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VARIABLE ANNUITY GUARANTEES

by

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1 Introduction

This note is intended to be read with Chapter 15 of *Actuarial Mathematics for Life Contingent Risks* (AMLCR) (Dickson *et al.*, 2019), where variable annuities (VAs) (also known as Segregated Funds in Canada) are listed as a type of equity-linked insurance that is particularly popular in North America. Typically, VAs offer more extensive and more complex guaranteed benefits than other equity-linked insurance, and these guarantees are the topic of this note.

Originally, VAs were designed to provide a deferred annuity, using the accumulated proceeds of a single premium. These assets are held in an account that is separate from the rest of the insurer's funds. At the end of the deferred (or accumulation) period, if they have survived, the policyholder may either convert the accumulated assets into a life annuity, or withdraw the assets at their market value. If the market value is below a specified minimum payout, then the insurer will add funds to cover the difference. If the annuitization rate (that is, the price per unit of annuity) is not guaranteed, then the value of the payout is the same whether the policyholder withdraws the cash value, or uses it to purchase a life annuity. Typically, in this case, the annuity option is ignored, so the VA becomes a variable payout endowment insurance, in that there is a death benefit payable if the policyholder dies before the end of the contract, and a maturity benefit payable if the policyholder survives. Both the death benefit and the maturity benefit will depend on the accumulated value of the invested premiums, but both will be subject to a guaranteed minimum payment.

Simple guaranteed minimum death benefits (GMDBs) and guaranteed minimum maturity benefits (GMMBs), where the guarantee is a proportion of the initial premium investment (typically $75\% - 100\%$) are described in AMLCR. In this note we describe more complex guarantees. In Section 3 we focus on GMDBs and GMMBs, which are cash payouts on death or maturity. In Section 4 we will extend the discussion to Guaranteed Minimum Income Benefits (GMIBs) and Guaranteed Minimum Withdrawal Benefits (GMWBs). GMIBs and GMWBs allow the policyholder to convert the proceeds of the VA into income at a prescribed future date, either through an annuity purchase (GMIB), or by allowing the policyholder to withdraw at least a minimum sum from the account each year, even if the account becomes fully depleted (GMWB). These are typically offered as riders to standard VA contracts, for an additional fee.

Our objective is to describe the payouts under different benefit design features. Illustrative payoff paths, similar to those discussed below, can be used to explore the contracts using stress testing, based on the profit testing techniques described in AMLCR, but it is important to note that these are highly complex and non-diversifiable risks. In most cases, for accurate valuation and risk management it is necessary to adapt some heavy duty mathematical and computational tools from quantitative and computational finance. These are well beyond the scope of this note, but some references are provided for interested readers.

2 Notation and Terminology

Policyholder's fund: F_t is the amount in the policyholder's separate account at time t, before any management charge deductions or cash withdrawals payable at t.

Premium: P is the total premium paid by the policyholder at the inception of the contract. This may be referred to as an initial investment, rather than a premium, emphasizing that this is predominantly an investment contract.

Maturity Date: n denotes the nominal term of the policy. At time n the guaranteed minimum maturity benefit may be withdrawn as a cash value, or the policyholder may elect to exercise a GMIB or GMWB.

Management charge rate: MC_t is the management charge deducted from the policyholder's fund at time t.

Guaranteed Minimum Maturity Benefit: G_t^M is the level of guaranteed maturity benefit that would be available as a cash value if the policy matured at t, but which is not actually available as a cash value until time n.

Guaranteed Minimum Death Benefit: G_t^D is typically set at the premium, but may increase at a fixed rate, and/or may increase as a function of the fund value.

Cash Value: CV_t denotes the cash value of the VA contract at t. At $t < n$, (i.e. before the maturity date) the cash value is the fund value minus any surrender charge applying at that time. At maturity, if the fund has fallen below the guarantee then the insurer must make up the difference. If the policyholder does not withdraw or annuitize their funds at maturity, then the insurer deposits sufficient funds in the separate account to make the fund value at n equal to CV_n, and the cash value thereafter is equal to the fund value. That is, if SC_t denotes the surrender charge applicable at t, then for

$$CV_t = \begin{cases} F_t - SC_t & t < n \\ \max\left(F_n, G_n^M\right) & t = n \\ F_t & t > n \end{cases}$$

Benefit Base: BB_t is a hypothetical valuation used to determine income benefits. See Section 4 for examples.

Annuitization rate: γ_y is the amount of annuity purchased per unit of single premium, for a life age y. That is, if (y) pays, say, 100,000, they would receive income of $B = 100,000\,\gamma_y$ per year in return.

If the income is paid as an annual, whole life annuity-due, then we have $\gamma_y = 1/\ddot{a}_y$. For other payment forms (e.g. payable monthly, guaranteed for r years, or payable on joint lives) the $\ddot{a}_y$ factor would be adjusted appropriately.

2.1 Some assumptions

In the following sections, we will present some examples of VA benefits. We will make the following simplifying assumptions that will not generally hold in practice.

1. We assume that all of the contracts are single premium contracts. In practice, policyholders may be permitted to invest additional premiums during the term of the contract.

2. We assume that there is a fixed maturity date n, at which the policyholder can decide whether to withdraw the maturity benefit, convert to income through the GMIB, or start annual withdrawals of the GMWB. In practice, the GMIB and GMWB exercise dates may be more flexible, after an initial minimum term.

3. We assume that, unless there is a GMWB, there are no elective withdrawals from the fund until maturity, when the full cash value is withdrawn or annuitized.

4. If a GMWB is purchased and exercised, we assume that the policyholder withdraws exactly the specified minimum benefit annually.

5. If a GMIB is purchased and exercised, we assume that the policyholder annuitizes their entire maturity cash value. In practice, they may withdraw some funds as a lump sum, and annuitize the rest.

6. We assume that management charges are deducted from the policyholder's fund at the start of each year. In practice, charges may be deducted more frequently. Unless specified, we assume there are no other expense deductions.

7. We assume that income/withdrawal payments under GMIBs and GMWBs are made annually in advance. In practice, payments may be more frequent.

3 GMMBs and GMDBs

The standard GMMB and GMDB are usually a fixed percentage of the premium investment. However, policyholders may be given the option to have the GMDB and the GMMB guarantee increase over the term of the contract. In this section we describe three types of increasing guarantees: the reset option, the step-up (or lookback) option, and the rollover (or guaranteed accumulation) option.

3.1 Reset option

If the fund value increases substantially above the initial guarantee, there is an incentive for the policyholder to terminate the policy and withdraw the cash value, especially if the policy is beyond the period where surrender charges are applied. They could then reinvest the proceeds in a new VA, with a new, higher guarantee. This is known as *lapse and re-entry* risk. The reset option mitigates the lapse and re-entry risk by allowing the policyholder to reset the guarantee from time to time to match the fund value at that time. The maturity date must be at least k-years from the reset date, where k is the minimum contract term; if the reset option is selected with less than k years to maturity, then the maturity date is changed to k years from the latest reset date.

It is worth considering why the insurer would offer the reset option if the same result could be achieved by a lapse-and-re-entry. Some insurer benefits are:

- The insurer saves on the operational and investment expenses of a lapse and re-entry.
- It can be a useful marketing feature for potential customers who are comparing the VA with a mutual fund (or with a VA that does not offer resets).
- It reduces the chances that the policyholder removes the funds and invests them elsewhere when a guarantee is far-out-of-the money.

However, there are additional risks involved for the insurer.

- If the insurer decides not to offer the policy any longer – for example because it is under-priced – the reset option extends the time until the portfolio can be closed.
- If markets rise quickly for a period, many policyholders will reset their guarantees at around the same time, potentially creating a sudden increase in the option liability across the portfolio.

- Additionally, if many policies reset at the same time then the revised maturity dates for all those policies will be set to the same date, creating a time-concentration risk that may impact the insurer's risk management.

Typically, where the GMMB is increased through a reset, the GMDB would be adjusted commensurately. For example, if a policy offers an initial GMMB of 90% of the premium, and a GMDB of 100% of the premium, the reset GMMB would be 90% of the fund value at reset, and the reset GMDB would be 100% of the fund value at reset.

3.2 Step-up Guarantees

Under the step-up model the insurer will specify step-up dates throughout the term of the contract. Typically, step-up dates are each year, or every five years, on the policy anniversary. At a step-up date the guarantee is increased to the level of the policyholder's fund at that date, if that is larger than the guarantee immediately before the step-up date. The outcome is that the guarantee at any point is the highest value that the policyholder's fund attained in each of the previous step-up dates. In financial mathematics this is known as a discrete lookback option. Step-up guarantees are more common for GMDBs than for GMMBs, and the model is also used for calculating the Benefit Base for income guarantees.

3.3 Guaranteed Rollover

Under the guaranteed rollover benefit, the insurer specifies rollover dates, typically at 5 or 10 year policy anniversaries. At each rollover date, if the fund value is greater than the guarantee in force, the guarantee is increased, exactly the same as a step-up. However, if the fund value is less than the guarantee, then the insurer will deposit additional cash into the policyholder's fund to bring it up to the guaranteed amount.

One interpretation of the guaranteed rollover is that it allows a policyholder to purchase a longer term policy - say, with a 30 year term, and have the same benefits as if they had purchased three consecutive 10-year policies (ignoring surrender charges and initial expense deductions).

The guaranteed rollover benefit is sometimes called a Guaranteed Minimum Accumulation Benefit (GMAB), but this term is now also used for a standard return of premium GMMB.

3.4 Comments and Example

Typically, insurers offer more opportunities to increase GMDBs than GMMBs, although historically reset and rollover GMMBs were common.

Resets and step-up guarantees can have a very similar effect, especially if step-ups are not permitted in the final k-years of the policy. Step-ups and rollovers can also have very similar impacts, as illustrated in the following example.

Example 3.1. An insurer issues a VA to (50), with a premium of $P = 100,000$. A management charge of $m = 2.5\%$ per year is deducted from the policyholder's fund annually at the start of each year. There are no other expense deductions. The initial GMMB and GMDB are both 100% of the premium.

The returns on the underlying fund assets in the t^{th} year, denoted R_t, are given in Table 1.

Calculate the cash payments to the policyholder, and the insurer's guarantee payouts (i.e. payments that are additional to the separate fund account) in each of the following cases. Assume the policyholder survives the 20-year period.

(a) The policyholder purchases a 20-year VA with no reset, step-up, or rollover options.

(b) The policyholder purchases a 15-year VA, resets the guarantees at time $t = 13$, with an extension to the policy term to time 20 (i.e. the minimum term is $k = 7$ years).

(c) The policyholder purchases a 20-year VA with a step-up guarantee, where the step-up dates are at each 5th policy anniversary.

(d) The policyholder purchases a 20-year VA with a rollover guarantee, where the rollover dates are at each 10th policy anniversary.

(e) The policyholder purchases a 20-year VA with a rollover guarantee, where the rollover dates are at each 5th policy anniversary.

Solution All values are in \$000's. Values given are rounded.
The policyholder's fund value at each year end is the same for parts (a), (b), and (c), as there is no potential for an additional injection of cash from the insurer during the term of the contract. In Table 2 we show the guarantee levels and the fund values for these cases. We have $F_0 = P = 100$,

Year t	Return R_t	Year t	Return R_t
1	-11.9%	11	15.3%
2	31.7%	12	28.8%
3	7.0%	13	-1.3%
4	-3.9%	14	2.5%
5	-8.2%	15	1.9%
6	-6.6%	16	-22.4%
7	17.1%	17	-17.6%
8	48.5%	18	17.4%
9	14.0%	19	8.4%
10	6.9%	20	-1.1%

Table 1: Returns used in Examples 3.1, 4.1, and 4.2.

and

$$MC_t = m \times F_t \qquad t = 0, 1, ..., 19$$
$$F_{t+1} = (F_t - MC_t)(1 + R_t) \qquad t = 0, 1, ..., 19$$
$$\text{Policyholder's payout} = \max\left(G_{20}^M,\, F_{20}\right) \quad \text{at } t = 20$$
$$\text{Insurer's payout} = \max\left(0, G_{20}^M - F_{20}\right) \quad \text{at } t = 20$$

(a) The guarantee at time $t = 20$ is $G_{20}^M = 100 < F_{20}$, so there is no guarantee payout to the insurer. The policyholder receives $F_{20} = 160.98$.

(b) The reset at time 13 increases the guarantee to $G_{20}^M = 228.64$, which is the payout to the policyholder, as it is greater than F_{20}. The insurer's guarantee cost is $G_{20}^M - F_{20} = 67.66$

(c) The step-ups increase the ultimate GMMB to $G_{20}^M = \max(F_0,\, F_5,\, F_{10},\, F_{15}) = F_{15} = 227.01$ which is the payout to the policyholder, as it is greater than F_{20}. The insurer's guarantee cost is $G_{20}^M - F_{20} = 66.04$.

(d) and (e) For the rollover guarantees, the fund value may be increased at the rollover dates, by an injection of cash from the insurer, so we have, in addition to the cashflows at time 20,

$$\text{Insurer's payout} = \max\left(0, G_{10}^M - F_{10}\right) \quad \text{at } t = 10 \quad \text{for (d)}$$
$$\text{Insurer's payout} = \max\left(0, G_t^M - F_t\right) \quad \text{at } t = 5, 10, 15 \quad \text{for (e).}$$

In Table 3 we show the fund values after the rollover payments (but before the deduction of management charges), along with the guarantees applying at each date. The rollover dates are underlined in the table. At each rollover date either the guarantee increases to match the fund value, or the fund value is increased to match the guarantee.

For the 10-yearly rollover, at time $t = 10$ the guarantee resets to $G_t^M = 168.29$, which is the payout at time 20 to the policyholder, as it is greater than F_{20}. The insurer's guarantee cost is $G_{20}^M - F_{20} = 7.32$ at $t = 20$.

For the 5-yearly rollover, at the first rollover date the insurer adds funds to the policyholder's account, to bring the fund value up to the guarantee at that time. At each subsequent rollover date the guarantee is reset to the fund value. The final guarantee is $G_{20}^M = \max(F_0, F_5, F_{10}, F_{15}) = F_{15} = 235.24$ which is the payout to the policyholder, as it is greater than F_{20}. The insurer's guarantee costs are therefore $G_5^M - F_5 = 3.50$ at $t = 5$ and $G_{20}^M - F_{20} = 68.43$ at $t = 20$.

Time t	Fund Value F_t	No resets G_t^M	Year 13 reset G_t^M	Step-up reset G_t^M
0	100.00	100.00	100.00	100.00
1	85.90	100.00	100.00	100.00
2	110.30	100.00	100.00	100.00
3	115.07	100.00	100.00	100.00
4	107.82	100.00	100.00	100.00
5	96.50	100.00	100.00	100.00
6	87.88	100.00	100.00	100.00
7	100.33	100.00	100.00	100.00
8	145.27	100.00	100.00	100.00
9	161.47	100.00	100.00	100.00
10	168.29	100.00	100.00	168.29
11	189.19	100.00	100.00	168.29
12	237.59	100.00	100.00	168.29
13	228.64	100.00	228.64	168.29
14	228.49	100.00	228.64	168.29
15	227.01	100.00	228.64	227.01
16	171.76	100.00	228.64	227.01
17	137.99	100.00	228.64	227.01
18	157.95	100.00	228.64	227.01
19	166.94	100.00	228.64	227.01
20	160.98	100.00	228.64	227.01

Table 2: Year end fund values (before deduction of management charge) and guarantees for Example 3.1, parts (a) (no resets), (b) (reset at $t = 13$), and (c) (5-yearly step-up resets).

Time	10-year rollover		5-year rollover	
t	F_t	G_t^M	F_t	G_t^M
0	100.00	100.00	100.00	100.00
1	85.90	100.00	85.90	100.00
2	110.30	100.00	110.30	100.00
3	115.07	100.00	115.07	100.00
4	107.82	100.00	107.82	100.00
5	96.50	100.00	100.00	100.00
6	87.88	100.00	91.07	100.00
7	100.33	100.00	103.97	100.00
8	145.27	100.00	150.54	100.00
9	161.47	100.00	167.32	100.00
10	168.29	168.29	174.40	174.40
11	189.19	168.29	196.05	174.40
12	237.59	168.29	246.20	174.40
13	228.64	168.29	236.93	174.40
14	228.49	168.29	236.78	174.40
15	227.01	168.29	235.24	235.24
16	171.76	168.29	177.99	235.24
17	137.99	168.29	142.99	235.24
18	157.95	168.29	163.68	235.24
19	166.94	168.29	172.99	235.24
20	160.98	168.29	166.81	235.24

Table 3: Year end fund values (before deduction of management charge) and guarantees for Example 3.1, parts (d) (10-yearly rollover guarantees) and (e) (5-yearly rollover guarantees).

4 Income Benefits

VAs have proven to be popular products for retirement planning, but uncertain cash payouts to survivors are not ideal for individuals who seek a predictable and secure income stream in retirement. Even if the VA proceeds are annuitized, without a guaranteed annuitization rate there is a lot of uncertainty as to the level of income provided, as the market annuitization rate can be volatile. To make the VA more attractive to policyholders who are looking for less uncertainty in their retirement income, whilst also wanting some upside investment opportunity, many insurers now offer guaranteed income benefits as an alternative to the cash maturity benefit. There are two broad forms of income benefits for VAs. We give a brief description here, and more detailed examples in subsequent subsections.

- The Guaranteed Minimum Income Benefit (GMIB) provides a specified guaranteed annuitization rate for the VA proceeds. In the simplest case, if the market annuitization rates at retirement are less favourable (i.e. more expensive) than the rate specified in the contract, then the policyholder can exercise their GMIB option to convert their VA proceeds at the guaranteed rate. If market annuitization rates are cheaper than the guaranteed rate, then the policyholder can withdraw their funds and annuitize at the market rate, with the same insurer, or with a different provider.

  GMIBs are usually more complex than this. Rather than annuitizing the cash proceeds of the VA, under the GMIB the amount annuitized is the Benefit Base. Typically this increases gradually over the accumulation (or deferred) period, so that, together with the guaranteed annuitization rate, the minimum level of income payable from the maturity date will also increase during the accumulation phase.

- The Guaranteed Minimum Withdrawal Benefit (GMWB) allows the policyholder to withdraw a specified minimum annual sum from the fund, after an initial waiting period. The minimum withdrawal is typically expressed as a percentage of the premium, or of a Benefit Base, with adjustments for any withdrawals during the waiting period, or for any withdrawals exceeding the minimum amount after the waiting period. The minimum withdrawal amount is payable even if there are no funds remaining in the policyholder's account. Most policies are sold with the minimum withdrawal benefits payable for life, although fixed term GMWBs are also available.

The main difference between these options is that under the GMIB the policyholder's funds are annuitized, so that the policyholder has no subsequent access to the capital. Under the

GMWB, the funds are not annuitized, so that the policyholder can access them at any time (provided the account is not fully depleted). When a policyholder with a GMIB dies, there is no residual payment to their estate (unless the annuity has a guaranteed term of payment). When a policyholder with a GMWB dies, any funds remaining in their account will be paid to their estate.

4.1 The Benefit Base

The amount of guaranteed income/withdrawal under a GMIB or GMWB may be based on the fund value at the maturity date (CV_n), but more commonly it is based on the value of the Benefit Base at maturity. We let BB_t denote the Benefit Base value if the policy were to mature at t. It is important to note that the Benefit Base cannot be withdrawn as a cash value, it is only used to determine the income amount under a GMIB or GMWB.

The initial Benefit Base is usually a specified percentage (typically, 100%) of the premium. Subsequently, the Benefit Base may increase at a guaranteed, fixed rate, or through a step-up process, or it may be the greater of these. Some examples are given in the following sections.

The Benefit Base may impact the management charges for the account; for example, the management charge at t may be of the form $MC_t = m \times F_t + m^b \times BB_t$.

In the following sections we describe the GMIB and GMWB in more detail, with examples.

4.2 Guarantee Minimum Income Benefit

Consider a VA with a term of n years, with premium P, all of which is invested in the policyholder's account. A GMIB rider offers a guaranteed annuitization rate that, if exercised, will be applied at the maturity date to annuitize the value of the Benefit Base.

The guaranteed annuitization rate specifies the amount of income provided per unit of lump sum; we use γ_y^g to denote the guaranteed annuitization rate for a life age y (at annuitization), and $\gamma_y(t)$ to denote the time t market annuitization rate for a life age y at that time. If the annuity is paid as an annual whole life annuity-due then $\gamma_y^g = 1/\ddot{a}_y^g$, where g indicates that the annuity factor is calculated using the GMIB interest and mortality assumptions. Similarly, we have $\gamma_y(t) = 1/\ddot{a}_y(t)$ where $\ddot{a}_y(t)$ denotes the market annuity factor at t.

For a policyholder who is age y at maturity (i.e. at time $t = n$), and who wishes to use the VA to provide income at that time, there are two options available. They could withdraw the cash proceeds, $CV_n = \max(G_n^M, F_n)$, from the VA, and convert them to an annuity-due at the market annuitization rate. Their income would be

$$B^{\text{mkt}} = F \times \gamma_y(n)$$

Alternatively, they could exercise their GMIB, which offers income of

$$B^{\text{GMIB}} = BB_n \times \gamma_y^g$$

The Benefit Base for a GMIB is always greater than or equal to the lump sum proceeds (i.e. $BB_n \geq CV_n$), but γ_y^g could be larger or smaller than $\gamma_y(n)$. If $\gamma_y(n) > \gamma_y^g$ then the optimal decision depends on how much greater the Benefit Base value is than the maturity cash value.

Example 4.1.

An insurer issues a 20-year VA policy to (50), with a GMIB rider. The premium is $P = 100,000$. You are given the following policy information.

(i) The GMMB is 100% of the premium, with a step-up at time 10. That is

$$G_t^M = \begin{cases} P & t \leq 9 \\ \max(P, F_{10}) & t \geq 10 \end{cases}$$

(ii) The Benefit Base at t is the greater of (a) the premium accumulated at 3.5% per year, and (b) a five-yearly step-up guarantee. That is

$$BB_t = \begin{cases} P(1.035)^t & t \leq 5 \\ \max\left(P(1.035)^t, F_5\right) & 5 \leq t < 10 \\ \max\left(P(1.035)^t, F_5, F_{10}\right) & 10 \leq t < 15 \\ \max\left(P(1.035)^t, F_5, F_{10}, F_{15}\right) & 15 \leq t < 20 \\ \max\left(P(1.035)^t, F_5, F_{10}, F_{15}, F_{20}\right) & t = 20 \end{cases}$$

(iii) The guaranteed minimum income is payable as a level annual whole life annuity, with first payment at age 70.

(iv) The guaranteed annuitization rate is calculated using the Standard Ultimate Life Table (SULT) at an interest rate of 4.0%. You are given that $\ddot{a}_{70|i=0.04} = 13.01704$.

(v) The management charge applied at the start of each year is 2.5% of the fund value, plus 0.3% of the Benefit Base.

(a) Calculate the guaranteed minimum annual income available from the policy, based on the information available at the policy inception.

(b) Calculate the guaranteed minimum annual income available from the policy, based on the information available at time 15, and assuming that the fund earned the returns given in Table 1.

(c) Assume that the policyholder survives to age 70, and that the fund earned the returns given in Table 1. You are also given that at maturity, the market annuitization rate is calculated using the SULT at an effective interest rate of 5.5% per year. You are given that $\ddot{a}_{70|i=0.055} = 11.5549$.

   (i) Calculate B^{mkt} and B^{GMIB}.

   (ii) Calculate the market value at time n of the insurer's guarantee costs.

Solution All values in 000's.

(a) The Benefit Base at maturity will be at least $P(1.035)^{20} = 198.98$. This can be annuitized at $\gamma_{70}^{g} = 1/\ddot{a}_{70|i=0.04} = 0.076822$, giving a guaranteed minimum income of $198.98 \times \gamma_{70}^{g} = 15.286$ per year.

(b) In Table 4 we show F_t, G_t, BB_t and MC_t, for $t = 0, \cdots, 20$. The Benefit Base at time 15 is 216.69, and since BB_{20} will not be smaller than this, we know that the guaranteed minimum income will be at least $B = 216.69 \times \gamma_{70}^{g} = 16.646$.

(c) from the table we see that the cash value of the policy at maturity is $CV_{20} = 162.68$, as the GMMB is in-the-money. From the SULT, we find that the market annuitization rate at maturity is $\gamma_{70}(20) = 1/\ddot{a}_{70|i=0.055} = 0.086543$. This gives us

$$B^{\text{mkt}} = 162.68 \times 0.086543 = 14.079$$
$$B^{\text{GMIB}} = 216.69 \times 0.076822 = 16.646.$$

In this case, even though the guaranteed annuitization rate is significantly less favourable to the policyholder than the market rate, the additional income created by using the Benefit Base makes it optimal to exercise the GMIB.

The overall cost to the insurer at maturity is the difference between the market value of the annuity and the policyholder's fund value, which is

$$16.646 \times \ddot{a}_{70|i=0.055} - F_{20} = 41.8932.$$

Year t	Guarantee G_t^M	Benefit Base BB_t	Fund Value F_t	Mgmt Charge MC_t
0	100.00	100.00	100.00	2.80
1	100.00	103.50	85.63	2.45
2	100.00	107.12	109.55	3.06
3	100.00	110.87	113.94	3.18
4	100.00	114.75	106.44	3.01
5	100.00	118.77	94.96	2.73
6	100.00	122.93	86.14	2.52
7	100.00	127.23	97.92	2.83
8	100.00	131.68	141.20	3.93
9	100.00	136.29	156.50	4.32
10	162.68	162.68	162.68	4.55
11	162.68	162.68	182.31	5.05
12	162.68	162.68	228.32	6.20
13	162.68	162.68	219.24	5.97
14	162.68	162.68	218.60	5.95
15	162.68	216.69	216.69	6.07
16	162.68	216.69	163.44	4.74
17	162.68	216.69	130.77	3.92
18	162.68	216.69	148.93	4.37
19	162.68	216.69	156.69	4.57
20	162.68	216.69	150.45	0.00

Table 4: Guarantee, Benefit Base, Fund Value, and Management Charge calculations for the GMIB contract in Example 4.1

4.3 Guaranteed Minimum Withdrawal Benefits

The GMWB guarantees that after an initial waiting period, the policyholder can withdraw a specified minimum amount each year, even if their withdrawals exceed the available funds. We will consider guaranteed lifetime withdrawal benefits, which continue as long as the policyholder survives, but some insurers also offer limited term GMWBs.

The minimum withdrawal is expressed as a percentage of the premium, or of the Benefit Base at the start of the withdrawal period. For example, a GMWB rider may provide for withdrawals starting at time n, of $c\%$ of BB_n each year. Typically, c ranges from 4% to 7%. The amount is usually fixed once the withdrawal period starts, as long as the policyholder does not withdraw more than the minimum amount. At any stage, provided $F_t > 0$, the policyholder may terminate the contract and withdraw the remaining funds. If the policyholder dies, then any remaining funds belong to their estate.

Year t	Return R_t	Year t	Return R_t
21	-10.7%	31	27.5%
22	11.4%	32	-6.1%
23	-3.5%	33	-0.5%
24	22.9%	34	-7.5%
25	-9.0%	35	11.2%
26	9.4%	36	-17.4%
27	7.5%	37	15.0%
28	-3.5%	38	9.4%
29	-5.5%	39	15.5%
30	-0.3%	40	5.2%

Table 5: Returns used in Example 4.2.

Example 4.2. An insurer issues a 20-year VA policy to (50), with a GMWB rider. The premium is $P = 100,000$. You are given the following policy information.

- The GMMB is 100% of the premium, with a step-up at time 10. That is

$$G_t^M = \begin{cases} P & t \le 9 \\ \max(P, F_{10}) & t \ge 10 \end{cases}$$

- The Benefit Base at $t < 10$ is the premium and at $t \ge 10$ is $\max(P, F_{10})$ (i.e. time 10 step-up).

- The guaranteed minimum withdrawal is 6.5% of the Benefit Base, payable from $t = 15$.
- If the fund is fully depleted at time T, say, then $F_t = 0$ for all $t > T$.
- The management charge applied at the start of each year is 2.5% of the fund value, plus 0.3% of the Benefit Base, conditional on $F_t > 0$. If $F_t = 0$ there is no management charge deduction.

(a) Calculate the guaranteed minimum annual withdrawal available from the policy, based on the information available at the policy inception.

(b) Assume that

- The policyholder dies immediately after her 90th birthday.
- She withdraws the guaranteed minimum amount each year from $t = 15$ until $t = 40$.
- Returns in the first 20 years are as given in Table 1.
- Returns in the 21st to 40th years are as given in Table 5.

(i) Calculate the withdrawal amount payable from $t = 15$.

(ii) Calculate the present value at $t = 15$ of the insurer's guarantee costs, at $i = 0.055$.

Solution All values in 000's.

(a) The Benefit Base at $t = 15$ will be at least 100, giving a guaranteed lifetime annual withdrawal of at least $0.065 \times 100 = 6.5$.

(b) In Tables 6 and 7 we show the Benefit Base, fund value (before withdrawals and management charges), withdrawal amount, management charge, and insurer guarantee costs for each year of the contract, split into the first 20 years and the last 20 years.

We see that the minimum withdrawal is 10.62, significantly greater than the initial guaranteed level, thanks to the year 10 step-up in the Benefit Base.

At $t = 28$ the policyholder's funds become insufficient to pay for the minimum withdrawal. After this point the policyholder's fund is fully depleted, and subsequent withdrawals are funded by the insurer. The present value of the insurer's costs at time 15, at $i = 5.5\%$, is 48.2.

Year	Benefit Base	Fund	Wdwl	Mgmt Charge	Insurer
t	BB_t	F_t	W_t	MC_t	Costs
0	100.00	100.00	0.00	2.80	0.00
1	100.00	85.63	0.00	2.44	0.00
2	100.00	109.56	0.00	3.04	0.00
3	100.00	113.98	0.00	3.15	0.00
4	100.00	106.51	0.00	2.96	0.00
5	100.00	95.06	0.00	2.68	0.00
6	100.00	86.28	0.00	2.46	0.00
7	100.00	98.16	0.00	2.75	0.00
8	100.00	141.68	0.00	3.84	0.00
9	100.00	157.13	0.00	4.23	0.00
10	163.46	163.46	0.00	4.58	0.00
11	163.46	183.19	0.00	5.07	0.00
12	163.46	229.41	0.00	6.23	0.00
13	163.46	220.29	0.00	6.00	0.00
14	163.46	219.65	0.00	5.98	0.00
15	163.46	217.72	10.62	5.93	0.00
16	163.46	156.11	10.62	4.39	0.00
17	163.46	116.2	10.62	3.40	0.00
18	163.46	120.02	10.62	3.49	0.00
19	163.46	114.80	10.62	3.36	0.00
20	163.46	99.71	10.62	2.98	0.00
⋮					

Table 6: Benefit base, fund values, withdrawals, management charges and insurer costs for the GMWB policy in Example 4.2; years 1-20.

Year t	Benefit Base BB_t	Fund F_t	Wdwl W_t	Mgmt Charge MC_t	Insurer Costs
⋮					
21	163.46	76.89	10.62	2.41	0.00
22	163.46	71.13	10.62	2.27	0.00
23	163.46	56.20	10.62	1.90	0.00
24	163.46	53.68	10.62	1.83	0.00
25	163.46	37.52	10.62	1.43	0.00
26	163.46	27.86	10.62	1.19	0.00
27	163.46	17.25	10.62	0.92	0.00
28	163.46	5.50	10.62	0.63	5.12
29	163.46	0.00	10.62	0.00	10.62
30	163.46	0.00	10.62	0.00	10.62
31	163.46	0.00	10.62	0.00	10.62
32	163.46	0.00	10.62	0.00	10.62
33	163.46	0.00	10.62	0.00	10.62
34	163.46	0.00	10.62	0.00	10.62
35	163.46	0.00	10.62	0.00	10.62
36	163.46	0.00	10.62	0.00	10.62
37	163.46	0.00	10.62	0.00	10.62
38	163.46	0.00	10.62	0.00	10.62
39	163.46	0.00	10.62	0.00	10.62
40	163.46	0.00	10.62	0.00	10.62

Table 7: Benefit base, fund values, withdrawals, management charges and insurer costs for the GMWB policy in Example 4.2; years 21-40.

5 Notes

We have focused in this note on some guaranteed benefit designs that have been popular in the recent past. New types of guarantees are constantly being developed, and some older styles are becoming much less common for new policies, though they remain important in legacy portfolios. A good way to get a feel for the types of guarantees offered currently is to download VA policy prospectuses from some major insurance companies.

Although the focus of this note is guaranteed benefits, there are other important design features of VAs. One feature that is important in practice concerns the investment options available. Typically, policyholders will be offered a wide range of mutual fund style options; riskier funds may not be allowed in conjunction with potentially costly benefit riders, such as the GMIB. The policy will also specify how often assets may be transferred between funds; typically, a small number of transfers are permitted without incurring charges.

Another design feature involves expenses. The examples in this note assumed annual management charges, determined by the sum of a multiple (m) of the fund value and a multiple (m^b) of the Benefit Base. In practice, the fee structure may be more complex than this, with front-end fees at inception, annual per-policy fees, and charges for early withdrawals. The value of m will vary by type of investment selected, both because the actual investment costs are higher for the most active funds, and because guarantee costs vary according to the volatility of the underlying assets.

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