1. Learning Objectives:

1. The candidate will understand the foundations of quantitative finance

Learning Outcomes:
(1c) Understand Ito integral and stochastic differential equations.

(1d) Understand and apply Ito’s Lemma.

(1j) Understand and apply Girsanov’s theorem in changing measures.

Sources:
Hirsa & Neftci 3rd, 2014, Ch. 9, Ch. 10

Chin et al., 2014, page 126-127

Commentary on Question:
Most candidates were able to work on part (a) and part (c) of this question using Ito’s Lemma. Many candidates were not able to figure out part (b) and part (c) of this question due to the lack of understanding of Girsanov’s theorem.

Solution:
(a) Verify that

\[ S_t = \exp \left( -\frac{1}{2} \int_0^t \sigma_u^2 \, du + \int_0^t \sigma_u \, dW_u \right) \]

Commentary on Question:
This part tests the application of Ito’s Lemma. Most candidates were able to earn full credits of this part.

Apply Ito’s lemma to \( \ln S_t \), we get

\[ d \ln S_t = S_t^{-1} \, dS_t - \frac{1}{2} S_t^{-2} (d \, S_t)^2 \]

\[ d \ln S_t = \sigma_t \, dW_t - \frac{1}{2} \sigma_t^2 \, dt \]

Integrating both sides gives

\[ \ln S_t - \ln S_0 = -\frac{1}{2} \int_0^t \sigma_u^2 \, du + \int_0^t \sigma_u \, dW_u \]
1. Continued

Noting $S_0 = 1$ and taking exponentials we have

$$S_t = \exp\left(-\frac{1}{2} \int_0^t \sigma_u^2 du + \int_0^t \sigma_u dW_u \right)$$

(b) Identify a probability measure $\mathbb{Q}$ that is equivalent to $\mathbb{P}$ such that the following process

$$W_t^Q = \int_0^t \sigma_u du - W_t$$

is a standard Wiener process with respect to $\mathbb{Q}$.

**Commentary on Question:**
Almost all candidates know that this part is related to Girsanov’s theorem. However, most candidates were not able to explicitly specify the measure $\mathbb{Q}$ in terms of measure $\mathbb{P}$.

Let $Q$ be a probability measure defined as

$$dQ = S_T dP = \exp\left(-\frac{1}{2} \int_0^T \sigma_u^2 du + \int_0^T \sigma_u dW_u \right) dP$$

Then by Girsanov’s theorem, the following process

$$W_t^Q = W_t - \int_0^t \sigma_u du$$

is a Wiener process. Hence $W_t^Q = -W_t^Q$ is a Wiener process.

(c) Show that $X_t$ is a martingale under $\mathbb{Q}$.

**Commentary on Question:**
This part can be solved in two ways. One can use the definition of martingale to prove the statement. The second way: one can use the property that a driftless stochastic differential equation leads to a martingale. Most candidates were able to derive the driftless stochastic differential equation and show that $X_t$ is a martingale under $\mathbb{Q}$. Only a few candidates used the first approach.
1. Continued

From part (b), we have
\[ dW_u^Q = -dW_u + \sigma_u du \]
Hence from part (a)
\[ X_t = S_t^{-1} = \exp \left( \frac{1}{2} \int_0^t \sigma_u^2 du - \int_0^t \sigma_u dW_u \right) \]
\[ = \exp \left( \frac{1}{2} \int_0^t \sigma_u^2 du + \int_0^t \sigma_u (dW_u^Q - \sigma_u du) \right) \]
\[ = \exp \left( -\frac{1}{2} \int_0^t \sigma_u^2 du + \int_0^t \sigma_u dW_u^Q \right) \]

Now for \( s < t \), we have
\[ E[X_t|I_s] = E^Q \left[ \exp \left( -\frac{1}{2} \int_s^t \sigma_u^2 du + \int_s^t \sigma_u dW_u^Q \right) | I_s \right] \]
\[ = X_s E^Q \left[ \exp \left( -\frac{1}{2} \int_s^t \sigma_u^2 du + \int_s^t \sigma_u dW_u^Q \right) | I_s \right] \]

By the properties of the Ito integral, we know that \( \int_s^t \sigma_u dW_u^Q \) is a normal variable with mean 0 and variance \( \int_s^t \sigma_u^2 du \). Hence
\[ E^Q \left[ \exp \left( -\frac{1}{2} \int_s^t \sigma_u^2 du + \int_s^t \sigma_u dW_u^Q \right) | I_s \right] = 1 \]

This proves that \( X_t \) is a martingale under \( Q \).

Alternative solution: Using Ito’s lemma
\[ dX_t = -S_t^{-2} dS_t + \frac{(\sigma_t S_t)^2}{2} (2S_t^{-3}) dt \]
\[ = -S_t^{-2} (\sigma_t S_t dW_t) + \sigma_t^2 S_t^{-1} dt \]
\[ = \sigma_t X_t (-dW_t + \sigma_t dt) = \sigma_t X_t dW_t^Q \]
where in the last equality we use the equation \( dW_u^Q = -dW_u + \sigma_u du \)

Therefore \( X_t = X_s + \int_s^t \sigma_u X_u dW_u^Q \) is a martingale under \( Q \).

(d) Show that for any constant \( K > 0 \)
\[ E^P[\max(0, S_T - K)] = KE^Q[\max(0, K^{-1} - X_T)] \]

Commentary on Question:
This part tests an application of changes of measures. Most candidates were able to do the change of measure.
1. Continued

From part (b), we have $dQ = S_T dP$. Then by change of measure, we have

$$KE^Q[\max(0, K^{-1} - X_T)] = E^Q[\max(0, 1 - KX_T)]$$

$$= E^P[S_T \max(0, 1 - KX_T)]$$

$$= E^P[\max(0, S_T - KX_T S_T)]$$

$$= E^P[\max(0, S_T - K)]$$
2. **Learning Objectives:**

1. The candidate will understand the foundations of quantitative finance.

**Learning Outcomes:**

(1a) Understand and apply concepts of probability and statistics important in mathematical finance.

(1c) Understand Ito integral and stochastic differential equations.

(1d) Understand and apply Ito’s Lemma.

(1h) Define and apply the concepts of martingale, market price of risk and measures in single and multiple state variable contexts.

(1i) Demonstrate understanding of the differences and implications of real-world versus risk-neutral probability measures, and when the use of each is appropriate.

(1j) Understand and apply Girsanov’s theorem in changing measures.

**Sources:**

Hira & Neftci 3rd, 2014, Chapter. 6, 13, 14

Chin et al. 2014, Chapter 4

**Commentary on Question:**

The candidates did well in part (a). The candidates demonstrated good understanding of Ito’s Lemma in part (b). Many candidates had trouble in part (c) due to the lack of understanding of Girsanov’s theorem and in part (d) due to the lack of understanding of the Tower property of expectation.

**Solution:**

(a) State the three conditions for a stochastic process to be a martingale.

**Commentary on Question:**

Most candidates were able to list the three conditions.

A stochastic process \( X_t \) is said to be a continuous-time martingale with respect to the family of Filtration \( F_s \) if it satisfies the following conditions:

(i) \( E(X_t | F_s) = X_s \), for all \( 0 \leq s \leq t \)

(ii) \( E(|X_t|) < \infty \)

(iii) \( X_t \) is \( F_t \)-adapted

(b) Determine the necessary condition on \( \mu, \sigma \), and \( \mathcal{R} \) in order for the stochastic process \( \tilde{Y}_t = \frac{B_t}{S_t} \) to be a martingale under \( \mathbb{P} \).
2. Continued

Commentary on Question:
The candidates demonstrated knowledge of how to apply Ito’s Lemma. However, some of them made mistakes in the calculation. A few candidates provided the alternative solution.

Note that $B_t$ is deterministic, so we can apply Ito’s Lemma:

$$dY_t = \frac{\partial Y_t}{\partial t} dt + \frac{\partial Y_t}{\partial S_t} dS_t + \frac{(\sigma S_t)^2}{2} \frac{\partial^2 Y_t}{\partial S_t^2} dt$$

$$= \frac{1}{S_t} dB_t - \frac{B_t}{S_t^2} dS_t + \frac{(\sigma S_t)^2}{2} \frac{\partial^2 Y_t}{\partial S_t^2} dt$$

$$= Y_t (r - \mu + \sigma^2) dt - \sigma Y_t dW_t$$

So, $r - \mu + \sigma^2 = 0$ is the condition for $Y_t$ to be a $\mathbb{P}$-martingale.

Alternative solution: Solve

$$S_t = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t}$$

$$B_t = B_0 e^{rt}$$

Thus

$$Y_t = \frac{B_t}{S_t} = \frac{B_0}{S_0} e^{(r-\mu+\sigma^2)t - \sigma W_t}$$

It follows that for $s < t$

$$E_s[Y_t] = \frac{B_0}{S_0} e^{(r-\mu+\sigma^2)(t-s)} E_s[e^{-\sigma (W_t - W_s)}]$$

$$= \frac{B_0}{S_0} e^{(r-\mu+\sigma^2)(t-s)} e^{\sigma^2(t-s)\frac{t-s}{2}}$$

$$= Y_s e^{(r-\mu+\sigma^2)(t-s)}$$

Therefore if $Y_t$ is a martingale, then we must have $r - \mu + \sigma^2 = 0$ in order to have $E_s[Y_t] = Y_s$.

(c) Find, using the Girsanov theorem, a probability measure $\overline{\mathbb{P}}$ and a standard Wiener process $\overline{W}_t$ under $\overline{\mathbb{P}}$ such that $Y_t$ is a martingale under $\overline{\mathbb{P}}$.

Commentary on Question:

Many candidates were not able to derive $dY_t$ with a new probability measure due to the lack of understanding of Girsanov’s theorem. Only a few candidates checked the Novikov condition.

Girsanov’s theorem states that if $W_t$ is a Wiener process under $\mathbb{P}$, there exists another $\overline{W}_t$ under the equivalent martingale measure $\overline{\mathbb{P}}$ defined as $\overline{W}_t = W_t + \int_0^t \theta_s ds$ for adapted process $\theta_s$.

Hence to find $\overline{W}_t$ such that $Y_t$ is a $\overline{\mathbb{P}}$-martingale, we have...
2. Continued

\[ dY_t = Y_t \left( r - \mu + \sigma^2 \right) dt - \sigma Y_t \left( d\tilde{\mathcal{W}}_t - \vartheta_t dt \right) \]

Equating the coefficients of the dt terms to 0, we have

\[ \vartheta_t = -\left( r - \mu + \sigma^2 \right) / \sigma \]

And clearly for finite \( t \),

\[ E \left[ e^{0.5 \times \int_0^t \vartheta_s^2 ds} \right] = e^{0.5 \times (r - \mu + \sigma^2)^2 t / \sigma^2} < \infty \]

(d) Prove that for \( t \leq T \) the process \( e^{(T-t)C_t} \) is a martingale under \( \mathbb{Q} \) by verifying that it satisfies the three conditions in part (a).

**Commentary on Question:**
Most candidates knew how to use part (a) to check if the process satisfies the three conditions to be martingale. Many candidates proved that the process is \( \mathcal{F}_t \)-adapted and the expectation is finite but had trouble in showing that the expectation of the process is equal to the initial value by using the Tower property of the expectation.

From risk-neutral pricing

\[ C_t = E_t^\mathbb{Q} \left[ e^{-r(T-t)} \max \left( \frac{1}{T} \int_0^T S_u du - K, 0 \right) \right] \]

Thus \( Z_t = e^{r(T-t)} C_t = E_t^\mathbb{Q} \left[ \max \left( \frac{1}{T} \int_0^T S_u du - K, 0 \right) \right] \)

Consider times \( t, s \) such that \( 0 < s < t < T \), then

\[ E^\mathbb{Q}[Z_t | \mathcal{F}_s] = E^\mathbb{Q} \left[ E^\mathbb{Q} \left[ \left( \frac{1}{T} \int_0^T S_u du - K \right)^+ \right] | \mathcal{F}_t \right] | \mathcal{F}_s \]

\[ = E^\mathbb{Q} \left[ \left( \frac{1}{T} \int_0^T S_u du - K \right)^+ \right] | \mathcal{F}_s \] by the Tower Property

\[ = e^{r(T-s)} E^\mathbb{Q} \left[ e^{-r(T-s)} \left( \frac{1}{T} \int_0^T S_u du - K \right)^+ \right] | \mathcal{F}_s \]

\[ = Z_s \]

Clearly \( Z_t \) is \( \mathcal{F}_t \)-adapted, and \( E^\mathbb{Q}[|Z_t|] \leq K + E^\mathbb{Q} \left[ \frac{1}{T} \int_0^T S_u du \right] \)

Since

\[ E^\mathbb{Q} \left[ \frac{1}{T} \int_0^T S_u du \right] = \frac{1}{T} \int_0^T S_0 e^{ru} du \]

we have

\[ E^\mathbb{Q}[|Z_t|] < \infty \]
3. **Learning Objectives:**
1. The candidate will understand the foundations of quantitative finance
2. The candidate will understand the fundamentals of fixed income markets and traded securities.
3. The candidate will understand:
   - The Quantitative tools and techniques for modeling the term structure of interest rates.
   - The standard yield curve models.
   - The tools and techniques for managing interest rate risk.

**Learning Outcomes:**
1a) Understand and apply concepts of probability and statistics important in mathematical finance.
1b) Understand the importance of the no-arbitrage condition in asset pricing.
1h) Define and apply the concepts of martingale, market price of risk and measures in single and multiple state variable contexts.
1j) Understand and apply Girsanov’s theorem in changing measures.
2a) Understand the characteristics of fixed rate, floating rate, and zero-coupon bonds.
2d) Understand the characteristics and uses of interest rate forwards, swaps, futures, and options.
3a) Understand and apply the concepts of risk-neutral measure, forward measure, normalization, and the market price of risk, in the pricing of interest rate derivatives.
3e) Demonstrate understanding of option pricing theory and techniques for interest rate derivatives.

**Sources:**

Problems and Solutions in Mathematical Finance: Stochastic Calculus, Chin, Eric, Nel, Dian and Olafsson, Sverrir, 2014 (Ch. 1)

Fixed Income Securities: Valuation, Risk, and Risk Management, Veronesi, Pietro, 2010 (Ch. 5, 21)
3. Continued

Commentary on Question:
Commentary listed underneath question component.

Solution:
(a) Explain why $F(t, S, T)$ is a $\mathbb{Q}^T$-martingale for any pair $S, T$ with $S < T$.

Commentary on Question:
Overall the candidates performed poorly on this part.

By definition, $F(t, S, T) = \frac{1}{T-S} \left( \frac{P(t, S)}{P(t, T)} - 1 \right) \equiv \frac{1}{T-S} (P(t, S) - P(t, T)) \frac{P(t, T)}{P(t, S)}$

So, the simply-compounded forward rate is a proportion of longing $P(t, S)$ and shorting $P(t, T)$, all normalized by the $P(t, T)$ numeraire. Hence, by construction it is a martingale under the measure $\mathbb{Q}^T$.

(b) Prove that at valuation time $t$ the price $SW_t$ of the swap is given by

$SW_t = E_t^Q \left[ \sum_{i=0}^{N-1} L(T_i, T_{i+1}) D(t, T_{i+1}) - K \sum_{i=0}^{N-1} P(t, T_{i+1}) \right]$

Commentary on Question:
Overall the candidates did poorly on this part. Most failed to show their understanding of swap valuation by providing a detailed proof.

Let $S_{T_0}$ be the payoff of the swap. Then under the risk-neutral measure, $\mathbb{Q}$, we have:

$SW_t = e^{\int_0^t r_s ds} E_t^Q \left[ e^{-\int_0^T r_s ds} S_{T_0} \right]$

Since

$S_{T_0} = \sum_{i=0}^{N-1} \left( L(T_i, T_{i+1}) - K \right) D(T_0, T_{i+1})$

we have

$SW_t = e^{\int_0^t r_s ds} E_t^Q \left[ e^{-\int_0^T r_s ds} \sum_{i=0}^{N-1} \left( L(T_i, T_{i+1}) - K \right) D(T_0, T_{i+1}) \right]$

$= E_t^Q \left[ D(t, T_0) \sum_{i=0}^{N-1} \left( L(T_i, T_{i+1}) - K \right) D(T_0, T_{i+1}) \right]$
3. Continued

\[ E_t^Q \left[ \sum_{i=0}^{N-1} L(T_i, T_{i+1}) D(t, T_{i+1}) \right] - K \sum_{i=0}^{N-1} E_t^Q[D(t, T_{i+1})] \]

\[ = E_t^Q \left[ \sum_{i=0}^{N-1} L(T_i, T_{i+1}) D(t, T_{i+1}) \right] - K \sum_{i=0}^{N-1} P(t, T_{i+1}) \]

(c) Derive the price \( SW_t \) in terms of \( K \) and \( P(t, T_0), ..., P(t, T_N) \) by evaluating the expectation in part (b).

Hint: Apply the aforementioned Equality (*) (with suitable choices for \( V_T \)) and use the result of part (a).

**Commentary on Question:**

*Overall the candidates performed poorly on this part. One common mistake was to forget to express \( F(t, T_i, T_{i+1}) \) as a simply-compounded forward rate by using default-free pure discount bonds.*

So we consider \( V_{T_{i+1}} = L(T_i, T_{i+1}) \) in each instance.

Re-arranging the \( E_t^Q[\] expression from the last equation in part (b), we obtain:

\[ SW_t = \sum_{i=0}^{N-1} P(t, T_{i+1}) E_t^{Q_{T+1}}[F(T_i, T_t, T_{i+1})] - K \sum_{i=0}^{N-1} P(t, T_{i+1}) \]

\[ = \sum_{i=0}^{N-1} P(t, T_{i+1}) F(t, T_i, T_{i+1}) - K \sum_{i=0}^{N-1} P(t, T_{i+1}) \]

\[ = \sum_{i=0}^{N-1} P(t, T_{i+1}) \left( \frac{P(t, T_i)}{P(t, T_{i+1})} - 1 \right) - K \sum_{i=0}^{N-1} P(t, T_{i+1}) \]

\[ = \sum_{i=0}^{N-1} (P(t, T_i) - P(t, T_{i+1})) - K \sum_{i=0}^{N-1} P(t, T_{i+1}) \]

\[ = P(t, T_0) - P(t, T_N) - K \sum_{i=0}^{N-1} P(t, T_{i+1}) \]
3. Continued

(d) Prove that

$$K(t) = \frac{P(t, T_0) - P(t, T_N)}{A(t)}$$

where and henceforth $A(t) = \sum_{i=0}^{N-1} P(t, T_{i+1})$

Commentary on Question:
Candidates who performed well in part (c) usually performed well on this part.

The forward swap rate is the value of the fixed rate that will set the price of the swap to 0 (otherwise an arbitrage opportunity would exist). Setting the expression in part (c) to 0, collapsing the telescoping series and re-arranging, we obtain that

$$K(t) = \frac{P(t, T_0) - P(t, T_N)}{\sum_{i=0}^{N-1} P(t, T_{i+1})} = \frac{P(t, T_0) - P(t, T_N)}{A(t)}$$

(e) Prove that the price $PS_t$ of the payer swaption at time $t \leq T_0$ is

$$PS_t = A(t)E_t^{Q_A}[(K(T_0) - K)^+]$$

Commentary on Question:
Many left this part blank and very few actually provided a coherent proof.

Using the derived value of $K(t)$ in part (d), we note that

$$K(T_0) = \frac{P(T_0, T_0) - P(T_0, T_N)}{A(T_0)} = \frac{1 - P(T_0, T_N)}{A(T_0)}$$

Thus

$$\frac{PS_{T_0}}{A(T_0)} = \frac{1}{A(T_0)} \left(1 - P(T_0, T_N) - K \sum_{i=0}^{N-1} P(T_0, T_{i+1})\right)^+ = (K(T_0) - K)^+$$

From risk-neutral pricing we have

$$PS_t = e^{\int_0^t r_s ds} E_t^Q \left[ e^{-\int_0^T r_s ds} PS_{T_0} \right] = E_t^Q [D(t,T_0) PS_{T_0}]$$

Using Equation (**), we obtain the price of the swaption

$$PS_t = A(t)E_t^{Q_A} \left[ \frac{PS_{T_0}}{A(T_0)} \right]$$

Thus

$$PS_t = A(t)E_t^{Q_A} [(K(T_0) - K)^+]$$
4. **Learning Objectives:**
1. The candidate will understand the foundations of quantitative finance

**Learning Outcomes:**
(1a) Understand and apply concepts of probability and statistics important in mathematical finance.
(1d) Understand and apply Ito’s Lemma.
(1e) Understand the Black Scholes Merton PDE (partial differential equation).

**Sources:**
An Introduction to the Mathematics of Financial Derivatives, Hirsa, Ali and Neftci, Salih N., 3rd Edition 2nd Printing, 2014, Ch. 5, 9, 10, 11, 13

**Commentary on Question:**
*This question tests the understanding of Ito’s Lemma and the Black-Scholes-Merton PDE. Overall, the candidates performed brilliantly on this question.*

**Solution:**
(a) Show that for $t > 0$

$$
\mathbb{E}[V_t] = (S_0)^k e^{k \left(k \mu + \frac{1}{2}k(k-1)\sigma^2 + c\right)}
$$

**Commentary on Question:**
The candidates performed brilliantly on this part. To receive full marks, candidates needed to show the result as intended.

$$
V_t = e^{ct}(S_t)^k \Rightarrow \mathbb{E}[V_t] = e^{ct} \mathbb{E}[(S_t)^k]
$$

$$
S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t} \Rightarrow (S_t)^k = (S_0)^k e^{k \left(\mu - \frac{1}{2}\sigma^2\right)t + k\sigma W_t}
$$

$$
\mathbb{E}[(S_t)^k] = (S_0)^k e^{k \left(\mu - \frac{1}{2}\sigma^2\right)t} \mathbb{E}[e^{k\sigma W_t}]
$$

$$
= (S_0)^k e^{k \left(\mu - \frac{1}{2}\sigma^2\right)t} \frac{1}{\sqrt{2\pi k\sigma^2t}}
$$

$$
= (S_0)^k e^{\left(k\mu + \frac{1}{2}k(k-1)\sigma^2\right)t}
$$

$$
\mathbb{E}[V_t] = e^{ct} \mathbb{E}[(S_t)^k] = e^{ct} \cdot (S_0)^k e^{k \left(\mu + \frac{1}{2}k(k-1)\sigma^2\right)t} = (S_0)^k e^{k \left(\mu + \frac{1}{2}k(k-1)\sigma^2 + c\right)t}
$$

(b) Show, using Ito’s lemma, that $V_t$ follows the process:

$$
\frac{dV_t}{V_t} = \left[k\mu + \frac{1}{2}k(k-1)\sigma^2 + c\right]dt + k\sigma dW_t
$$
Commentary on Question:
The candidates performed brilliantly on this part. To receive full marks, candidates needed to apply Ito’s Lemma correctly.

Find the partial derivatives:
- \( \frac{\partial V}{\partial S} = e^{ct}(S)^{k-1}k = VS^{-1}k \)
- \( \frac{\partial^2 V}{\partial S^2} = e^{ct}(S)^{k-2}k(k-1) = VS^{-2}k(k-1) \)
- \( \frac{\partial V}{\partial t} = e^{ct}(S)^k c = Vc \)

Apply Ito’s Lemma:
\[
dV = \frac{\partial V}{\partial S}(dS) + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} (dS)^2 + \frac{\partial V}{\partial t} (dt)
\]

Plug in the partial derivatives and expressions for \( dS \) and \( (dS)^2 \):
\[
dV = (VS^{-1}k)(\mu S dt + \sigma S dW_t) + \frac{1}{2} (VS^{-2}k(k-1))(\sigma S dW_t)^2 + Vc(dt)
\]

By Ito’s Isometry \((dW_t)^2 = dt\):
\[
dV = (Vk)(\mu dt + \sigma dW_t) + \frac{1}{2} (Vk(k-1))(\sigma^2 dt) + Vc(dt)
\]

Identify the coefficients of \( dt \) and \( dW_t \):
\[
dV = V \left[ k\mu + \frac{1}{2} k(k-1)\sigma^2 + c \right] dt + Vk\sigma dW_t
\]

Show \( \frac{dV}{V} \):
\[
\frac{dV}{V} = \left[ k\mu + \frac{1}{2} k(k-1)\sigma^2 + c \right] dt + k\sigma dW_t
\]

(c) Show that
\[
c = (1-k) \left( r + \frac{\sigma^2}{2} \right)
\]

Commentary on Question:
The candidates performed well on this part. To receive full marks, candidates may use the Black-Scholes PDE or other no-arbitrage conditions (e.g., driftless RN dynamics).
4. Continued

Apply the Black-Scholes PDE:
\[-rV + rS \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} = 0\]

Plug in the partial derivatives:
\[-rV + rS(VS^{-1}k) + \frac{1}{2} \sigma^2 S^2 (VS^{-2}k(k - 1)) + Vc = 0\]

Simplify the equation:
\[-r + r(k) + \frac{1}{2} \sigma^2 (k^2 - k) + c = 0\]

Thus:
\[c = (1 - k)(r + \frac{\sigma^2 }{2} k)\]
5. **Learning Objectives:**

2. The candidate will understand the fundamentals of fixed income markets and traded securities.

**Learning Outcomes:**

(2a) Understand the characteristics of fixed rate, floating rate, and zero-coupon bonds.

(2b) Bootstrap a yield curve.

**Sources:**


**Commentary on Question:**

This question tests candidates on the various methods of extracting the discount factors from coupon bonds, including bootstrapping using matrix notation, regression, and curve fitting. The candidates did well overall.

**Solution:**

(a) Express the relation among \( Z(0), c \) and \( P(0) \) in matrix format.

**Commentary on Question:**

Most candidates did well on this part. Some failed to provide the matrix equation in the correct order.

The price of a coupon bond can be, in general, written using vectors as:

\[
P_c(0, T) = C \times Z(0)
\]

Given the cash flow matrix, we can obtain a system of \( n \) equations with \( n \) unknowns.

\[
P(0) = C \times Z(0)
\]

\[
\begin{pmatrix}
P_c(0, T_1) \\
P_c(0, T_2) \\
\vdots \\
P_c(0, T_n)
\end{pmatrix} = \begin{pmatrix}
c^1(T_1) & c^1(T_2) & \cdots & c^1(T_n) \\
c^2(T_1) & c^2(T_2) & \cdots & c^2(T_n) \\
\vdots & \vdots & \ddots & \vdots \\
c^n(T_1) & c^n(T_2) & \cdots & c^n(T_n)
\end{pmatrix} \begin{pmatrix}
Z(0, T_1) \\
Z(0, T_2) \\
\vdots \\
Z(0, T_n)
\end{pmatrix}
\]
5. Continued

(b) Identify all elements of \( C \) and \( P(0) \).

**Commentary on Question:**
Most candidates did well on this part. Some provided incorrect cash flows.

\[
C = \begin{pmatrix}
100 & 0 & 0 \\
2 & 102 & 0 \\
3 & 3 & 103 \\
\end{pmatrix}
\]

\[
P(0) = \begin{pmatrix}
98.50 \\
97.25 \\
96.00 \\
\end{pmatrix}
\]

(c) Express \( Z(0) \) in terms of \( C \) and \( P(0) \) from parts (a) and (b).

**Commentary on Question:**
Most of candidates did well on this part.

\[
C^{-1} = \begin{pmatrix}
100 & 0 & 0 \\
2 & 102 & 0 \\
3 & 3 & 103 \\
\end{pmatrix}^{-1}
\]

\[
Z(0) = \begin{pmatrix}
Z(0,0.5) \\
Z(0,1.0) \\
Z(0,1.5) \\
\end{pmatrix} = C^{-1} \times P(0)
\]

\[
= \begin{pmatrix}
100 & 0 & 0 \\
2 & 102 & 0 \\
3 & 3 & 103 \\
\end{pmatrix}^{-1} \begin{pmatrix}
98.50 \\
97.25 \\
96.00 \\
\end{pmatrix}
\]

(d)

(i) Describe a regression-based method to extract the discount factors from all these Treasury bonds available in the market.

(ii) Express \( Z(0) \) in matrix format using the augmented \( C \) and \( P(0) \).

**Commentary on Question:**
For part (i), candidates who described a regression-based method received at least partial credits. Most candidates did poorly for part (ii).

To use all the available Treasury bonds provided, we cannot use the bootstrapping method to extract discount factors because the method involves inverting the cash flow matrix \( C \). We now have 5 rows (number of bonds) but only 3 columns (3 maturity points at 0.5y, 1.0y, and 1.5y, respectively).
5. Continued

(i)

For every bond, we can use this

\[ P^i_c(0, T^i) = C^i \times Z(0) + \varepsilon^i \]

where \( \varepsilon^i \) is a random error term that captures any factor that generates the “mispricing.”

We use a regression equation (or an Ordinary Least Squares formula):

\[ y^i = \alpha + \sum_{j=1}^{n} \beta^j x^{ij} + \varepsilon^i \]

where the data are \( y^i = P^i_c(0, T^i) \) and \( x^{ij} = C_{ij} \), and the regressors are \( \beta^j = Z(0, T_j) \).

From basic Ordinary Least Squares (OLS) formulas, we then find

\[ Z(0) = (C' \times C)^{-1} C' \times P(0) \]

(ii)

\[
C = \begin{pmatrix}
100 & 0 & 0 \\
2 & 102 & 0 \\
3 & 3 & 103 \\
2.5 & 102.5 & 0 \\
3.5 & 3.5 & 103.5
\end{pmatrix}
\]

Thus

\[
C' = \begin{pmatrix}
100 & 0 & 0 \\
2 & 102 & 0 \\
3 & 3 & 103 \\
2.5 & 102.5 & 0 \\
3.5 & 3.5 & 103.5
\end{pmatrix}
\]

Based on the bond prices given,

\[
P(0) = \begin{pmatrix}
98.50 \\
97.25 \\
96.00 \\
98.75 \\
97.50
\end{pmatrix}
\]
5. Continued

Thus

\[ Z(0) = \begin{pmatrix} Z(0, 0.5) \\ Z(0, 1.0) \\ Z(0, 1.5) \end{pmatrix} \]

\[ = (C' \times C)^{-1}C' \times P(0) = \begin{pmatrix} 100 & 0 & 0 \\ 2 & 102 & 0 \\ 3 & 3 & 103 \\ 2.5 & 102.5 & 0 \\ 3.5 & 3.5 & 103.5 \end{pmatrix}^{-1} \times \begin{pmatrix} 100 & 0 & 0 \\ 2 & 102 & 0 \\ 3 & 3 & 103 \\ 2.5 & 102.5 & 0 \\ 3.5 & 3.5 & 103.5 \end{pmatrix} = \begin{pmatrix} 98.5 & 0 & 0 \\ 97.25 & 0 & 0 \\ 96.00 & 0 & 0 \\ 98.75 & 0 & 0 \\ 97.50 & 0 & 0 \end{pmatrix} \]

(e)

(i) Describe the Nelson Siegel curve fitting method to extract discount factors from coupon bonds.

(ii) Describe one advantage and one disadvantage of the Nelson Siegel method relative to the bootstrapping based methodology used in part (c).

**Commentary on Question:**
*Overall most of candidates did well on part (e). Many were able to describe the Nelson Siegel method and its advantages and disadvantages. Alternative answers were acceptable for part (ii).*

(i)

Nelson Siegel method is a curve fitting approach that postulate a parametric functional form for the discount factor as a function of maturity and use the current bond prices to *estimate* the parameters of this functional form.

The discount factor is posited to be given by:

\[ Z(0, T) = e^{-r(0, T)T} \]
5. Continued

where the continuously compounded yield with maturity $T$ is given by:

$$\tau(0, T) = \theta_0 + (\theta_1 + \theta_2) \frac{1 - e^{-\frac{T}{\lambda}}}{\frac{T}{\lambda}} - \theta_2 e^{-\frac{T}{\lambda}}$$

where $\theta_0$, $\theta_1$, $\theta_2$ and $\lambda$ are parameters to be estimated from the current bond data.

The parameter estimation is an optimization / minimization process in which the bond prices are calculated with the model parameters iteratively until the estimation error (the difference between the model bond price and the actual bond price) squared is minimized.

(ii)

Advantage (any of these):
(1) The Nelson Siegel method can deal with the case where there are less bonds than number of maturities.
(2) The Nelson Siegel method can still work when the bootstrap methodology fails at a particular maturity due to bad data or an aberration that should be corrected in the data source; Such a correction is not easy if we use the bootstrap methodology.
(3) The Nelson Siegel method might suggest a potential trading opportunity due to temporary “mispricing” or market inefficiency.

Disadvantage (any of these):
(1) The Nelson Siegel method is to “minimize” mismatches, not to find perfect matching, over the entire yield curve.
(2) or a particular maturity, its derived discount factor might be slightly off.
(3) The Nelson Siegel method does not work well with non-linear yield curves.
6. **Learning Objectives:**

2. The candidate will understand the fundamentals of fixed income markets and traded securities.

**Learning Outcomes:**

(2a) Understand the characteristics of fixed rate, floating rate, and zero-coupon bonds.

(2e) Describe the issues in modeling low to negative interest rates.

**Sources:**

QFIC-116-17 Low Yield Curves and Absolute Volatilities – Moody’s Analytics.

**Commentary on Question:**

*The question is relatively straightforward - it is mostly based on a good comprehension of an important subject that is the differentiation in the application the Lognormal/Black, the Normal/Absolute and the Displaced/Lognormal swaption formula in the context of the applicable yield curve, the volatility and the level of the forward strike prices.*

**Solution:**

(a) Compare the stochastic differential equation (SDE) for the forward swap rates underlying the absolute/normal swaption formula with the SDE for the forward swap rates underlying the Black swaption formula.

The Lognormal/Black swaption SDE specifies a different underlying stochastic process for the forward swap rates from the stochastic process specified by the Normal/Absolute swaption SDE, as shown below:

\[
\frac{dF_t^T}{F_t^T} = \sigma dW_t
\]

\(F_t^T\) is the forward swap rate of a swap with maturity at \(T\), at time of \(t\). \(W(t)\) is a Weiner process and \(\sigma\) is the **lognormal** volatility of the forward swap rate (or the forward swap rate is assumed to be lognormally distribute and evolves according to a geometric Brownian motion).

\[
dF_t^T = \sigma dW_t
\]

\(F_t^T\) is the forward swap rate of a swap with maturity at \(T\), at time of \(t\). \(W(t)\) is a Weiner process and \(\sigma\) is the **normal** volatility of the forward swap rate.
6. Continued

(b) 
(i) Compare behaviors of absolute/normal volatility and Black volatility exhibited at different strikes.

(ii) Explain the reason for the differences.

(i) When there is a skew of swaption volatilities for different strikes away-from-the-money, Black volatilities have shown a marked dependence on the strike level whereas by contrast the absolute/normal volatilities are relatively constant.

Or put in another way -

The Absolute/Normal volatility does not exhibit any pronounced dependency on the underlying interest rate strike levels, while the Black volatility increases significantly once the interest rate level decreases beyond a certain level.

(ii) The reason for the difference –

Because of the lognormal nature, Black volatility measures volatilities in the relative changes of the forward swap rates, while the Absolute/Normal volatility measures volatilities in the absolute changes of the forward swap rates.

(c) Describe the adjustment in the SDE for the Displaced Lognormal swaption formula.

\[
\frac{d\widehat{F}_t^T}{\widehat{F}_t^T} = \sigma dW_t, \quad \text{where} \quad \widehat{F}_t^T := F_t^T + \delta
\]

\(F_t^T\) is the forward swap rate of a swap with maturity at \(T\), at time of \(t\). \(W(t)\) is a Weiner process and \(\sigma\) is the lognormal volatility of the forward swap rate (or the forward swap rate is assumed to be lognormally distribute and evolves according to a geometric Brownian motion).

\(\widehat{F}_t^T\) has a displacement \(\delta\) added for calibration purpose.

(d) Compare and contrast swaption implied volatility modeling by giving:

(i) One advantage of the Displaced Lognormal formula over the Black formula

(ii) One advantage of the Black formula over the Displaced Lognormal formula
6. **Continued**

(i) **Advantage** (Displaced Lognormal w.r.t Black’s formula):

- Severe limitations with Black’s formula have become apparent as rates have fallen close to or below zero. Because Black’s formula assumes that rates are lognormally distributed, the formula becomes infinitely sensitive to price changes as rates tend to zero. The Black’s formula cannot be solved at all for negative strikes or forward rates. Displaced lognormal can bypass this limitation. There is an extra degree of freedom in specifying the displacement parameter which can be used to vary the underlying rate distribution.

(ii) **Disadvantage** (Displaced Lognormal w.r.t Black’s formula):

- There is significant subjectivity in choosing the displacement, there is no solid theoretical foundation for choosing the displacement, but simply a convenience adopted by some market participants to overcome the limitation of Black’s formula.

<table>
<thead>
<tr>
<th>Implied Volatility</th>
<th>Stable?</th>
<th>Objective?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Black</td>
<td>✗</td>
<td>✓</td>
</tr>
<tr>
<td>Absolute/Normal</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Displaced Lognormal</td>
<td>✓</td>
<td>✗</td>
</tr>
</tbody>
</table>

(e) Assess the use of absolute/normal volatilities and the Black volatilities from the following perspectives:

(i) Correlation with yield curve

(ii) Evaluating and validating quality of fit to market prices

- **Correlation with yield curve**

- Correlations between yield curve levels and Black volatilities are typically strongly negative. Correlations between yield curve levels and absolute/normal volatilities are weakly positive (**2GP**).

- **Evaluating and validating the quality of fit to market**

- Evaluating and validating the quality of fit to market is likely to be easier with absolute/lognormal volatilities rather than Black volatilities. This is particularly true when Black volatilities could not be quoted for many swaption contracts due to negative strikes or forward rates and the entire surface was extremely unstable.

- Validating fits in terms of prices can help to evaluate the fit to market in this type of situation. However, when extrapolating surfaces either to longer maturities or away-from-themoney, working with absolute/normal volatilities can simplify the process and produce much more robust and justifiable results.
7. **Learning Objectives:**

4. The candidate will understand:
   - How to apply the standard models for pricing financial derivatives.
   - The implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory.
   - How to evaluate risk exposures and the issues in hedging them.

**Learning Outcomes:**

(4a) Demonstrate an understanding of option pricing techniques and theory for equity derivatives.

(4d) Demonstrate an understanding of how to delta hedge, and the interplay between hedging assumptions and hedging outcomes.

(4h) Compare and contrast the various kinds of volatility, e.g., actual, realized, implied and forward, etc.

**Sources:**
The Volatility Smile, Derman, Miller, and Park, 2016, Ch 5

**Commentary on Question:**

This question tests candidates’ understanding of delta hedging strategy under Black-Scholes-Merton framework. In fact, the question is directly from the syllabus material: The Volatility Smile, Derman, Miller, and Park, 2016, Chapter 5.

**Solution:**

(a) Determine the least upper bound $SUP_t$ of $PV_t$ for a given time $t \leq T$. Justify your answer.

**Commentary on Question:**

The candidates performed as expected. Partial credits were given for the correct answer that did not have proper justification.

Since $\sigma_R > \sigma_I$, we have $V_{R,t} \geq V_{I,t}$ because option value is an increasing function of volatility. Thus $PV_t \leq V_{R,0} - V_{I,0}$ for $t \leq T$

The upper bound of $PV_t$ occurs when $V_{R,t} = V_{I,t}$, which holds when the stock price drops to zero (and thus $V_{R,t} = V_{I,t} = 0$) or when the stock price goes to infinity (and thus $V_{R,t} = V_{I,t} =$ option’s intrinsic value).

Thus, the least upper bound of $SUP_t$ of $PV_t = V_{R,0} - V_{I,0}$ for a given time $t \leq T$
7. Continued

(b) Prove that the largest lower bound \( \text{INF}_i \) of \( PV_i \) for a given time \( t \leq T \) is

\[
\text{INF}_i = V_{R,0} - V_{I,0} + K e^{-r(T-t)} \left[ 1 - 2N \left( \frac{\sigma_R - \sigma_I}{2} \sqrt{T-t} \right) \right]
\]

(Hint: Use the first and second derivatives for extrema.)

**Commentary on Question:**
The candidates performed below expectations. Some candidates were able to solve for the correct stock price that makes \( V_R, t - V_I, t \) reach its maximum value, the key part of the proof. Partial credits were given for each correct step shown in the proof.

The lower bound occurs when \( V_R, t - V_I, t \) is maximum.

From the first derivative test, the derivative of \( V_R, t - V_I, t \) with respect to the stock price \( S \) (i.e., the delta) is zero at some value \( \hat{S} \).

\[
\frac{\ln \left( \frac{\hat{S}}{K} \right) + \left( r + \frac{\sigma_R^2}{2} \right)(T-t)}{\sigma_R \sqrt{T-t}} = \frac{\ln \left( \frac{\hat{S}}{K} \right) + \left( r + \frac{\sigma_I^2}{2} \right)(T-t)}{\sigma_I \sqrt{T-t}} = \frac{\sigma_R \sqrt{T-t}}{\sigma_I \sqrt{T-t}}
\]

Solving for \( \hat{S} \) from the above equation, we have \( \hat{S} = K e^{\left( \frac{\sigma_R \sigma_I}{2} - r \right)(T-t)} \)

Using \( PV_t \) to denote \( PV_i \) when \( = \hat{S} \), we have

\[
P V_{R,0} - V_{I,0} - e^{-r(T-t)} \left[ \hat{S} N(d_{1,R}) - Ke^{-r(T-t)} N(d_{2,R}) - \hat{S} N(d_{1,I}) \right] + Ke^{-r(T-t)} N(d_{2,I})
\]

\[
P V_{R,0} - V_{I,0} + Ke^{-2r(T-t)} \left[ N(d_{2,R}) - N(d_{2,I}) \right]
\]

Where \( d_{2,I} = \frac{\ln \left( \frac{\hat{S}}{K} \right) + \left( r - \frac{\sigma_I^2}{2} \right)(T-t)}{\sigma_I \sqrt{T-t}} = \frac{\left( \frac{\sigma_R \sigma_I}{2} - r \right)(T-t) + \left( r - \frac{\sigma_I^2}{2} \right)(T-t)}{\sigma_I \sqrt{T-t}} = \left( \frac{\sigma_R - \sigma_I}{2} \right) \sqrt{T-t} \)

\[
d_{2,R} = \frac{\ln \left( \frac{\hat{S}}{K} \right) + \left( r - \frac{\sigma_R^2}{2} \right)(T-t)}{\sigma_R \sqrt{T-t}} = \frac{\left( \frac{\sigma_R \sigma_I}{2} - r \right)(T-t) + \left( r - \frac{\sigma_R^2}{2} \right)(T-t)}{\sigma_R \sqrt{T-t}} = \left( \frac{\sigma_R - \sigma_I}{2} \right) \sqrt{T-t}
\]

Thus: \( d_{2,R} = -d_{2,I} \)
7. Continued

\[ \bar{V}_t = V_{R,0} - V_{I,0} + K e^{-2r(T-t)} \left[ N \left( -d_{2,I} \right) - N \left( d_{2,I} \right) \right] \]

\[ \bar{V}_t = V_{R,0} - V_{I,0} + K e^{-2r(T-t)} \left[ 1 - 2N \left( d_{2,I} \right) \right] \]

\[ \bar{V}_t = V_{R,0} - V_{I,0} + K e^{-2r(T-t)} \left[ 1 - 2N \left( \frac{\sigma_R - \sigma_I}{2} \sqrt{T - t} \right) \right] \]

Moreover, using the second derivative (the gamma) we find that

\[ \frac{\partial^2 (V_{R,t} - V_{I,t})}{\partial S^2} \bigg|_{S=S} = \frac{N'(d_{1,R})}{S \sigma_R \sqrt{T-t}} - \frac{N'(d_{1,I})}{S \sigma_I \sqrt{T-t}} \]

Utilizing \( d_{1,R} = d_{1,I} \) when \( S = \bar{S} \) we have

\[ \frac{\partial^2 (V_{R,t} - V_{I,t})}{\partial S^2} \bigg|_{S=\bar{S}} = \frac{N'(d_{1,R})}{\bar{S} \sqrt{T-t}} \left( \frac{1}{\sigma_R} - \frac{1}{\sigma_I} \right) < 0 \]

This proves that the largest lower bound \( INF_{t} \) of \( PV_{t} \) for a given time \( t \leq T \) is \( \bar{PV}_t \)

That is: \( INF_{t} = V_{R,0} - V_{I,0} + K e^{-2r(T-t)} \left[ 1 - 2N \left( \frac{\sigma_R - \sigma_I}{2} \sqrt{T - t} \right) \right] \)
8. **Learning Objectives:**

4. The candidate will understand:
   - How to apply the standard models for pricing financial derivatives.
   - The implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory.
   - How to evaluate risk exposures and the issues in hedging them.

**Learning Outcomes:**

(4a) Demonstrate an understanding of option pricing techniques and theory for equity derivatives.

(4d) Demonstrate an understanding of how to delta hedge, and the interplay between hedging assumptions and hedging outcomes.

(4e) Analyze the Greeks of common option strategies.

**Sources:**

Pricing and Hedging of Financial Derivatives, Marroni and Perdomo, Ch 6

The Volatility Smile, Derman, Miller, and Park, 2016, Ch. 7

**Commentary on Question:**

This question tests candidates’ understanding of option Greeks and delta hedging strategy. The candidates performed as expected. Partial credits were given for each step that was completed correctly.

**Solution:**

(a) Determine which figure corresponds to Delta, Gamma, Vega, and Theta, respectively. Justify your answers.

**Commentary on Question:**

Candidates performed well on this part.

Figure 1 = vega
Figure 2 = gamma
Figure 3 = delta
Figure 4 = theta

Call delta is bounded between 0 and 1. It approaches 1 when the option is deep in-the-money and 0 when it is deep out-of-the-money. Only Figure 3 fits this profile.

Call theta cannot be positive. Only Figure 4 fits this profile.
8. Continued

Since both gamma and vega exhibit a bell shape, one way to differentiate them is to recognize that vega diminishes progressively with the reduction of time-to-maturity regardless of the stock price level. This is not the case for gamma: (i) When the stock price is close to the strike price near the option expiry, a small change in stock price could quickly result in call option delta flipping between 0 and 1, that is, gamma is highly unstable in this situation. (ii) When the stock price is away from the strike price, gamma diminishes with the reduction of the time-to-maturity, similar to vega. Therefore, it is clear that only Figure 1 fits vega profile and Figure 2 fits gamma profile.

(b) Determine which line (dotted, solid or dashed) corresponds to time-to-maturity of 1 year, 1 month, and 1 week, respectively, by analyzing any one of the four figures. Briefly justify your answers based on each Greek’s characteristics.
Commentary on Question:
The candidates performed well on this part. Though the solution includes justifications for all Figures, the candidates needed to justify only one of the four Figures to earn full marks.

For all four Figures:
- \( X = 1 \) week (dotted line)
- \( Y = 1 \) month (solid line)
- \( Z = 1 \) year (dashed line)

Vega (Figure 1): the shorter the time-to-maturity, the smaller the vega impact is regardless of the stock price level. Following this logic, \( x = 1 \) week, \( y = 1 \) month, \( z = 1 \) year.

Gamma (Figure 2): The shorter the time-to-maturity, the more unstable the gamma becomes when the stock price is near the strike price. Following this logic, \( x = 1 \) week, \( y = 1 \) month, \( z = 1 \) year.

Delta (Figure 3): the shorter the time-to-maturity, the more unstable the delta becomes when the stock price is near the strike price. Following this logic, \( x = 1 \) week, \( y = 1 \) month, \( z = 1 \) year.
8. Continued

Theta (Figure 4): the shorter the time-to-maturity, the more unstable the theta becomes when the stock price is near the strike price. Following this logic, x = 1 week, y = 1 month, z = 1 year.

(c) You are given the following for a European call option on a non-dividend-paying stock:

- The implied volatility (standard deviation) = 20%
- The time-to-maturity = 6 months
- The stock price = $90
- Vega = 0.207, expressed as the sensitivity of option price change with respect to 1 percentage point change in the level of the volatility (from 20% to 21%)

Calculate the option’s Gamma.

**Commentary on Question:**
The candidates performed well on this part.

\[
Vega = 0.01 \cdot S \sqrt{T - t} \cdot N'(d_1)
\]

\[
Gamma = \frac{N'(d_1)}{\sigma S \sqrt{T - t}}
\]

\[
Gamma = 100 \cdot \frac{Vega}{\sigma S^2 (T - t)} = 100 \cdot \frac{0.207}{20\% \cdot 90^2 \cdot 0.5} = 0.0256
\]

(d) Estimate the profit or loss from delta-hedging strategy at the end of two weeks if

(i) You rebalanced your hedge only at the end of each week

(ii) You never rebalanced your hedge

**Commentary on Question:**
The candidates performed below expectations on this part. The key for subpart (i) is to estimate an updated delta for week 2 based on week 1 delta and gamma.

Since you sold 100 put options at the end of week 0, you need to be short 100*0.26 = 26 shares to start the delta hedge program immediately.

After one week, the estimated delta for the 100 put options is

\[
100[-0.26 + 0.0255*(92 - 90)] = -20.9
\]
8. Continued

(i) If you rebalance the hedge position, you need to buy 5 shares to bring the total short position from 26 shares to 21 shares at the end of week 1. Since the delta was –26 at the end of week 2, you need to sell 5 shares to bring the total short position from 21 shares to 26 shares at the end of week 2.

\[
\text{Profit w/ rebalance} = -100[2.03 - 2.17] - 26[92 - 90] - 20.9[90 - 92] = 3.8
\]

(ii) If you don’t rebalance the hedge position, then you are still short 26 shares.

\[
\text{Profit w/o rebalance} = -100[2.03 - 2.17] - 26[90 - 90] = 14
\]

(e) Explain why the option price changed even though the stock price was unchanged from the end of week 0 to the end of week 2.

**Commentary on Question:**
*The candidates performed well on this part.*

Let \( P \) = put option price  
Let \( S \) = stock price  
\[
\Delta P = \frac{\partial P}{\partial S} \Delta S + \frac{\partial P}{\partial t} \Delta t + \frac{\partial^2 P}{\partial S^2} \Delta S^2
\]

Since \( \Delta S = 0 \) from week 0 to week 2, \( \Delta P = \frac{\partial P}{\partial t} \Delta t \)  
the option price change is due to theta impact (time decay)
9. **Learning Objectives:**

1. The candidate will understand the foundations of quantitative finance.

3. The candidate will understand:
   - The Quantitative tools and techniques for modeling the term structure of interest rates.
   - The standard yield curve models.
   - The tools and techniques for managing interest rate risk.

**Learning Outcomes:**

(1c) Understand Ito integral and stochastic differential equations.

(1d) Understand and apply Ito’s Lemma.

(3b) Understand and apply various one-factor interest rate models.

(3c) Calibrate a model to observed prices of traded securities.

(3d) Describe the practical issues related to calibration, including yield curve fitting.

(3g) Understand and apply the techniques of interest rate risk hedging.

**Sources:**

Fixed Income Securities: Valuation, Risk, and Risk Management, Veronesi, Pietro, 2010 - Ch. 15, 16, 22

Problems and Solutions in Mathematical Finance: Stochastic Calculus, Chin, Eric, Nel, Dian and Olafsson, Sverrir, 2014 - Ch 3

**Commentary on Question:**

*Commentary listed underneath question component.*

**Solution:**

(a) Describe method to estimate parameters $\gamma^*, \overline{F}$ and $\sigma$.

**Commentary on Question:**

*The candidates did not do well on this part. Many candidates knew that the parameters were estimated by minimizing the difference between market bond price vs. modeled bond price or historical data, however, most weren’t clear on which parameters should be estimated by which data. In addition, a lot of candidates did not show their knowledge that the estimation should be based on the term structure of interest rates, and weren’t aware that $\sigma$ is volatility of short rate.*
9. Continued

The parameters $\gamma^*$, $\bar{r}^*$ of the Vasicek model can be estimated from the term structure of interest rates.

The methodology consists of computing the difference between the model prices and the data.

If the model works well, the sum of squares of these “pricing errors” are minimized.

The minimization procedure called nonlinear least squares yields estimates of the two parameters of interest using information from cross section of bonds at time 0 and minimization has to be performed numerically.

$\sigma$ can be estimated directly from the time series of interest rates $r_t$. We should use data on overnight government bonds or 1-months T-Bill rate to estimate $\sigma$.

(b) Show that

$$ r_t = \bar{r} + (r_0 - \bar{r}) e^{-\gamma^* t} + \sigma e^{-\gamma^* t} \int_0^t e^{\gamma^* s} dX_s. $$

Commentary on Question:

Most candidate did well on this part and were able to integrate the term to prove the equation. Some candidates got few points deducted as too many steps were skipped.

Multiplying by $e^{\gamma^* s}$ on $dr_s$ yields

$$ e^{\gamma^* s} dr_s = e^{\gamma^* s} (\bar{r} - r_s) ds + \sigma e^{\gamma^* s} dX_s $$

$$ e^{\gamma^* s} dr_s + e^{\gamma^* s} \gamma^* r_s ds = \gamma^* \bar{r} e^{\gamma^* s} ds + \sigma e^{\gamma^* s} dX_s $$

$$ d(e^{\gamma^* s} r_s) = \gamma^* \bar{r} e^{\gamma^* s} ds + \sigma e^{\gamma^* s} dX_s $$

Alternatively,

Set $F_s = r_s e^{\gamma^* s}$

$$ \frac{\partial F_s}{\partial s} = \gamma^* r_s e^{\gamma^* s}, \quad \frac{\partial F_s}{\partial r_s} = e^{\gamma^* s}, \quad \frac{\partial F_s}{\partial rs} = e^{\gamma^* s}, \quad \frac{\partial^2 F_s}{\partial^2 rs} = 0 $$

$$ d(F_s) = \frac{\partial F_s}{\partial s} ds + \frac{\partial F_s}{\partial r_s} dr_s + \frac{1}{2} \frac{\partial^2 F_s}{\partial^2 r_s} \sigma ds $$

$$ d(e^{\gamma^* s} r_s) = \gamma^* \bar{r} e^{\gamma^* s} ds + \sigma e^{\gamma^* s} dX_s $$
9. Continued

Integrating between 0 and \( t \) and dividing by \( e^{\gamma^* t} \), we get

\[
\frac{r_t}{e^{\gamma^* t}} = \frac{r_0}{e^{\gamma^* t}} + \sigma e^{\gamma^* t} \int_0^t e^{\gamma^* s} dX_s
\]

\[
r_t = \bar{r} + (r_0 - \bar{r})e^{-\gamma^* t} + \sigma e^{-\gamma^* t} \int_0^t e^{\gamma^* s} dX_s
\]

(c)

(i) Calculate \( Z^{Vasicek}(\bar{r}_0; 0, 1.139) \).

(ii) Calculate \( A(0, 3.139) \) and \( B(0, 3.139) \) based on the estimates given above.

Commentary on Question:

The candidates did well at this part. Some candidates failed to identify the short rate used in calculation, or used the wrong sign in calculation.

(i) \( Z^{Vasicek}(r_0; 0, 1.139) = e^{A(0, 1.139) - B(0, 1.139) \cdot 0.018 \cdot 100} = 97.9728 \)

(ii) \( B(0, \tau) = B(\tau) = \frac{1 - e^{-\gamma^* \tau}}{\gamma^*} \)

\[
B(3.139) = \frac{1 - e^{-0.029(3.139)}}{0.029} = 3.000365
\]

\[
A(\tau) = A(0, \tau) = (B(\tau) - (\tau - 0))(\bar{r}^* - \frac{\sigma^2}{2 \gamma^*}) - \frac{\sigma^2 B(\tau)^2}{4 \gamma^*}
\]

\[
A(3.139) = (3.000365 - 3.139)(0.0209 - \frac{0.0178^2}{2 \cdot 0.029^2}) - \frac{0.0178^2 \cdot 3.000365^2}{4 \cdot 0.0209} = -0.00137
\]

Alternatively,

\[
Z^{Vasicek}(r_0; 0, 3.139) = e^{A(0, 3.139) - B(0, 3.139) \cdot 0.018 \cdot 100} = 94.6128
\]

\[
\ln\left(\frac{Z^{Vasicek}(r_0; 0, 3.139)}{100}\right) = \ln(0.946128) = A(3.139) - 3.000365 \cdot 0.018
\]

\[
= A(3.139) = -0.055377 + 3.000365 \cdot 0.018 = -0.00137
\]
9. Continued

(d) Prove that if $\Delta$ makes the portfolio insensitive to $r_t$ at time $t$, then

$$\Delta = \frac{B(t,T_1)Z^{\text{Vasicek}}(r_t,t;T_1)}{B(t,T_2)Z^{\text{Vasicek}}(r_t,t;T_2)}$$

where $B(t,T) = \frac{1 - e^{-\gamma(T-t)}}{\gamma*}$. 

Commentary on Question:
The candidate did well at this part. Most candidates were able to show that the key to have portfolio insensitive to interest rate is to have the $dr_t$ term equal to 0 (or $dX_t$). No or few points was given if candidates failed to demonstrate their knowledge on this.

$$\Pi(r, t) = Z^{\text{Vasicek}}(r_t, t; T_1) - \Delta Z^{\text{Vasicek}}(r_t, t; T_2)$$

We want to choose $\Delta$ so that the sensitivity of portfolio $\Pi(r, t)$ to changes in interest rate $r_t$ is equal to zero:

$$\frac{\partial \Pi(r_t, t)}{\partial r_t} = 0$$

$$\frac{\partial Z^{\text{Vasicek}}(r_t, t; T_1)}{\partial r_t} - \Delta \frac{\partial Z^{\text{Vasicek}}(r_t, t; T_2)}{\partial r_t} = 0$$

$$\Delta = \frac{\partial Z^{\text{Vasicek}}(r_t, t; T_1)}{\partial r_t} / \frac{\partial Z^{\text{Vasicek}}(r_t, t; T_2)}{\partial r_t}$$

Since $\frac{\partial Z}{\partial r} = -B(t,T)Z(r, t, T)$ under Vasicek model, we have

$$\Delta = \frac{B(t,T_1)Z^{\text{Vasicek}}(r_t,t;T_1)}{B(t,T_2)Z^{\text{Vasicek}}(r_t,t;T_2)}$$

where

$$B(t,T) = \frac{1 - e^{-\gamma(T-t)}}{\gamma*}$$
9. Continued

(e) Calculate the $\Delta$ in part (d) at time 0 when $T_1 = 1.139$ and $T_2 = 3.139$, using the data given and results derived earlier.

**Commentary on Question:**
The candidates did extremely well at this part.

$$B(t, T) = \frac{1 - e^{-\gamma^*(T-t)}}{\gamma^*}$$

$B(0, 3.139) = 3.000365, B(0, 1.139) = 1.12039$

Hedge ratio $\Delta = \frac{B(1, 139) + 2Vasicek(r_0, 0.139)}{B(3, 139) + 2Vasicek(r_0, 0.3139)}$

$\Delta = \frac{1.12039 \times 97.9728}{3.000365 \times 94.6128}$

$\Delta = 0.3867$
10. **Learning Objectives:**
1. The candidate will understand the foundations of quantitative finance.

3. The candidate will understand:
   - The Quantitative tools and techniques for modeling the term structure of interest rates.
   - The standard yield curve models.
   - The tools and techniques for managing interest rate risk.

**Learning Outcomes:**
(1a) Understand and apply concepts of probability and statistics important in mathematical finance.
(1c) Understand Ito integral and stochastic differential equations.
(1d) Understand and apply Ito’s Lemma.
(1k) Understand the importance of the Feynman-Kac Theorem.
(3b) Understand and apply various one-factor interest rate models.
(3f) Apply the models to price common interest sensitive instruments including: callable bonds, bond options, caps, floors, and swaptions.

**Sources:**
Fixed Income Securities: Valuation, Risk, and Risk Management, Veronesi, Pietro, 2010 (Ch. 17,19, 21)

Problems and Solutions in Mathematical Finance: Stochastic Calculus, Chin, Eric, Nel, Dian and Olafsson, Sverrir, 2014 (Ch 3)

An Introduction to the Mathematics of Financial Derivatives, Hirsa, Ali and Neftci, Salih N., 3rd Edition 2nd Printing, 2014 (Ch. 9,12, 21)

**Commentary on Question:**
This question tested candidates’ knowledge of Ito integral and stochastic differential equations, Ito’s Lemma, the importance of the Feynman-Kac Theorem, one-factor interest rate model such as Ho-Lee model and application of the model to price common interest sensitive instruments such as callable bond on coupon bond. In general, the candidates performed as expected on this question.

**Solution:**
(a) Calculate the mean and the variance of \[ M_t = \int_0^t X_s ds . \]
10. Continued

Commentary on Question:
In general, the candidates performed well on this question. A lot of candidates provided the alternate solution to calculate the mean of $M_t$ and some candidates provided the alternate solution to calculate the variance of $M_t$. A few candidates applied integrating by parts incorrectly and stated $E[X_t] = tX_t$ and consequently earned no credits.

Using integrating by parts,
$$
\int_0^t X_s \, ds = sX_s|_0^t - \int_0^t sdX_s = tX_t - \int_0^t sdX_s = \int_0^t (t-s) \, dX_s
$$

By taking expectations,
$$
E(M_t) = E\left(\int_0^t X_s \, ds\right) = E(\int_0^t (t-s) \, dX_s) = 0
$$

Alternatively,
$$
E(M_t) = E\left(\int_0^t X_s \, ds\right) = \left(\int_0^t E[X_s] \, ds\right) = 0
$$

$$
E(M_t^2) = E\left(\left(\int_0^t (t-s) \, dX_s\right)^2\right)
$$

By Ito Isometry,
$$
E(M_t^2) = E(\int_0^t (t-s)^2 \, ds) = \frac{t^3}{3}
$$

Alternatively:
$$
E(M_t^2) = E\left(\left(\int_0^t X_s \, ds\right)^2\right) = E\left(\int_{s=0}^{s=t} \int_{u=0}^{u=t} X_s X_u du ds\right)
$$

$$
= \int_{s=0}^{s=t} \int_{u=0}^{u=t} E(X_s X_u) du + \int_{u=s}^{u=t} E(X_s X_u) du \, ds
$$

Note that for $t_2 > t_1$,
$$
E(X_{t_2} X_{t_1}) = E(E(X_{t_2} X_{t_1})|F_{t_1}) = E(E(X_{t_2} | F_{t_1}) X_{t_1}) = E(X_{t_1} X_{t_1}) = t_1
$$

Thus
$$
E(M_t^2) = \int_{s=0}^{s=t} \left(\int_{u=0}^{u=s} u \, du + \int_{u=s}^{u=t} s \, du\right) ds
$$

$$
= \int_{s=0}^{s=t} \left(\frac{s^2}{2} + s(t-s)\right) ds = t \frac{t^2}{2} - \frac{t^3}{6} = \frac{t^3}{3}
$$
10. Continued

(b) Show that $Z(r_t, t; T) = e^{-\tau(T-t)t} \int_t^T \theta_u du + \sigma^2(T-t)^3 / 6$ by using the Feynman-Kac Theorem.

**Commentary on Question:**
In general, the candidates performed as expected on this question. A few candidates did not make use of Feynman-Kac Theorem to do the proof. They simply stated $Z(r_t, t; T)$ in terms of $A(t, T)$ and $B(t, T)$. As a result, no credits were given. In addition, some candidates stated Feynman-Kac Theorem correctly but did not provide any steps of proving the result. As a result, partial credits were given.

$d r_t = \theta_t^* dt + \sigma d X_t$

Integrating on both sides,

$$\int_t^s d r_u = \int_t^s \theta_u^* du + \int_t^s \sigma d X_u du$$

$r_s - r_t = \int_t^s \theta_u^* du + \sigma (X_s - X_t)$

$r_s = r_t + \int_t^s \theta_u^* du + \sigma (X_s - X_t)$

By Feynman-Kac Theorem,

$$Z(r_t, t; T) = V(r_t, t) = E^* \left[ e^{-\int_t^T r_s ds} g(r_T, T) \right] r_t = E^* \left[ e^{-\int_t^T r_s ds} 1 \right] r_t$$

The rate $r_s$ conditional on $r_t = r$ is normally distributed with mean $r + \int_t^s \theta_u^* du$ and variance $\sigma^2(s - t)$.

Since $X_{s-t} = X_s - X_t$ is independent of $F_t$ for every $s > t$, the random variable $\int_t^T X_{s-t} ds = \int_0^{T-t} X_u du \sim N(0, (T-t)^3 / 3)$ is independent of $F_t$, i.e. it is independent of $r_t$.

$$Z(r_t, t; T) = e^{-\tau(T-t)t} \int_t^T \theta_u du E \left[ e^{-\sigma \int_t^T X_{s-t} ds} \right]$$

$$Z(r_t, t; T) = e^{(-\tau(T-t)t) \int_t^T \theta_u du + \sigma^2(T-t)^3 / 6}$$
10. Continued

(c)

(i) Identify the distribution of \( Z(r_t, t; T) \) and \( \frac{\ln Z(r_t, t; T)}{T-t} \).

(ii) Prove that the volatility of the changes \( dr_t(\tau) \) in long-term bond yields equals the volatility of the changes \( dr_t \) in short-term interest rates.

**Commentary on Question:**
*In general, the candidates performed as expected on this question. Most candidates answered correctly on part (i), but some candidates did not provide expression for \( r_t(\tau) \) or even did not attempt on part (ii).*

(i) As \( r_t \) is normally distributed under Ho-Lee model

\[
Z(r_t, t; T) = e^{A(t,T)-(T-t)r_t}
\]

is log normally distributed

\[
\frac{\ln Z(r_t, t; T)}{T-t}
\]

is normally distributed

(ii) \( r_t(\tau) = -\frac{\ln Z(r_t, t; T)}{T-t} = -\frac{A(t,T)}{T-t} + r_t \)

As \( -\frac{A(t,T)}{T-t} \) is a non stochastic term, this implies

\[
\text{Variance}\ (dr_t(\tau)) = \text{Variance}\ (dr_t) = \sigma^2 dt
\]

(d) Outline the steps of pricing the above call option, assuming the pricing takes place on an ex-coupon date.

**Commentary on Question:**
*In general, the candidates performed poorly on this question.*

Most candidates did not identify the first step is to compute the \( A(T_0, T_i) \) and \( B(T_0, T_i) \) and then evaluate \( A(T_0, T_i) \) and \( B(T_0, T_i) \) based on the given parameter in the Ho-Lee model. Only a few candidates mentioned the second step is to look for \( r^* \) that indeed makes the coupon bond price \( T_0 \) equal to strike price \( K \) and to compute \( K_1 = Z(r^*, T_0; T_i) \).

Partial credits were given if candidates stated the expression of \( S_Z(T_0, T_i) \), \( d_1(i) \) and \( d_2(i) \) for every maturity and illustrated that call price is the sum of individual calls.

A few candidates misunderstood the question and provided steps of using Monte Carlo simulation for estimating the price of the above call option. As a result, no credits were given.
10. Continued

The first step is to compute the $A(T_0, T_i)$ and $B(T_0, T_i)$ for every coupon date $T_i$ after the maturity of the option where

\[ A(T_0, T_i) = \int_{T_0}^{T_i} \theta (s - T_i) ds + \frac{\sigma^2 (T_i - T_0)^3}{6} = -\frac{\theta (T_i - T_0)^2}{2} + \frac{\sigma^2 (T_i - T_0)^3}{6} \]
\[ B(T_0, T_i) = T_i - T_0 \]

The second step is to look for $r^*$ that indeed makes the coupon bond price at $T_0$ equal to strike price $K$ by solving the equation

\[ \frac{c}{2} \sum_{i=1}^{n} Z(r^*, T_0; T_i) + Z(r^*, T_0; T_n) = K \]

The third step is to compute $K_i = Z(r^*, T_0; T_i)$

The fourth step is to compute $S_2(T_0, T_i)$, $d_1(i)$ and $d_2(i)$ for every maturity

Call price (x 100) = $\frac{c}{2} \sum_{i=1}^{n} Call_i + Call_n$
11. **Learning Objectives:**

The candidate will understand:

- How to apply the standard models for pricing financial derivatives.
- The implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory.
- How to evaluate risk exposures and the issues in hedging them.

**Learning Outcomes:**

(4i) Define and explain the concept of volatility smile and some arguments for its existence.

(4k) Describe and contrast several approaches for modeling smiles, including: stochastic volatility, local-volatility, jump-diffusions, variance-gamma, and mixture models.

**Sources:**

Pricing and Hedging Financial Derivatives, Marroni and Perdomo, 2014, Ch 7

**Commentary on Question:**

*Overall the performance was below expectation, and not many candidates earned full marks. Parts (c) and (e) were the worst performed, while part (a) seemed to be the part that were answered best.*

**Solution:**

(a) Describe the position of a long Straddle and a long Strangle and their payoffs with respect to underlying spot price.

**Commentary on Question:**

*Part (a) was straightforward, if candidates knew what Straddle and Strangle were, they gave correct results as well as payoff pattern (description, formula, graph were all good). But some candidates got confused between Straddle and Strangle, in that case, half credits were given.*

Straddle: purchase a call and put simultaneously with same maturity and strike.

Strangles: purchase a call and put with same maturity but different strikes.

When stock is in the range of two strikes, both options are out of the money, and investor will lose the entire purchase premium. When stock price increases, Call becomes in the money and pay off increases accordingly while Put becomes out of the money and may end up with 0 value. When stock price decreases, Call becomes out of the money, and Put becomes in the money and payoff increases.
11. Continued

(b) Plot the graph of Delta of a long Strangle and justify your answers.

Commentary on Question:
Common mistake was describing delta of Straddle instead of Strangle or didn't realize the delta neutral position when underlying was sitting between two strikes. To receive full marks, candidates must also indicate the converging values.

Delta of a long Strangle approaching maturity:

When stock is in the range of two strikes, both options are out of the money, so deltas are converging to zero.

When stock price increases (larger than Call strike), Call becomes in the money with delta approaching to 1 and Put is out of the money. When stock price decreases (small than Put strike), Call becomes out of the money and Put becomes in the money with delta approaching to -1.

(c) Describe the dynamics of Vega of a long Straddle and show why it can be used to hedge volatility risks.

Commentary on Question:
Common mistake was expressing Vega itself (increase by underlying) instead of dynamics of Vega versus underlying (graph was also acceptable). Also, many candidates just answered the first half of the question without analyzing why it could be used to hedge volatility risk (level).

Vega of both calls and put are positive and Vega of the Straddle is just sum of the two Vegas for their call and put components, so the dynamics will still show a bell-shaped curve.
11. Continued

When underlying is close to the strike of Straddle, both call and put are becoming very volatile so the Vega of Straddle is at the peak; while underlying is moving away from the strike, the Vega is decreasing and approaching to none.

By making delta neutral, Straddle allows to take advantage of underlying changes. Given the Vega exposure, when underlying price moves away from delta neutral position, Vega decreases. Therefore, regardless of which direction the price moves, a long Straddle position will help to hedge it.

(d) Describe the Delta of risk reversals.

Commentary on Question:
Many candidates forgot or didn't address the relationship between features of delta vs hedging opportunities; delta neutral can also be expressed as insensitive to underlying changes or similar as holding an underlying directly. But most candidates were able to describe what Risk Reversal is and its payoff pattern.

Risk reversals are combinations of purchasing a call option while simultaneous sale of a put option with a lower strike.

Delta of risk reversal is just the sum of a long call delta and a short put delta so end up with a long position of underlying.

Since the two options have different strikes, for spot within the two strikes, both options are out of the money, so risk reversals achieve a neutral position.

(e) Plot the dynamics of Vega with respect to underlying spot and assess if the actuarial student’s suggestion is appropriate for your company.

Commentary on Question:
This was the hardest part of the question; many candidates didn't show the dynamics of Vega and thus couldn't judge if the suggestion was appropriate or not. Even when they explained properly about the change of Vega, they didn't know how to use the feature to arrive the solution.

Vega of both call and put are positive and converges to zero for deeply out of the money option and unstable near their respective strikes. So, when underlying is closer to 110, the dynamic looks more like the one for call while when underlying is approaching to 90, it looks more like that of put, giving a shape with two peaks.
Given above, risk reversals have sensitive Vega with small changes of underlying spots while keeps delta neutral position within some range (between two strikes). Therefore, it can take positions on the steepness/flatness of volatility skew (from one strike to another) and then hedge your liabilities with high sensitivity of dVega/dSpot.
12. **Learning Objectives:**

1. The candidate will understand the foundations of quantitative finance.

3. The candidate will understand:
   - The Quantitative tools and techniques for modeling the term structure of interest rates.
   - The standard yield curve models.
   - The tools and techniques for managing interest rate risk.

**Learning Outcomes:**

(1d) Understand and apply Ito’s Lemma.

(1j) Understand and apply Girsanov’s theorem in changing measures.

(3a) Understand and apply the concepts of risk-neutral measure, forward measure, normalization, and the market price of risk, in the pricing of interest rate derivatives.

(3i) Understand and apply the Heath-Jarrow-Morton approach including the Libor Market Model.

**Sources:**

Pietro – Ch. 21 – pages 710-711, 717, 730-731

Hirsa-Neftci 3rd Ed - Ch. 19 – pages 325-326

**Commentary on Question:**

Most candidate attempted parts (a) to (c), and only a few tried part (d) and (e). Some candidates only got partial credits because they failed to work out the integrals.

**Solution:**

(a) Calculate the price of the power call option using the LIBOR market model.

Hint: If a variable $x$ has a lognormal distribution with mean $\bar{x} = \mathbb{E}[x]$ and variance $\text{var}(\ln(x)) = \sigma^2_x$, then for any constant $\alpha$, $x^\alpha$ also has a lognormal distribution with mean $E[x^\alpha] = \bar{x}^\alpha e^{(\alpha-1)\frac{\alpha^2}{2}}\sigma^2_x$ and variance $\text{Var}(\ln(x^\alpha)) = \alpha^2\sigma^2_x$.

**Commentary on Question:**

Most of the candidates who attempted this question were able to demonstrate that they knew the steps to solve the question.
12. Continued

We should choose $\sigma_f(t) = \sigma_f^{w}(T)$ for the volatility. (See page 716.)

Then $\text{var}\left(\ln(r_n(\tau,T))\right) = \int_0^\tau \sigma_f(t)^2dt = \left(\sigma_f^{w}(T)\right)^2 \tau$

The mean and variance of the T-forward risk neutral expected 3-month LIBOR raised to the 2th can be written as:

$$g(0,\tau,T) = f_n(0,\tau,T)^2 e^{\frac{(2-1)^2}{2}\left(\sigma_f^{w}(T)\right)^2 \tau}$$

$$\sigma_f^2 = 2^2 \left(\sigma_f^{w}(T)\right)^2 \tau$$

While intended to give away that $f_n(0,\tau,T)$ is the mean of $r_n(\tau,T)$, the equation provided in the exam might have been understood that $f_n(0,\tau,T)$ was the mean of $\ln(r_n(\tau,T))$ instead. Both understandings were accepted and given proper credits for Part (a), and the calculated price had no bearing on the other parts.

When identifying all the numbers to apply the Hint, we have

$$\alpha = 2, \quad \sigma_x^2 = \sigma_f(T)^2 \tau = (35\%)^2 \cdot 0.75 = 0.091875$$

Two different numbers for $\bar{x} = E_f^*[r_n(\tau,T)]$ as below were accepted.

**Case 1:** Simply taking $\bar{x} = f_n(0,\tau,T) = 3.42\%$ as intended. Then

$$g(0,\tau,T) = \bar{x}^{(\alpha)} e^{\frac{(\alpha-1)\alpha}{2}\sigma_x^2} = (3.42\%)^2 \cdot e^{-\frac{(\alpha-1)\alpha}{2}\sigma_x^2} = 0.001282192.$$ 

Plug in the values given, we can calculate

$$g(0,\tau,T) = 0.00128$$

$$\sigma_f^2 = 0.3675$$

Power call: $NZ(0,T)[g(0,\tau,T)N(d_1) - KN(d_2)]$

$$d_1 = \frac{1}{\sigma_T} \log\left(\frac{g(0,\tau,T)}{K}\right) + \frac{1}{2} \sigma_T$$

$$d_2 = d_1 - \sigma_T$$

Plug in the values required, we get

$d_1 = 0.7547$

$d_2 = 0.1485$
12. Continued

Look up the values in the normal distribution table,
N(d1) = 0.7734*(1-0.479) + 0.7764*(0.479) = 0.7748
N(d2) = 0.5557*(1-0.848) + 0.5596*(0.848) = 0.559

Power call price = $426.5

Case 2: Calculating \( \bar{x} \) using mean of \( \ln(r_n(\tau, T)) \) as \( f_n(0, \tau, T) \)

From \( \ln(r_n(\tau, T)) \sim \text{Normal}(3.42\%, 0.091875) \) we have
\[
\bar{x} = \exp \left( 3.42\% + \frac{0.091875}{2} \right) = 1.08343603.
\]

Then
\[
g(0, \tau, T) = \bar{x} e^{\frac{(\alpha-1)\alpha}{2} \sigma_t^2} = (1.08343603)^2 * e^{\frac{(2-1)2}{2} \sigma_t^2} = 1.286789041.
\]

\[
d_1 = \frac{1}{\sigma_T} \log \left( \frac{g(0, \tau, T)}{K} \right) + \frac{1}{2} \sigma_T = 12.15829456, \quad N(d_1) = 1.
\]

\[
d_2 = d_1 - \sigma_T = 12.15829456 - 11.55207678, \quad N(d_2) = 1.
\]

\[
Price = NZ(0, T)[g(0, \tau, T)N(d_1) - KN(d_2)]
\]
\[
= 0.9530[1.286789041 - (3.12\%)^2] = 1.225382268 \text{ million}
\]

Note that in both cases \( \sigma_T^2 = \alpha^2 \sigma_\tau^2 = 0.3675 \)

(b) Derive an expression for the drift \( m(t, T) \).

**Commentary on Question:**

*Most candidates got this one right.*

The HJM no arbitrage condition prescribes the relationship between drift and the volatility term in the risk neutral process of the instantaneous forward rate.

\[
m(t, T) = \sigma_f(t, T) \int_t^T \sigma_f(t, \tau) d\tau \quad \text{Formula 21.55}
\]
\[
= \sigma e^{-\alpha(T-t)} \int_t^T \sigma e^{-\alpha(\tau-t)} d\tau
\]
\[
= \sigma e^{-\alpha(T-t)} \frac{\sigma}{\alpha} (1 - e^{-\alpha(T-t)})
\]
12. Continued

(c) Prove that the short rate $r_t$ is given by

$$r_t = f(0,t) + \frac{\sigma^2}{2\alpha^2}(1 - e^{-\alpha t})^2 + \sigma \int_0^t e^{-\alpha(t-\tau)} dW_\tau$$

**Commentary on Question:**
Most candidates were able to show that

$$\int_0^t df(\tau, T) = \int_0^t m(t, T) \, dt + \int_0^t \sigma e^{-\alpha(T-\tau)} dW_\tau,$$

however, some did not get full marks because they failed to expand and simplify the integrals.

$$f(t,T)$$

$$= f(0,T) + \int_0^t df(\tau, T)$$

$$= f(0,T) + \int_0^t \frac{\sigma^2}{\alpha} \left( e^{-\alpha(T-\tau)} - e^{-2\alpha(T-\tau)} \right) d\tau + \int_0^t \sigma e^{-\alpha(T-\tau)} dW_\tau$$

$$= f(0,T) + \frac{\sigma^2}{\alpha} \left( e^{-\alpha T} t - \frac{\sigma^2}{\alpha} \int_0^t e^{2\alpha \tau} d\tau + \int_0^t \sigma e^{-\alpha(T-\tau)} dW_\tau \right)$$

$$= f(0,T) + \frac{\sigma^2}{\alpha^2} \left( e^{-\alpha(T-t)} - e^{-\alpha t} - \frac{1}{2} e^{-2\alpha(T-t)} + \frac{1}{2} e^{-2\alpha T} \right) + \sigma \int_0^t e^{-\alpha(T-\tau)} dW_\tau$$

Since $r_t = f(t,t)$,

$$r_t = f(0,t) + \frac{\sigma^2}{2\alpha^2}(1 - e^{-\alpha t})^2 + \sigma \int_0^t e^{-\alpha(t-\tau)} dW_\tau$$

(d) Derive the risk neutral dynamics $\frac{dZ(t,T)}{Z(t,T)}$ in terms of $r_t$ and the parameters $\sigma$ and $\alpha$.

**Commentary on Question:**
Approximately 1/4 of the candidate attempted this question. Most of them were able to state the correct relationship, but failed to work out the final answer. Only a few candidates got full marks on this one.
12. Continued

\[
\frac{dZ(t, T)}{Z(t, T)} = r_t dt + \sigma_Z(t, T) dW_t
\]

Equating Formula 21.53 and 21.54:

\[
\sigma_Z(t, T) \frac{\partial \sigma_Z(t,T)}{\partial T} = m(t, T) = \sigma e^{-\alpha (T-t)} \int_t^T \sigma e^{-\alpha (\tau-t)} d\tau
\]

Equating Formula 21.53 and 21.54, and using result from (b):

\[
\sigma_f(t, T) = - \frac{\partial \sigma_Z(t,T)}{\partial T} = \sigma e^{-\alpha (T-t)}
\]

\[
\sigma_Z(t, T) = - \int_t^T \sigma e^{-\alpha (\tau-t)} d\tau
\]

\[
\sigma_Z(t, T) = - \frac{\sigma}{\alpha} (1 - e^{-\alpha (T-t)})
\]

\[
\frac{dZ(t, T)}{Z(t, T)} = r_t dt - \frac{\sigma}{\alpha} (1 - e^{-\alpha (T-t)}) dW_t
\]

(e) Determine the \( Q^S \) dynamics of \( P(t,T;S) \) where \( Q^S \) is the equivalent measure associated with \( Z(t,S) \) as the numeraire.

**Commentary on Question:**

*Most candidates did not answer this question, and none got full marks. Partial credits were given to those mentioned the martingale, and the use of Girsanov’s theorem to eliminate the drift term.*

Under the T–forward risk neutral dynamics, the forward price for delivery at S is a martingale. (page 710 of chapter 21):

\[
P(S,T) = \frac{Z(t,S)}{Z(t,T)} \text{ is a martingale with respect to } Q^T.
\]

We can always use Girsanov’s theorem to eliminate the drift term.
12. Continued

Using Ito’s lemma,

\[
\frac{dP(S,T)}{P(S,T)} = \frac{dZ(t,S)}{Z(t,S)} - \frac{dZ(t,T)}{Z(t,T)} - \frac{dZ(t,T) dZ(t,S)}{Z(t,T) Z(t,S)} + \left(\frac{dZ(t,T)}{Z(t,T)}\right)^2
\]

Only the first two terms generate terms outside of the drift, use the result from above:

\[
dP(S,T) = drift + \left(-\frac{\sigma}{\alpha}(1 - e^{-\alpha(S-t)})dW_t\right) - \left(-\frac{\sigma}{\alpha}(1 - e^{-\alpha(T-t)})dW_t\right)
\]

\[
= drift - \frac{\sigma}{\alpha}(e^{-\alpha(T-t)} - e^{-\alpha(S-t)})dW_t
\]

For some Brownian motion \(W^T\) respect to \(Q^T\),

\[
dP(S,T) = -\frac{\sigma}{\alpha}(e^{-\alpha(T-t)} - e^{-\alpha(S-t)})dW^T_t
\]
13. **Learning Objectives:**

3. The candidate will understand:
   - The Quantitative tools and techniques for modeling the term structure of interest rates.
   - The standard yield curve models.
   - The tools and techniques for managing interest rate risk.

**Learning Outcomes:**

(3d) Describe the practical issues related to calibration, including yield curve fitting.

(3f) Apply the models to price common interest sensitive instruments including: callable bonds, bond options, caps, floors, and swaptions.

**Sources:**


**Commentary on Question:**

*Commentary listed underneath question component.*

**Solution:**

(a) Identify which models give positive probability to negative rates.

**Commentary on Question:**

*This is a basic recall question, and most candidates understood which models were normal vs lognormal.*

Vasicek, Ho-Lee, and one factor Hull-White give positive probability to negative rates.

(b) Identify which models could not be fitted to the term structure of interest rates exactly and explain why.

**Commentary on Question:**

*This was another basic recall question, and most candidates understood the limitations of the models. However, some candidates had difficulty explaining why fitting was not possible.*

The Vasicek model does not perfectly fit the term structure of rates.

This is because it is a simple model with only three parameters and one driving source of risk.
13. Continued

(c) Write down an expression for the annualized volatility $\sigma_t(\tau)$ of the changes $dr_t(\tau)$ in long-term bond yield $r_t(\tau)$ for the one-factor Hull-White model.

Commentary on Question:
This was a basic recall question and most candidates did well.

The volatility of long term bond yield in one-factor Hull-White model is given by:

$$\sigma_t(\tau) = \frac{B(\tau)}{\tau} \sigma$$

Where $B(\tau) = \frac{1 - \exp(-\gamma^* \tau)}{\gamma^*}$

(d) Show that $\sigma_t(\tau)$ in part (c) increases as $\tau$ increases when $\gamma^* < 0$ and it decreases as $\tau$ increases when $\gamma^* > 0$.

Commentary on Question:
There are many possible solutions, and full marks were awarded for any proof that was complete and correct.

Most candidates argued heuristically (i.e. a word argument only) or did not have a sound mathematical/logical approach. As a result, most candidates did poorly on this question.

Showing that a function increases or decreases can be demonstrated by taking a derivative, i.e.

$$\frac{d}{d\tau} \sigma_t(\tau) = \left(\frac{B'(\tau)}{\tau} - \frac{B(\tau)}{\tau^2}\right) \sigma$$

$$\frac{d}{d\tau} \sigma_t(\tau) = \left(\frac{e^{-\gamma^* \tau} - \frac{1}{\gamma^*} (1 - e^{-\gamma^* \tau})}{\tau^2}\right) \sigma$$

$$\frac{d}{d\tau} \sigma_t(\tau) = \left(\frac{\gamma^* e^{-\gamma^* \tau} - 1 + e^{-\gamma^* \tau}}{\gamma^* \tau^2}\right) \sigma$$
13. Continued

\[ \frac{d}{d\tau} \sigma_t(\tau) = \left( \frac{e^{-\gamma^*\tau}(\gamma^*\tau - e^{\gamma^*\tau} + 1)}{\gamma^*\tau^2} \right) \sigma \]

Clearly \( e^{-\gamma^*\tau} > 0 \) and \( \frac{1}{\tau^2} > 0 \)

The other remaining term \( \gamma^*\tau - e^{\gamma^*\tau} + 1 \) is always negative. This can be easily shown using a first and second derivative test, or more simply, by recognizing that the Taylor series expansion of \( e^{\gamma^*\tau} \) will negate the other two terms and continue with a leading negative.

The last component of the derivative is \( \frac{1}{\gamma^*} \).

Given the above arguments, when \( \gamma^* < 0 \), the derivative is always positive, hence \( \sigma_t(\tau) \) increases.

Likewise, when \( \gamma^* > 0 \), the derivative is always negative, hence \( \sigma_t(\tau) \) decreases.

(e)

(i) Plot the volatility term structure for the above two models on a reproduction of the above graph.

(ii) Explain the relative positions of the three curves.

(iii) Describe the expected behavior of long-term bond yield under the one-factor Hull-White model.

Commentary on Question:

This question was designed to assess candidates’ understanding of model fitting. Note that the volatility data given in this question is an inversion of the example volatility data given in Veronesi, p 661.

For part (i), despite the inverted data, most candidates responded with the original model fittings as seen in the Veronesi text (e.g. downward sloping HW curve). As a result, most candidates did poorly.

For part (ii), most candidates could not justify their answers (as there would be no adequate justification for using the fittings from the Veronesi text) and did poorly.

For part iii), most candidates responded with a memorized answer (i.e. mean-reversion); however, this failed to recognize the connection that this question has to part d), and as a result, most candidates did poorly.
13. Continued

Roughly speaking, the two models would produce term structures as follows:

![Graph showing term structures for Ho-Lee and Hull-White models]

The Ho-Lee model assumes the volatility of long-term rates is equal to the short term rate. This is why it is flat, and at 1.4.

The Hull-White model has extra parameters to allow fitting (though differences need to be minimized, hence a mid-fit).

As the HW fitting is upward sloping, by part d), it must be the case that $\gamma^* < 0$. In other words, this process is mean-averting, or an explosive process.
14. Learning Objectives:
1. The candidate will understand the foundations of quantitative finance.

4. The candidate will understand:
   • How to apply the standard models for pricing financial derivatives.
   • The implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory.
   • How to evaluate risk exposures and the issues in hedging them.

Learning Outcomes:
(1b) Understand the importance of the no-arbitrage condition in asset pricing.

(4a) Demonstrate an understanding of option pricing techniques and theory for equity derivatives.

(4b) Identify limitations of the Black-Scholes-Merton pricing formula

(4d) Demonstrate an understanding of how to delta hedge, and the interplay between hedging assumptions and hedging outcomes.

(4e) Analyze the Greeks of common option strategies.

Sources:
QFIC 114-17

The Volatility Smile, Derman, Ch. 5

Pricing and Hedging Financial Derivatives, Marroni, Ch. 6

Commentary on Question:
This question intends to test candidates’ understanding of no-arbitrage condition in asset pricing, as well as the concept of delta hedging and Greeks of common option strategies.

Solution:
(a) Calculate the installment of flat fees the company should charge the policyholder at the beginning of each year. Describe the assumptions you used for the calculation.

Commentary on Question:
The candidates did not do well on this question. Most of those who attempted this question were able to identify that GMAB is equivalent to put option. Few of them calculated the installment correctly.
Very few candidates listed the assumption for Black-Scholes framework.
14. Continued

Use Black-Scholes pricing formula to calculate the put option price and list the assumptions for Black Scholes framework,

\[ S = X = 2500, T = 5, r = 3\% \text{ and Implied Vol} = \sigma = 15\% \]

\[
d_1 = \frac{\ln \left( \frac{S}{X} \right) + \left( r + \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}}
\]

\[
d_2 = d_1 - \sigma \sqrt{T}
\]

\[
P = X e^{-rT} N(-d_2) - SN(-d_1) = 165.77
\]

\[ Annuity \ Factor(r = 3\%, T = 5) = \frac{1 - e^{-rT}}{1 - e^{-r}} = 4.71 \]

\[
Premium = \frac{P}{Annuity \ Factor(r = 3\%, T = 5)} = 35.17 \text{ for each share}
\]

So the installment of flat fees \( = 35.17 \times \frac{100,000}{2,500} = 1406.93 \)

The assumptions used to calculate the premium

a) Hedging is continuous
b) There are no transaction costs
c) Volatility is constant
d) There are no arbitrage opportunities
e) The underlying is lognormally distributed
f) There are no costs associated with borrowing stock for going short

(b) Show that

\[
\frac{\partial P}{\partial S} = -N(-d_1)
\]

where \( N(\cdot) \) is the cumulative normal distribution.

**Commentary on Question:**

*The candidates did well on this question. Most candidates could utilize put call parity and chain rule while taking the partial derivative w.r.t. \( S \). Full marks could be obtained even if the steps were not as detailed as the solution given here.*
14. Continued

According to put call parity, \( C + Ke^{-rT} = P + S \)

\[
\frac{\partial P}{\partial S} = \frac{\partial}{\partial S} \left[ Ke^{-rT}N(-d_2) - SN(-d_1) \right]
\]

\[
= Ke^{-rT} \frac{\partial N(-d_2)}{\partial S} - N(-d_1) - S \frac{\partial N(-d_1)}{\partial S}
\]

\[
= -Ke^{-rT} \frac{\partial N(-d_2)}{\partial d_2} \frac{\partial d_2}{\partial S} - N(-d_1) + S \frac{\partial N(-d_1)}{\partial d_1} \frac{\partial d_1}{\partial S}
\]

\[
= -Ke^{-rT} N'(-d_2) \frac{1}{S\sigma\sqrt{T}} - N(-d_1) + SN'(-d_1) \frac{1}{S\sigma\sqrt{T}}
\]

\[
= -N(-d_1) + \frac{1}{S\sigma\sqrt{T}} \left[ SN'(-d_1) - Ke^{-rT} N'(-d_2) \right]
\]

\[
Ke^{-rT} N'(-d_2) = Ke^{-rT} \frac{1}{\sqrt{2\pi}} e^{-0.5d_2^2}
\]

\[
= Ke^{-rT} \frac{1}{\sqrt{2\pi}} e^{-0.5(d_1-\sigma\sqrt{T})^2}
\]

\[
= Ke^{-rT} \frac{1}{\sqrt{2\pi}} e^{-0.5(d_1^2-2d_1\sigma\sqrt{T}+\sigma^2T)} = Ke^{-rT} \frac{1}{\sqrt{2\pi}} e^{-0.5d_1^2+2d_1\sigma\sqrt{T}-0.5\sigma^2T}
\]

\[
= Ke^{-rT} N'(-d_1)e^{d_1\sigma\sqrt{T}} e^{-0.5\sigma^2T}
\]

\[
= Ke^{-rT} N'(-d_1)e^{\ln(S/K)+(r+0.5\sigma^2)T} e^{-0.5\sigma^2T}
\]

\[
= Ke^{-rT} N'(-d_1) \frac{S}{K} e^{(r+0.5\sigma^2)T} e^{-0.5\sigma^2T}
\]

\[
= SN'(-d_1)
\]

Therefore, \( \frac{\partial P}{\partial S} = -N(-d_1) \)

(c) Describe the five Greeks given above and their contributions to the P&L.

Commentary on Question:
Most candidates were able to describe the five Greeks. However, not many candidates described the Greeks contributions to the P&L well, especially Gamma and Theta. Among those who attempted this question, most of them failed to evaluate the Greeks contribution to P&L from the option writer’s perspective, therefore, the positive/negative contributions from Gamma and Theta were reversed.
14. Continued

Delta, measure the change of option value respective to the underlying security value change, or delta is the first derivative of the price of an option with respect to the price of the underlying security, since delta is hedged, deltal contribution is 0.

Gamma, measure the sensitivity of delta respective to the price of underlying security, since the hedging is performed on daily basis rather than continuously as assumed in Black Scholes world, the change of delta day to day will have an impacts on hedge effectiveness. The insurance company is the option writer, therefore delta hedging the product is equivalent to shorting a put option, in which case Gamma is always negative.

Rho, the first derivative of the option price with respect to the change of interest rate, since interest rate remained constant over the month, it has 0 impacts on the P&L.

Vega, the first derivative(sensitivity) of the option price with respect to the volatility of the underlying asset. Since implied volatility remained constant over the month, it has 0 impact on the P&L.

Theta, the first derivative(sensitivity) of the option price with respect to the time to maturity of the option. For the insurance company, as an option writer, Theta is typically positive.

(d) Identify the two Greeks which contributed the above amounts to the P&L.

Commentary on Question:
Among the candidates who attempted this question, the positive/negative contributions from Gamma and Theta were often reversed by remembering the sign of Greeks for a long position of the option instead of a short position.

Gamma contribution is -180. Theta is 574.

(e) Identify whether each of the following statements is true or false. Briefly justify your answer.

(i) The Delta of a European put option that is out-of-the-money will converge to 0 as the expiration date approaches.

(ii) For an out-of-the-money option with the underlying having low volatility, if the volatility increases, the Delta of the option increases.

Commentary on Question:
The candidates did well on this question. To get full marks, candidate should clearly state whether each of the two statements is true or false, and provide explanations to justify the statements.
14. Continued

(i) True.
If the put option is out-of-the-money, the delta will tend towards 0. In this case, the option will not be exercised, and consequently, there will be no dependency of the option price on the price of the underlying security.

(ii) True.
Due to the low volatility, the probability that the underlying price will cross the strike before the expiry is relatively low. If the volatility starts to increase, we would expect a higher probability of the option expiring in-the-money. This increases the dependency of the option price to the price of the underlying, and thus it increases the delta of the option.
15. **Learning Objectives:**
   1. The candidate will understand the foundations of quantitative finance.
   5. The candidate will understand important quantitative techniques relating to financial time series, volatility modeling, and stochastic modeling.

**Learning Outcomes:**
(1h) Define and apply the concepts of martingale, market price of risk and measures in single and multiple state variable contexts.

(1i) Demonstrate understanding of the differences and implications of real-world versus risk-neutral probability measures, and when the use of each is appropriate.

(5e) Demonstrate an understanding of the general uses and techniques of stochastic modeling.

**Sources:**
Economic Scenario Generators: A Practical Guide, SOA, 2016, Chapters 1, 5-7, 9, 10

**Commentary on Question:**
*Commentary listed underneath question component.*

**Solution:**
(a) Determine the type(s) of ESG simulation needed for each of the three applications above, and justify your choices.

**Commentary on Question:**
The candidates received credits for determining the right type of scenario and for providing the justification for each application. Most candidates did well on determining the right type of scenario for each application, but some struggled on the justifications.

Application 1:
- The fair value of the embedded investment guarantee is the market-consistent value of a derivative (usually a put type option) instrument
- So risk-neutral/market-consistent ESG is required

Application 2:
- The measurement of the effectiveness of a hedging program will often involve the application of both real-world and market-consistent simulations. In the outer loop of a nested stochastic model for the economic capital calculation, the hedging strategy needs to be modeled along each real-world path which aims to represent the state of the world. While the inner loop of the nested stochastic model, cash flows of hedging are priced in that state of the world using risk-neutral scenarios projected forward from the current node of the simulation
15. Continued

- In this case, the real-world simulation is used to assess the overall risk of the variable annuity book and to measure the effectiveness of the hedging strategy.

Application 3:
- A traded American put option is a financial derivatives that has investment guarantees and its historical and current market prices are observables.
- So, a risk-neutral ESG is required.

(b) The average present values of the equity accumulation factors are as below:

<table>
<thead>
<tr>
<th>Year 1</th>
<th>Year 2</th>
<th>Year 3</th>
<th>Year 4</th>
<th>Year 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k$</td>
<td>80.02%</td>
<td>89.26%</td>
<td>90.17%</td>
<td>86.55%</td>
</tr>
</tbody>
</table>

Calculate $k$ in the table above.

**Commentary on Question:**
Most candidates did well on this part.

$K$ is the average of the present value of asset price at end of year 1 discounted back to time zero, across all the scenarios.

\[ k = \frac{1}{M} \sum_{i=1}^{M} X_{\tau}(i) v_{\tau}(i) \]

\[ k = \frac{(95.2\% \times 93.7\% + 100.8\% \times 95.9\% + 119.5\% \times 97.5\% + 75.5\% \times 98.8\% + 90.2\% \times 97.1\%)}{5} \]

\[ = 92.61\% \]

(c) Assess whether this scenario set passes the martingale test.

**Commentary on Question:**
In general, the candidates did well on this part. Some did conclude failing of the martingale test but did not provide any explanation.

- The series of expected time zero asset price is as below: 92.61%, 80.02%, 89.26%, 90.17%, 86.55%
- If the series is martingale, the expected present value of the asset prices should be equal to the price at time zero
- The time zero price of the asset is 1.
- Since the martingale test is based on random sampling, it is acceptable that there is random fluctuation, but the average of the discounted asset prices at all periods should be close to one
- Therefore the scenario is not martingale (or it fails the martingale test).
15. Continued

(d) Determine \( s(5) \) and explain its purpose.

**Commentary on Question:**

*In general, the candidates who knew how to calculate the asset price at year 5 could explain the purpose. Partial credit was deducted if candidates did not provide the purpose of \( s(5) \).*

- The accumulation factor can be expressed as

\[
g'(t) = e^{\mu(t)T} = e^{(\mu(T) + s(T))T} = e^{s(T)T}g(T)
\]

- The present value of asset price at end of year 5 is thus 

\[e^{s(5)\times 5 \times 86.55\%} = 100\%\]

- \( s(5) = \frac{1}{5} \ln\left(\frac{1}{86.55\%}\right) = 2.89\% \)

- The purpose of \( S(5) \) is to adjust the equity accumulation factors so that the new factors pass the Martingale test, and can be used for pricing in a risk neutral fashion

(e) Determine the time zero price of the option with \( K = 1 \) using the appropriate scenario data above.

**Commentary on Question:**

*Most candidates had problems with this part. Very few of them did the calculations correctly. Majority of the candidates did not know how to adjust the accumulation factors, and just used the average of the unadjusted factors; therefore, they received partial credits. Quite a few candidates just used the average of the accumulation factors from the new set.*

- The option pricing should be using risk-neutral scenario, which passes the martingale test
- The new equity accumulation factors should be used
- The adjustment on the accumulation factor for year 4 is \( 100.20\% / 90.17\% = 1.1112 \)
- The adjusted accumulation factors for year 4 are calculated as:
  - \( 141.90\% \times 1.1112 = 157.68\% \),
  - \( 100.70\% \times 1.1112 = 111.90\% \),
  - \( 99.80\% \times 1.1112 = 110.90\% \),
  - \( 75.50\% \times 1.1112 = 83.90\% \),
  - \( 105.70\% \times 1.1112 = 117.46\% \)
15. Continued

- The adjustment on the accumulation factor for year 5 is 100% / 86.55% = 1.1554
- The adjusted accumulation factors for year 5 are calculated as:
  - 116.40% * 1.1554 = 134.49%,
  - 125.40% * 1.1554 = 144.89%,
  - 98.20% * 1.1554 = 113.46%,
  - 90.2% * 1.1554 = 104.22%,
  - 93.00% * 1.1554 = 107.45%
- The payoffs are calculated as 46.09%, 28.40%, 12.18%, 0%, 12.46%
- The present value of the payoff is thus calculated as
  \[
  C_0 = \frac{1}{5} (0.46087473 \times 82.70\% + 0.283953678 \times 78.70\% + 0.121816511 \times 84.40\%
  + 0 \times 83.00\% + 0.124557176 \times 86.07\%)
  = 0.1629
  \]
16. **Learning Objectives:**
5. The candidate will understand important quantitative techniques relating to financial time series, volatility modeling, and stochastic modeling.

**Learning Outcomes:**
(5c) Apply various techniques for analyzing factor models including principal component analysis (PCA) and statistical factor analysis.

**Sources:**
Fixed Income Securities: Valuation, Risk, and Risk Management, Veronesi, Pietro, 2010, Ch. 4.6

QFIA-125-16 (or QFIQ-119-19): Market Models: A Guide for Financial Data Analysis, Ch. 6, Principal Component Analysis

**Commentary on Question:**
The question tests candidates’ understanding of PCA from the basics to more sophisticated knowledge utilization. Overall the candidates did well in parts (a) to (d), demonstrating a fairly good idea of what PCA does and key components it’s used to identify. Later parts were more challenging, where part (f) saw the lowest marks awarded.

**Solution:**
(a) Identify two benefits of PCA.

**Commentary on Question:**
The candidates generally did well on part (a). A large portion of candidates were able to identify the dimension reduction benefit. In light of part (e), most candidates identified one of the benefits as “to resolve multicollinearity,” which was accepted as a correct answer, as they were able to relate to orthogonal/uncorrelated nature of the principal components.

The benefits of PCA include the following:
(to earn full marks, 2 out of the 3 will suffice)

1. PCA provides an intuitive way to explain the change in the financial data using the trend, tilt and convexity factors;
2. PCA allows to explain a large portion of the variation in the financial data using fewer factors (reduced dimensions);
3. The principal components are uncorrelated of each other, which makes risk management easier to implement. (Resolution to multicollinearity issue, orthogonal components are both acceptable here)
16. Continued

(b) Calculate $\lambda_4$ and $\lambda_5$.

Commentary on Question:
The candidates generally did well on part (b). Partial credits were awarded in the case where candidates did not get the correct final answer but demonstrated knowledge of the relationship between the proportion of variation explained and the sum of the corresponding eigenvalues.

The proportion of variation explained by the first $n$ principal components together is $\sum_{i=1}^{n} \frac{\lambda_i}{k}$

Thus, $\lambda_4 = k \left(0.9924 - \left(\sum_{i=1}^{3} \frac{\lambda_i}{k}\right)\right) = 8(0.9924 - 0.9831) = 0.0742$

To find $\lambda_5$ we use the fact that $\sum_{i=1}^{k} \frac{\lambda_i}{k} = 1$, that is, $\sum_{i=1}^{8} \lambda_i = 8$

Therefore

$$\lambda_5 = 8 - (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4) - (\lambda_6 + \lambda_7 + \lambda_8)$$

$$= 8 - 8 \times 0.9924 - (0.0221 + 0.0027 + 0.0003) = 0.0357$$

(c) Determine the minimum number of principal components for meeting the given objective.

Commentary on Question:
Most candidates were able to identify that the first two components explained over 96% variation. Partial credits were awarded when candidates did not verify that the first component did not achieve the 96% target.

Solution 1:
The variation explained by the first component/factor is its eigenvalue divided by the total number of variables (8), thus $6.7432/8 = 0.8429$ or 84.29%

The variation explained by the second component/factor is $0.9823/8 = 0.1228$ or 12.28%

The variation explained by the first and second components $= 84.29\% + 12.28\% = 96.57\% > 96\%$

As a result the number of principal components required for meeting the given objective is 2.
16. Continued

Solution 2:
The proportion of variation explained by the first n principal components together is \[ \sum_{i=1}^{n} \frac{\lambda_i}{K} \]

For a required portion of variance explained of 96%, the minimum required sum of the eigen values is \( 8 \times 0.96 = 7.68 \)

Start from the first PC: \( \lambda_1 = 6.7432 < 7.68 \)

Adding the second PC: \( \lambda_1 + \lambda_2 = 6.7432 + 0.9823 = 7.7255 > 7.68 \)

Therefore the number of PC required to meet the object is 2.

(d) Interpret the meanings of the first and second principal components of your yield curve factor model.

**Commentary on Question:**
Most candidates were able to identify the names of the first and second components. The candidates were expected to elaborate on the behavior of the yield curve in the question to earn full marks.

The first principal component is the trend component, or level factor and also called a “shift”. (Note: any of the three names is acceptable.)

It represents parallel increase in all rates on the yield curve since the weights in first column are positive and in relatively similar scale;

The second principal component is the tilt component, or slope factor.

It represents a drop in the short-term rates and rise in the longer-term rates since the weights in second column are first negative and then positive.

(e) Describe the next steps of the orthogonalization of variables using PCA to deal with multicollinearity issue.

**Commentary on Question:**
Most candidates who attempted to solve this question were able to identify the key steps in descriptive terms without expressing the formulas, in which case partial credits were awarded.
16. Continued

The following key steps need to be taken:

Step 1: Perform an OLS on the principal components \( P_1, P_2, P_3 \) to obtain the intercept \( a \) and slope coefficients \( b = (b_1, b_2, b_3)' \)

\[
\Delta (\sigma_K - \sigma_{ATM}) = a + Pb + \varepsilon
\]

where \( P = (P_1, P_2, P_3) \)

Step 2: Convert the estimates back to express the dependent variable in form of original explanatory variables \( X_i \)

\[
\Delta (\sigma_K - \sigma_{ATM}) = c +Xd + \varepsilon
\]

where \( X = (X_{0.9}, X_{1.1}, X_{1.2}) \)

\[ d = \Sigma W b \]

\( \Sigma \) is the diagonal matrix with \( \frac{1}{\sigma_i} \), where \( \sigma_i \) is the standard deviation of \( X_i \)

\( W \) is the matrix of the factor weights

\[ c = a + \mu \] where \( \mu = (\mu_1, \mu_2, \mu_3)' \) and \( \mu_i \) is the mean of \( X_i \)

(f) Determine \( \gamma \).

Commentary on Question:
The candidates did poorly on part (f). The question is not as computational intense as it seems, as most terms were designed to cancel out. Unfortunately, only a few candidates attempted to work out this part. Most of the candidates who attempted the question failed to recognize the correct formula to solve.

Following the expression in (e)

\[
\beta_1 = \frac{1}{\sigma_{0.9}} * (w_{0.9,1}, w_{0.9,2}, w_{0.9,3}) * b
\]

\[
b = (1, \left( \frac{1-e^{-\gamma \tau}}{\gamma \tau} \right) - \gamma, 0.4 \left( \frac{1-e^{-\gamma \tau}}{\gamma \tau} - \gamma \right) - e^{-\gamma \tau})'
\]

where \( b \) is the OLS coefficients

Given that

\[
(w_{0.9,1}, w_{0.9,2}, w_{0.9,3}) = (0.02, 0.064, -0.16)
\]

\[
\frac{1}{10\%} * \left( 0.02 + 0.064 \left( \frac{1-e^{-\gamma \tau}}{\gamma \tau} \right) - \gamma \right) - 0.16 \left[ 0.4 \left( \frac{1-e^{-\gamma \tau}}{\gamma \tau} - \gamma \right) - e^{-\gamma \tau} \right]
\]

\[ = 1.2 \]
Continued

Solving for $\gamma$ gives

$$\left(1.6e^{-2\gamma}\right) = 1$$

And thus $\gamma = 0.235$

(g) Critique your supervisor’s suggestion and explain why your model should be used.

**Commentary on Question:**

*The candidates did not generally do well on this part. Some of those who tried were able to identify the noise issue and earned partial credits.*

The alternatives provided by the supervisor is not as robust as the fixed strike deviations because of the following two benefits:

1. Empirically, time series data on fixed strikes or fixed delta volatilities often have very negative autocorrelation. On the other hand, daily variations in fixed strike deviations are much less noisy.

2. Theoretically, it has been shown by Derman’s model that the relationship between fixed strike deviations and underlying price did not depend on the market regime (whether the market was trending, range-bounded, or jumpy).