SPRING 2005
EXAM M SOLUTIONS

Question #1
Key: B

Let K be the curtate future lifetime of (x + k)

$${}_kL = 1000v^{K+1} - 1000P_{x:\overline{3}|} \times \ddot{a}_{\overline{K+1}|}$$

When (as given in the problem), (x) dies in the second year from issue, the curtate future lifetime of $(x+1)$ is 0, so

$${}_1L = 1000v - 1000\,P_{x:\overline{3}|}\,\ddot{a}_{\overline{1}|}$$
$$= \frac{1000}{1.1} - 279.21$$
$$= 629.88 \approx 630$$

The premium came from

$$P_{x:\overline{3}|} = \frac{A_{x:\overline{3}|}}{\ddot{a}_{x:\overline{3}|}}$$
$$A_{x:\overline{3}|} = 1 - d\,\ddot{a}_{x:\overline{3}|}$$
$$P_{x:\overline{3}|} = 279.21 = \frac{1 - d\,\ddot{a}_{x:\overline{3}|}}{\ddot{a}_{x:\overline{3}|}} = \frac{1}{\ddot{a}_{x:\overline{3}|}} - d$$

Question #2
Key: E

Note that above 40, decrement 1 is DeMoivre with omega = 100; decrement 2 is DeMoivre with omega = 80.

That means $\mu_{40}^{(1)}(20) = 1/40 = 0.025$; $\mu_{40}^{(2)}(20) = 1/20 = 0.05$

$\mu_{40}^{(\tau)}(20) = 0.025 + 0.05 = 0.075$

Or from basic definition of μ ,

${}_t p_{40}^{(\tau)} = \frac{60-t}{60} \times \frac{40-t}{40} = \frac{2400 - 100t + t^2}{2400}$

$d\left({}_t p_{40}^{(\tau)}\right) / dt = (-100 + 2t) / 2400$

at $t = 20$ gives $-60/2400 = 0.025$

${}_{20} p_{40}^{(\tau)} = (2/3) * (1/2) = 1/3$

$\mu_{40}^{(\tau)}(20) = \left[-d\left({}_t p_{40}^{(\tau)}\right) / dt\right] / {}_{20} p_{40}^{(\tau)} = 0.025/(1/3) = 0.075$

Question #3
Key: B

$$
\begin{aligned}
{}_{5|}q_{\overline{35:45}} &= {}_{5|}q_{35} + {}_{5|}q_{45} - {}_{5|}q_{35:45} \\
&= {}_5p_{35}q_{40} + {}_5p_{45}q_{50} - {}_5p_{35:45}q_{40:50} \\
&= {}_5p_{35}q_{40} + {}_5p_{45}q_{50} - {}_5p_{35} \times {}_5p_{45}\left(1 - p_{40:50}\right) \\
&= {}_5p_{35}q_{40} + {}_5p_{45}q_{50} - {}_5p_{35} \times {}_5p_{45}\left(1 - p_{40}\,p_{50}\right) \\
&= (0.9)(.03) + (0.8)(0.05) - (0.9)(0.8)\left[1 - (0.97)(0.95)\right] \\
&= 0.01048
\end{aligned}
$$

Alternatively,

$$
{}_6p_{35} = {}_5p_{35} \times p_{40} = (0.90)(1 - 0.03) = 0.873
$$

$$
{}_6p_{45} = {}_5p_{45} \times p_{50} = (0.80)(1 - 0.05) = 0.76
$$

$$
\begin{aligned}
{}_{5|}q_{\overline{35:45}} &= {}_5p_{\overline{35:45}} - {}_6p_{\overline{35:45}} \\
&= \left({}_5p_{35} + {}_5p_{45} - {}_5p_{35:45}\right) - \left({}_6p_{35} + {}_6p_{45} - {}_6p_{35:45}\right) \\
&= \left({}_5p_{35} + {}_5p_{45} + {}_5p_{35} \times {}_5p_{45}\right) - \left({}_6p_{35} + {}_6p_{45} - {}_6p_{35} \times {}_6p_{45}\right) \\
&= \left(0.90 + 0.80 - 0.90 \times 0.80\right) - \left(0.873 + 0.76 - 0.873 \times 0.76\right) \\
&= 0.98 - 0.96952 \\
&= 0.01048
\end{aligned}
$$

Question #4
Key: D

Let G be the expense-loaded premium.
Actuarial present value (APV) of benefits = $100,000 A_{35}$
APV of premiums = $G\ddot{a}_{35}$
APV of expenses = $\left[0.1G + 25 + (2.50)(100)\right]\ddot{a}_{35}$
Equivalence principle:

$$G\ddot{a}_{35} = 100,000 A_{35} + (0.1G + 25 + 250)\ddot{a}_{35}$$

$$G = 100,000\frac{A_{35}}{\ddot{a}_{35}} + 0.1G + 275$$

$$0.9G = 100,000 P_{35} + 275$$

$$G = \frac{(100)(8.36) + 275}{0.9}$$

$$= 1234$$

Question #5
Key: D

Poisoned wine glasses are drunk at a Poisson rate of 2×0.01 = 0.02 per day.
Number of glasses in 30 days is Poisson with $\lambda = 0.02 \times 30 = 0.60$

$$f(0) = e^{-0.60} = 0.55$$

Question #6
Key: E

View the compound Poisson process as two compound Poisson processes, one for smokers and one for non-smokers. These processes are independent, so the total variance is the sum of their variances.

For smokers, $\lambda = (0.2)(1000) = 200$

$$\begin{aligned} \text{Var(losses)} &= \lambda\left[\text{Var}(X) + \left(E(X)\right)^2\right] \\ &= 200\left[5000 + (-100)^2\right] \\ &= 3,000,000 \end{aligned}$$

For non-smokers, $\lambda = (0.8)(1000) = 800$

$$\begin{aligned} \text{Var(losses)} &= \lambda\left[\text{Var}(X) + \left(E(X)\right)^2\right] \\ &= 800\left[8000 + (-100)^2\right] \\ &= 14,400,000 \end{aligned}$$

$$\begin{aligned} \text{Total variance} &= 3{,}000{,}000 + 14{,}400{,}000 \\ &= 17{,}400{,}000 \end{aligned}$$

Question #7
Key: E

$E[Z] = b\bar{A}_x$

since constant force $\bar{A}_x = \mu / \mu + \delta$

$$E(Z) = \frac{b\mu}{\mu+\delta} = \frac{b(0.02)}{(0.06)} = b/3$$

$$\text{Var}[Z] = \text{Var}\left[b v^{\mathrm{T}}\right] = b^2 \text{Var}\left[v^{\mathrm{T}}\right] = b^2\left({}^2\bar{A}_x - \bar{A}_x^2\right)$$

$$= b^2\left(\frac{\mu}{\mu+2\delta} - \left(\frac{\mu}{\mu+\delta}\right)^2\right)$$

$$= b^2\left[\frac{2}{10} - \frac{1}{9}\right] = b^2\left(\frac{4}{45}\right)$$

$$\text{Var}(Z) = E(Z)$$

$$b^2\left[\frac{4}{45}\right] = \frac{b}{3}$$

$$b\left[\frac{4}{45}\right] = \frac{1}{3} \Rightarrow b = 3.75$$

Question #8
Key: A

$$A^{1}_{30:\overline{3}|} = 1000vq_{30} + 500v^2\ {}_{1|}q_{30} + 250v^3\ {}_{2|}q_{30}$$

$$= 1000\left(\frac{1}{1.06}\right)\left(\frac{1.53}{1000}\right) + 500\left(\frac{1}{1.06}\right)^2(0.99847)\left(\frac{1.61}{1000}\right) + 250\left(\frac{1}{1.06}\right)^3(0.99847)(0.99839)\left(\frac{1.70}{1000}\right)$$

$$= 1.4434 + 0.71535 + 0.35572 = 2.51447$$

$$\ddot{a}^{(2)}_{30:\overline{1}|} = \tfrac{1}{2} + \tfrac{1}{2}\left(\frac{1}{1.06}\right)^{1/2}\left(1 - \tfrac{1}{2}q_{30}\right) = \frac{1}{2} + \frac{1}{2}(0.97129)\left(1 - \frac{0.00153}{2}\right)$$

$$= \frac{1}{2} + \frac{1}{2}(0.97129)(0.999235)$$

$$= 0.985273$$

Annualized premium $= \dfrac{2.51447}{0.985273}$

$= 2.552$

Each semiannual premium $= \dfrac{2.552}{2}$

$= 1.28$

Question #9
Key: B

$E\left[x-d \middle| x>d\right]$ is the expected payment per payment with an ordinary deductible of d

It can be evaluated (for Pareto) as

$$\frac{E(x)-E(x\wedge d)}{1-F(d)}=\frac{\dfrac{\theta}{\alpha-1}-\dfrac{\theta}{\alpha-1}\left[1-\left(\dfrac{\theta}{d+\theta}\right)^{\alpha-1}\right]}{1-\left[1-\left(\dfrac{\theta}{d+\theta}\right)^{\alpha}\right]}$$

$$=\frac{\dfrac{\theta}{\alpha-1}\left(\dfrac{\theta}{d+\theta}\right)^{\alpha-1}}{\left(\dfrac{\theta}{d+\theta}\right)^{\alpha}}$$

$$=\frac{d+\theta}{\alpha-1}$$

$=d+\theta$ in this problem, since $\alpha=2$

$$E\left[x-100 \middle| x>100\right]=\tfrac{5}{3}E\left[x-50 \middle| x>50\right]$$

$$100+\theta=\tfrac{5}{3}(50+\theta)$$

$$300+3\theta=250+5\theta$$

$$=\theta=25$$

$$E\left[x-150 \middle| x>150\right]=150+\theta$$

$$=150+25$$

$$=175$$

Question #10
Key: D

Let S = score

$$E(S) = E\left(E(S|\theta)\right) = E(\theta) = 75$$

$$\begin{aligned} Var(S) &= E\left[Var(S|\theta)\right] + Var\left[E(S|\theta)\right] \\ &= E\left(8^2\right) + Var(\theta) \\ &= 64 + 6^2 \\ &= 100 \end{aligned}$$

S is normally distributed (a normal mixture of normal distributions with constant variance is normal; see Example 4.30 in Loss Models for the specific case, as we have here, with a normally distributed mean and constant variance)

$$\Pr ob\left[S < 90 | S > 65\right] = \frac{F(90) - F(65)}{1 - F(65)}$$

$$= \frac{\Phi\left(\frac{90-75}{10}\right) - \Phi\left(\frac{65-75}{10}\right)}{1 - \Phi\left(\frac{65-75}{10}\right)}$$

$$\frac{\Phi(1.5) - \Phi(-1.0)}{1 - \Phi(-1.0)} = \frac{0.9332 - (1 - 0.8413)}{1 - (1 - 0.8413)} = \frac{0.7745}{0.8413} = 0.9206$$

Note that (though this insight is unnecessary here), this is equivalent to per payment model with a franchise deductible of 65.

Question #11
Key: C

Ways to go $0 \to 2$ in 2 years

$$0-0-2;\ p=(0.7)(0.1)=0.07$$
$$0-1-2;\ p=(0.2)(0.25)=0.05$$
$$0-2-2;\ p=(0.1)(1)=0.1$$

Total = 0.22
Binomial m = 100 q = 0.22
Var = (100) (0.22) (0.78) = 17

Question #12
Key: A

For death occurring in year 2

$$APV=\frac{0.3\times 1000}{1.05}=285.71$$

For death occurring in year 3, two cases:

(1) State 2 $\to$ State 1 $\to$ State 4:	(0.2×0.1)	= 0.02
(2) State 2 $\to$ State 2 $\to$ State 4:	(0.5×0.3)	= 0.15
Total		0.17

$$\text{APV} = \frac{0.17\times 1000}{1.05^2}=154.20$$

Total. APV = 285.71 + 154.20 = 439.91

Question #13
Key: C

$$\left({}_9V + P\right)(1.03) = q_{x+9}b + \left(1 - q_{x+9}\right){}_{10}V$$
$$= q_{x+9}\left(b - {}_{10}V\right) + {}_{10}V$$
$$(343)(1.03) = 0.02904(872) + {}_{10}V$$
$$\Rightarrow {}_{10}V = 327.97$$
$$b = \left(b - {}_{10}V\right) + {}_{10}V = 872 + 327.97 = 1199.97$$
$$P = b\left(\frac{1}{\ddot{a}_x} - d\right) = 1200\left(\frac{1}{14.65976} - \frac{0.03}{1.03}\right)$$
$$= 46.92$$
${}_9V =$ initial reserve – P = 343 – 46.92 = 296.08

Question #14
Key: B

$d = 0.06 \Rightarrow V = 0.94$

Step 1 Determine p_x

$$668 + 258vp_x = 1000vq_x + 1000v^2 p_x\left(p_{x+1} + q_{x+1}\right)$$
$$668 + 258(0.94)\,p_x = 1000(0.94)(1 - p_x) + 1000(0.8836)\,p_x(1)$$
$$668 + 242.52\,p_x = 940(1 - p_x) + 883.6\,p_x$$
$$p_x = 272/298.92 = 0.91$$

Step 2 Determine $1000P_{x:\overline{2}|}$

$$668 + 258(0.94)(0.91) = 1000P_{x:\overline{2}|}\left[1 + (0.94)(0.91)\right]$$
$$1000P_{x:\overline{2}|} = \frac{[220.69 + 668]}{1.8554} = 479$$

Question #15
Key: D

$$100,000\left(IA\right)^1_{40:\overline{10}|} = 100,000\, v\, p_{40}\left[\left(IA\right)^1_{41:\overline{10}|} - 10v^{10}\,{}_9p_{41}\,q_{50}\right] + A^1_{40:\overline{10}|}\left(100,000\right) \quad \left[\text{see comment}\right]$$

$$= 100,000\frac{0.99722}{1.06}\left[0.16736 - \frac{10\left(\dfrac{8,950,901}{9,287,264}\right)}{1.06^{10}}\times\left(0.00592\right)\right]$$

$$+\left(0.02766\times 100,000\right)$$

$$=15,513$$

Where $A^1_{40:\overline{10}|} = A_{40} - {}_{10}E_{40}\,A_{50}$

$$= 0.16132 - \left(0.53667\right)\left(0.24905\right)$$

$$= 0.02766$$

Comment: the first line comes from comparing the benefits of the two insurances. At each of age 40, 41, 42,…,49 $\left(IA\right)^1_{40:\overline{10}|}$ provides a death benefit 1 greater than $\left(IA\right)^1_{41:\overline{10}|}$. Hence the $A^1_{40:\overline{10}|}$ term. But $\left(IA\right)^1_{41:\overline{10}|}$ provides a death benefit at 50 of 10, while $\left(IA\right)^1_{40:\overline{10}|}$ provides 0. Hence a term involving ${}_{9|}q_{41} = {}_9p_{41}\,q_{50}$. The various v's and p's just get all actuarial present values at age 40.

Question #16
Key: A

$$1000\,{}_1V_x = \pi(1+i) - q_x(1000 - 1000\,{}_1V_x)$$
$$40 = 80(1.1) - q_x(1000 - 40)$$
$$q_x = \frac{88-40}{960} = 0.05$$

$$_1AS = \frac{(\text{G} - \text{ expenses})(1+i) - 1000q_x}{p_x}$$
$$= \frac{(100-(0.4)(100))(1.1)-(1000)(0.05)}{1-0.05}$$
$$= \frac{60(1.1)-50}{0.95} = 16.8$$

Question #17
Key: B

(Referring to the number of losses, *X*, was a mistake. *X* is the random variable for the loss amount, the severity distribution).

Losses in excess of the deductible occur at a Poisson rate
of $\lambda^* = (1-F(30))\lambda = 0.75\times 20 = 15$

$$E(X-30|X>30) = \frac{70-25}{0.75} = \frac{45}{0.75} = 60$$
$$Var(S) = \lambda^* \times E\left((X-30)^2|X>30\right)$$
$$= 15E\left(X^2 - 60X + 900|X>30\right) = 15E\left(X^2 - 60(X-30) - 900|X>30\right)$$
$$= 15(9{,}000 - 60\times 60 - 900)$$
$$= 67{,}500$$

Question #18
Key: A

S	$(S-3)_+$
0	0
1	0
2	0
3	0
4	1
5	2
6	3
$\vdots$	$\vdots$

$$E\left[(S-3)_+\right] = E[S] - 3 + 3f_S(0) + 2f_S(1) + 1f_S(2)$$

$$E[S] = 2\times[0.6 + 2\times 0.4] = 2.8$$

$$f_S(0) = e^{-2}$$

$$f_S(1) = e^{-2}\times 2\times(0.6) = 1.2e^{-2}$$

$$f_S(2) = e^{-2}\times 2(0.4) + \frac{e^{-2}2^2}{2}\times(0.6)^2 = 1.52e^{-2}$$

$$E\left[(S-3)_+\right] = 2.8 - 3 + 3\times e^{-2} + 2\times 1.2e^{-2} + 1\times 1.52e^{-2}$$

$$= -0.2 + 6.92e^{-2}$$

$$= 0.7365$$

Question #19
Key: C

Write (i) as $\frac{p_k}{p_{k-1}} = c + \frac{c}{k}$

This is an (a, b, 0) distribution with $a = b = c$.

Which?

1. If Poisson, $a = 0$, so $c = 0$ and $b = 0$

   $p_1 = p_2 = ... = 0$

   $p_0 = 0.5$

   p_k's do not sum to 1. Impossible. Thus not Poisson

2. If Geometric, $b = 0$, so $c = 0$ and $a = 0$

   By same reasoning as #1, impossible, so not Geometric.

3. If binomial, a and b have opposite signs. But here $a = b$, so not binomial.

4. Thus negative binomial.

   $$1 = \frac{a}{b} = \frac{\beta/(1+\beta)}{(r-1)\beta/(1-\beta)} = \frac{1}{r-1}$$

   so $r = 2$

   $p_0 = 0.5 = (1+\beta)^{-r} = (1+\beta)^{-2}$

   $1+\beta = \sqrt{2} = 1.414$

   $\beta = \sqrt{2} - 1 = 0.414$

   $c = a = \beta/(1+\beta) = 0.29$

Question #20
Key: C

At any age, $p_x'^{(1)} = e^{-0.02} = 0.9802$
$q_x'^{(1)} = 1 - 0.9802 = 0.0198$, which is also $q_x^{(1)}$, since decrement 2 occurs only at the end of the year.

Actuarial present value (APV) at the start of each year for that year's death benefits
= 10,000*0.0198 v = 188.1

$p_x^{(\tau)} = 0.9802 * 0.96 = 0.9410$
$E_x = p_x^{(\tau)} v = 0.941\ v = 0.941 * 0.95 = 0.8940$

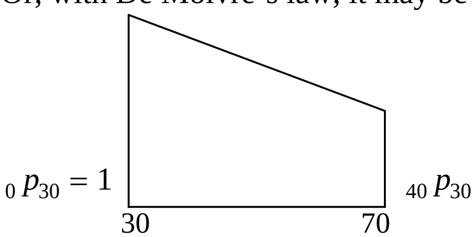
APV of death benefit for 3 years $188.1 + E_{40} * 188.1 + E_{40} * E_{41} * 188.1 = 506.60$

Question #21
Key: B

$$\overset{\circ}{e}_{30:\overline{40|}} = \int_0^{40} {}_t p_{30} dt$$
$$= \int_0^{40} \frac{\omega - 30 - t}{\omega - 30} dt$$
$$= t - \frac{t^2}{2(\omega - 30)} \Big|_0^{40}$$
$$= 40 - \frac{800}{\omega - 30}$$
$$= 27.692$$

$\omega = 95$

Or, with De Moivre's law, it may be simpler to draw a picture:



$$\overset{\circ}{e}_{30:\overline{40|}} = \text{area } = 27.692 = 40 \frac{\left(1 + {}_{40} p_{30}\right)}{2}$$
$${}_{40} p_{30} = 0.3846$$
$$\frac{\omega - 70}{\omega - 30} = 0.3846$$
$$\omega = 95$$
$${}_t p_{30} = \frac{65 - t}{65}$$

$$Var = E(T)^2 - \left(E(T)\right)^2$$

One way to evaluate this expression is based on Equation 3.5.4 in <u>Actuarial Mathematics</u>

$$Var(T) = \int_0^{\infty} 2t\, {}_t p_x dt - \overset{\circ}{e}_x^2$$
$$= 2\int_0^{65} t \bullet \left(1 - \frac{t}{65}\right) dt - \Bigg(\int_0^{65} \left(1 - \frac{t}{65}\right) dt^2$$

$$= 2*(2112.5 - 1408.333) - (65 - 65/2)^2$$
$$= 1408.333 - 1056.25 = 352.08$$

Another way, easy to calculate for De Moivre's law is

$$Var(T) = \int_0^\infty t^2 \, {}_t p_x \mu_x(t)\,dt - \left(\int_0^\infty t \, {}_t p_x \mu_x(t)\,dt\right)^2$$
$$= \int_0^{65} t^2 \times \frac{1}{65}\,dt - \left(\int_0^{65} t \times \frac{1}{65}\,dt\right)^2$$
$$= \frac{t^3}{3 \times 65}\Big|_0^{65} - \left(\frac{t^2}{2 \times 65}\Big|_0^{65}\right)^2$$
$$= 1408.33 - (32.5)^2 = 352.08$$

With De Moivre's law and a maximum future lifetime of 65 years, you probably didn't need to integrate to get $E(T(30)) = \mathring{e}_{30} = 32.5$

Likewise, if you realize (after getting $\omega = 95$) that $T(30)$ is uniformly on (0, 65), its variance is just the variance of a continuous uniform random variable:

$$Var = \frac{(65-0)^2}{12} = 352.08$$

Question #22
Key: E

$$ {}_1V = \frac{218.15(1.06) - 10{,}000(0.02)}{1 - 0.02} = 31.88 $$
$$ {}_2V = \frac{(31.88 + 218.15)(1.06) - (9{,}000)(0.021)}{1 - 0.021} = 77.66 $$

Question #23
Key: D

$$ e_x = e_y = \sum_{k=1}^{\infty} {}_t p_x = 0.95 + 0.95^2 + \ldots $$
$$ = \frac{0.95}{1 - 0.95} = 19 $$
$$ e_{xy} = p_{xy} + {}_2 p_{xy} + \ldots $$
$$ = 1.02(0.95)(0.95) + 1.02(0.95)^2 (0.95)^2 + \ldots $$
$$ = 1.02\left[0.95^2 + 0.95^4 + \ldots\right] = \frac{1.02(0.95)^2}{1 - 0.95^2} = 9.44152 $$
$$ e_{\overline{xy}} = e_x + e_y - e_{xy} = 28.56 $$

Question #24
Key: A

Local comes first. I board

So I get there first if he waits more than 28 – 16 = 12 minutes after the local arrived.

His wait time is exponential with mean 12
The wait before the local arrived is irrelevant; the exponential distribution is memoryless
Prob(exp with mean 12>12) $= e^{-12/12} = e^{-1} = 36.8\%$

Question #25
Key: E

This problem is a direct application of Example 5.18 in Probability Models (p. 308); it follows from proposition 5.3 (p. 303).
Deer hit at time s are found by time t (here, t = 10) with probability $F(t – s)$, where F is the exponential distribution with mean 7 days.
We can split the Poisson process “deer being hit” into “deer hit, not found by day 10” and “deer hit, found by day 10”. By proposition 5.3, these processes are independent Poisson processes. Deer hit, found by day 10, at time s has Poisson rate $20\times F(t-s)$. The expected number hit and found by day 10 is its integral from 0 to 10.

$$E\left(N\left(t\right)\right)=20\int_0^t F\left(t-s\right)ds$$

$$E\left(N\left(10\right)\right)=20\int_0^{10} 1-e^{\frac{-(10-s)}{7}}ds$$

$$=20\left(10-7e^{\frac{s-10}{7}}\Big|_0^{10}\right)$$

$$=20\left(10-7+7e^{-10/7}\right)=94$$

Question #26
Key: E

$$\bar{a}_x = \int_0^\infty e^{-0.08t} dt = 12.5$$

$$\bar{A}_x = \int_0^\infty e^{-0.08t}(0.03)\,dt = \frac{3}{8} = 0.375$$

$$^2\bar{A}_x = \int_0^\infty e^{-0.13t}(0.03)\,dt = \frac{3}{13} = 0.23077$$

$$\sigma\left(\bar{a}_{\overline{T|}}\right) = \sqrt{\text{Var}\left[\bar{a}_{\overline{T|}}\right]} = \sqrt{\frac{1}{\delta^2}\left[{}^2\bar{A}_x - \left(\bar{A}_x\right)^2\right]} = \sqrt{400\left[0.23077 - (0.375)^2\right]} = 6.0048$$

$$\Pr\left[\bar{a}_{\overline{T|}} > \bar{a}_x - \sigma\left(\bar{a}_{\overline{T|}}\right)\right] = \Pr\left[\bar{a}_{\overline{T|}} > 12.5 - 6.0048\right]$$

$$= \Pr\left[\frac{1-v^T}{0.05} > 6.4952\right] = \Pr\left[0.67524 > e^{-0.05T}\right]$$

$$= \Pr\left[T > \frac{-\ln 0.67524}{0.05}\right] = \Pr\left[T > 7.85374\right]$$

$$= e^{-0.03\times 7.85374} = 0.79$$

Question #27
Key: A

$${}_5p_{50}^{(\tau)} = e^{-(0.05)(5)} = e^{-0.25} = 0.7788$$

$${}_5q_{55}^{(1)} = \int_0^5 \mu_{55}^{(1)}(t) \times e^{-(0.03+0.02)t} dt = -(0.02/0.05)e^{-0.05t}\Big|_0^5$$

$$= 0.4\left(1 - e^{-0.25}\right)$$

$$= 0.0885$$

Probability of retiring before $60 = {}_5p_{50}^{(\tau)} \times {}_5q_{55}^{(1)}$
= 0.7788*0.0885
= 0.0689

Question #28
Key: C

Complete the table:

$l_{81} = l_{[80]} - d_{[80]} = 910$

$l_{82} = l_{[81]} - d_{[81]} = 830$ (not really needed)

$\mathring{e}_x = e_x + \frac{1}{2} \qquad \left(\frac{1}{2} \text{ since UDD}\right)$

$\mathring{e}_{[x]} = e_{[x]} + 1/2$

$$\mathring{e}_{[x]} = \left[\frac{l_{81} + l_{82} + l_{83} + \dots}{l_{[80]}}\right] + \frac{1}{2}$$

$\left[\mathring{e}_{[80]} - \frac{1}{2}\right] l_{[80]} = l_{81} + l_{82} + \dots$ Call this equation (A)

$\left[\mathring{e}_{[81]} - \frac{1}{2}\right] l_{[81]} = l_{82} + \dots$ Formula like (A), one age later. Call this (B)

Subtract equation (B) from equation (A) to get

$$l_{81} = \left[\mathring{e}_{[80]} - \frac{1}{2}\right] l_{[80]} - \left[\mathring{e}_{[81]} - \frac{1}{2}\right] l_{[81]}$$

$$910 = [8.5 - 0.5]1000 - \left[\mathring{e}_{[81]} - 0.5\right]920$$

$$\mathring{e}_{[81]} = \frac{8000 + 460 - 910}{920} = 8.21$$

Alternatively, and more straightforward,

$$p_{[80]} = \frac{910}{1000} = 0.91$$

$$p_{[81]} = \frac{830}{920} = 0.902$$

$$p_{81} = \frac{830}{910} = 0.912$$

$$\overset{\circ}{e}_{[80]} = \frac{1}{2} q_{[80]} + p_{[80]} \left(1 + \overset{\circ}{e}_{81} \right)$$

where $q_{[80]}$ contributes $\frac{1}{2}$ since UDD

$$8.5 = \frac{1}{2}(1 - 0.91) + (0.91)\left(1 + \overset{\circ}{e}_{81} \right)$$

$$\overset{\circ}{e}_{81} = 8.291$$

$$\overset{\circ}{e}_{81} = \frac{1}{2} q_{81} + p_{81} \left(1 + \overset{\circ}{e}_{82} \right)$$

$$8.291 = \frac{1}{2}(1 - 0.912) + 0.912\left(1 + \overset{\circ}{e}_{82} \right)$$

$$\overset{\circ}{e}_{82} = 8.043$$

$$\begin{aligned}\overset{\circ}{e}_{[81]} &= \frac{1}{2} q_{[81]} + p_{[81]} \left(1 + \overset{\circ}{e}_{82} \right) \\ &= \frac{1}{2}(1 - 0.902) + (0.902)(1 + 8.043) \\ &= 8.206\end{aligned}$$

Or, do all the recursions in terms of *e*, not $\overset{\circ}{e}$, starting with $e_{[80]} = 8.5 - 0.5 = 8.0$, then final step

$$\overset{\circ}{e}_{[81]} = e_{[81]} + 0.5$$

Question #29
Key: A

t	p_{x+t}	${}_t p_x$	v^t	$v^t\ {}_t p_x$
0	0.7	1	1	1
1	0.7	0.7	0.95238	0.6667
2	–	0.49	0.90703	0.4444
3	–	–	–	–

From above $\ddot{a}_{x:\overline{3}|} = \sum_{t=0}^{2} v^t\ {}_t p_x = 2.1111$

$$1000\,{}_2V_{x:\overline{3}|} = 1000\left(1 - \frac{\ddot{a}_{x+2:\overline{1}|}}{\ddot{a}_{x:\overline{3}|}}\right) = 1000\left(1 - \frac{1}{2.1111}\right) = 526$$

Alternatively,

$$P_{x:\overline{3}|} = \frac{1}{\ddot{a}_{x:\overline{3}|}} - d = 0.4261$$

$$\begin{aligned} 1000\,{}_2V_{x:\overline{3}|} &= 1000\left(v - P_{x:\overline{3}|}\right) \\ &= 1000(0.95238 - 0.4261) \\ &= 526 \end{aligned}$$

You could also calculate $A_{x:\overline{3}|}$ and use it to calculate $P_{x:\overline{3}|}$.

Question #30
Key: E

Let G be the expense-loaded premium.
Actuarial present value (APV) of benefits = $1000A_{50}$.

APV of expenses, except claim expense = $15+1\times\ddot{a}_{50}$

APV of claim expense = $50A_{50}$ (50 is paid when the claim is paid)

APV of premiums = $G\,\ddot{a}_{50}$

Equivalence principle: $G\ddot{a}_{50} = 1000A_{50} + 15 + 1\times\ddot{a}_{50} + 50A_{50}$

$$G = \frac{1050A_{50} + 15 + \ddot{a}_{50}}{\ddot{a}_{50}}$$

For De Moivre's with ω = 100, x = 50 $\quad A_{50} = \frac{a_{\overline{50}|}}{50} = 0.36512$

$$\ddot{a}_{50} = \frac{1-A_{50}}{d} = 13.33248$$

Solving for G, G = 30.88

Question #31
Key: B

The variance calculation assumes independence, which should have been explicitly stated.

$E(S) = E(N)E(X)$

$\mathrm{Var}(S) = E(N)\mathrm{Var}(X) + E^2(X)\mathrm{Var}(N)$

	$E(N)$	$\mathrm{Var}(N)$	$E(X)$	$\mathrm{Var}(X)$	E(S)	$\mathrm{Var}(S)$
P.B	30	21	300	10,000	9,000	2.19×10^6
S.B	30	27	1000	400,000	30,000	39×10^6
L.Y	30	12	5000	2,000,000	150,000	360×10^6
					189,000	400×10^6 (rounded)

Standard deviation = $\sqrt{400\times10^6} = 20,000$

189,000 + 20,000 = 209,000

Question #32
Key: B

$S_X(150) = 1 - 0.2 = 0.8$

$f_{Y^p}(y) = \dfrac{f_X(y+150)}{S_X(150)}$ So $f_{Y^p}(50) = \dfrac{0.2}{0.8} = 0.25$

$f_{Y^p}(150) = \dfrac{0.6}{0.8} = 0.75$

$E\left(Y^p\right) = (0.25)(50) + (0.75)(150) = 125$

$E\left[\left(Y^p\right)^2\right] = (0.25)\left(50^2\right) + (0.75)(150)^2 = 17{,}500$

$\mathrm{Var}\left(Y^p\right) = E\left[\left(Y^p\right)^2\right] - \left[E\left(Y^p\right)\right]^2 = 17{,}500 - 125^2 = 1875$

Slight time saver, if you happened to recognize it:

$Var\left(Y^p\right) = Var\left(Y^p - 50\right)$ since subtracting a constant does not change variance, regardless of the distribution

But $Y^p - 50$ takes on values only 0 and 100, so it can be expressed as 100 times a binomial random variable with n = 1, q = 0.75

$Var = \left(100^2\right)(1)(0.25)(0.75) = 1875$

Question #33
Key: A

${}_4p_{50} = e^{-(0.05)(4)} = 0.8187$

${}_{10}p_{50} = e^{-(0.05)(10)} = 0.6065$

${}_8p_{60} = e^{-(0.04)(8)} = 0.7261$

${}_{18}p_{50} = \left({}_{10}p_{50}\right)\left({}_8p_{60}\right) = 0.6065 \times 0.7261$

$= 0.4404$

${}_{4|14}q_{50} = {}_4p_{50} - {}_{18}p_{50} = 0.8187 - 0.4404 = 0.3783$

Question #34
Key: A

Model Solution:

X denotes the loss variable.
X_1 denotes Pareto with $\alpha = 2$; X_2 denotes Pareto with $\alpha = 4$

$$F_X(200) = 0.8\,F_{X_1}(200) + 0.2\,F_{X_2}(200)$$

$$= 0.8\left[1 - \left(\frac{100}{200+100}\right)^2\right] + 0.2\left[1 - \left(\frac{3000}{3000+200}\right)^4\right]$$

$$= 1 - 0.8\left(\frac{1}{3}\right)^2 - 0.2\left(\frac{15}{16}\right)^4$$

$$= 0.7566$$

Question #35
Key: D

$$\ddot{a}_{40:\overline{5}|} = \ddot{a}_{40} - {}_5E_{40}\,\ddot{a}_{45}$$
$$= 14.8166 - (0.73529)(14.1121)$$
$$= 4.4401$$

$$\pi\ddot{a}_{40:\overline{5}|} = 100{,}000 A_{45}\ v^5\ {}_5p_{40} + \pi\,(IA)^1_{40:\overline{5}|}$$

$$\pi = 100{,}000 A_{45} \times {}_5E_{40} \Big/ \left(\ddot{a}_{40:\overline{5}|} - (IA)^1_{40:\overline{5}|}\right)$$
$$= 100{,}000(0.20120)(0.73529)/(4.4401 - 0.04042)$$
$$= 3363$$

Question #36
Key: B

Calculate the probability that both are alive or both are dead.
P(both alive) = ${}_k p_{xy} = {}_k p_x \cdot {}_k p_y$
P(both dead) = ${}_k q_{\overline{xy}} = {}_k q_x \, {}_k q_y$
P(exactly one alive) = $1 - {}_k p_{xy} - {}_k q_{\overline{xy}}$
Only have to do two year's worth so have table

Pr(both alive)	Pr(both dead)	Pr(only one alive)
1	0	0
(0.91)(0.91) = 0.8281	(0.09)(0.09) = 0.0081	0.1638
(0.82)(0.82) = 0.6724	(0.18)(0.18) = 0.0324	0.2952

$$APV\ Annuity = 30{,}000\left(\frac{1}{1.05^0} + \frac{0.8281}{1.05^1} + \frac{0.6724}{1.05^2}\right) + 20{,}000\left(\frac{0}{1.05^0} + \frac{0.1638}{1.05^1} + \frac{0.2952}{1.05^2}\right) = 80{,}431$$

Alternatively,

$$\ddot{a}_{xy} = 1 + \frac{0.8281}{1.05} + \frac{0.6724}{1.05^2} = 2.3986$$

$$\ddot{a}_x = \ddot{a}_y = 1 + \frac{0.91}{1.05} + \frac{0.82}{1.05^2} = 2.6104$$

$$APV = 20{,}000\,\ddot{a}_x + 20{,}000\,\ddot{a}_y - 10{,}000\,\ddot{a}_{xy}$$

(it pays 20,000 if *x* alive and 20,000 if y alive, but 10,000 less than that if both are alive)

$$= (20{,}000)(2.6104) + (20{,}000)(2.6104) - (10{,}000)2.3986$$

$$= 80{,}430$$

Other alternatives also work.

Question #37
Key: C

Let *P* denote the contract premium.

$$P = \bar{a}_x = \int_0^\infty e^{-\delta t} e^{-\mu t} dt = \int_0^\infty e^{-0.05t} dt = 20$$

$$E[L] = \bar{a}_x^{IMP} - P$$

$$\bar{a}_x^{IMP} = \int_0^{10} e^{-0.03t} e^{-0.02t} dt + e^{-0.03(10)} e^{-0.02(10)} \int_0^\infty e^{-0.03t} e^{-0.01t} dt$$

$$= \frac{l - e^{-0.5}}{0.05} + \frac{e^{-0.5}}{0.04} = 23$$

$$E[L] = 23 - 20 = 3$$

$$\frac{E[L]}{P} = \frac{3}{20} = 15\%$$

Question #38
Key: C

$$A^1_{30:\overline{2}|} = 1000 v q_{30} + 500 v^2 {}_{1|}q_{30}$$

$$= 1000\left(\frac{1}{1.06}\right)(0.00153) + 500\left(\frac{1}{1.06}\right)^2 (0.99847)(0.00161)$$

$$= 2.15875$$

Initial fund $= 2.15875 \times 1000$ participants = 2158.75

Let F_n denote the size of Fund 1 at the end of year *n*.

$$F_1 = 2158.75(1.07) - 1000 = 1309.86$$

$$F_2 = 1309.86(1.065) - 500 = 895.00$$

Expected size of Fund 2 at end of year 2 = 0 (since the amount paid was the single benefit premium). Difference is 895.

Question #39
Key: B

Let *c* denote child; *ANS* denote Adult Non-Smoker; *AS* denote Adult Smoker.

$$P(3|c)P(c) = \frac{3^3 e^{-3}}{3!} \times 0.3 = 0.067$$

$$P(3|ANS)P(ANS) = \frac{1e^{-1}}{3!} \times 0.6 = 0.037$$

$$P(3|AS)P(AS) = \frac{4^3 e^{-4}}{3!} \times 0.1 = 0.020$$

$$P(AS|N=3) = \frac{0.020}{(0.067 + 0.037 + 0.020)} = 0.16$$

Question #40
Key: C

$$E[S] = E[N]E[X] = 3 \times 10 = 30$$

$$Var(S) = E[N]Var(X) + E[X]^2 Var(N)$$

$$= 3 \times \frac{400}{12} + 100 \times 3.6 = 100 + 360 = 460$$

For 95th percentile, $E[S] + 1.645\sqrt{Var(S)} = 30 + \sqrt{460} \times 1.645 = 65.28$