr>
</tbody>
</table>

Then,

\[
f(1) = \frac{23,175(0.87381)(8,000) + 22,098(0.87381)(10,000) + 22,765(12,000) + 28,994(14,000)}{[23,175(0.87381)]^2 + [22,098(0.87381)]^2 + 22,765^2 + 28,994^2} = 0.48285
\]

\[
f(2) = \frac{23,175(0.87381)(7,000) + 22,098(5,000) + 22,765(3,000)}{[23,175(0.87381)]^2 + 22,098^2 + 22,765^2} = 0.22627
\]

\[
f(3) = \frac{23,175(6,000) + 22,098(3,000)}{23,175^2 + 22,098^2} = 0.20026
\]

\[
f(4) = \frac{23,175(2,100)}{23,175^2} = 0.09061
\]

(e) Estimate the unpaid claims for each of accident years 2, 3 and 4 using the parameterized BF model with diagonal effects.

AY2: 22,098(0.09061) = 2,002
AY3: 22,765(0.20026 + 0.09061) = 6,622
AY4: 28,994(0.22627 + 0.20026 + 0.09061) = 14,994
5. **Learning Objective:**
5. The candidate will understand methodologies for determining an underwriting profit margin.

**Learning Outcome:**
(5d) Allocate an underwriting profit margin (risk load) among different accounts.

**Source:**
An Application of Game Theory: Property Catastrophe Risk Load, Mango

**Solution:**
(a) Calculate the total premium to be received by ExRay.

\[
\text{Mean} = 0.05(1500) + 0.03(3000) + 0.02(5000) = 265 \\
\text{Variance} = 0.05(1500)^2 + 0.03(3000)^2 + 0.02(5000)^2 - 265^2 = 812,275 \\
\text{Total Premium} = 265 + 0.0001(812,275) = 346.23
\]

(b) Calculate the premium for each company using the Shapley method.

First obtain the mean for each company’s loss:
\[
E(M) = 0.05(500) + 0.03(2000) = 85 \\
E(T) = 0.05(1000) + 0.02(3000) = 110 \\
E(B) = 0.03(1000) + 0.02(2000) = 70
\]

Next obtain the variance for each company’s loss:
\[
Var(M) = 0.05(500)^2 + 0.03(2000)^2 - 85^2 = 125,275 \\
Var(T) = 0.05(1000)^2 + 0.02(3000)^2 - 110^2 = 217,900 \\
Var(B) = 0.03(1000)^2 + 0.02(2000)^2 - 70^2 = 105,100
\]

Then obtain the covariances:
\[
Cov(M,T) = 0.05(500)(1000) - 85(110) = 15,650 \\
Cov(M,B) = 0.03(2000)(1000) - 85(70) = 54,050 \\
Cov(T,B) = 0.02(3000)(2000) - 110(70) = 112,300
\]

The three premiums are:
\[
M : 85 + 0.0001(125,275 + 15,650 + 54,050) = 104.50 \\
T : 110 + 0.0001(217,900 + 15,650 + 112,300) = 144.59 \\
B : 70 + 0.0001(105,100 + 54,050 + 112,300) = 97.15
\]
6. Learning Objective:
2. The candidate will understand the considerations in selecting a risk margin for unpaid claims.

Learning Outcome:
(2c) Describe methods to assess this uncertainty.

Source:

Solution:
(a) Calculate the internal systemic risk coefficient of variation for each of:

(i) Motor
(ii) Liability

The squared coefficients of variation are:
Motor: \(0.06^2(10 / 50) + 0.05^2(40 / 50) + 2(0.5)(0.06)(0.05)(10 / 50)(40 / 50) = 0.002224\)
Liability: \(0.05^2(30 / 50) + 0.04^2(20 / 50) + 2(0.5)(0.05)(0.04)(30 / 50)(20 / 50) = 0.001636\)
The coefficients of variation are:
Motor: \(0.002224^{1/2} = 0.0472\)
Liability: \(0.001636^{1/2} = 0.0404\)

(b) Calculate the internal systemic risk coefficient of variation for both lines combined.

The calculations from Part (a) include correlations within each line. The two lines each have a weight of 50% and so begin with
\(0.002224(50 / 100)^2 + 0.001636(50 / 100)^2 = 0.000965\).
The additional covariance terms for correlations between lines are
\(0.25(0.06)(0.05)(10 / 100)(30 / 100)\)
\(+0.25(0.06)(0.04)(10 / 100)(20 / 100)\)
\(+0.25(0.05)(0.05)(40 / 100)(30 / 100)\)
\(+0.25(0.05)(0.04)(40 / 100)(20 / 100) = 0.0001495.\)
The total is \(0.000965 + 2(0.0001495) = 0.001264\) and the answer is the square root, 0.0356.

(c) Calculate the consolidated coefficient of variation from the three sources of uncertainty. Assume independence between each of the sources of uncertainty.

The total is \(0.03^2 + 0.07^2 + 0.001264 = 0.007064.\) The answer is the square root, 0.0840.
7. Learning Objective:
3. The candidate will understand excess of loss coverages and retrospective rating.

Learning Outcome:
(3d) Explain retrospective rating in graphical terms.

Source:
The Mathematics of Excess of Loss Coverages and Retrospective Rating – A Graphical Approach, Lee

Solution:
(a) Identify the quantity that should be between the brackets.

The equation is \( H = b + CEr_H \) (eq. 4.29) and so the quantity in brackets is \( r_H \).

(b) Identify the quantity that should be between the brackets.

The equation is \( G = b + CEr_G \) (eq. 4.14 and 4.15) and so the quantity in brackets is \( r_G \).

(c) Provide the expression that should be between the brackets. (It should not contain any of quantities (i) through (vi) listed above.)

The equation is \( e + E = b + CE[r_H + \varphi(r_H) - \varphi(r_G)] \) (eq. 4.29 and 4.32) and so the expression in brackets is \( r_H + \varphi(r_H) - \varphi(r_G) \).

(d) Show how taking differences between the three equations above produces the two equations that can be used to determine the rating values for a retrospective rating plan.

\[(b) - (a) \text{ yields } G - H = b + CEr_G - b - CEr_H = CE(r_G - r_H).\]

\[(c) - (a) \text{ yields } e + E - H = b + CE[r_H + \varphi(r_H) - \varphi(r_G)] - b - CEr_H = CE[\varphi(r_H) - \varphi(r_G)].\]
8. **Learning Objective:**
4. The candidate will understand how to apply the fundamental techniques of reinsurance pricing.

**Learning Outcome:**
(4d) Apply an aggregate distribution model to a reinsurance pricing scenario.

**Source:**
Basics of Reinsurance Pricing, Clark

**Solution:**
(a) Demonstrate that the mean and coefficient of variation of aggregate losses are 1.95 billion and 0.961, respectively.

\[
\begin{align*}
E(N) &= 1.3 \\
E(S) &= 0.6(1) + 0.3(2) + 0.1(3) = 1.5 \\
E(A) &= 1.3(1.5) = 1.95 \\
E(S^2) &= 0.6(1) + 0.3(4) + 0.1(9) = 2.7 \\
Var(A) &= 1.3(2.7) = 3.51 \\
CoV(A) &= 3.51^{1/2} / 1.95 = 0.961
\end{align*}
\]

(b) Calculate the probability that aggregate losses will be:

(i) 1 billion

(ii) 6 billion

**Commentary on Question:**
This solution uses the recursive formula. The answer can also be obtained by enumerating all the ways in which the indicated aggregate loss can occur.

\[
\begin{align*}
p(1) &= \frac{1.3}{1}[l(0.6)(0.2725)] = 0.2126 \\
p(6) &= \frac{1.3}{6}[l(0.6)(0.0498) + 2(0.3)(0.0849) + 3(0.1)(0.1399)] = 0.0266
\end{align*}
\]

(c) Demonstrate that the method of moments estimates are \( \mu = 0.341 \) and \( \sigma^2 = 0.654 \).

**Commentary on Question:**
This solution assumes units are in billions of dollars. Candidates who obtained different values by working with dollars received full credit.
8. Continued

The estimates are:
\[ \sigma^2 = \ln(0.961^2 + 1) = 0.654 \]
\[ \mu = \ln(\text{mean}) - \frac{\sigma^2}{2} = \ln(1.95) - \frac{0.654}{2} = 0.341. \]

(d) Calculate the retention for a treaty covering aggregate catastrophe losses that is expected to be triggered once in 20 years, using the lognormal model.

The normal distribution 95th percentile is \( 0.341 + 1.645(0.654)^{1/2} = 1.6713 \). The lognormal value is \( \exp(1.6713) = 5.319 \) billion.