Exam MFE/3F Spring 2009

Answer Key

Question #	*Answer*
1	**E**
2	**B**
3	**B**
4	**D**
5	**E**
6	**D**
7	**D**
8	**B**
9	**E**
10	**C**
11	**C**
12	**E**
13	**A**
14	**E**
15	**C**
16	**A**
17	**A**
18	**A**
19	**D**
20	**C**

1. Answer: E

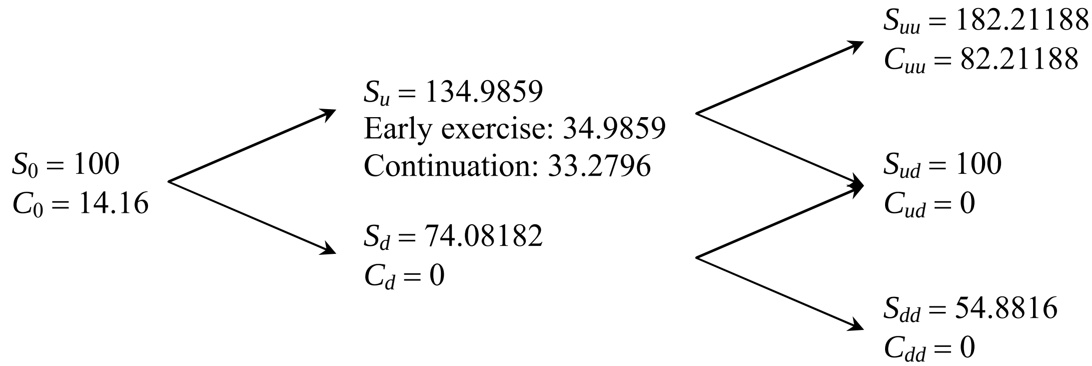
We have $S_0 = 10$, $\delta = 0.05$, $\sigma = 0.3$, $r = 0.05$, and $h = 1$. By (10.10),

$$\begin{cases} u = \exp[(r-\delta)h + \sigma\sqrt{h}] = \exp[(0.05-0.05)\times 1 + 0.3\sqrt{1}] = e^{0.3} \\ d = \exp[(r-\delta)h - \sigma\sqrt{h}] = \exp[(0.05-0.05)\times 1 - 0.3\sqrt{1}] = e^{-0.3} \end{cases}.$$

By (10.5),

$$p^* = \frac{e^{(r-\delta)h} - d}{u-d} = \frac{e^{(0.05-0.05)\times 1} - e^{-0.3}}{e^{0.3} - e^{-0.3}} = 0.42556\,.$$

The stock prices and call prices are listed at each node below:



For the calculation of C_u, we have

$$C_u = e^{-0.05}[p^* C_{uu} + (1 - p^*)C_{ud}] = 33.2796,$$

but early exercise would be optimal at a value of 34.9859. The time-0 price of the call is

$$C = e^{-0.05}[p^* C_u + (1 - p^*)C_d] = 14.1624.$$

Remark:

For a given volatility σ, if u and d are computed using the method of forward tree, then

$$p^* = \frac{e^{(r-\delta)h} - d}{u-d} = \frac{e^{(r-\delta)h} - e^{(r-\delta)h-\sigma\sqrt{h}}}{e^{(r-\delta)h+\sigma\sqrt{h}} - e^{(r-\delta)h-\sigma\sqrt{h}}} = \frac{1 - e^{-\sigma\sqrt{h}}}{e^{\sigma\sqrt{h}} - e^{-\sigma\sqrt{h}}} = \frac{e^{\sigma\sqrt{h}} - 1}{e^{2\sigma\sqrt{h}} - 1} = \frac{1}{1 + e^{\sigma\sqrt{h}}}\,,$$

and hence

$$1 - p^* = \frac{1}{1 + e^{-\sigma\sqrt{h}}}\,.$$

As a result, $p^* < \frac{1}{2}$; this provides a check for p^*.

2. Answer: B

(i) The average of the stock price is:

$$100 + \frac{1}{12}(5 + 20 + 15 + 10 + 15 + 10 + 0 - 10 + 5 + 25 + 10 + 15)$$

$$= 100 + \frac{1}{12} \times 120$$

$$= 100 + 10$$

The payoff is thus 10.

(ii) The call is knocked-out on Oct 31, 2008, when the stock price is 125.
The payoff is thus 0.

(iii) The call is knocked-in on Feb 29, 2008, when the stock price is 120.
The payoff is thus max(115 – 110, 0) = 5.

The maximum difference is 10 – 0 = 10.

Remark: While it is incorrect to say that an option that goes out of existence has an undefined payoff, some statements in the text can be confusing. For the May 2009 exam, (A) was also accepted as a correct answer.

3. Answer: B

Since

$$u = 55/50 = 1.1 \quad \text{and} \quad d = 40/50 = 0.8,$$

we have, by (10.5),

$$p^* = \frac{e^{(r-\delta)h} - d}{u - d} = \frac{e^{-0.05} - 0.8}{1.1 - 0.8} = 0.5041.$$

The no-arbitrage price of the call is

$$C_0 = e^{-rh}[p^* C_u + (1 - p^*)C_d] = e^{-0.05}(0.5041 \times 5) = 2.3976 > 1.9.$$

As a result, an arbitrageur would buy the underpriced call and then hedge the risk of the stock in order to obtain riskless arbitrage profit. This rules out (A), (D) and (E).

Since the delta of the call is positive (see Figure 10.2 on page 320), the arbitrageur must short sell shares to eliminate the stock price risk. This rules out (C).

Alternative method:

We determine the replicating portfolio of the call option. Suppose that at $t = 0$, the replicating portfolio has Δ shares and B dollars in a bank account earning a risk-free rate of interest. Since the stock pays dividends, by investing all dividends in the stock, the number of shares would grow to $\Delta e^{\delta h}$ after h years.

$$\begin{cases} \Delta e^{\delta} S_u + B e^{r} = C_u \\ \Delta e^{\delta} S_d + B e^{r} = C_d \end{cases}$$

$$\begin{bmatrix} 55e^{0.1} & e^{0.05} \\ 40e^{0.1} & e^{0.05} \end{bmatrix} \begin{bmatrix} \Delta \\ B \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 15e^{0.1} & 0 \\ 40e^{0.05} & 1 \end{bmatrix} \begin{bmatrix} \Delta \\ B \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \Delta \\ B \end{bmatrix} = \begin{bmatrix} 0.3016 \\ -12.6831 \end{bmatrix}$$

The no-arbitrage price of the call is $C = \Delta S + B = 2.4 > 1.9$.

Therefore, Michael can make an arbitrage profit by purchasing the call option at $1.9 and short selling the replicating portfolio. Since $\Delta > 0$ and $B < 0$, shorting the replicating portfolio, in this case, means shorting 0.3016 shares of the stock and lending $12.6831 at the risk-free rate.

Remark: In a binomial model, $\Delta = e^{-\delta h} \dfrac{C_u - C_d}{S_u - S_d}$.

In the Black-Scholes model, $\Delta = e^{-\delta h} N(d_1)$.

4. Answer: D

We can construct an ordinary K-strike European put by buying K units of a K-strike European cash-or-nothing put and selling a K-strike European asset-or-nothing put:

$$\text{Ordinary Put} = K \times (\text{Cash-or-nothing Put}) - \text{Asset-or-nothing Put.}$$

The two terms on the right-hand side above correspond to the two terms in the Black-Scholes formula

$$P = Ke^{-rT} N(-d_2) - S(0)e^{-\delta T} N(-d_1).$$

The price of the asset-or-nothing put is $S(0)e^{-\delta T} N(-d_1)$, which is a formula that can also be found in the middle of page 706.

We are given $S(0) = 1000$, $K = 600$ (not 400), $\delta = 0.02$, $r = 0.025$, $\sigma = 20\%$, and $T = 1$.

Thus,

$$d_1 = \frac{\ln(1000/600) + (0.025 - 0.02 + \tfrac{1}{2} \times 0.2^2) \times 1}{0.2\sqrt{1}} = 2.679128 \approx 2.68 .$$

From the normal table, $N(-2.68) = 1 - N(2.68) = 1 - 0.9963 = 0.0037$

Price of Puts = 1,000,000 × 1,000 × $e^{-0.02}$ × 0.0037 = 3,626,735 ≈ 3.6 million

5. Answer: E

Path	risk-neutral probability	time-2 price of the bond	time-2 payoff
↑↑	0.7 × 0.7 = 0.49	$e^{-0.18} = 0.83527$	0.9 – 0.83527 = 0.06473
↑↓	0.7 × 0.3 = 0.21	$e^{-0.12} = 0.88692$	0.9 – 0.88692 = 0.01308
↓↑	0.3 × 0.7 = 0.21	0.88692	0.01308
↓↓	0.3 × 0.3 = 0.09	$e^{-0.06} = 0.94176$	0

The put price is $0.49 \times \frac{0.06473}{e^{0.12+0.15}} + 0.21 \times \frac{0.01308}{e^{0.12+0.15}} + 0.21 \times \frac{0.01308}{e^{0.12+0.09}} = 0.02854$.

6. Answer: D

Let $f(x, t) = 1/x$ so that $Y(t) = f(X(t), t)$.

Then $f_x(x, t) = -1/x^2$, $f_{xx}(x, t) = 2/x^3$, and $f_t(x, t) = 0$.

By formulas (20.17a, b, c),

$$[dX(t)]^2 = [(8 - 2X(t))dt + 8dZ(t)]^2 = 8^2[dZ(t)]^2 = 64dt.$$

By Itô's lemma,

$$\begin{aligned} dY(t) &= -\frac{1}{X^2(t)}dX(t) + \frac{1}{2}\frac{2}{X^3(t)}[dX(t)]^2 + 0dt \\ &= -\frac{1}{X^2(t)}[(8 - 2X(t))dt + 8dZ(t)] + \frac{64}{X^3(t)}dt \\ &= \left(\frac{64}{X^3(t)} - \frac{8}{X^2(t)} + \frac{2}{X(t)}\right)dt - \frac{8}{X^2(t)}dZ(t) \\ &= [64Y^3(t) - 8Y^2(t) + 2Y(t)]dt - 8Y^2(t)dZ(t) \end{aligned}$$

which means that $\alpha(y) = 64y^3 - 8y^2 + 2y$ and $\beta(y) = -8y^2$.

Thus, $\alpha(½) = 64\left(\frac{1}{8}\right) - 8\left(\frac{1}{4}\right) + 2\left(\frac{1}{2}\right) = 7$.

7. Answer: D

Let Q_u (Q_d) be the price of a security that pays \$1 when the up (down) state occurs, and r be the continuously compounded risk-free interest rate. Then

$$\begin{cases} Q_u + Q_d = e^{-r} \\ 12Q_u + 8Q_d = 10 \\ 2Q_u = 1.13 \end{cases}$$

By using the 3rd equation, we get $Q_u = 0.565$. Putting back into the 2nd equation, we get

$$Q_d = \frac{1}{8}(10 - 12 \times 0.565) = 0.4025,$$

and hence it follows from the 1st equation that

$$e^{-r} = 0.565 + 0.4025 = 0.9675.$$

Now if S_d is 6 instead of 8, then because r and S_0 are unchanged, the system of simultaneous equations becomes

$$\begin{cases} Q_u + Q_d = 0.9675 \\ 12Q_u + 6Q_d = 10 \\ 2Q_u = C_0 \end{cases}$$

Eliminating Q_d from the 2nd equation by using the 1st equation, we get

$$Q_u = \frac{1}{6}(10 - 6 \times 0.9675) = 0.69917.$$

Thus, $C_0 = 2Q_u = 1.398$.
Remark: The result is independent of the true probability of an up-move. Analogously, the Black-Scholes equation and formulas do not depend on α.

Alternative method:

The time-0 price of the call option is

$$C_0 = e^{-rh}[p^* \times C_u + (1 - p^*) \times C_d] = e^{-r}\left[\frac{e^r - 0.8}{1.2 - 0.8} \times 2 + (1 - p^*) \times 0\right] = \frac{1 - 0.8e^{-r}}{0.2}.$$

Setting $\frac{1 - 0.8e^{-r}}{0.2} = 1.13$, we get $e^{-r} = 0.9675$ (or $r = 3.3\%$).

If $S_d = 6$, then $d = 0.6$, and

$$C_0 = e^{-r} \times \frac{e^r - 0.6}{1.2 - 0.6} \times 2 = \frac{1 - 0.6e^{-r}}{0.3} = \frac{1 - 0.6 \times 0.9675}{0.3} = 1.398.$$

8. Answer: B

The pricing formula for the derivative security, V, must satisfy the Black-Scholes partial differential equation (21.11)

$$\frac{\partial V}{\partial t} + (r - \delta)S\frac{\partial V}{\partial s} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial s^2} = rV \, .$$

For $V(s, t) = e^{rt} \ln s$, we have

$$V_t = re^{rt} \ln s = rV, \quad V_s = \frac{1}{s}e^{rt}, \quad V_{ss} = -\frac{1}{s^2}e^{rt}.$$

Thus, the PDE becomes

$$rV + (r-\delta)S\left(\frac{1}{S}e^{rt}\right) + \frac{1}{2}\sigma^2 S^2\left(-\frac{1}{S^2}e^{rt}\right) = rV$$

$$(r-\delta)e^{rt} - \frac{1}{2}\sigma^2 e^{rt} = 0$$

$$r - \delta - \frac{1}{2}\sigma^2 = 0,$$

yielding

$$\delta = r - \frac{1}{2}\sigma^2 = 0.055 - 0.5(0.3)^2 = 0.01$$

Alternative method:

As the derivative security does not pay dividends, we have, for $t \le T$,

$$V[S(t), t] = F_{t,T}^{P}(V[S(T), T]) \, .$$

In particular,

$$V[S(0), 0] = F_{0,T}^{P}(V[S(T), T]) \, ,$$

which, in this problem, means

$$\ln[S(0)] = \mathrm{E}^*[e^{-rT}e^{rT} \ln S(T)] = \mathrm{E}^*[\ln S(T)].$$

Under the risk-neutral probability measure,

$$\ln S(T) \sim \mathcal{N}(\ln S(0) + (r - \delta - ½\sigma^2)T, \; \sigma^2 T),$$

which is a result given at the top of page 650, with α replaced by $r - \delta$.

Thus, the condition $\ln[S(0)] = \mathrm{E}^*[\ln S(T)]$ means that $(r - \delta - ½\sigma^2)T = 0$, yielding the same solution as before.

Third method (not in the syllabus):

By the *fundamental theorem of asset pricing*, the stochastic process $\{e^{-rt} V[S(t), t]\}$ is a martingale with respect to the risk-neutral probability measure, yielding the condition

$$\mathrm{E}^*[\ln S(t)] = \ln S(0).$$

9. Answer: E

By put-call parity (equation (9.4) on page 286, but replacing S_0 by the foreign exchange rate x_0 and the dividend yield δ by the foreign risk-free interest rate r_f), the price of a 4-year dollar-denominated European call option on yens with a strike price of $0.008 is

$$\begin{aligned}&P + x_0 \exp(-r_f T) - K \exp(-rT)\\&= 0.0005 + 0.011e^{-0.015\times 4} - 0.008e^{-0.03\times 4}\\&= 0.003764\end{aligned}$$

Note that 125 = 1/0.008. By currency option put-call duality (equation (9.7) on page 292), the price of a 4-year yen-denominated European put option on dollars with a strike price of ¥(1/0.008) is

$$0.003764 \times \frac{1}{0.008} \times \frac{1}{0.011} = 42.77325.$$

Alternative method:

Note that 125 = 1/0.008. By currency option put-call duality (equation (9.7) on page 292), the price of a 4-year yen-denominated European call option on dollars with a strike price of ¥(1/0.008) is

$$0.0005 \times \frac{1}{0.008} \times \frac{1}{0.011} = 5.68182.$$

By put-call parity, the price of a 4-year yen-denominated European put option on dollars with a strike price of ¥125 is

$$\begin{aligned}&C - F_{0,T}^{P}(\frac{1}{x}) + K \exp(-r_f T)\\&= 5.68182 - \frac{1}{0.011}e^{-0.03\times 4} + 125e^{-0.015\times 4}\\&= 42.77326\end{aligned}$$

10. Answer: C

Observe that

$$\begin{cases} dS_1(t) = 0.08S_1(t)dt + 0.2S_1(t)dZ(t) \\ dS_2(t) = 0.0925S_2(t)dt - 0.25S_2(t)dZ(t) \end{cases}.$$

If we long 1 unit of S_1, then we must long $\Delta = \frac{0.2S_1}{0.25S_2} = \frac{0.8S_1}{S_2}$ shares of S_2 so that

$$dS_1(t) + \Delta dS_2(t)$$

has no $dZ(t)$ term.

In terms of dollar amount,

"1 share of S_1" to "$0.8S_1/S_2$ shares of S_2"
= "S_1 dollars invested in stock 1" to "$0.8S_1$ dollars invested in stock 2"
= 1 : 0.8.

Thus, the percentage in stock 1 is $\frac{1}{1+0.8} = 55.556\%$.

Remarks: (i) The two expected rates of return, 0.08 and 0.0925, are not used in determining the proportion. They are needed for determining *r*. (ii) The proportion is independent of *t*.

11. Answer: C

To find the price, we need to first determine the negative constant *a*.

From (ii), we know that true stock price process is a geometric Brownian motion with α = 0.05 and σ = 0.2. By (20.35) (but replace *r* by α) or by the moment-generating function formula for a normal random variable, the expected value of the contingent claim at time *T* is

$$\mathrm{E}[S(T)^a] = S(0)^a \exp\{[a(\alpha - \delta) + \frac{1}{2}a(a-1)\sigma^2]T\}.$$

Substituting $T = 1$, $S(0) = 0.5$, $\delta = 0$, $\alpha = 0.05$ and $\sigma = 0.2$ into the equation above,

$$0.5^a \exp[0.05a + \frac{1}{2}a(a-1)(0.2)^2] = 1.4$$

$$a \ln 0.5 + 0.05a + 0.02a(a-1) = \ln 1.4$$

$$0.02a^2 + (0.03 + \ln 0.5)a - \ln 1.4 = 0$$

$$a = -0.49985 \text{ or } 33.66 \text{ (rejected)}$$

By the first part of Proposition 20.3 on page 667, the time-0 price of the contingent claim is

$F_{0,1}^P[S(1)^a]$

$= e^{-r} S(0)^a \exp[ar + \frac{1}{2}a(a-1)\sigma^2]$

$= e^{-r}\mathrm{E}[S(1)^a]\exp[a(r-\alpha)]$

$= e^{-0.03} \times 1.4 \times e^{+0.01}$

$= 1.372$

Alternatively, one can calculate the time-0 price using the formula $\mathrm{E}^*[e^{-r} S(1)^a]$, where the asterisk signifies that the expectation is taken with respect to the risk-neutral probability measure.

12. Answer: E

By (9.13), call price is a decreasing function of K. Thus, $C(50, T) \geq C(55, T)$.
By the footnote on page 300,

$$C(50, T) - C(55, T) \leq (55 - 50)e^{-rT}.$$

Thus, (I) is correct.

For (II) and (III), we start with their middle expression:

$$P(45, T) - C(50, T) + S.$$

While there is not a direct relation between $P(45, T)$ and $C(50, T)$, we can use put-call parity to express $P(45, T)$ in terms of $C(45, T)$,

$$\begin{aligned} P(45, T) - C(50, T) + S &= [C(45, T) - S + 45e^{-rT}] - C(50, T) + S \\ &= C(45, T) - C(50, T) + 45e^{-rT}. \end{aligned}$$

Similar to (I), we have

$$0 \leq C(45, T) - C(50, T) \leq (50 - 45)e^{-rT},$$

which is equivalent to

$$45e^{-rT} \leq C(45, T) - C(50, T) + 45e^{-rT} \leq 50e^{-rT}.$$

Thus, (III) is correct.

Since (III) is correct, (II) must be incorrect.

13. Answer: A

8 months after purchasing the option, the remaining time to expiration = 4 months.

$$d_1 = \frac{\ln(85/75) + (0.05 - 0 + \frac{1}{2} \times 0.26^2) \times 4/12}{0.26\sqrt{4/12}} = 1.019888 \approx 1.02, \ N(d_1) \approx 0.8461,$$

$$d_2 = d_1 - \sigma\sqrt{T} = 1.019888 - 0.26\sqrt{4/12} = 0.869777 \approx 0.87, \ N(d_2) \approx 0.8078$$

At time of purchase,

$$C = SN(d_1) - Ke^{-rT}N(d_2) \approx 85 \times 0.8461 - 75e^{-0.05 \times (4/12)} \times 0.8078 = 12.3349$$

Hence, 8-month holding profit is $12.3349 - 8e^{0.05\times8/12} = 4.0637 \approx 4.06$.

14. Answer: E

By (24.31), $F_{0,2}[P(2, 3)] = \dfrac{P(0,3)}{P(0,2)}$.

In a Black-Derman-Toy model, r_{dd}, r_{ud} and r_{uu} are in a geometric progression. Thus,

$$\frac{r_{uu}}{r_{ud}} = \frac{r_{ud}}{r_{dd}} \Rightarrow \frac{0.8}{r_{ud}} = \frac{r_{ud}}{0.2} \Rightarrow r_{ud} = \sqrt{0.8 \times 0.2} = 40\% .$$

Because the risk-neutral probability of an "up" and a "down" move are both 0.5,

$$P(0,2) = 0.5 \times \left(\frac{1}{(1+r_0)(1+r_u)} + \frac{1}{(1+r_0)(1+r_d)} \right)$$
$$= \frac{0.5}{1+r_0}\left(\frac{1}{1.6} + \frac{1}{1.3} \right) = \frac{0.69712}{1+r_0}$$

and

$$P(0,3) = 0.25 \times \left(\frac{1}{(1+r_0)(1+r_u)(1+r_{uu})} + \frac{1}{(1+r_0)(1+r_u)(1+r_{ud})} + \frac{1}{(1+r_0)(1+r_d)(1+r_{ud})} + \frac{1}{(1+r_0)(1+r_d)(1+r_{dd})} \right)$$
$$= \frac{0.25}{1+r_0}\left(\frac{1}{1.6 \times 1.8} + \frac{1}{1.6 \times 1.4} + \frac{1}{1.3 \times 1.4} + \frac{1}{1.3 \times 1.2} \right) = \frac{0.49603}{1+r_0}$$

Thus,

$$F_{0,2}[P(2, 3)] = \frac{0.49603}{0.69712} = 0.7115 .$$

15. Answer: C

The short-rate process is a Vasicek model with $a = 0.1$, $b = 0.08$, and $\sigma = 0.05$.

We first determine the Sharpe ratio $\phi(r, t)$. By (24.2) and (24.19), if the true short-rate process is

$$\mathrm{d}r(t) = a(r(t))\,\mathrm{d}t + \sigma(r(t))\,\mathrm{d}Z(t),$$

then the risk-neutral short-rate process is

$$\mathrm{d}r(t) = [a(r(t)) + \sigma(r(t))\phi(r(t), t)]\mathrm{d}t + \sigma(r(t))\mathrm{d}\tilde{Z}(t),$$

where $\mathrm{d}\tilde{Z}(t) = \mathrm{d}Z(t) - \phi(r(t), t)\mathrm{d}t$. The stochastic process $\{\tilde{Z}(t)\}$ is a standard Brownian motion under the risk-neutral probability measure. By comparing the drift of the true process with that of the risk-neutral process, we get

$$\sigma(r)\phi(r, t) = 0.005.$$

Since $\sigma(r) = 0.05$, we have $\phi(r, t) = 0.1$ for all r and t.

Now it follows from (24.17) that

$$\frac{\alpha(0.04, 2, 5) - 0.04}{q(0.04, 2, 5)} = 0.1.$$

So we need to find $q(0.04, 2, 5)$. By (24.12),

$$q(r,\ t,\ T) = -\frac{P_r(r,\ t,\ T)}{P(r,\ t,\ T)} \times \sigma(r).$$

When the bond price has an affine structure (as in the case of Vasicek and CIR models), we have

$$-\frac{P_r(r,\ t,\ T)}{P(r,\ t,\ T)} = B(t, T),$$

or

$$q(r,\ t,\ T) = B(t,\ T) \times \sigma(r).$$

For the Vasicek model,

$$B(t,T) = \bar{a}_{\overline{T-t|}\text{force of interest}=a} = \frac{1 - \exp[-a(T-t)]}{a} = \frac{1 - e^{-0.1\times 3}}{0.1} = 2.591818.$$

Hence,

$$\alpha(0.04, 2, 5) = 0.04 + (0.1 \times 2.591818 \times 0.05) = 0.05296.$$

16. Answer: A

By line –3 on page 704, the risk-neutral probability that $S(T) > K$ is $N(d_2)$, where

$$d_2 = \frac{\ln(S(0)/K) + (r - \delta - ½\sigma^2)T}{\sigma\sqrt{T}}.$$

As a result, the true probability that $S(T) > K$ is $N(\hat{d}_2)$, where

$$\hat{d}_2 = \frac{\ln(S(0)/K) + (\alpha - \delta - ½\sigma^2)T}{\sigma\sqrt{T}},$$

which is (18.24).

Now, $S(0) = 100$, $\alpha = 0.1$, $\sigma = 0.3$, $\delta = 0$, $T = 0.75$, and $K = 125$, giving

$$\hat{d}_2 = \frac{\ln(100/125) + (0.1 - 0 - ½ \times 0.3^2) \times 0.75}{0.3\sqrt{0.75}} = -0.700109.$$

The answer is $N(-0.7) = 1 - N(0.7) = 1 - 0.7580 = 0.242$.

Alternative method:

Under the true probability measure,

$$\ln S(T) \sim \mathcal{N}(\ln S(0) + (\alpha - \delta - ½\sigma^2)T,\ \sigma^2 T),$$

which is a result given at the top of page 650.

$\Pr(S(T) > K)$

$= \Pr(\ln S(T) > \ln K)$

$= \Pr(Z > \dfrac{\ln K - [\ln S(0) + (\alpha - \delta - ½\sigma^2)T]}{\sigma\sqrt{T}})$ where $Z \sim N(0, 1)$

$= \Pr(Z > \dfrac{\ln 125 - \ln 100 - (0.1 - 0 - ½ \times 0.3^2) \times 0.75}{0.3\sqrt{0.75}})$

$= \Pr(Z > \dfrac{\ln 1.25 - 0.055 \times 0.75}{0.3\sqrt{0.75}})$

$= \Pr(Z > 0.700109)$

$= 1 - 0.7580$

$= 0.242$

17. Answer: A

(i) By the Black-Scholes formula, $P(S(0),T) = Ke^{-rT}N(-d_2) - S(0)e^{-\delta T}N(-d_1)$.
Since the 1-year put is at-the-money and the stock is nondividend-paying, we have $S(0) = K$ and $\delta = 0$. This yields

$$\frac{P(S(0),T)}{S(0)} = e^{-rT}N(-d_2) - N(-d_1) = e^{-0.012}N(-d_2) - N(-d_1),$$

where $d_1 = \dfrac{\ln[S(0)/K] + (r - \delta + ½\sigma^2)T}{\sigma\sqrt{T}} = \dfrac{0.012 + ½\sigma^2}{\sigma}$ and $d_2 = d_1 - \sigma$.

(ii) Delta of a put option is $-e^{-\delta T}N(-d_1) = -[1 - N(d_1)]$.

As a result, we have

$$e^{-0.012}N(-d_2) - N(-d_1) < 0.05 \qquad (1)$$

and

$$1 - N(d_1) = 0.4364. \qquad (2)$$

Equation (2) implies that $N(d_1) = 0.5636$, or $d_1 = 0.16$, which means

$$\frac{0.012 + ½\sigma^2}{\sigma} = 0.16$$

$$\sigma^2 - 0.32\sigma + 0.024 = 0$$

$$\sigma = 0.12 \quad \text{or} \quad \sigma = 0.2$$

Equation (1) implies that

$$N(-d_2) < e^{0.012}(0.05 + 0.4364) = 0.4923,$$

or $N(d_2) > 0.5077$, or $d_2 > 0.02$.

Since $d_2 = d_1 - \sigma$, and $d_1 = 0.16$, we must have $\sigma < 0.14$. So, $\sigma = 0.12$.

18: Answer: A

Let $\mathrm{d}S(t) = \alpha S(t)\mathrm{d}t + \sigma S(t)\mathrm{d}Z(t)$. Then

$$S(t) = S(0)\exp[(\alpha - 0.5\sigma^2)t + \sigma Z(t)].$$

Thus, for stock 1, $\sigma = 0.2$ and $\alpha = 0.1 + 0.5 \times 0.2^2 = 0.12$. For stock 2, $\sigma = 0.3$ and $\alpha = 0.125 + 0.5 \times 0.3^2 = 0.17$.

Because of the no-arbitrage constraint, (at each point of time) the Sharpe ratios $\dfrac{\alpha - r}{\sigma}$ of the two stocks must be equal:

$$\frac{0.12 - r}{0.2} = \frac{0.17 - r}{0.3}$$

$$0.36 - 3r = 0.34 - 2r$$

$$r = 0.02$$

19. Answer: D

The question asks for the put-option version of formula (12.5) on page 380. As pointed out in the last sentence of the first paragraph on page 381, σ is "the volatility of the prepaid forward." The formula for the unconditional variance in (iii) means that $\sigma^2 = 0.01$.

The time-0 prepaid forward price for time-1 delivery of the stock is

$$F_{0,1}^P(S) = S(0) - \mathrm{PV}_{0,1}(\mathrm{Div}) = 50 - 5e^{-0.12\times 0.75} = 45.4303$$

The prepaid forward price of the strike is its discounted value,

$$F_{0,1}^P(K) = 45e^{-0.12} = 39.9114.$$

Thus,

$$\begin{aligned} d_1 &= \frac{\ln[F_{0,1}^P(S)/F_{0,1}^P(K)] + ½\sigma^2 T}{\sigma\sqrt{T}} \\ &= \frac{\ln(45.4303/39.9114) + ½\times 0.1^2 \times 1}{0.1\sqrt{1}} \\ &= 1.34518 \approx 1.35 \end{aligned}$$

$$d_2 = d_1 - \sigma\sqrt{T} = 1.34518 - 0.1\sqrt{1} = 1.24518 \approx 1.25$$

The price of the one unit of the put option is

$$F_{0,1}^P(K)N(-d_2) - F_{0,1}^P(S)N(-d_1) = 39.9114(1 - 0.8944) - 45.4303(1 - 0.9115) = 0.1941$$

The price of 100 units of the put option is 19.41.

Remarks To derive (12.5), one assumes that the prepaid forward price process, $\{F_{t,T}^P(S)\,;\,t \le T\}$, is a geometric Brownian motion with volatility σ, i.e., one assumes that

$$\frac{\mathrm{d}F_{t,T}^P(S)}{F_{t,T}^P(S)} = \mu\mathrm{d}t + \sigma\mathrm{d}Z(t), \qquad t \le T,$$

or

$$F_{t,T}^P(S) = F_{0,T}^P(S)\exp[(\mu - \tfrac{1}{2}\sigma^2)t + \sigma Z(t)], \qquad t \le T.$$

Thus,

$$\mathrm{Var}[\ln F_{t,T}^P(S)] = \mathrm{Var}[\sigma Z(t)] = \sigma^2 t, \qquad t \le T,$$

which is condition (iii) in the question. For a derivation of (12.5), see Proposition 6.2.3 in the book *Martingale Methods in Financial Modelling* by M. Musiela and M. Rutkowski (1997).

The textbook treats two cases of dividend payments:

(i) The dividends are deterministic. That is, their amounts and when they are paid are known and fixed.

(ii) The stock pays dividends continuously at a rate proportional to its price. Because the stock price is stochastic, the dividends are stochastic.

Case (i): With deterministic dividends, the stock price is

$$S(t) = F_{t,T}^P(S) + \mathrm{PV}_{t,T}(\mathrm{Div}), \qquad t \le T,$$

which is equivalent to formula (5.3) on page 131 of McDonald (2006). Differentiating the equation with respect to t yields

$$\begin{aligned}\mathrm{d}S(t) &= \mathrm{d}F_{t,T}^P(S) + \mathrm{dPV}_{t,T}(\mathrm{Div}) \\ &= F_{t,T}^P(S)\,[\mu\mathrm{d}t + \sigma\mathrm{d}Z(t)] + \mathrm{dPV}_{t,T}(\mathrm{Div}).\end{aligned}$$

If t is not a dividend-payment date, then $\mathrm{dPV}_{t,T}(\mathrm{Div}) = \mathrm{PV}_{t,T}(\mathrm{Div})\,(r\mathrm{d}t)$. If t is a dividend-payment date, then the differential $\mathrm{dPV}_{t,T}(\mathrm{Div})$ is the negative of the amount of dividend paid at that time. Because of the stock price jumps downward at each dividend-payment date, the stock price process $\{S(t)\}$ does not have continuous sample paths and hence cannot be a geometric Brownian motion. It follows from

$$\begin{aligned}\frac{\mathrm{d}S(t)}{S(t)} &= \frac{F_{t,T}^P(S)[\mu\mathrm{d}t + \sigma\mathrm{d}Z(t)] + \mathrm{dPV}_{t,T}(\mathrm{Div})}{S(t)} \\ &= \frac{F_{t,T}^P(S)\mu\mathrm{d}t + \mathrm{dPV}_{t,T}(\mathrm{Div})}{S(t)} + \frac{F_{t,T}^P(S)}{S(t)}\sigma\mathrm{d}Z(t)\end{aligned}$$

that the volatility of the stock is $\dfrac{F_{t,T}^P(S)}{S(t)}\sigma$, which is a function of t, not a constant.

The expression $\dfrac{F_{t,T}^P(S)}{S(t)}\sigma$ gives a motivation for the "approximate correction" formula at the top of page 365 in McDonald (2006).

Case (ii): The time-t prepaid forward price is

$$F_{t,T}^{P}(S) = e^{-\delta(T-t)}S(t), \qquad t \le T.$$

It follows from Itô's Lemma that

$$\mathrm{d}\,F_{t,T}^{P}(S) = e^{-\delta(T-t)}S(t)\delta\mathrm{d}t + e^{-\delta(T-t)}\mathrm{d}S(t) + 0,$$

or

$$\frac{\mathrm{d}F_{t,T}^{P}(S)}{F_{t,T}^{P}(S)} = \delta\mathrm{d}t + \frac{\mathrm{d}S(t)}{S(t)}.$$

Hence,

$$\frac{\mathrm{d}S(t)}{S(t)} = (\alpha - \delta)\mathrm{d}t + \sigma\mathrm{d}Z(t)$$

if and only if

$$\frac{\mathrm{d}F_{t,T}^{P}(S)}{F_{t,T}^{P}(S)} = \alpha\mathrm{d}t + \sigma\mathrm{d}Z(t).$$

This means that the prepaid forward price process, $\{F_{t,T}^{P}(S)\,;\, t \le T\}$, is a geometric Brownian motion if and only if the stock price process, $\{S(t)\}$, is a geometric Brownian motion; both stochastic processes have the same parameter σ.

In case (i), the time-t price of the (deterministic) dividends paid between t and T is

$$S(t) - F_{t,T}^{P}(S) = \mathrm{PV}_{t,T}(\mathrm{Div}).$$

In case (ii), the time-t price of the (stochastic) dividends paid between t and T is

$$S(t) - F_{t,T}^{P}(S) = S(t) - e^{-\delta(T-t)}S(t) = S(t)[1 - e^{-\delta(T-t)}] = S(t)\delta\,\bar{a}_{\overline{T-t|}},$$

where the annuity-certain $\bar{a}_{\overline{T-t|}}$ is calculated using the dividend yield δ, not the risk-free rate r, as the force of interest.

20. Answer: C

According to equation (13.5) (or Taylor series expansion), for a small move of size ε in the stock price,

$$V(S+\varepsilon) \approx V(S) + \frac{1}{1\,!}V'(S)\varepsilon + \frac{1}{2\,!}V''(S)\varepsilon^2 = V(S) + \Delta(S)\varepsilon + \frac{1}{2}\Gamma(S)\varepsilon^2.$$

With $V(S) = 2.34$, $\Delta(S) = -0.181$, and $\Gamma(S) = 0.035$, the equation above becomes

$$2.21 = 2.34 + (-0.181)\varepsilon + \frac{1}{2}(0.035)\varepsilon^2,$$

or

$$0.0175\varepsilon^2 - 0.181\varepsilon + 0.13 = 0,$$

whose solutions are

$$\varepsilon = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{0.181 \pm \sqrt{(-0.181)^2 - 4\times 0.0175 \times 0.13}}{2\times 0.0175} = 9.566324 \text{ or } 0.776534$$

The first solution $\varepsilon = 9.566324$ is not a small move in the stock price. Thus,

$\varepsilon = 0.776534$ and $S(0) + \varepsilon = 86 \;\Rightarrow\; S(0) = 86 - 0.776534 = 85.223466 \approx 85.20.$