

SOCIETY OF ACTUARIES

EXAM FM FINANCIAL MATHEMATICS

EXAM FM SAMPLE SOLUTIONS

Interest Theory

This page indicates changes made to Study Note FM-09-05.

January 14, 2014:

Questions and solutions 58–60 were added.

June, 2014

Question 58 was moved to the Derivatives Markets set of sample questions.

Questions 61-73 were added.

Many of the questions were re-worded to conform to the current style of question writing. The substance was not changed.

Some of the questions in this study note are taken from past SOA/CAS examinations.

These questions are representative of the types of questions that might be asked of candidates sitting for the Financial Mathematics (FM) Exam. These questions are intended to represent the depth of understanding required of candidates. The distribution of questions by topic is not intended to represent the distribution of questions on future exams.

The following model solutions are presented for educational purposes. Alternate methods of solution are, of course, acceptable.

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1. Solution: C

Given the same principal invested for the same period of time yields the same accumulated value, the two measures of interest $i^{(2)} = 0.04$ and δ must be equivalent, which means:

$\left(1 + \frac{i^{(2)}}{2}\right)^2 = e^\delta$ over a one-year period. Thus,

$$e^\delta = \left(1 + \frac{i^{(2)}}{2}\right)^2 = 1.02^2 = 1.0404$$

$$\delta = \ln(1.0404) = 0.0396.$$

2. Solution: E

From basic principles, the accumulated values after 20 and 40 years are

$$100[(1+i)^{20} + (1+i)^{16} + \dots + (1+i)^4] = 100 \frac{(1+i)^4 - (1+i)^{24}}{1 - (1+i)^4}$$

$$100[(1+i)^{40} + (1+i)^{36} + \dots + (1+i)^4] = 100 \frac{(1+i)^4 - (1+i)^{44}}{1 - (1+i)^4}.$$

The ratio is 5, and thus (setting $x = (1+i)^4$)

$$5 = \frac{(1+i)^4 - (1+i)^{44}}{(1+i)^4 - (1+i)^{24}} = \frac{x - x^{11}}{x - x^6}$$

$$5x - 5x^6 = x - x^{11}$$

$$5 - 5x^5 = 1 - x^{10}$$

$$x^{10} - 5x^5 + 4 = 0$$

$$(x^5 - 1)(x^5 - 4) = 0.$$

Only the second root gives a positive solution. Thus

$$x^5 = 4$$

$$x = 1.31951$$

$$X = 100 \frac{1.31951 - 1.31951^{11}}{1 - 1.31951} = 6195.$$

Annuity symbols can also be used. Using the annual interest rate, the equation is

$$100 \frac{s_{\overline{40}|}}{a_{\overline{4}|}} = 5(100) \frac{s_{\overline{20}|}}{a_{\overline{4}|}}$$

$$\frac{(1+i)^{40} - 1}{i} = 5 \frac{(1+i)^{20} - 1}{i}$$

$$(1+i)^{40} - 5(1+i)^{20} + 4 = 0$$

$$(1+i)^{20} = 4$$

and the solution proceeds as above.

3. Solution: C

Eric's (compound) interest in the last 6 months of the 8th year is $100 \left(1 + \frac{i}{2}\right)^{17} \frac{i}{2}$.

Mike's (simple) interest for the same period is $100 \frac{i}{2}$.

Thus,

$$100 \left(1 + \frac{i}{2}\right)^{15} \frac{i}{2} = 200 \frac{i}{2}$$

$$\left(1 + \frac{i}{2}\right)^{15} = 2$$

$$1 + \frac{i}{2} = 1.047294$$

$$i = 0.09459 = 9.46\%$$

4. Solution: A

The periodic interest is $0.10(10,000) = 1000$. Thus, deposits into the sinking fund are $1627.45 - 1000 = 627.45$.

Then, the amount in sinking fund at end of 10 years is $627.45 s_{\overline{10}|0.14} = 12,133$. After repaying the loan, the fund has 2,133, which rounds to 2,130.

5. Solution: E

The beginning balance combined with deposits and withdrawals is $75 + 12(10) - 5 - 25 - 80 - 35 = 50$. The ending balance of 60 implies 10 in interest was earned.

The denominator is the average fund exposed to earning interest. One way to calculate it is to weight each deposit or withdrawal by the remaining time:

$$75(1) + 10\left(\frac{11}{12} + \frac{10}{12} + \cdots + \frac{0}{12}\right) - 5\frac{10}{12} - 25\frac{6}{12} - 80\frac{5}{24} - 35\frac{2}{12} = 90.833.$$

The rate of return is $10/90.833 = 0.11009 = 11.0\%$.

6. Solution: C

$$\begin{aligned} 77.1 &= v(Ia)_{\overline{n}|} + \frac{nv^{n+1}}{i} \\ &= v\left[\frac{\ddot{a}_{\overline{n}|} - nv^n}{i}\right] + \frac{nv^{n+1}}{i} \\ &= \frac{a_{\overline{n}|}}{i} - \frac{nv^{n+1}}{i} + \frac{nv^{n+1}}{i} \\ &= \frac{a_{\overline{n}|}}{i} = \frac{1-v^n}{i^2} = \frac{1-v^n}{0.011025} \end{aligned}$$

$$0.85003 = 1 - v^n$$

$$1.105^{-n} = 0.14997$$

$$n = -\frac{\ln(0.14997)}{\ln(1.105)} = 19.$$

To obtain the present value without remembering the formula for an increasing annuity, consider the payments as a perpetuity of 1 starting at time 2, a perpetuity of 1 starting at time 3, up to a perpetuity of 1 starting at time $n + 1$. The present value one period before the start of each perpetuity is $1/i$. The total present value is $(1/i)(v + v^2 + \cdots + v^n) = (1/i)a_{\overline{n}|}$.

7. Solution: C

The interest earned is a decreasing annuity of 6, 5.4, etc. Combined with the annual deposits of 100, the accumulated value in fund Y is

$$\begin{aligned} & 6(Ds)_{\overline{10}|0.09} + 100s_{\overline{10}|0.09} \\ &= 6 \left(\frac{10(1.09)^{10} - s_{\overline{10}|0.09}}{0.09} \right) + 100(15.19293) \\ &= 565.38 + 1519.29 \\ &= 2084.67. \end{aligned}$$

8. Deleted

9. Solution: D

For the first 10 years, each payment equals 150% of interest due. The lender charges 10%, therefore 5% of the principal outstanding will be used to reduce the principal.

At the end of 10 years, the amount outstanding is $1000(1-0.05)^{10} = 598.74$.

Thus, the equation of value for the last 10 years using a comparison date of the end of year 10 is

$$598.74 = Xa_{\overline{10}|10\%} = 6.1446X$$

$$X = 97.44.$$

10. Solution: B

The book value at time 6 is the present value of future payments:

$$BV_6 = 10,000v^4 + 800a_{\overline{4}|0.06} = 7920.94 + 2772.08 = 10,693.$$

The interest portion is $10,693(0.06) = 641.58$.

11. Solution: A

The value of the perpetuity after the fifth payment is $100/0.08 = 1250$. The equation to solve is:

$$1250 = X(v + 1.08v^2 + \cdots + 1.08^{24}v^{25})$$

$$= X(v + v + \cdots + v) = X(25)/1.08$$

$$X = 50(1.08) = 54.$$

12. Solution: C

Equation of value at end of 30 years:

$$10(1-d/4)^{-40}(1.03)^{40} + 20(1.03)^{30} = 100$$

$$10(1-d/4)^{-40} = [100 - 20(1.03)^{30}] / 1.03^{40} = 15.7738$$

$$1-d/4 = 1.57738^{-1/40} = 0.98867$$

$$d = 4(1-0.98867) = 0.0453 = 4.53\%$$

13. Solution: E

The accumulation function is $a(t) = \exp\left[\int_0^t (s^2/100)ds\right] = \exp(t^3/300)$.

The accumulated value of 100 at time 3 is $100\exp(3^3/300) = 109.41743$.

The amount of interest earned from time 3 to time 6 equals the accumulated value at time 6 minus the accumulated value at time 3. Thus

$$(109.41743 + X)[a(6)/a(3) - 1] = X$$

$$(109.41743 + X)(2.0544332/1.0941743 - 1) = X$$

$$(109.41743 + X)0.877613 = X$$

$$96.026159 = 0.122387X$$

$$X = 784.61.$$

14. Solution: A

$$167.50 = 10a_{\overline{5}|9.2\%} + 10(1.092)^{-5} \sum_{t=1}^{\infty} \left[\frac{(1+k)}{1.092} \right]^t$$

$$167.50 = 38.86955 + 6.44001 \frac{(1+k)/1.092}{1 - (1+k)/1.092}$$

$$(167.50 - 38.86955)[1 - (1+k)/1.092] = 6.44001(1+k)/1.092$$

$$128.63045 = 135.07046(1+k)/1.092$$

$$1+k = 1.0399$$

$$k = 0.0399 \Rightarrow K = 3.99\%$$

15. Solution: B

$$\text{Option 1: } 2000 = Pa_{\overline{10}|0.0807}$$

$$P = 299 \Rightarrow \text{Total payments} = 2990$$

Option 2: Interest needs to be $2990 - 2000 = 990$

$$990 = i[2000 + 1800 + 1600 + \dots + 200]$$

$$= 11,000i$$

$$i = 0.09 = 9.00\%$$

16. Solution: B

Monthly payment at time t is $1000(0.98)^{t-1}$.

Because the loan amount is unknown, the outstanding balance must be calculated prospectively.

The value at time 40 months is the present value of payments from time 41 to time 60:

$$\begin{aligned} OB_{40} &= 1000[0.98^{40}v^1 + \dots + 0.98^{59}v^{20}] \\ &= 1000 \frac{0.98^{40}v^1 - 0.98^{60}v^{21}}{1 - 0.98v}, v = 1/(1.0075) \\ &= 1000 \frac{0.44238 - 0.25434}{1 - 0.97270} = 6888. \end{aligned}$$

17. Solution: C

The equation of value is

$$98S_{\overline{3n}|} + 98S_{\overline{2n}|} = 8000$$

$$\frac{(1+i)^{3n} - 1}{i} + \frac{(1+i)^{2n} - 1}{i} = 81.63$$

$$(1+i)^n = 2$$

$$\frac{8-1}{i} + \frac{4-1}{i} = 81.63$$

$$\frac{10}{i} = 81.63$$

$$i = 12.25\%$$

18. Solution: B

Convert 9% convertible quarterly to an effective rate of j per month:

$$(1+j)^3 = \left(1 + \frac{0.09}{4}\right) \text{ or } j = 0.00744.$$

Then

$$2(Ia)_{\overline{60}|0.00744} = 2 \frac{\ddot{a}_{\overline{60}|0.00744} - 60v^{60}}{0.00744} = 2 \frac{48.6136 - 38.4592}{0.00744} = 2729.7.$$

19. Solution: C

For Account K, the amount of interest earned is $125 - 100 - 2X + X = 25 - X$.

The average amount exposed to earning interest is $100 - (1/2)X + (1/4)2X = 100$. Then

$$i = \frac{25 - X}{100}.$$

For Account L, examine only intervals separated by deposits or withdrawals. Determine the interest for the year by multiplying the ratios of ending balance to beginning balance. Then

$$i = \frac{125}{100} \frac{105.8}{125 - X} - 1.$$

Setting the two equations equal to each other and solving for X ,

$$\frac{25 - X}{100} = \frac{13,225}{100(125 - X)} - 1$$

$$(25 - X)(125 - X) = 13,225 - 100(125 - X)$$

$$3,125 - 150X + X^2 = 13,225 - 12,500 + 100X$$

$$X^2 - 250X + 2,400 = 0$$

$$X = 10.$$

Then $i = (25 - 10)/100 = 0.15 = 15\%$.

20. Solution: A

Equating present values:

$$100 + 200v^n + 300v^{3n} = 600v^{10}$$

$$100 + 200(0.76) + 300(0.76)^2 = 600v^{10}$$

$$425.28 = 600v^{10}$$

$$0.7088 = v^{10}$$

$$0.96617 = v$$

$$1.03501 = 1 + i$$

$$i = 0.035 = 3.5\%.$$

21. Solution: A

The accumulation function is:

$$a(t) = e^{\int_0^t \frac{1}{8+r} dr} = e^{\ln(8+r)|_0^t} = \frac{8+t}{8}.$$

Using the equation of value at end of 10 years:

$$\begin{aligned} 20,000 &= \int_0^{10} (8k + tk) \frac{a(10)}{a(t)} dt = k \int_0^{10} (8+t) \frac{18/8}{(8+t)/8} dt = k \int_0^{10} 18 dt \\ &= 180k \Rightarrow k = \frac{20,000}{180} = 111. \end{aligned}$$

22. Solution: D

Let C be the redemption value and $v = 1/(1+i)$. Then

$$\begin{aligned} X &= 1000ra_{\overline{2n}|i} + Cv^{2n} \\ &= 1000r \frac{1-v^{2n}}{i} + 381.50 \\ &= 1000(1.03125)(1-0.5889^2) + 381.50 \\ &= 1055.11. \end{aligned}$$

23. Solution: D

Equate net present values:

$$-4000 + 2000v + 4000v^2 = 2000 + 4000v - Xv^2$$

$$\frac{4000 + X}{1.21} = 6000 + \frac{2000}{1.1}$$

$$X = 5460.$$

24. Solution: E

For the amortization method, the payment is determined by

$$20,000 = Xa_{\overline{20}|0.065} = 11.0185, \quad X = 1815.13.$$

For the sinking fund method, interest is $0.08(2000) = 1600$ and total payment is given as X , the same as for the amortization method. Thus the sinking fund deposit $= X - 1600 = 1815.13 - 1600 = 215.13$.

The sinking fund, at rate j , must accumulate to 20000 in 20 years. Thus, $215.13s_{\overline{20}|j} = 20,000$, which yields (using calculator) $j = 14.18\%$.

25. Solution: D

The present value of the perpetuity $= X/i$. Let B be the present value of Brian's payments.

$$B = Xa_{\overline{n}|} = 0.4 \frac{X}{i}$$

$$a_{\overline{n}|} = \frac{0.4}{i} \Rightarrow 0.4 = 1 - v^n \Rightarrow v^n = 0.6$$

$$K = v^{2n} \frac{X}{i}$$

$$K = 0.36 \frac{X}{i},$$

Thus the charity's share is 36% of the perpetuity's present value.

26. Solution: D

The given information yields the following amounts of interest paid:

$$\text{Seth} = 5000 \left(\left(1 + \frac{0.12}{2} \right)^{10} - 1 \right) = 8954.24 - 5000 = 3954.24$$

$$\text{Janice} = 5000(0.06)(10) = 3000.00$$

$$\text{Lori} = P(10) - 5000 = 1793.40 \text{ where } P = \frac{5000}{a_{\overline{10}|6\%}} = 679.35$$

The sum is 8747.64.

27. Solution: E

For Bruce, $X = 100[(1+i)^{11} - (1+i)^{10}] = 100(1+i)^{10}i$. Similarly, for Robbie, $X = 50(1+i)^{16}i$. Dividing the second equation by the first gives $1 = 0.5(1+i)^6$ which implies $i = 2^{1/6} - 1 = 0.122462$. Thus $X = 100(1.122462)^{10}(0.122462) = 38.879$.

28. Solution: D

$$\text{Year } t \text{ interest is } ia_{\overline{n-t+1}|i} = 1 - v^{n-t+1}.$$

$$\text{Year } t+1 \text{ principal repaid is } 1 - (1 - v^{n-t}) = v^{n-t}.$$

$$X = 1 - v^{n-t+1} + v^{n-t} = 1 + v^{n-t}(1 - v) = 1 + v^{n-t}d.$$

29. Solution: B

For the first perpetuity,

$$32 = 10(v^3 + v^6 + \dots) = 10v^3 / (1 - v^3)$$

$$32 - 32v^3 = 10v^3$$

$$v^3 = 32 / 42.$$

For the second perpetuity,

$$X = v^{1/3} + v^{2/3} + \dots = v^{1/3} / (1 - v^{1/3}) = (32 / 42)^{1/9} / [1 - (32 / 42)^{1/9}] = 32.599.$$

30. Solution: D

Under either scenario, the company will have $822,703(0.05) = 41,135$ to invest at the end of each of the four years. Under Scenario A these payments will be invested at 4.5% and accumulate to $41,135s_{\overline{4}|0.045} = 41,135(4.2782) = 175,984$. Adding the maturity value produces 998,687 for a loss of 1,313. Note that only answer D has this value.

The Scenario B calculation is

$$41,135s_{\overline{4}|0.055} = 41,135(4.3423) = 178,621 + 822,703 - 1,000,000 = 1,324.$$

31. Solution: D.

The present value is

$$\begin{aligned} & 5000[1.07v + 1.07^2v^2 + \dots + 1.07^{20}v^{20}] \\ &= 5000 \frac{1.07v - 1.07^{21}v^{21}}{1 - 1.07v} = 5000 \frac{1.01905 - 1.48622}{1 - 1.01905} = 122,617. \end{aligned}$$

32. Solution: C.

The first cash flow of 60,000 at time 3 earns 2400 in interest for a time 4 receipt of 62,400. Combined with the final payment, the investment returns 122,400 at time 4. The present value is $122,400(1.05)^{-4} = 100,699$. The net present value is 699.

33. Solution: B.

Using spot rates, the value of the bond is:

$$60/1.07 + 60/1.08^2 + 1060/1.09^3 = 926.03.$$

34. Solution: E.

Using spot rates, the value of the bond is:

$$60/1.07 + 60/1.08^2 + 1060/1.09^3 = 926.03. \text{ The annual effective rate is the solution to } 926.03 = 60a_{\overline{3}|i} + 1000(1+i)^{-3}. \text{ Using a calculator, the solution is 8.9\%.}$$

35. Solution: C.

Duration is the negative derivative of the price multiplied by one plus the interest rate and divided by the price. Hence, the duration is $-(-700)(1.08)/100 = 7.56$.

36. Solution: C

The size of the dividend does not matter, so assume it is 1. Then the duration is

$$\frac{\sum_{t=1}^{\infty} tv^t}{\sum_{t=1}^{\infty} v^t} = \frac{(Ia)_{\infty|}}{a_{\infty|}} = \frac{\ddot{a}_{\infty|}/i}{1/i} = \frac{1/(di)}{1/i} = \frac{1}{d} = \frac{1.1}{0.1} = 11.$$

37. Solution: B

$$\text{Duration} = \frac{\sum_{t=1}^{\infty} tv^t R_t}{\sum_{t=1}^{\infty} v^t R_t} = \frac{\sum_{t=1}^{\infty} tv^t 1.02^t}{\sum_{t=1}^{\infty} v^t 1.02^t} = \frac{(Ia)_{\infty|j}}{a_{\infty|j}} = \frac{\ddot{a}_{\infty|j}/j}{1/j} = \frac{1}{d}.$$

The interest rate j is such that $(1+j)^{-1} = 1.02v = 1.02/1.05 \Rightarrow j = 0.03/1.02$. Then the duration is $1/d = (1+j)/j = (1.05/1.02)/(0.03/1.02) = 1.05/0.03 = 35$.

45. Solution: A

For the time weighted return the equation is:

$$1+0 = \frac{12}{10} \frac{X}{12+X} \Rightarrow 120+10X = 12X \Rightarrow 120 = 2X \Rightarrow X = 60.$$

Then the amount of interest earned in the year is $60 - 60 - 10 = -10$ and the weighted amount exposed to earning interest is $10(1) + 60(0.5) = 40$. Then $Y = -10/40 = -25\%$.

46. Solution: A

The outstanding balance is the present value of future payments. With only one future payment, that payment must be $559.12(1.08) = 603.85$. The amount borrowed is $603.85a_{4|0.08} = 2000$. The first payment has $2000(0.08) = 160$ in interest, thus the principal repaid is $603.85 - 160 = 443.85$.

Alternatively, observe that the principal repaid in the final payment is the outstanding loan balance at the previous payment, or 559.12. Principal repayments form a geometrically decreasing sequence, so the principal repaid in the first payment is $559.12/1.08^3 = 443.85$.

47. Solution: B

Because the yield rate equals the coupon rate, Bill paid 1000 for the bond. In return he receives 30 every six months, which accumulates to $30s_{\overline{20}|j}$ where j is the semi-annual interest rate. The equation of value is $1000(1.07)^{10} = 30s_{\overline{20}|j} + 1000 \Rightarrow s_{\overline{20}|j} = 32.238$. Using a calculator to solve for the interest rate produces $j = 0.0476$ and so $i = 1.0476^2 - 1 = 0.0975 = 9.75\%$.

48. Solution: A

To receive 3000 per month at age 65 the fund must accumulate to $3,000(1,000/9.65) = 310,880.83$. The equation of value is $310,880.83 = X\ddot{s}_{\overline{300}|0.08/12} = 957.36657X \Rightarrow 324.72$.

49. Solution: D

(A) The left-hand side evaluates the deposits at age 0, while the right-hand side evaluates the withdrawals at age 17.

(B) The left-hand side has 16 deposits, not 17.

(C) The left-hand side has 18 deposits, not 17.

(D) The left-hand side evaluates the deposits at age 18 and the right-hand side evaluates the withdrawals at age 18.

(E) The left-hand side has 18 deposits, not 17 and 5 withdrawals, not 4.

50. Deleted

51. Solution: D

Because only Bond II provides a cash flow at time 1, it must be considered first. The bond provides 1025 at time 1 and thus $1000/1025 = 0.97561$ units of this bond provides the required cash. This bond then also provides $0.97561(25) = 24.39025$ at time 0.5. Thus Bond I must provide $1000 - 24.39025 = 975.60975$ at time 0.5. The bond provides 1040 and thus $975.60975/1040 = 0.93809$ units must be purchased.

52. Solution: C

Because only Mortgage II provides a cash flow at time two, it must be considered first. The mortgage provides $Y / a_{\overline{20}|0.07} = 0.553092Y$ at times one and two. Therefore, $0.553092Y = 1000$ for $Y = 1808.02$. Mortgage I must provide $2000 - 1000 = 1000$ at time one and thus $X = 1000/1.06 = 943.40$. The sum is 2751.42.

53. Solution: A

Bond I provides the cash flow at time one. Because 1000 is needed, one unit of the bond should be purchased, at a cost of $1000/1.06 = 943.40$.

Bond II must provide 2000 at time three. Therefore, the amount to be reinvested at time two is $2000/1.065 = 1877.93$. The purchase price of the two-year bond is $1877.93/1.07^2 = 1640.26$.

The total price is 2583.66.

54. Solution: C

Given the coupon rate is greater than the yield rate, the bond sells at a premium. Thus, the minimum yield rate for this callable bond is calculated based on a call at the earliest possible date because that is most disadvantageous to the bond holder (earliest time at which a loss occurs). Thus, X , the par value, which equals the redemption value because the bond is a par value bond, must satisfy

$$\text{Price} = 1722.25 = 0.04Xa_{\overline{30}|0.03} + Xv_{0.03}^{30} = 1.196X \Rightarrow X = 1440.$$

55. Solution: B

Because $40/1200$ is greater than 0.03 , for early redemption the earliest redemption should be evaluated. If redeemed after 15 years, the price is $40a_{\overline{30}|0.03} + 1200/1.03^{30} = 1278.40$. If the bond is redeemed at maturity, the price is $40a_{\overline{40}|0.03} + 1100/1.03^{40} = 1261.80$. The smallest value should be selected, which is 1261.80.

56. Solution: E

Given the coupon rate is less than the yield rate, the bond sells at a discount. Thus, the minimum yield rate for this callable bond is calculated based on a call at the latest possible date because that is most disadvantageous to the bond holder (latest time at which a gain occurs). Thus, X , the par value, which equals the redemption value because the bond is a par value bond, must satisfy

$$\text{Price} = 1021.50 = 0.02Xa_{\overline{20}|0.03} + Xv_{0.03}^{20} = 0.851225X \Rightarrow X = 1200.$$

57. Solution: B

Given the price is less than the amount paid for an early call, the minimum yield rate for this callable bond is calculated based on a call at the latest possible date. Thus, for an early call, the effective yield rate per coupon period, j , must satisfy $\text{Price} = 1021.50 = 22a_{\overline{19}|j} + 1200v_j^{19}$. Using the calculator, $j = 2.67\%$. We also must check the yield if the bond is redeemed at maturity. The equation is $1021.50 = 22a_{\overline{20}|j} + 1100v_j^{20}$. The solution is $j = 2.46\%$. Thus, the yield, expressed as a nominal annual rate of interest convertible semiannually, is twice the smaller of the two values, or 4.92%.

58. Moved to Derivatives section

59. Solution: C

First, the present value of the liability is $PV = 35,000a_{\overline{15}|6.2\%} = 335,530.30$.

The duration of the liability is:

$$\bar{d} = \frac{\sum tv^t R_t}{\sum v^t R_t} = \frac{35,000v + 2(35,000)v^2 + \dots + 15(35,000)v^{15}}{335,530.30} = \frac{2,312,521.95}{335,530.30} = 6.89214.$$

Let X denote the amount invested in the 5 year bond.

$$\text{Then, } \frac{X}{335,530.30}(5) + \left(1 - \frac{X}{335,530.30}\right)(10) = 6.89214 \Rightarrow X = 208,556.$$

60. Solution: A

The present value of the first eight payments is:

$$PV = 2000v + 2000(1.03)v^2 + \dots + 2000(1.03)^7 v^8 = \frac{2000v - 2000(1.03)^8 v^9}{1 - 1.03v} = 13,136.41.$$

The present value of the last eight payments is:

$$\begin{aligned} PV &= 2000(1.03)^7 0.97v^9 + 2000(1.03)^7 (0.97)^2 v^{10} + \dots + 2000(1.03)^7 (0.97^8)v^{16} \\ &= \frac{2000(1.03)^7 0.97v^9 - 2000(1.03)^7 (0.97)^9 v^{17}}{1 - 0.97v} = 7,552.22. \end{aligned}$$

Therefore, the total loan amount is $L = 20,688.63$.

61. Solution: E

$$2000 = 500 \exp \left(\int_0^t \frac{\frac{r^2}{100}}{3 + \frac{r^3}{150}} dr \right)$$
$$4 = \exp \left(0.5 \int_0^t \frac{\frac{r^2}{50}}{3 + \frac{r^3}{150}} dr \right) = \exp \left[0.5 \ln \left(3 + \frac{r^3}{150} \right) \Big|_0^t \right]$$
$$4 = \exp \left[0.5 \ln \left(1 + \frac{t^3}{450} \right) \right] = \left(1 + \frac{t^3}{450} \right)^{\frac{1}{2}}$$
$$16 = \left(1 + \frac{t^3}{450} \right)$$
$$t = 18.8988$$

62. Solution: E

Let F , C , r , and i have their usual interpretations. The discount is $(Ci - Fr)a_{\overline{n}|}$ and the discount in the coupon at time t is $(Ci - Fr)v^{n-t+1}$. Then,

$$194.82 = (Ci - Fr)v^{26}$$
$$306.69 = (Ci - Fr)v^{21}$$
$$0.63523 = v^5 \Rightarrow v = 0.91324 \Rightarrow i = 0.095$$
$$(Ci - Fr) = 194.82(1.095)^{26} = 2062.53$$
$$\text{Discount} = 2062.53 a_{\overline{40}|0.095} = 21,135$$

63. Solution: A

$$699.68 = Pv^{8-5+1}$$
$$P = 842.39 \text{ (annual payment)}$$
$$P_1 = \frac{699.68}{1.0475^4} = 581.14$$
$$I_1 = 842.39 - 581.14 = 261.25$$
$$L = \frac{261.25}{0.0475} = 5500 \text{ (loan amount)}$$
$$\text{Total interest} = 842.39(8) - 5500 = 1239.12$$

64. Solution: D

$$OB_{18} = 22,000(1.007)^{18} - 450.30s_{\overline{18}|0.007} = 16,337.10$$

$$16,337.10 = Pa_{\overline{24}|0.004}$$

$$P = 715.27$$

65. Solution: C

If the bond has no premium or discount, it was bought at par so the yield rate equals the coupon rate, 0.038.

$$d = \frac{\frac{1}{2} \left(1(190)v + 2(190)v^2 + \dots + 14(190)v^{14} + 14(5000)v^{14} \right)}{190v + 190v^2 + \dots + 190v^{14} + 5000v^{14}}$$

$$d = \frac{95 \left\{ (Ia)_{\overline{14}|} + 14(5000)v^{14} \right\}}{190a_{\overline{14}|} + 5000v^{14}}$$

$$d = 5.5554$$

Or, taking advantage of a shortcut:

$$d = \ddot{a}_{\overline{14}|0.038} = 11.1107. \text{ This is in half years, so dividing by two, } d = \frac{11.1107}{2} = 5.5554.$$

66. Solution: A

$$\bar{v} = \frac{7.959}{1.072} = 7.425$$

$$P(0.08) = P(0.072)[1 - (\Delta i)\bar{v}]$$

$$P(0.08) = 1000[1 - (0.008)(7.425)] = 940.60$$

67. Solution: E

$$(1 + s_3)^3 = (1 + s_2)^2(1 + {}_2f_1)$$

$$0.85892 = \frac{1}{(1 + s_3)^3}, s_3 = 0.052$$

$$0.90703 = \frac{1}{(1 + s_2)^2}, s_2 = 0.050$$

$$1.052^3 = 1.050^2(1 + {}_2f_1)$$

$${}_2f_1 = 0.056$$

68. Solution: C

Let d_0 be the Macaulay duration at time 0.

$$d_0 = \ddot{a}_{\overline{8}|0.05} = 6.7864$$

$$d_1 = d_0 - 1 = 5.7864$$

$$d_2 = \ddot{a}_{\overline{7}|0.05} = 6.0757$$

$$\frac{d_1}{d_2} = \frac{5.7864}{6.0757} = 0.9524$$

This solution employs the fact that when a coupon bond sells at par the duration equals the present value of an annuity-due. For the duration just before the first coupon the cash flows are the same as for the original bond, but all occur one year sooner. Hence the duration is one year less.

Alternatively, note that the numerators for d_1 and d_2 are identical. That is because they differ only with respect to the coupon at time 1 (which is time 0 for this calculation) and so the payment does not add anything. The denominator for d_2 is the present value of the same bond, but with 7 years, which is 5000. The denominator for d_1 has the extra coupon of 250 and so is 5250. The desired ratio is then $5000/5250 = 0.9524$.

69. Solution: A

Let N be the number of shares bought of the bond as indicated by the subscript.

$$N_C(105) = 100, N_C = 0.9524$$

$$N_B(100) = 102 - 0.9524(5), N_B = 0.9724$$

$$N_A(107) = 99 - 0.9524(5), N_A = 0.8807$$

70. Solution: B

All are true except B. Immunization requires frequent rebalancing.

71. Solution: D

Set up the following two equations in the two unknowns:

$$A(1.05)^2 + B(1.05)^{-2} = 6000$$

$$2A(1.05)^2 - 2B(1.05)^{-2} = 0.$$

Solving simultaneously gives:

$$A = 2721.09$$

$$B = 3307.50$$

$$|A - B| = 586.41.$$

72. Solution: A

Set up the following two equations in the two unknowns.

$$(1) \quad 5000(1.03)^3 + B(1.03)^{-b} = 12,000 \Rightarrow$$

$$5463.635 + B(1.03)^{-b} = 12,000 \Rightarrow B(1.03)^{-b} = 6536.365$$

$$(2) \quad 3(5000)(1.03)^3 - bB(1.03)^{-b} = 0 \Rightarrow 16,390.905 - b6536.365 = 0$$

$$b = 2.5076$$

$$B = 7039.27$$

$$\frac{B}{b} = 2807.12$$

73. Solution: D

$$P_A = A(1+i)^{-2} + B(1+i)^{-9}$$

$$P_L = 95,000(1+i)^{-5}$$

$$P'_A = -2A(1+i)^{-3} - 9B(1+i)^{-10}$$

$$P'_L = -5(95,000)(1+i)^{-6}$$

Set the present values and derivatives equal and solve simultaneously.

$$0.92456A + 0.70259B = 78,083$$

$$-1.7780A - 6.0801B = -375,400$$

$$B = \frac{78,083(1.7780 / 0.92456) - 375,400}{0.70259(1.7780 / 0.92456) - 6.0801} = 47,630$$

$$A = [78,083 - 0.70259(47,630)] / 0.92456 = 48,259$$

$$\frac{A}{B} = 1.0132$$