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ACTUARIAL MODELS—FINANCIAL ECONOMICS SEGMENT

SOME REMARKS ON *DERIVATIVES MARKETS*

by

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The parameter δ in the Black-Scholes formula

The Black-Scholes formula for pricing a European call option on a stock is introduced in equation (12.1) on page 377 of the textbook without proof. It is important to understand the meaning of each parameter in the formula. The meaning of the *dividend yield*, δ, is not well explained in the book. The assumption on dividend payments is given in the first sentence on page 132: dividends are paid continuously at a rate that is proportional to the stock price. More precisely, for each share of the stock the amount of dividends paid between time t and $t+dt$ is assumed to be $S(t)\delta dt$. Here, $S(t)$ denotes the price of one share of the stock at time t, $t \geq 0$. (Note that the book also writes $S(t)$ as S_t; the symbol S in equation (12.1) is the same as $S(0)$ and S_0.) This is not exactly a reasonable assumption, but it is needed to obtain (12.1).

It is indicated on page 132 that, if all dividends are re-invested immediately, then one share of the stock at time 0 will grow to $e^{\delta t}$ shares at time t, $t \geq 0$. A calculus proof of this fact is as follows. Let $n(t)$ denote the number of shares of the stock at time t under this immediate reinvestment policy. Thus, $n(0) = 1$. Because the additional number of shares purchased between time t and $t+dt$ is $dn(t)$, we have

$$n(t)S(t)\delta dt \;=\; S(t)dn(t),$$

or

$$\frac{d}{dt}n(t) \;=\; n(t)\delta.$$

Rewriting the last equation as

$$\frac{d}{dt}\ln[n(t)] \;=\; \delta,$$

integrating both sides, and applying the condition $n(0) = 1$, we obtain the result $n(t) \;=\; e^{\delta t}$.

Thus, if we want one share of the stock at time T, we can buy $e^{-\delta T}$ share at time 0 and reinvest all dividends between time 0 and time T. This gives a meaning to the quantity $Se^{-\delta T}$ in formula (12.1).

The parameter σ in the Black-Scholes formula

The symbol σ in (12.1) is usually called *volatility*. In the finance literature, the term "volatility" does not always have the same meaning. The quantity σ enters the Black-Scholes model via the stochastic differential equation (20.1) on page 649,

$$\frac{dS(t)}{S(t)} \;=\; \alpha dt \;+\; \sigma dZ(t).$$

Because $dS(t)$ is the instantaneous change in the stock price, the left-hand side of the equation, $dS(t)/S(t)$, is the *instantaneous rate of return* due to changes in the stock price. As the first quantity on the right-hand side, αdt, is deterministic, the variance of $dS(t)/S(t)$ is the variance of $\sigma dZ(t)$, which is σ^2 times the variance of $dZ(t)$. As pointed out on page 650, $dZ(t)$ is a normal random variable with variance dt. Thus, the variance of the instantaneous rate of return, $dS(t)/S(t)$, is

$$\sigma^2\, dt,$$

which gives an interpretation for σ.

The stochastic differential equation (20.1) has an explicit solution,

$$S(t) \;=\; S(0)\exp[(\alpha - \frac{\sigma^2}{2})t \;+\; \sigma Z(t)];$$

see (20.13) and (20.29). By means of Itô's Lemma, we can check that the stochastic differential equation (20.1) is indeed satisfied. The exponent, $(\alpha - \frac{\sigma^2}{2})t \;+\; \sigma Z(t)$, is the *continuously compounded return* from time 0 to time t (as defined on page 353) due to stock price changes; its variance is

$$\mathrm{Var}[\sigma Z(t)] \;=\; \sigma^2 \mathrm{Var}[Z(t)] \;=\; \sigma^2 t.$$

In other words, $\sigma\sqrt{t}$ is the standard deviation of the continuously compounded return over the time interval $[0, t]$ (due to changes in the stock price). On page 919 of the book, volatility is defined as "[t]he standard deviation of the continuously compounded return on an asset." This is not quite correct, because it has not specified that the length of the time interval is one.

The total return of a stock has two components: return from capital gains (or loss) and return from dividends. The dividend yield assumption means that the instantaneous rate of return from dividends is the constant δ.

The parameters δ and σ in the binomial model

The quantities, δ and σ, also appear in Chapters 10 and 11, which are on binomial models. On page 316, δ is called the continuous dividend yield, and on page 321, σ is called the annualized standard deviation of the continuously compounded stock return. Because binomial models are discrete models, it seems strange that these "continuous-time" concepts appear. The motivation for incorporating δ and σ in binomial models is sketched in Section 11.3. By letting the length of each time period, h, tend to zero (and the number of periods tend to infinity), we can obtain the *risk-neutral* geometric Brownian motion for stock price movements with the dividend yield δ and volatility σ. Note that the textbook has suggested three pairs of formulas for u and d,

$$u \;=\; e^{\alpha(h)+\sigma\sqrt{h}}, \qquad d \;=\; e^{\alpha(h)-\sigma\sqrt{h}}.$$

In (10.10), $\alpha(h) = (r - \delta)h$. In (11.18), $\alpha(h) \equiv 0$. In (11.19), $\alpha(h) = (r - \delta - \tfrac{1}{2}\sigma^2)h$. They yield the same limit as $h \to 0$. Because of the no-arbitrage condition (10.4), $\alpha(h) = (r - \delta)h$ is the preferred formula in the textbook.

Greeks and the Black-Scholes Partial Differential Equation

Greeks are partial derivatives of the option price formula. Because the author does not want to use calculus in the first half of the book, the definitions given on pages 382 and 383 are numerical approximations. "The actual formulas for the Greeks appear in Appendix 12.B."

Equation (13.10) on page 430, given in terms of three Greeks, is the celebrated *Black-Scholes partial differential equation*. The version in terms of derivatives (and with the added assumption that the stock pays dividends continuously at a rate proportional to its price) can be found in (21.11) on page 682. Note from page xxiii that "[a]lthough the Black-Scholes *formula* is famous, the Black-Scholes differential *equation* … is the more profound result."

Perpetual Options

The parameters h_1 and h_2 in Section 12.6 are the positive and negative roots, respectively, of the quadratic equation

$$\frac{\sigma^2}{2} h^2 + (r - \delta - \frac{\sigma^2}{2})h - r = 0.$$

Formula (12.17), $[S(0)/H]^{h_1}$, gives the current price for the payment of $1 which will be paid as soon as the stock price rises to level H, $H > S(0)$. For $S(0) < H < K$, the identity

$$[S(0)/H]^{h_1} [H/K]^{h_1} = [S(0)/K]^{h_1}$$

and its interpretation may remind students of *life contingencies* of the pure endowment formula

$$_mE_x \, _nE_{x+m} = \, _{m+n}E_x.$$