QFI Quantitative Finance Exam

Exam QFIQF

MORNING SESSION

Date: Tuesday, April 28, 2020
Time: 8:30 a.m. – 11:45 a.m.

INSTRUCTIONS TO CANDIDATES

General Instructions

1. This examination has a total of 100 points. It consists of a morning session (worth 60 points) and an afternoon session (worth 40 points).

   a) The morning session consists of 9 questions numbered 1 through 9.

   b) The afternoon session consists of 6 questions numbered 10 through 15.

   The points for each question are indicated at the beginning of the question.

2. Failure to stop writing after time is called will result in the disqualification of your answers or further disciplinary action.

3. While every attempt is made to avoid defective questions, sometimes they do occur. If you believe a question is defective, the supervisor or proctor cannot give you any guidance beyond the instructions on the exam booklet.

Written-Answer Instructions

1. Write your candidate number at the top of each sheet. Your name must not appear.

2. Write on only one side of a sheet. Start each question on a fresh sheet. On each sheet, write the number of the question that you are answering. Do not answer more than one question on a single sheet.

3. The answer should be confined to the question as set.

4. When you are asked to calculate, show all your work including any applicable formulas. When you are asked to recommend, provide proper justification supporting your recommendation.

5. When you finish, insert all your written-answer sheets into the Essay Answer Envelope. Be sure to hand in all your answer sheets because they cannot be accepted later. Seal the envelope and write your candidate number in the space provided on the outside of the envelope. Check the appropriate box to indicate morning or afternoon session for Exam QFIQF.

6. Be sure your written-answer envelope is signed because if it is not, your examination will not be graded.

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****BEGINNING OF EXAMINATION****
Morning Session

1. (*5 points*) You are working in the product development department of an insurance company that sells equity-linked annuities. Your manager assigns you to investigate exotic options.

(a) (*1 point*) Describe three common differences between exotic options and standard options.

Assume that X_t and Y_t are stochastic processes satisfying the following stochastic differential equations:

$$dX_t = \mu_1 X_t dt + X_t \left(\alpha \, dW_t^{(1)} + \beta \, dW_t^{(2)}\right), \; X_0 = 1$$
$$dY_t = \mu_2 Y_t dt + Y_t \left(\beta \, dW_t^{(1)} + \gamma \, dW_t^{(2)}\right), \; Y_0 = 1$$

where $\mu_1, \mu_2, \alpha, \beta, \gamma$ are positive constant parameters, $W_t^{(1)}$ and $W_t^{(2)}$ are two independent standard Wiener processes.

Let

$$B_t = \frac{(\alpha+\beta)W_t^{(1)} + (\beta+\gamma)W_t^{(2)}}{\sqrt{(\alpha+\beta)^2 + (\beta+\gamma)^2}}$$

$$S_t = \sqrt{X_t Y_t}$$

(b) (*1.5 points*) Derive the stochastic differential equation of S_t.

(c) (*1.5 points*) Show that B_t is a standard Wiener process.

(d) (*1 point*) Derive the condition on the parameters under which S_t is a martingale with respect to the filtration generated by B_t.

2. (*5 points*) Let $W(t)$ be a standard Wiener process. Also let f and g be two square integrable adapted processes such that the two Ito integrals $\int_0^t f(u)\,dW(u)$ and $\int_0^t g(u)\,dW(u)$ are well-defined.

(a) (*1.5 points*) Apply Ito's isometry to prove that

$$E\left[\left(\int_0^t f(u)\,dW(u)\right)\left(\int_0^t g(u)\,dW(u)\right)\right] = E\left[\int_0^t f(u)\,g(u)\,du\right].$$

Now, denote $X(t) = \int_0^t sgn\big(W(s)\big)\,dW(s)$, where $sgn(x) = \begin{cases} 1, & if\ x \geq 0 \\ -1, & if\ x < 0 \end{cases}$.

(b) (*2.5 points*) Show using the result in part (a) or otherwise, that

(i) $E\left[X(t)W(t)\right] = 0$.

(ii) $E\left[X(t)W^2(t)\right] = \frac{4}{3}\sqrt{\frac{2t^3}{\pi}}$.

Hint: $E\left[\left|W(s)\right|\right] = \sqrt{\frac{2s}{\pi}}$.

(c) (*1 point*) Show that $E\left[\left|X(t)W(t)\right|\right] \leq t$.

3. (*7 points*) Let $\left(\Omega, \{\mathcal{F}_t\}, \mathcal{F}, \mathbb{P}\right)$ be a filtered probability space, $W(t)$ be a standard Brownian motion, and $f(t)$ be a continuous and deterministic function such that the Ito integral

$$I(t)=\int_0^t f(s)\, dW(s)$$

is well defined.

For any real number u and $0 \le t_1 < t_2$, let

$$M_u(t_1,t_2)=\exp\left\{u\left[I(t_2)-I(t_1)\right]-\frac{1}{2}u^2\int_{t_1}^{t_2} f^2(s)ds\right\}.$$

(a) (*2.5 points*)

(i) State the three conditions for a process to be a martingale.

(ii) Show that, for a fixed u, $\{M_u(0,t)\}$ satisfies those conditions.

(b) (*2 points*) Show that $\mathbb{E}\left[M_u(t_1,t_2)\middle|\mathcal{F}_{t_1}\right]=1$.

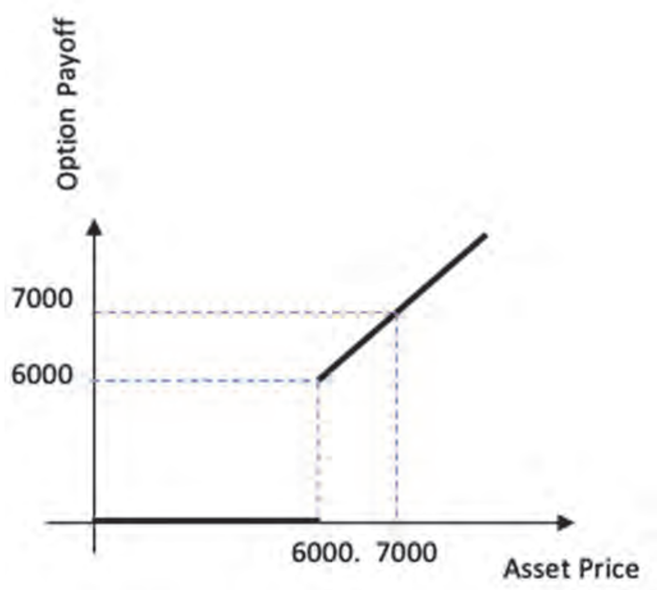
(c) (*2 points*) Show that $I(t_1)$ and $I(t_2)-I(t_1)$ are independent normal random variables. Identify their means and variances.

(d) (*0.5 points*) Find $\text{Cov}\left(I(t_1), I(t_2)\right)$.

4. (*7 points*) You are considering buying an asset-or-nothing European call option with the following characteristics:

- Underlying asset's spot price S_0 is 1,000
- Strike price K of the option is 6,000
- Annualized volatility σ of the underlying asset is 40%
- Risk-free rate r is 2% per annum
- Time-to-expiration T is 1 year

The annualized rate of return of the underlying asset is 7% and no dividend is paid. The payoff diagram of the asset-or-nothing European call option is given below:



(a) (*3 points*)

(i) Write the arbitrage-free option price C as an expectation with respect to the risk-neutral probability measure $\mathbb{Q}$.

(ii) Derive the formula $C = S_0 N(d_1)$ where $d_1 = \dfrac{\ln\left(\dfrac{S_0}{K}\right) + \left(r + \dfrac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$.

(iii) Calculate C using the above formula.

4. **Continued**

Your coworker thinks that she can accurately price the option by changing the probability measure $\mathbb{Q}$ to an equivalent measure $\mathbb{P}$, under which the option becomes more likely to be in-the-money.

(b) (*1.5 points*)

(i) State how Girsanov's Theorem can be used in this case.

(ii) Show that the adapted process on the probability space $(\Omega, I_t, \mathbb{Q})$ defined by $X_t = d_1$ satisfies the Novikov condition with d_1 being the constant calculated in part (a).

(c) (*1 point*)

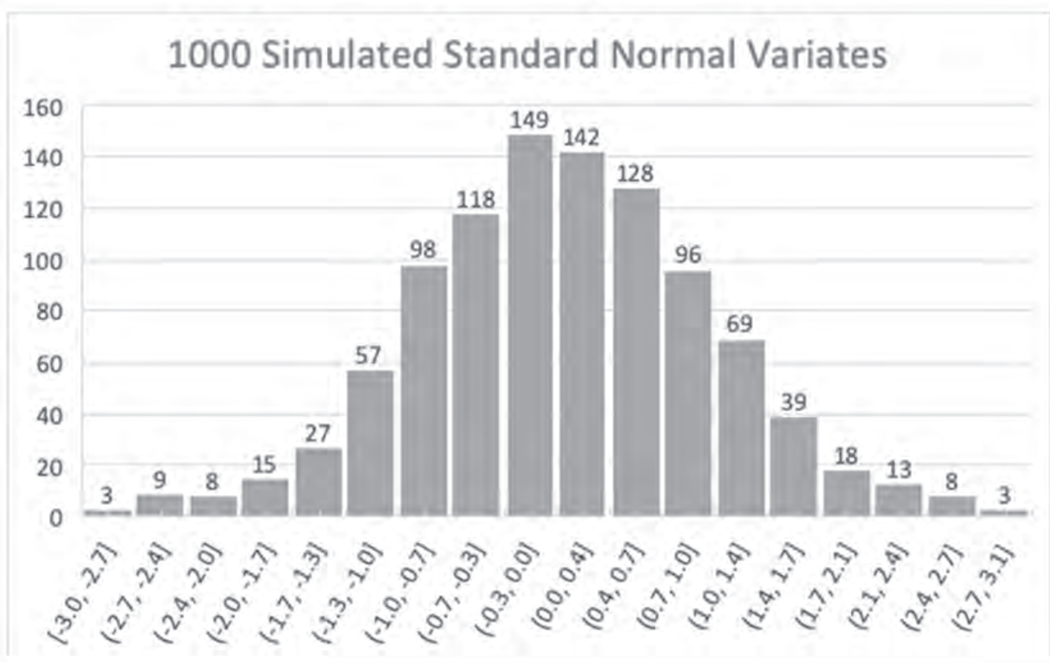
(i) Define the probability measure $\mathbb{P}$.

(ii) Express $N(d_1)$ as an expectation with respect to $\mathbb{P}$.

Question 4 continued on the next page.

4. **Continued**

Your coworker has obtained the simulation results under $\mathbb{P}$ as shown in the graph below and partially completed the calculations to estimate C as shown in the table below.



Normal Variates Set	Left Endpoint	Right Endpoint	Contribution to C
1	-3	-2.7	2.28002E-12
2	-2.7	-2.4	2.43277E-11
3	-2.4	-2	9.50235E-11
4	-2	-1.7	7.82913E-10
5	-1.7	-1.3	6.19252E-09
6	-1.3	-1	5.74461E-08
7	-1	-0.7	3.5128E-07
8	-0.7	-0.3	y
9	-0.3	0	z

(d) (*1.5 points*) Calculate y and z and estimate C using the simulation results and the table above.

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5. (*7 points*) ABC Insurance Company (AIC) has issued a Guaranteed Investment Certificate (GIC) with payments schedule described below.

GIC payments schedule

Time T_i	1	2	3	4	5
Payments (in millions)	1.5	1.5	1.5	1.5	1.5

Current Market Conditions (continuously compounded interest rate of 4%)

Time T_i	Cash flows (CF) (in millions)	Discount factor $Z(0,T_i)$	Discounted cash flows $CF*Z(0,T_i)$	Weight by discounted cashflows W_i	W_i*T_i	$W_i*T_i^2$
1	1.5	0.961	1.441	0.216	0.216	0.216
2	1.5	0.923	1.385	0.208	0.416	0.831
3	1.5	0.887	1.330	0.200	0.599	1.797
4	1.5	0.852	1.278	0.192	0.767	3.070
5	1.5	0.819	1.228	0.184	0.922	4.609

(a) (*1 point*) Calculate the duration and the convexity of the GIC, assuming the term structure of interest rates currently is flat at a continuously compounded rate of 4%.

AIC's Chief Investment Officer (CIO) is worried about losses from nonparallel shifts in the term structure of interest rates. She suggests a hedging strategy of using two zero-coupon bonds (a 2-year zero-coupon bond and a 5-year zero-coupon bond) by matching both the dollar duration and the dollar convexity of the GIC simultaneously.

(b) (*2.5 points*) Construct a hedging portfolio based on the CIO's suggestion.

5. Continued

Revised Market Conditions (continuously compounded interest rate of 3.75%)

Time T_i	Cash flows CF (in millions)	Discount factor $Z(0,T_i)$	Discounted cash flows $CF * Z(0,T_i)$	Weight by discounted cashflows W_i	$W_i * T_i$	$W_i * T_i^2$
1	1.5	0.963	1.445	0.215	0.215	0.215
2	1.5	0.928	1.392	0.207	0.415	0.829
3	1.5	0.894	1.340	0.200	0.599	1.797
4	1.5	0.861	1.291	0.192	0.769	3.078
5	1.5	0.829	1.244	0.185	0.926	4.632

(c) (*1.5 points*) Assess the gains or losses of the hedging portfolio constructed in part (b), if interest rates were to decrease by 25 basis points from 4% to 3.75%.

One analyst working in AIC recommends a barbell-bullet bond portfolio to achieve positive portfolio returns. He also claims that this barbell-bullet trading strategy represents a short-term arbitrage opportunity if interest rates do not move significantly over time.

(d) (*1 point*) Explain how to construct a barbell-bullet bond portfolio.

(e) (*0.5 points*) Explain whether the barbell-bullet bond portfolio can achieve a positive portfolio return when the moves of interest rates are small and in parallel.

(f) (*0.5 points*) Critique the analyst's claim.

6. (*6 points*) You are evaluating two zero-coupon bonds using the Vasicek model. The following market data is given:

Bond	**Face Value**	**Duration**	**Convexity**	**Theta**	**Price**
1-Year	100	0.906	0.821	0.791	98.91
10-Year	100	1.427	2.037	0.178	59.17

The standard deviation of the 1-Year bond rate changes is 2.2%. The current overnight deposit interest rate is 1% per annum.

(a) (*1.5 points*) Show that these values are not consistent with a single factor arbitrage free model of the short rate.

The parameters of the Vasicek model are estimated as below:

$$\bar{r}^* = 0.06$$
$$\gamma^* = 0.7$$

(b) (*2 points*) Calculate the pricing errors of the bonds.

You have decided to execute a relative value trade to exploit the arbitrage opportunity.

(c) (*1.5 points*) Determine the optimal hedge ratio, instrument(s) to trade and position (long or short).

(d) (*0.5 points*) Calculate the initial profit.

(e) (*0.5 points*) Critique the following statement:

"By executing the arbitrage strategy, we are hedged and will reap a guaranteed profit from the position."

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7. (*9 points*) ACL is a public company that pays no dividend. Its end of day stock prices in the past week are given below:

Day	**0**	**1**	**2**	**3**	**4**	**5**
Stock price S	\$1000.00	\$1005.00	\$1025.10	\$1004.60	\$1004.60	\$999.58

Your firm believes that ACL's stock price, S, follows the geometric Brownian motion as:

$$dS = \mu S dt + \sigma S dW \qquad (1)$$

where μ = 8.35% and σ = 16.20%.

(a) (*0.5 points*) Define actual volatility and realized volatility.

(b) (*1.5 points*) Estimate ACL's realized volatility for the 5-day period expressed as annualized volatility assuming 252 trading days per year.

On Day 5, you are given the following information on a European call option on ACL's stock price:

Option's market price	**Option's strike price**	**Option's time-to-maturity**
\$77	\$1,000	1 year

The continuously compounded risk-free interest rate is 2.00% per annum for all maturities.

(c) (*1.5 points*) Determine if ACL's implied volatility is greater than, equal to, or less than its actual volatility.

On Day 5, your firm sold 100 ACL's call options as listed above at the market price and immediately started delta hedging these options by buying 58 shares of ACL stock.

(d) (*1.5 points*) Determine if ACL's implied volatility is greater than, equal to, or less than the volatility your firm used in its delta hedging strategy.

(e) (*1 point*) Outline implications for your firm's hedging profit/loss given your firm's choice of the volatility in its delta hedging strategy.

7. Continued

Assume that

- There is no change in your firm's delta hedging strategy until option maturity.
- Your firm's hedging position is adjusted continuously without any transaction costs.
- ACL's stock price behavior as shown in Equation (1) turns out to be correct.

(f) (*1 point*) Determine the present value of your firm's hedging profit/loss.

Your firm uses Monte Carlo simulations to value a guaranteed benefit that relates to the performance of an equity index (hereafter, the Index). Your firm assumes that the Index value follows a geometric Brownian motion and generates scenarios such that (i) the scenarios reproduce the market price of at-the-money options on the Index as of the valuation date and (ii) the scenario's time step is annual. On a valuation day, you are given:

- The Index value = 1000.
- At-the-money implied volatility of the Index was 17%, 19% and 20% for time-to-maturity of 1 year, 2 years and 3 years, respectively.
- The continuously compounded risk-free interest rate was 2% for all maturities.
- Randomly generated numbers from the standard normal distribution in scenarios were 0.1 for year 1, -0.1 for year 2, and 0 for year 3.

(g) (*2 points*) Determine the simulated Index value at the end of year 3.

8. (*9 points*) As a derivative strategy analyst, you are asked to examine the empirical volatility movements and the corresponding derivative strategies.

(a) (*1 point*) Explain how a straddle strategy and a strangle strategy provide Vega protection with minimal exposure to other market risks.

The current continuous risk-free interest rate is 3% and the time-to-maturity of options is 6 months.

You are given the following information.

Underlying S	100	100	100
Strike K	80	100	120
Volatility σ	28%	20%	25%
d_1	1.302	0.177	-0.858
d_2	1.104	0.035	-1.035
$N(d_1)$	0.904	0.570	0.195
$N(d_2)$	0.865	0.514	0.150
$N(-d_1)$	0.096	0.430	0.805
$N(-d_2)$	0.135	0.486	0.850

Vanilla option			
Underlying S	100	100	100
Strike K	80	100	120
Call Price	22.168	6.371	1.767
Put Price	0.977	4.882	19.980
Gamma Γ	0.009	0.028	0.016
Vega $\mathcal{V}$	12.089	27.772	19.521
Theta Θ_{Call}	-5.430	-7.074	-5.413
Theta Θ_{Put}	-1.340	-4.035	-4.347

(b) (*1.5 points*)

(i) Calculate the cost of each of the two strategies in part (a).

(ii) Calculate Delta, Gamma, Vega, and Theta of each of the two strategies in part (a).

(c) (*1.5 points*) Estimate Delta and Vega of the two strategies in part (a) when

(i) Underlying price moves from 100 to 50;

(ii) Underlying price moves from 100 to 200.

(d) (*1 point*) Describe how Vega changes with:

(i) The time to maturity;

(ii) The underlying value.

8. Continued

Your manager wants you to analyze the volatilities further. You are asked to model the volatility smile.

You are particularly interested in local volatility models. Assume that the non-dividend paying underlying price is $S_0 = 100$ and the volatility varies according to the following formula.

$$\sigma(S) = \frac{(S - S_0)^2}{S^2} + 0.2$$

(e) (*3 points*) Calculate the cost of the two strategies in part (a) using a tree assuming that $\Delta t = 0.25$ and the continuous risk-free interest rate is 3%.

(f) (*1 point*) Describe advantages and disadvantages of the local volatility model.

9. (*5 points*) Bank ABC is exploring ways to replicate a wide range of payoffs using puts, calls, and riskless bonds.

(a) (*1 point*) Compare and contrast static and dynamic replication.

(b) (*0.5 points*) List four limitations of replication.

Bank ABC currently owns an index fund with a current share price of $100. It seeks to have its loss limited to $10 per share while accepting its gain capped at $10 per share.

Option with maturity at t=1	**Price at time 0**
Put, strike = 90	2.0
Put, strike = 100	6.0
Call, strike = 90	12.0
Call, strike = 100	6.0
Call, strike = 110	1.9
Call, strike = 115	1.5

(c) (*1 point*) Construct a structured product using the options above to meet ABC's goal.

(d) (*0.5 points*) Calculate the price at time 0 of the structured product constructed in part (c).

Bank ABC would like to consider an alternative product with the gain capped at $15 per share, and 50% of any loss absorbed (e.g., if the index drops to $60 per share, the holder will lose $20 per share).

Assume that the riskless rate is 0%.

(e) (*1 point*) Replicate the payoff of the alternative product using only riskless bonds, the stock, and calls/puts on the stock.

(f) (*1 point*) Sketch the payoffs of the strategies in part (c) and part (e) and compare their costs.

****END OF EXAMINATION****
Morning Session

USE THIS PAGE FOR YOUR SCRATCH WORK

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