FALL 2005
EXAM C SOLUTIONS

Question #1
Key: D

$\hat{S}(300) = 3/10$ (there are three observations greater than 300)
$\hat{H}(300) = -\ln[\hat{S}(300)] = -\ln(0.3) = 1.204$.

Question #2
Key: A

$$E(X \mid \lambda) = Var(X \mid \lambda) = \lambda$$
$$\mu = v = E(\lambda) = \alpha\theta;\ a = Var(\lambda) = \alpha\theta^2;\ k = v/a = 1/\theta$$
$$Z = \frac{n}{n+1/\theta} = \frac{n\theta}{n\theta+1}$$
$$0.15 = \frac{\theta}{\theta+1}(1) + \frac{1}{\theta+1}\mu = \frac{\theta+\mu}{\theta+1}$$
$$0.20 = \frac{2\theta}{2\theta+1}(2) + \frac{1}{2\theta+1}\mu = \frac{4\theta+\mu}{2\theta+1}$$
From the first equation,
$0.15\theta + 0.15 = \theta + \mu$ and so $\mu = 0.15 - 0.85\theta$
Then the second equation becomes
$$0.4\theta + 0.2 = 4\theta + 0.15 - 0.85\theta$$
$$0.05 = 2.75\theta;\ \theta = 0.01818$$

Question #3
Key: E

$$0.75 = \frac{1}{1+(100/\theta)^\gamma};\ 0.25 = \frac{1}{1+(500/\theta)^\gamma}$$
$$(100/\theta)^\gamma = 1/3;\ (500/\theta)^\gamma = 3$$
Taking the ratio of these two equalities produces $5^\gamma = 9$. From the second equality,
$9 = [(500/\theta)^2]^\gamma = 5^\gamma;\ (500/\theta)^2 = 5;\ \theta = 223.61$

Question #4
Key: B

$f(x) = a + bx + cx^2 + dx^3$

$0 = f(0) = a$

$2 = f(2) = a + 2b + 4c + 8d$

$1 = f'(0) = b$

$-24 = f''(2) = 2c + 12d;\ c = -12 - 6d$

Insert the values for a, b, and c into the second equation to obtain
$2 = 2 + 4(-12 - 6d) + 8d;\ 48 = -16d;\ d = -3$

Then $c = 6$ and $f(x) = x + 6x^2 - 3x^3;\ f(1) = 4$

Question #5
Key: E

Begin with

y	350	500	1000	1200	1500
s	2	2	1	1	1
r	10	8	5	2	1

Then $\hat{S}_1(1250) = \frac{8}{10}\frac{6}{8}\frac{4}{5}\frac{1}{2} = 0.24$

The likelihood function is

$$L(\theta) = \left[\theta^{-1}e^{-350/\theta}\right]^2 \left[\theta^{-1}e^{-500/\theta}\right]^2 e^{-500/\theta}\theta^{-1}e^{-1000/\theta}\left[e^{-1000/\theta}\right]^2 \theta^{-1}e^{-1200/\theta}\theta^{-1}e^{-1500/\theta}$$
$$= \theta^{-7}e^{-7900/\theta}$$

$$l(\theta) = -7\ln\theta - \frac{7900}{\theta};\ l'(\theta) = -\frac{7}{\theta} + \frac{7900}{\theta^2} = 0;\ \hat{\theta} = 7900/7$$

$\hat{S}_2(1250) = e^{-1250(7)/7900} = 0.33$

The absolute difference is 0.09.

Question #6
Key: E

$$f(x) = -S'(x) = \frac{4x\theta^4}{(\theta^2 + x^2)^3}$$

$$L(\theta) = f(2)f(4)S(4) = \frac{4(2)\theta^4}{(\theta^2 + 2^2)^3}\frac{4(4)\theta^4}{(\theta^2 + 4^2)^3}\frac{\theta^4}{(\theta^2 + 4^2)^2} = \frac{128\theta^{12}}{(\theta^2 + 4)^3(\theta^2 + 16)^5}$$

$$l(\theta) = \ln 128 + 12\ln\theta - 3\ln(\theta^2 + 4) - 5\ln(\theta^2 + 16)$$

$$l'(\theta) = \frac{12}{\theta} - \frac{6\theta}{\theta^2 + 4} - \frac{10\theta}{\theta^2 + 16} = 0; 12(\theta^4 + 20\theta^2 + 64) - 6(\theta^4 + 16\theta^2) - 10(\theta^4 + 4\theta^2) = 0$$

$$0 = -4\theta^4 + 104\theta^2 + 768 = \theta^4 - 26\theta^2 - 192$$

$$\theta^2 = \frac{26 \pm \sqrt{26^2 + 4(192)}}{2} = 32; \theta = 5.657$$

Question #7
Key: A

$$E(X \mid \theta) = \int_0^\theta x\frac{2x}{\theta^2}dx = \frac{2\theta}{3}; Var(X \mid \theta) = \int_0^\theta x^2\frac{2x}{\theta^2}dx - \frac{4\theta^2}{9} = \frac{\theta^2}{2} - \frac{4\theta^2}{9} = \frac{\theta^2}{18}$$

$$\mu = (2/3)E(\theta) = (2/3)\int_0^1 4\theta^4 d\theta = 8/15$$

$$EVPV = v = (1/18)E(\theta^2) = (1/18)\int_0^1 4\theta^5 d\theta = 1/27$$

$$VHM = a = (2/3)^2 Var(\theta) = (4/9)\left[4/6 - (4/5)^2\right] = 8/675$$

$$k = \frac{1/27}{8/675} = 25/8; Z = \frac{1}{1 + 25/8} = 8/33$$

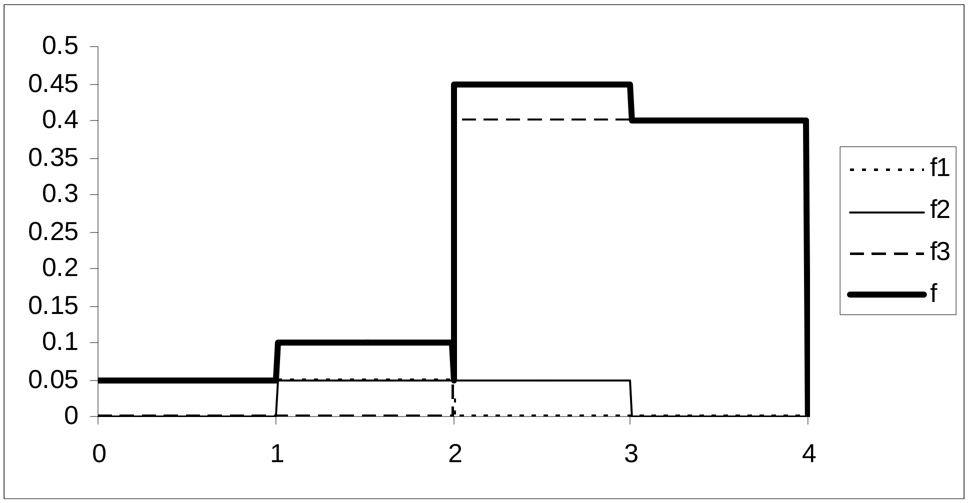
Estimate is $(8/33)(0.1) + (25/33)(8/15) = 0.428$.

Question #8
Key: D

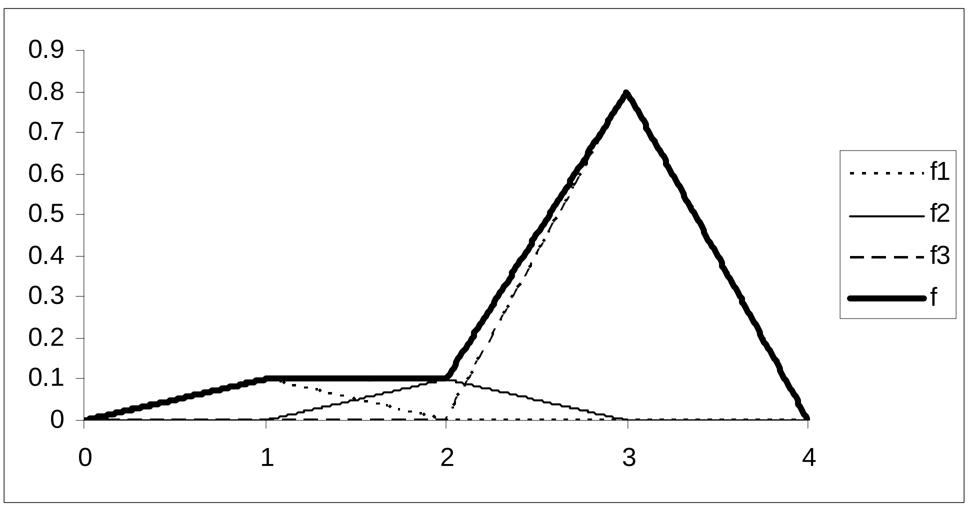
From the Poisson(4) distribution the probabilities at 0, 1, and 2 are 0.0183, 0.0733, and 0.1463. The cumulative probabilities are 0.0183, 0.0916, and 0.2381. Because 0.0916 < 0.13 < 0.2381 the simulated number of claims is 2. Claim amounts are simulated from solving $u = 1 - e^{-x/1000}$ for $x = -1000\ln(1-u)$. The two simulated amounts are 51.29 and 2995.73 for a total of 3047.02

Question #9
Key: B

It may be easiest to show this by graphing the density functions. For the first function the three components are each constant. One is of height 1/20 from 0 to 2 (representing the empirical probability of 1/10 at 1, one is height 1/20 from 1 to 3 and one is height 8/20 from 2 to 4. The following figure shows each of them and their sum, the kernel density estimator.



The triangular one is similar. For the triangle from 0 to 2, the area must be 1/10. With a base of 2, the height is 1/10. the same holds for the second triangle. The third has height 8/10. When added they look as follows;



The question asks about cumulative probabilities. From 0 to 1 the first is linear and the second is quadratic, but by x = 1 both have accumulated 0.05 of probability. Because the cumulative distribution functions are the same at 1 and the density functions are identical from 1 to 2, the distribution functions must be identical from 1 to 2.

Question #10
Key: D and E

For the Poisson distribution, the mean, λ, is estimated as 230/1000 = 0.23.

# of Days	Poisson Probability	Expected # of Workers	Observed # of Workers	χ^2
0	0.794533	794.53	818	0.69
1	0.182743	182.74	153	4.84
2	0.021015	21.02	25	0.75
3 or more	0.001709	1.71	4	3.07
Total			1000	9.35

The χ^2 distribution has 2 degrees of freedom because there are four categories and the Poisson parameter is estimated (d.f. = 4 – 1 – 1 = 2).

The critical values for a chi-square test with two degrees of freedom are shown in the following table.

Significance Level	Critical Value
10%	4.61
5%	5.99
2.5%	7.38
1%	9.21

9.35 is greater than 9.21 so the null hypothesis is rejected at the 1% significance level.

Question #11
Key: D

$EVPV = \hat{v} = \dfrac{25(480-472.73)^2 + 30(466.67-472.73)^2}{2-1} = 2423.03$ where 480 = 12,000/25, 466.67 = 14,000/30, and 472.73 = 26,000/55.

$k = 2423.03/254 = 9.54;\ Z = \dfrac{55}{55+9.54} = 0.852$

Question #12
Key: C

$$f''(x) = \begin{cases} 1.6x, & 0 < x < 1 \\ 1.6 - 0.8(x-1) = 2.4 - 0.8x, & 1 < x < 3 \end{cases} \quad S = \int_0^1 (1.6x)^2\,dx + \int_1^2 (2.4-0.8x)^2\,dx = 2.56$$

Question #13
Key: C

Relative risk = $e^{-\beta_1-\beta_2}$
which has partial derivatives $-e^{-0.2}$ at $\hat{\beta}_1 = 0.05$ and $\hat{\beta}_2 = 0.15$
Using the delta method, the variance of the relative risk is

$$\frac{1}{10,000}\begin{pmatrix}-e^{-0.2} & -e^{-0.2}\end{pmatrix}\begin{pmatrix}2 & 1\\ 1 & 3\end{pmatrix}\begin{pmatrix}-e^{-0.2}\\ -e^{-0.2}\end{pmatrix} = \frac{7e^{-0.4}}{10,000} = 0.000469$$

Std dev = 0.0217
upper limit = $e^{-0.2} + 1.96(0.0217)$
= 0.8613

Alternatively, consider the quantity $\beta_1 + \beta_2$. The variance is
$\frac{1}{10,000}\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}2 & 1\\ 1 & 3\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix} = \frac{7}{10,000} = 0.0007$. The lower limit for this quantity is
$0.2 - 1.96\sqrt{0.0007} = 0.1481$ and the upper limit for the relative risk is $e^{-0.1481} = 0.8623$.

Question #14
Key: C

The quantity of interest is $P = \Pr(X \le 5000) = \Phi\left(\frac{\ln 5000 - \mu}{\sigma}\right)$. The point estimate is
$\Phi\left(\frac{\ln 5000 - 6.84}{1.49}\right) = \Phi(1.125) = 0.87$.
For the delta method:
$\frac{\partial P}{\partial \mu} = \frac{-\phi(1.125)}{1.49} = -0.1422; \frac{\partial P}{\partial \sigma} = \frac{-1.125\phi(1.125)}{1.49} = -0.1600$ where $\phi(z) = \frac{1}{\sqrt{2\pi}}e^{-z^2/2}$.
Then the variance of $\hat{P}$ is estimated as $(-0.1422)^2 0.0444 + (-0.16)^2 0.0222 = 0.001466$ and the lower limit is $P_L = 0.87 - 1.96\sqrt{0.001466} = 0.79496$.

Question #15
Key: A

$$\Pr(\theta = 0.1 \mid X_1 = 1) = \frac{\Pr(X_1 = 1 \mid \theta = 0.1)\Pr(\theta = 0.1)}{\Pr(X_1 = 1 \mid \theta = 0.1)\Pr(\theta = 0.1) + \Pr(X_1 = 1 \mid \theta = 0.3)\Pr(\theta = 0.3)}$$

$$= \frac{0.1(0.8)}{0.1(0.8) + 0.3(0.2)} = \frac{4}{7}$$

Then,

$$E(X_2 \mid \theta = 0.1) = 0(0.2) + 1(0.1) + 2(0.7) = 1.5$$

$$E(X_2 \mid \theta = 0.3) = 0(0.6) + 1(0.3) + 2(0.1) = 0.5$$

$$E(X_2 \mid X_1 = 1) = (1.5)\frac{4}{7} + (0.5)\frac{3}{7} = 1.071$$

Question #16
Key: D

The requirement is that

$$0.01\hat{F}(1500) \geq 1.96\sqrt{\frac{\hat{F}(1500)\hat{S}(1500)}{N}}$$

$$0.0001\frac{P^2}{N^2} \geq 3.8416\frac{P(N-P)}{N^3}$$

$$\frac{NP}{N-P} \geq 38,416.$$

For the five answer choices, the left hand side is 34,364, 15,000, 27,125, 39,243, and 37,688. Only answer D meets the condition.

Question #17
Key: D

$$\frac{s_4}{r_4} = \hat{H}(y_4) - \hat{H}(y_3) = 0.5691 - 0.4128 = 0.1563.$$

$$\frac{s_4}{r_4^2} = \hat{V}[\hat{H}(y_4)] - \hat{V}[\hat{H}(y_3)] = 0.014448 - 0.009565 = 0.004883.$$

Therefore, $s_4 = \dfrac{(s_4 / r_4)^2}{s_4 / r_4^2} = \dfrac{0.1563^2}{0.004833} = 5.$

Question #18
Key: A

$$\ln f(x) = \ln\theta - 2\ln(\theta + x)$$
$$\frac{\partial \ln f(x)}{\partial\theta} = \frac{1}{\theta} - \frac{2}{\theta + x}$$
$$\frac{\partial^2 \ln f(x)}{\partial\theta^2} = -\frac{1}{\theta^2} + \frac{2}{(\theta + x)^2}$$
$$E\left[\frac{\partial^2 \ln f(x)}{\partial\theta^2}\right] = -\frac{1}{\theta^2} + \int_0^\infty \frac{2\theta}{(\theta + x)^4}dx = -\frac{1}{\theta^2} + \left[-\frac{2\theta}{3(\theta + x)^3}\right]\Bigg|_0^\infty = -\frac{1}{\theta^2} + \frac{2}{3\theta^2} = -\frac{1}{3\theta^2}$$
$$I(\theta) = \frac{n}{3\theta^2};\quad Var = \frac{3\theta^2}{n}$$

Question #19
Key: B

$$\mu = E[E(X \mid \lambda)] = E(\lambda) = 1(0.9) + 10(0.09) + 20(0.01) = 2$$
$$EVPV = v = E[Var(X \mid \lambda)] = E(\lambda) = 2$$
$$VHM = a = Var[E(X \mid \lambda)] = Var(\lambda) = 1(0.9) + 100(0.09) + 400(0.01) - 2^2 = 9.9$$
$$Z = \frac{1}{1 + 2/9.9} = 0.83193;\quad 11.983 = 0.83193x + 0.16807(2);\quad x = 14$$

Question #20
Key: A

The given interval for H can be written as $0.775 \pm 1.96\sqrt{0.063}$ and therefore the estimated variance of $\hat{H}$ is 0.063. To apply the delta method,

$$S = e^{-H};\quad \frac{dS}{dH} = -e^{-H};\quad V\hat{a}r(\hat{S}) \doteq (-e^{-\hat{H}})^2 Var(\hat{H}) = (-e^{-0.775})^2(0.063) = 0.134\,.$$

The point estimate of S is $e^{-0.775} = 0.4607$ and the confidence interval is $0.4607 \pm 1.96\sqrt{0.0134} = 0.2269$ or (0.23, 0.69).

Question #21
Key: B

The first step is to trend the year 1 data by 1.21 and the year 2 data by 1.1. The observations are now 24.2, 48.4, 60.5, 33, 44, 99, and 132.
The first two sample moments are 63.014 and 5262.64. The equations to solve are

$$63.014 = e^{\mu+0.5\sigma^2}; \quad 4.14336 = \mu + 0.5\sigma^2$$

$$5262.64 = e^{2\mu+2\sigma^2}; \quad 8.56839 = 2\mu + 2\sigma^2.$$

Taking four times the first equation and subtracting the second gives 2μ and therefore

$$\mu = \frac{4(4.14336) - 8.56839}{2} = 4.00.$$

Question #22
Key: A

$$\hat{\mu} = \bar{x} = 12/60 = 0.2, \; EVPV = \hat{v} = \bar{x} = 0.2$$

$$VHM = \hat{a} = \frac{10(0.4-0.2)^2 + 20(0.25-0.2)^2 + 30(0.1-0.2)^2 - (3-1)(0.2)}{60 - \frac{10^2+20^2+30^2}{60}} = 0.009545$$

$$\hat{k} = 20.9524; \quad Z = \frac{10}{10+20.9524} = 0.323$$

Question #23
Key: E

By elimination, (A) is incorrect because $f''(3) = -1.833 \neq 0$, (B) is incorrect because $f''(0) = -2 \neq 0$, (C) is incorrect because $f''(0) = -1 \neq 0$, and (D) is incorrect because $f''(0) = -1 \neq 0$. Therefore (E) must be correct. Also, this function does meet all the requirements:

$$f''(0) = 0; \quad f_0(1) = f_1(1) = 2; \quad f_0'(1) = f_1'(1) = 0; \quad f_0''(1) = f_1''(1) = 3; \quad f(3) = 6; \quad f''(3) = 0$$

Question #24
Key: B and C

For males, $c_j = 1$ and for females, $c_j = e^{0.27} = 1.31$. Then,

$$\hat{H}(20) = \frac{1}{3+2(1.31)} + \frac{1}{2+2(1.31)} + \frac{1}{2+1.31} = 0.6965 \text{ and } \hat{S}_{female}(20) = \left(e^{-0.6965}\right)^{1.31} = 0.402.$$

.

Question #25
Key: C

$l(\tau,\theta)=\sum_{j=1}^{5}\ln f(x_j)=\sum_{j=1}^{5}\ln\tau+(\tau-1)\ln x_j-\tau\ln\theta-(x_j/\theta)^{\tau}$. Under the null hypothesis it is $l(2,\theta)=\sum_{j=1}^{5}\ln 2+\ln x_j-2\ln\theta-(x_j/\theta)^2$. Inserting the maximizing value of 816.7 for θ gives -35.28. The likelihood ratio test statistic is $2(-33.05+35.28)=4.46$. There is one degree of freedom. At a 5% significance level the critical value is 3.84 and at a 2.5% significance level it is 5.02.

Question #26
Key: C

It is given that n = 4, v = 8, and Z = 0.4. Then, $0.4=\dfrac{4}{4+\frac{8}{a}}$ which solves for a = 4/3. For the covariance,

$$\begin{aligned}Cov(X_i,X_j)&=E(X_iX_j)-E(X_i)E(X_j)\\&=E[E(X_iX_j\mid\theta)]-E[E(X_i\mid\theta)]E[E(X_j\mid\theta)]\\&=E[\mu(\theta)^2]-E[\mu(\theta)]^2=Var[\mu(\theta)]=a=4/3.\end{aligned}$$

Question #27
Key: A

u	z	x	lognormal	with deductible
0.6217	0.31	5.8325	341.21	241.21
0.9941	2.52	7.49	1790.05	1690.05
0.8686	1.12	6.44	626.41	526.41
0.0485	-1.66	4.355	77.87	0
			Average	614.42

The value of z is obtained by inversion from the standard normal table. That is, $u=\Pr(Z\le z)$. The value of x is obtained from $x=0.75z+5.6$. The lognormal value is obtained by exponentiating x and the final column applies the deductible.

Question #28
Key: B

$$MSE = E[(X^2 - \theta^2)^2] = E(X^4 - 2X^2\theta^2 + \theta^4)$$
$$= 24\theta^4 - 2(2\theta^2)\theta^2 + \theta^4 = 21\theta^4$$

Question #29
Key: C

The sample mean of $\dfrac{157(0)+66(1)+19(2)+4(3)+2(4)}{248} = 0.5$ is the maximum likelihood estimate of the geometric parameter β as well as the method of moments estimate of the Poisson parameter λ. Then, $P = (1+0.5)^{-1} = 0.6667$ and $Q = e^{-0.5} = 0.6065$. The absolute difference is 0.0602.

Question #30
Key: D

$$\bar{x} = \frac{5000(0)+2100(1)+750(2)+100(3)+50(4)}{8000} = 0.5125 \text{ and}$$
$$s^2 = \frac{5000(0.5125)^2 + 2100(0.4875)^2 + 750(1.4875)^2 + 100(2.4875)^2 + 50(3.4875)^2}{7999} = 0.5874.$$
Then, $\hat{\mu} = \hat{v} = \bar{x} = 0.5125$ and $\hat{a} = s^2 - \bar{x} = 0.0749$. The credibility factor is
$$Z = \frac{1}{1+0.5125/0.0749} = 0.1275 \text{ and the estimate is } 0.1275(1)+0.8725(0.5125) = 0.5747.$$

Question #31
Key: B

$s = F_n(3000) = 4/8 = 0.5$ because for the *p-p* plot the denominator is *n*+1.
$t = F(3000) = 1 - e^{-3000/3300} = 0.59711$. For the difference plot, *D* uses a denominator of *n* and so $D = 4/7 - 0.59711 = -0.02568$ and the answer is $0.5 - 0.59711 + 0.02568 = -0.071$.

Question #32
Key: B

$\pi(q\,|\,2,2) \propto f(2\,|\,q)f(2\,|\,q)\pi(q) = q(q)(q^2/0.039) \propto q^4$. Because $\int_{0.2}^{0.5} q^4 dq = 0.006186$,
$\pi(q\,|\,2,2) = q^4/0.006186$. Given *q*, the expected number of claims is
$E(N\,|\,q) = 0(0.1) + 1(0.9-q) + 2q = 0.9+q$. The Bayesian estimate is

$$E(N\,|\,2,2) = \int_{0.2}^{0.5}(0.9+q)\frac{q^4}{0.006186}dq = 1.319.$$

Question #33
Key: E

$$0.689 = F_{500}(1500) = 0.5F_{500}(1000) + 0.5F_{500}(2000) = 0.5\left(\frac{200+110}{500} + \frac{310+x}{500}\right) \Rightarrow x = 69$$

$$0.839 = F_{500}(3500) = 0.5F_{500}(2000) + 0.5F_{500}(5000) = 0.5\left(\frac{310+69}{500} + \frac{379+y}{500}\right) \Rightarrow y = 81$$

Question #34
Key: A

A is false because the test works best when the expected number of observations is about the same from interval to interval. B is true (*Loss Models*, 427-8), C is true (*Loss Models*, 428), and D is true (*Loss Models*, 430).

Question #35
Key: E

$$n\lambda \ge \lambda_0\left[1+\left(\frac{\sigma_Y}{\theta_Y}\right)^2\right];\ \theta_Y = \alpha\theta = 10{,}000\alpha;\ \sigma_Y^2 = \alpha\theta^2 = 10^8\alpha$$

$$n\lambda \ge \left(\frac{1.96}{0.1}\right)^2\left[1+\frac{10^8\alpha}{10^8\alpha^2}\right] = 384.16(1+\alpha^{-1})$$

Because α is needed, but not given, the answer cannot be determined from the information given.